# THE HEIGHT OF CM POINTS ON ORTHOGONAL SHIMURA VARIETIES AND COLMEZ'S CONJECTURE

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## 1. INTRODUCTION

These are the notes for a 6 hours course given by the author during the summer school "Géométrie d'Arakelov", organized by the Institut Fourier (Grenoble) in the summer of 2017. They are based on the paper "Faltings' Heights of Abelian Varieties with Complex Multiplication" [AGHMP2] by myself, Eyal Goren, Ben Howard and Keerthi Madapusi Pera and on notes by myself and Eyal Goren. No new results are presented. The goal is to describe the strategy to reduce the proof of an averaged version of Colmez's conjecture to a conjecture of Bruinier, Kudla and Yang, an instance of what is known as the *Kudla's program*. Infact a weak version of this conjecture, which is proven in [AGHMP2] and is briefly discussed in the present text, implies this weaker version of Colmez's conjecture. Note that the latter has been used by Tsimerman [Ts] to provide an unconditional proof of the André-Oort conjecture for abelian varieties of Hodge type. Around the same time as [AGHMP2] also X. Yuan and S.-W. Zhang [YZ] proved, using different techniques, the averaged form of Colmez's conjecture.

## 2. THE AVERAGED COLMEZ'S CONJECTURE

Let $E \subset \mathbb{C}$ be a CM field of degree $2d$ with a totally real subfield $F$ of degree $d$. Let $A$ be an abelian variety over $\mathbb{C}$ of dimension $d$ with an action of the ring of integers $\mathcal{O}_E$ of $E$. A theorem of Shimura and Taniyama, see [CCO, Thm. 1.7.2.1], implies that $A$ can be defined over a number field $K$. Furthermore $A$ has potentially everywhere good reduction thanks to [ST, Thm. 6]. Thus, possibly after enlarging $K$, we may further assume that $A$ extends to an abelian scheme $\mathcal{A}$ over the ring of integers $\mathcal{O}_K$ of $K$.

### 2.1. **Faltings' height.**
We denote by $\omega_{\mathcal{A}}$ the $\wedge^d$-power of the relative invariant differentials $\mathrm{H}^0(\mathcal{A}, \Omega^1_{\mathcal{A}/\mathcal{O}_K})$ of $\mathcal{A}$. It is a projective $\mathcal{O}_K$-module of rank 1. Given a generator $s$ of $\omega_{\mathcal{A}} \otimes_{\mathcal{O}_K} K$ define

$$h_\infty^{\mathrm{Falt}}(A, s) = \frac{-1}{2[K:\mathbb{Q}]} \sum_{\sigma \colon K \to \mathbb{C}} \log |\int_{A^\sigma(\mathbb{C})} s^\sigma \wedge \overline{s^\sigma}|,$$

where for every embedding $\sigma \colon K \to \mathbb{C}$ we let $A^\sigma$ and $s^\sigma$ be the base change of $A$ and the section $s$ to $\mathbb{C}$. When we write $A^\sigma(\mathbb{C})$ we consider the underlying complex analytic structure so that the integral makes sense. Define

$$h_f^{\mathrm{Falt}}(A, s) = \frac{1}{[K:\mathbb{Q}]} \sum_{\mathfrak{p} \subset \mathcal{O}_K} \mathrm{ord}_{\mathfrak{p}}(s) \cdot \log \mathrm{N}(\mathfrak{p}),$$

where $\mathrm{ord}_{\mathfrak{p}}(s)$ is defined as the order with respect to a generator of $\omega_{\mathcal{A}} \otimes_{\mathcal{O}_K} \mathcal{O}_{K,\mathfrak{p}}$ as $\mathcal{O}_{K,\mathfrak{p}}$-module. Finally define

$$h^{\mathrm{Falt}}(A) = h_f^{\mathrm{Falt}}(A,s) + h_\infty^{\mathrm{Falt}}(A,s),$$

the *Faltings' height* or *modular height* of $A$.

*Remark* 2.1. As the notation suggests the quantity $h^{\mathrm{Falt}}(A)$ is independent of the choice of the section $s$ thanks to the product formula. Thanks to the normalization factor $\frac{1}{[K:\mathbb{Q}]}$ it is also invariant under field extension.

The Faltings' height can also be defined as the arithmetic degree of the metrized line bundle $\widehat{\omega}_{\mathcal{A}}$ over $\mathrm{Spec}(\mathcal{O}_K)$ as in Example 1.8, Ch. VII, where $\omega_{\mathcal{A}}$ is the underlying $\mathcal{O}_K$-module and the metrics at infinity are defined using integration as above.

2.2. **Colmez's theorem.** The fact that $A$ has an action of $\mathcal{O}_E$ singles out a subset $\Phi \subset \mathrm{Hom}(E, \mathbb{C})$: the action of $E$ on $\mathrm{H}^0(A, \Omega^1_{A/\mathbb{C}})$ decomposes into a sum of $d$ one dimensional eigenspaces for the action of $E$ and on each of them $E$ acts via an embedding $E \to \mathbb{C}$. We let $\Phi$ be the subset of embeddings $E \subset \mathbb{C}$ appearing in this way. It is called the *CM type* of $A$. It is a subset of cardinality $d$ and $\mathrm{Hom}(E, \mathbb{C}) = \Phi \amalg \overline{\Phi}$ (here $\overline{\Phi}$ stands for the image of $\Phi$ under complex conjugation on $\mathbb{C}$) . We then have the following Theorem of Colmez, [Col, Théorème 0.3]:

**Theorem 2.2.** *Under the assumption that $A$ is an abelian variety of dimension $d$ with action of the ring of integers $\mathcal{O}_E$ of $E$ and with CM type $\Phi$, the Faltings' height $h^{\mathrm{Falt}}(A)$ depends only on the pair $(E, \Phi)$, and not on the choice of the abelian variety $A$.*

We write $h_{(E,\Phi)}^{\mathrm{Falt}}$ for the quantity $h^{\mathrm{Falt}}(A)$ as it is independent of the choice of $A$.

In the same paper Colmez provided a conjectural formula, see [Col, Conj. 0.4], that computes $h_{(E,\Phi)}^{\mathrm{Falt}}$ in terms special values of $L$-functions of Artin characters. When $d = 1$, so that $E$ is a quadratic imaginary field, Colmez's conjecture is a form of the famous *Chowla-Selberg formula*:

$$h_{(E,\Phi)}^{\mathrm{Falt}} = -\frac{1}{2}\frac{L'(\varepsilon,0)}{L(\varepsilon,0)} - \frac{1}{2}\log(2\pi) - \frac{1}{4}\log|D_E|,$$

where $D_E$ is the discriminant of $E$ and $L(\varepsilon, s)$ is the Hecke $L$-function associated to the quadratic character $\varepsilon$ defined by the quadratic extension $\mathbb{Q} \subset E$.

In these lectures we'll be interested in an averaged form of Colmez's conjecture where we fix the CM field $E$ but we sum over the set $\mathrm{CM}(E)$ of all CM types $\Phi$ (i.e., subsets $\Phi$ of $\mathrm{Hom}(E, \mathbb{C})$ of cardinality $d$ such that $\mathrm{Hom}(E, \mathbb{C}) = \Phi \amalg \overline{\Phi}$). His conjectural formula amounts to the following:

**Colmez's averaged conjecture:**

$$\frac{1}{2^d} \sum_{\Phi \in \mathrm{CM}(E)} h_{(E,\Phi)}^{\mathrm{Falt}} = -\frac{1}{2} \cdot \frac{L'(\chi,0)}{L(\chi,0)} - \frac{1}{4} \cdot \log\left|\frac{D_E}{D_F}\right| - \frac{d}{2} \cdot \log(2\pi)$$

$$= -\frac{1}{2} \cdot \frac{\Lambda'(\chi,0)}{\Lambda(\chi,0)} - \frac{d}{4}\log(16\pi^3 e^\gamma),$$

where $\chi \colon \mathbb{A}_F^\times \to \{\pm 1\}$ is the quadratic Hecke character determined by the extension $E/F$, $L(\chi, s)$ is the associated $L$-function, $\Lambda(\chi, s) = |\frac{D_E}{D_F}|^{s/2} \Gamma_{\mathbb{R}}(s+1)^d L(\chi, s)$ is the completed $L$-function (so that $\Lambda(\chi, 1-s) = \Lambda(\chi, s)$), $D_E$ and $D_F$ are the discriminants of $E$ and $F$ respectively and $\gamma = -\Gamma'(1)$ is the Euler-Mascheroni constant.

### 2.3. Colmez's conjecture.

Let us reformulate Colmez's conjecture more precisely. Let $\mathbb{Q}^{\mathrm{CM}} \subset \mathbb{C}$ be the composite of all algebraic CM extensions of $\mathbb{Q}$. It is a Galois extension of $\mathbb{Q}$, containing the composite of all algebraic abelian extensions of $\mathbb{Q}$. Denote by $\mathcal{G} = \mathrm{Gal}(\mathbb{Q}^{\mathrm{CM}}/\mathbb{Q})$ its Galois group; write $c \in \mathcal{G}$ for the complex conjugation. Let $\mathcal{CM}^0$ be the $\mathbb{Q}$-vector space of locally constant, *central* functions (i.e., functions constant on conjugacy classes) $a \colon \mathcal{G} \to \mathbb{Q}$ such that the function $\mathcal{G} \ni g \mapsto a(g) + a(cg)$ is constant. Any such $a$ is a $\mathbb{C}$-linear combination $a = \sum_\eta a(\eta)\eta$ of Artin characters. Since $c \in \mathcal{G}$ is a central element any such character satisfies $\eta(c) = \pm\eta(\mathrm{Id})$. The assumption that $a(g) + a(cg)$ is constant implies that for all non-trivial $\eta$ for which $a(\eta) \neq 0$ we have $\eta(c) = -\eta(\mathrm{Id})$ so that $L(\eta, 0) \neq 0$. In particular to $a \in \mathcal{CM}^0$ we can associate the complex number

$$Z(a) = -\sum_\eta a(\eta) \left( \frac{L'(\eta, 0)}{L(\eta, 0)} + \frac{\log(f_\eta)}{2} \right),$$

where $f_\eta$ is the Artin conductor of $\eta$.

Now start with a CM field $E \subset \mathbb{Q}^{\mathrm{CM}}$ and a CM type $\Phi$. Define the locally constant function on $\mathcal{G}$:

$$a_{(E,\Phi)}(\sigma) = |\Phi \cap \sigma \circ \Phi| \quad \forall \sigma \in \mathcal{G}.$$

The average

$$a^0_{(E,\Phi)} = \frac{1}{[\mathcal{G} : \mathrm{Stab}(\Phi)]} \sum_{\tau \in \mathcal{G}/\mathrm{Stab}(\Phi)} a_{(E,\tau\Phi)}$$

lies in $\mathcal{CM}^0$. In fact

$$a_{(E,\Phi)}(\sigma) + a_{(E,\Phi)}(c\sigma) = |\Phi|$$

is independent of $\sigma \in \mathcal{G}$ and hence also $a^0_{(E,\Phi)}(\sigma) + a^0_{(E,\Phi)}(c\sigma)$. Here $\mathrm{Stab}(\Phi) \subset \mathcal{G}$ is the subgroup stabilizing the CM type $\Phi$. We then have the following conjecture.

**Colmez's conjecture:** If $A$ is an abelian variety with CM by the ring of integers $\mathcal{O}_E$ of $E$ and with CM type $\Phi$, we have

$$h^{\mathrm{Falt}}_{(E,\Phi)} = Z(a^0_{(E,\Phi)}).$$

Colmez verified the correctness of his conjecture, up to rational multiples of $\log(2)$, for $E$ an abelian extension of $\mathbb{Q}$. Obus [Ob] removed this error term for abelian extensions. When $d = 2$, Yang [Ya] was the first to prove the formula for non-abelian extensions.

### 2.4. Some consequences and reduction steps.

We will use this (conjectural) combinatorial expression in two ways. First of all one can compute the following equality of virtual representations

$$\frac{1}{[E:\mathbb{Q}]} \sum_\Phi a^0_{(E,\Phi)} = 2^{d-2} \left( \mathbf{1} + \frac{1}{d} \mathrm{Ind}^{\mathcal{G}}_{\mathcal{G}_F}(\chi) \right),$$

which provides upon taking $Z(\_)$ the expression appearing on the right hand side of Colmez's averaged conjecture. Here $\mathcal{G}_F = \mathrm{Gal}(\mathbb{Q}^{\mathrm{CM}}/F)$. In fact

- $\log(2\pi) = \frac{\zeta'(0)}{\zeta(0)}$ is the value at $s = 0$ of the logarithmic derivative of the Riemann zeta function which is the $L$-function associated to the trivial character;
- the $L$-function of the induced representation $\mathrm{Ind}^{\mathcal{G}}_{\mathcal{G}_F}(\chi)$ is the $L$-function of $\chi$ (over the field $F$);
- the Artin conductor of $\mathrm{Ind}^{\mathcal{G}}_{\mathcal{G}_F}(\chi)$ is $|D_E/D_F|$.

Secondly, we relate the averaged sum over the CM types to the height of the total reflex algebra $E^\sharp$ and the reflex CM type $\Phi^\sharp$. This will play a crucial role in the sequel. We let $E^\sharp$ be the étale $\mathbb{Q}$-algebra defined, via Grothendieck's formalism of Galois theory, by the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-set $\mathrm{CM}(E)$ of all CM types on $E$, i.e., $E^\sharp$ is characterized by the fact that $\mathrm{Hom}(E^\sharp, \overline{\mathbb{Q}}) \cong \mathrm{CM}(E)$ as $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-sets. We let $\Phi^\sharp \subset \mathrm{Hom}(E^\sharp, \overline{\mathbb{Q}})$ consist of all $\Phi \in \mathrm{CM}(E)$ such that the given embedding $\iota_0 \colon E \to \mathbb{C}$ lies in $\Phi$. In fact one can prove that $E^\sharp = \prod_i E'_i$ is a product of CM fields (as many as the orbits of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\mathrm{CM}(E)$) and $\Phi^\sharp \subset \mathrm{Hom}(E^\sharp, \overline{\mathbb{Q}}) = \mathrm{II}_i \mathrm{Hom}(E'_i, \overline{\mathbb{Q}})$ is the disjoint union $\Phi^\sharp = \mathrm{II}_i \Phi'_i$ of CM types for the $E'_i$. Furthermore if $E'_i$ corresponds to the orbit of $\Phi_i \in \mathrm{CM}(E)$ then $(E, \Phi_i)$ and $(E'_i, \Phi'_i)$ is a reflex pair. Then

$$(1) \qquad a^0_{(E^\sharp, \Phi^\sharp)} = \frac{1}{[E:\mathbb{Q}]} \sum_{\Phi \in \mathrm{CM}(E)} a^0_{(E,\Phi)}.$$

Colmez further proves, see [Col, Théorème 0.3], the following result:

**Proposition 2.3.** *There exists a unique $\mathbb{Q}$-linear map* $\mathrm{ht} \colon \mathcal{CM}^0 \to \mathbb{R}$ *such that* $h^{\mathrm{Falt}}_{(E,\Phi)} = \mathrm{ht}(a^0_{(E,\Phi)})$.

We deduce that $h^{\mathrm{Falt}}_{(E^\sharp, \Phi^\sharp)} = \frac{1}{[E:\mathbb{Q}]} \sum_{\Phi \in \mathrm{CM}(E)} h^{\mathrm{Falt}}_{(E,\Phi)}$. Colmez's averaged conjecture amounts then to prove the following:

**Colmez's averaged conjecture revisited:**

$$[E:\mathbb{Q}] h^{\mathrm{Falt}}_{(E^\sharp, \Phi^\sharp)} = -\frac{1}{2} \cdot \frac{\Lambda'(\chi, 0)}{\Lambda(\chi, 0)} - \frac{d}{4} \log(16\pi^3 e^\gamma).$$

**Our strategy:** The Chowla-Selberg formula implies Colmez's averaged conjecture for $d = 2$. Therefore we may and will assume through this text that $d \geq 2$. We will define a certain normal scheme $\mathcal{Y}_0$, finite over $\mathrm{Spec}(\mathcal{O}_E)$, carrying an abelian scheme $\mathcal{A}^\sharp$ with action of $\mathcal{O}_{E^\sharp}$ and CM type $\Phi^\sharp$. In particular $\mathcal{Y}_0$ will carry the metrized line bundle $\widehat{\omega}_{\mathcal{A}^\sharp}$ whose degree, divided by the degree of $\mathcal{Y}_0$, will compute $h^{\mathrm{Falt}}_{(E^\sharp, \Phi^\sharp)}$ by definition.

On the other hand, for $L \subset E$ a suitable lattice, we will also define auxiliary morphisms $\mathcal{Y}_L \to \mathcal{M}_L$ where $\mathcal{Y}_L \to \mathcal{Y}_0$ is a finite morphism and $\mathcal{M}_L$ are certain models of Shimura varieties of orthogonal type. The key point is that the metrized line bundle $\omega_{\mathcal{A}^\sharp}$ over $\mathcal{Y}_L$ is the pull–back of a combination of *arithmetic special divisors* on $\mathcal{M}_L$, at least away from a finite set of primes depending on $L$ that we

will denote $D_L$. Using this and work of Bruinier, Kudla and Yang [BKY] we will get a way to compute $h^{\mathrm{Falt}}_{(E^\sharp, \Phi^\sharp)}$ as the arithmetic intersection between these special divisors and $\mathcal{Y}_L$. This provides the RHS in the formula for Colmez's averaged conjecture (revisited), up to rational linear combination of $\log p$ for $p \in D_L$. As we may choose lattices so that $\cap_L D_L = \emptyset$ and logarithms of primes are linearly independent, Colmez's averaged conjecture follows.

## 3. Shimura varieties of orthogonal type and CM cycles

3.1. **GSpin Shimura varieties.** Let $V$ be a $\mathbb{Q}$-vector space of dimension $n + 2$ with $n \geq 0$, equipped with a quadratic form

$$Q \colon V \to \mathbb{Q}$$

which is non degenerate, of signature $(n, 2)$. Consider the associated bilinear form

$$[\_, \_] \colon V \times V \to \mathbb{Q}, \quad [x, y] = Q(x + y) - Q(x) - Q(y).$$

We have the associated Clifford algebra $C(V) = C(V, Q)$. It is a $\mathbb{Q}$-algebra, with an inclusion

$$V \hookrightarrow C(V)$$

satisfying the following universal property: for any $\mathbb{Q}$ algebra $R$ with a $\mathbb{Q}$-linear map $j \colon V \to R$ such that

$$j(v)j(v) = Q(v)$$

there exists a unique homomorphism of $\mathbb{Q}$-algebras

$$C(V) \to R$$

such that the composite with the inclusion $V \subset C(V)$ is $j$. In particular for any $v$ and $w \in V$, we have

$$v \cdot w + w \cdot v = [v, w] \in C(V),$$

where $v \cdot w$ (and $w \cdot v$) is the product in $C(V)$.

The construction of the Clifford algebra is quite straightforward. In fact,

$$C(V) := (\bigoplus_{n=0}^{\infty} V^{\otimes m})/(v \otimes v - Q(v) | v \in V),$$

the quotient of the tensor algebra of $V$ by the two sided ideal generated by the elements $v \otimes v - Q(v)$ for $v \in V$. As such an ideal is generated by elements lying in even degree (in the tensor algebra considered with its natural grading), the $\mathbb{Z}/2\mathbb{Z}$-grading on the tensor algebra (into even and odd tensors) passes to the Clifford algebra that correspondingly splits into a direct sum

$$C(V) = C^+(V) \oplus C^-(V).$$

Note that $C^+(V)$ is a subalgebra while $C^-(V)$ is just a two-sided module for $C^+(V)$. Furthermore we have the following formulas:

$$\dim_{\mathbb{Q}}(C(V)) = 2^{n+2}, \quad \dim^+_{\mathbb{Q}}(C(V)) = \dim_{\mathbb{Q}}(C^-(V)) = 2^{n+1}.$$

(Recall that $V$ has dimension $n + 2$.)

Next we construct the algebraic group $\mathrm{GSpin}(V, Q)$. Given a commutative $\mathbb{Q}$-algebra $R$, its $R$-valued points are

$$\mathrm{GSpin}(V)(R) := \{x \in (C^+(V) \otimes_{\mathbb{Q}} R)^* : x(V \otimes_{\mathbb{Q}} R)x^{-1} \subset V \otimes_{\mathbb{Q}} R\}.$$

In particular any $x \in \mathrm{GSpin}(V)(R)$ acts on $V \otimes_{\mathbb{Q}} R$ and, since for any $y \in V \otimes_{\mathbb{Q}} R$

$$Q(y) = Q(xyx^{-1}),$$

this action factors through the orthogonal group $\mathrm{O}(V, Q)(R)$. Notice that the units of $R$ form a subgroup of the center of $\mathrm{GSpin}(V)(R)$ and in particular they act trivially on $V \otimes_{\mathbb{Q}} R$. Furthermore, one can prove that the action of $\mathrm{GSpin}(V, Q)$ on $V$ factors through the special orthogonal group $\mathrm{SO}(V, Q)$ and one gets an exact sequence of algebraic groups:

$$(2) \qquad\qquad 0 \longrightarrow \mathbb{G}_m \longrightarrow \mathrm{GSpin}(V) \longrightarrow \mathrm{SO}(V, Q) \longrightarrow 0$$

## 3.2. Examples of GSpin groups.

3.2.1. *Example I: the case $n = 0$.* In this case

$$V = \mathbb{Q}e_1 \oplus \mathbb{Q}e_2,$$

So $C^-(V) = V$ and $C^+(V)$ is an algebra of dimension 2:

$$C^+(V) = \mathbb{Q} \oplus \mathbb{Q}e_1 \cdot e_2.$$

Denote $x = e_1 \cdot e_2$, let $a_1 = Q(e_1)$ and $a_2 = Q(e_2)$. They are negative rational numbers and if we write $b = [e_1, e_2] \in \mathbb{Q}$, we have

$$x^2 = e_1 e_2 e_1 e_2 = -e_1^2 e_2^2 + [e_1, e_2]e_1 e_2 = -a_1 a_2 + bx,$$

so we have

$$x^2 - bx + a_1 a_2 = 0.$$

As an algebra, this gives

$$C^+(V) = \mathbb{Q}[x]/(x^2 - bx + a_1 a_2),$$

which is an imaginary quadratic field $K$, and we see that

$$\mathrm{GSpin}(V) = C^+(V)^\times = \mathrm{Res}_{K/\mathbb{Q}} \mathbb{G}_m.$$

In particular its base change to $\mathbb{R}$ is the so-called Deligne torus $\mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m,\mathbb{R}}$.

3.2.2. *Example II: the case $n = 1$.* Consider the $\mathbb{Q}$-vector space

$$V \subset M_{2 \times 2}(\mathbb{Q}) = \{x \in M_{2 \times 2} : Tr(x) = 0\}.$$

Fix some $N \in \mathbb{N}$ such that $N \geq 1$. Let $Q_N$ be the quadratic form

$$A \mapsto N \cdot \det A.$$

Then

$$\mathrm{GSpin}(V) \cong \mathrm{GL}_2,$$

where $\mathrm{GL}_2$ acts on $V$ by conjugation.

3.3. **Hermitian symmetric spaces.** Fix the algebraic group $G := \mathrm{GSpin}(V, Q)$. As a first step in order to construct a Shimura variety, we need to construct a Hermitian symmetric space; see §2 of Ch. X for this notion. It admits several realizations:

1. as a complex manifold $D_{\mathbb{C}} = \{z \in V_{\mathbb{C}} \backslash \{0\} : Q(z) = 0, [z, \bar{z}] < 0\}/\mathbb{C}^* \subset \mathbb{P}(V_{\mathbb{C}})$
2. as a Riemannian manifold $D_{\mathbb{R}} = \{$ Negative definite oriented planes $H \subset V_{\mathbb{R}}\}$
3. using Deligne torus $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{R}}$, and realizing
$$D = G(\mathbb{R}) \text{ conjugacy class of } h \colon \mathbb{S} \to G_{\mathbb{R}}.$$

In fact, there are natural bijections between $D_{\mathbb{C}}$, $D_{\mathbb{R}}$ and $D$. Let us explain how we can go back and forth between these incarnations.

Given $H = \mathbb{R}e_1 \oplus \mathbb{R}e_2$ as in 2), with $\{e_1, e_2\}$ an orthogonal basis with $Q(e_1) = Q(e_2)$, we let $z = e_1 + ie_2$, then take the line $[z]$ to get the realization 1). To get realization 3), we simply take $\mathbb{S} \cong \mathrm{GSpin}H \hookrightarrow G_{\mathbb{R}}$ using that $H \subset V_{\mathbb{R}} := V \otimes_{\mathbb{Q}} \mathbb{R}$.

**Example I.** In the case discussed in §3.2.1 the Hermitian space consists of two points
$$D_{\mathbb{R}} = \{ \text{ two possible orientations on } V_{\mathbb{R}} = \mathbb{R}e_1 \oplus \mathbb{R}e_2\}.$$

**Example II.** In the case of §3.2.2 we have
$$D_{\mathbb{R}} \cong \mathbb{H}^+ \sqcup \mathbb{H}^- \subset \mathbb{C}$$
which are the Poincaré upper and lower half planes. The inverse of the map is given as
$$\mathbb{R} \cdot Re \begin{pmatrix} z & -z^2 \\ 1 & -z \end{pmatrix} \oplus \mathbb{R} \cdot \mathrm{im} \begin{pmatrix} z & -z^2 \\ 1 & -z \end{pmatrix} \leftarrow z = x + iy.$$

Pick $[z] \in D_{\mathbb{C}}$, then
$$V_{\mathbb{C}} = \mathbb{C}z \oplus (\mathbb{C}z \oplus \mathbb{C}\bar{z})^{\perp} \oplus \mathbb{C}\bar{z}$$
and the tangent space of $Q$ in $\mathbb{P}(V_{\mathbb{C}})$ at $[z]$ can be computed as the Zariski tangent space at $[z]$, namely the set of lines $[z + \delta\epsilon + \gamma\epsilon\bar{z}]$, with $\delta \in (\mathbb{C}z \oplus \mathbb{C}\bar{z})^{\perp}$ and $\epsilon$ a formal variable with square $\epsilon^2 = 0$, such that $Q([z + \delta\epsilon + \gamma\epsilon\bar{z}]) = 0$, i.e, if and only if $\gamma = 0$. Thus the tangent space of $D_{\mathbb{C}}$ at $[z]$ is isomorphic to $(\mathbb{C}z \oplus \mathbb{C}\bar{z})^{\perp}$ and $\dim D_{\mathbb{C}} = n$.

3.4. **GSpin-Shimura varieties.** Given $V$ and $D$ and the algebraic group $G := \mathrm{GSpin}(V, Q)$ as in the previous section, define the complex manifold:
$$M_K(\mathbb{C}) = G(\mathbb{Q})\backslash D \times G(\mathbb{A}_f)/K = \amalg_{g \in G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K}\Gamma_g\backslash D$$

for some compact open subgroup $K$ of the adelic points $G(\mathbb{A}_f)$ of $G$. Here $\Gamma_g := G(\mathbb{Q}) \cap (gKg^{-1})$ is an arithmetic subgroup of $G(\mathbb{Q})$. It has a natural structure of algebraic variety over $\mathbb{C}$.

Given a quadratic lattice $L \subset V$, i.e., a lattice on which $Q$ is integral valued, one can construct a compact open subgroup $K_L$ by taking
$$K_L = G(\mathbb{A}_f) \cap C^+(\widehat{L})^{\times} \subset C^+(V)^{\times}(\mathbb{A}_f),$$

where $\widehat{L} := L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$. We will be especially interested in the case that $L$ is maximal among the integral lattices. This will guarantee the existence of good integral

models over $\mathbb{Z}$ for the Shimura variety $M_{K_L}(\mathbb{C})$. If the compact open subgroup is of the type $K_L$ for some lattice $L$, we simply write $M_L(\mathbb{C})$.

Now let us look at the examples again.

**Example I.** $M(\mathbb{C})$ consists of finitely many points.

**Example II.** We have

$$V = M_{2 \times 2}(\mathbb{Q})^{Tr=0}$$

and using $Q_N$ we get $\mathrm{GSpin}(V) \cong \mathrm{GL}_2$. Take

$$L := \left\{ \begin{pmatrix} a & \frac{-b}{N} \\ c & -a \end{pmatrix} : a, b, c \in \mathbb{Z} \right\},$$

one can check that

$$K_L \cong \prod_p \widetilde{K}_p,$$

where

$$\widetilde{K}_p = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL_2(\mathbb{Z}_p) : \gamma \in N\mathbb{Z}_p \right\}.$$

In this case one sees that

$$M_2(\mathbb{C}) \cong Y_0(N)(\mathbb{C})$$

which is the modular curve of level $\Gamma_0(N)$, classifying cyclic isogenies $\rho \colon E \to E'$ of degree $N$ of elliptic curves.

**Warning:** The case of elliptic curves is misleading as it might appear that $M_L(\mathbb{C})$ has a natural moduli interpretation. This is not the case if the dimension $n + 2$ of $V$ is large. In this case $M_L(\mathbb{C})$ does *not* have, in general, a natural interpretation as a PEL type moduli problem, that classifies abelian varieties with given polarization, endomorphisms and level structures. As we will see, this is the source of complications when one attempts to provide integral models for $M_L(\mathbb{C})$.

## 4. Extra structures on GSpin-Shimura Varieties

Recall the notation from the previous sections: we fixed a vector space $V$ over $\mathbb{Q}$ of dimension $n + 2$, and a quadratic form $Q \colon V \to \mathbb{Q}$ of signature $(n, 2)$, with a maximal quadratic lattice $L \subset V$. We let $G$ be the algebraic group $\mathrm{GSpin}(V)$ and $D$ the associated Hermitian symmetric space. Then we defined

$$M_L(\mathbb{C}) = G(\mathbb{Q}) \backslash D \times G(\mathbb{A}_f) / K_L$$

for a particular choice of compact open subgroup $K_L$ associated to the lattice $L$.

We have a natural functor

{ Algebraic Representations of $G$} $\to$ { Local Systems of $\mathbb{Q}$-vector spaces on $M_L(\mathbb{C})$},

given by

$$(G \to GL(W)) \mapsto (W_{\mathrm{Betti},\mathbb{Q}} \longrightarrow M_L(\mathbb{C})),$$

where

$$W_{\mathrm{Betti},\mathbb{Q}} := G(\mathbb{Q}) \backslash (W \times D) \times G(\mathbb{A}_f) / K_L.$$

Note that this gives us a unique pair $(W_{\mathrm{dR}}, \nabla)$ of a locally free $\mathcal{O}_{M_L(\mathbb{C})}$-module with integral connection. Namely $W_{\mathrm{dR}} := W_{\mathrm{Betti},\mathbb{Q}} \otimes_{\mathbb{Q}} \mathcal{O}_{M_L(\mathbb{C})}$ with the connection

$\nabla = 1 \otimes d$. Then $(W_{\mathrm{dR}}, \nabla)$ is characterized as the vector bundle with integrable connection such that

$$W_{\mathrm{dR}}^{\nabla=0} = W_{\mathrm{Betti},\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}.$$

We further have the following extra properties:

a. For any $z$ in the symmetric space $D$, the map

$$h_z \colon \mathbb{S} \to G_{\mathbb{R}} \to GL(W_{\mathbb{R}}),$$

where $G_{\mathbb{R}} \to GL(W_{\mathbb{R}})$ is defined by the representation $G \to GL(W)$, induces a map

$$\mathbb{S}(\mathbb{C}) = \mathbb{C}^* \times \mathbb{C}^* \to \mathrm{GL}(W_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}) = \mathrm{GL}(W_{\mathbb{C}})$$

and the fiber $W_{\mathrm{dR},z}$ at $z$ has a bigraduation $\oplus_{p,q} W_{\mathrm{dR},z}^{p,q}$ obtained by the decomposition of $W_{\mathbb{C}}$ according to the action of $\mathbb{C}^* \times \mathbb{C}^*$.

b. $W_{\mathrm{dR}}$ is endowed with a decreasing filtration $\mathrm{Fil}^J(W_{\mathrm{dR}}) \subset W_{\mathrm{dR}}$ by holomorphic sub-bundles of $W_{\mathrm{dR}}$, defined pointwise by

$$\mathrm{Fil}^J(W_{\mathrm{dR},z}) := \oplus_{p \geq J} W_{\mathrm{dR},z}^{p,q}.$$

A $\mathbb{Q}$-local system on $M_L(\mathbb{C})$ with these properties is called a *variation of $\mathbb{Q}$-Hodge structures*. In particular, $W_{\mathrm{Betti},\mathbb{Q}}$ is a variation of $\mathbb{Q}$-Hodge structures.

4.1. **Example A: the representation $V$.** Consider in particular the homomorphism

$$G \to \mathrm{SO}(V),$$

given by

$$x \mapsto \{y \mapsto xyx^{-1}\}$$

as a map $V \to V$ inside $\mathrm{SO}(V)$. Now we get as before a variation of $\mathbb{Q}$-Hodge structures $V_{\mathrm{Betti}}$ and even a variation of $\mathbb{Z}$-Hodge structures $\mathbb{V}_{\mathrm{Betti}}$. In particular we get a vector bundle with connection $V_{\mathrm{dR}}$. They are all endowed with a quadratic form $Q_{\mathrm{Betti}}$ and $Q_{\mathrm{dR}}$ respectively.

For any $(z, g) \in M_L(\mathbb{C})$ with $z \in D_{\mathbb{C}}$ and $g \in G(\mathbb{A}_f)$, where $D_{\mathbb{C}}$ is the incarnation of the symmetric space as the isotropic lines in $V_{\mathbb{C}}$, the morphism $h_z$ defines a decomposition

$$V_{\mathbb{C}} = \mathbb{C}_z \oplus (\mathbb{C}_z \oplus \mathbb{C}_{\bar{z}})^{\perp} \oplus \mathbb{C}_{\bar{z}} \subset \mathrm{End}(H_{1,\mathrm{dR}}(A_z)).$$

The filtration is given by

$$\mathrm{Fil}^1(\mathbb{V}_{\mathrm{dR},z}) = \mathbb{C}z$$

$$\mathrm{Fil}^0(\mathbb{V}_{\mathrm{dR},z}) = \mathbb{C}z \oplus (\mathbb{C}z \oplus \mathbb{C}\bar{z})^{\perp}$$

$$\mathrm{Fil}^{-1}(\mathbb{V}_{\mathrm{dR},z}) = \mathbb{C}z \oplus (\mathbb{C}z \oplus \mathbb{C}\bar{z})^{\perp} \oplus \mathbb{C}\bar{z}.$$

4.2. **Example B: the Kuga-Satake abelian scheme.** Consider the representation $C(V)$ of $G = \mathrm{GSpin}(V)$ via the inclusion $G \subset C^+(V)^*$ and the map $C^+(V)^* \subset \mathrm{GL}(C(V))$ provided by left multiplication. It provides a variation of Hodge structures: for any $z \in D$, we have

$$C(V)_z = C(V)_z^{-1,0} \oplus C(V)_z^{0,-1};$$

this is equivalent to giving a complex structure on $C(V_{\mathbb{R}})$ and we obtain a complex abelian variety

$$A_z := C(V_{\mathbb{R}})/C(L),$$

called the *Kuga-Satake abelian variety*. In this way one gets an abelian scheme

$$\pi \colon A \longrightarrow M_L(\mathbb{C}).$$

The associated vector bundle with connection $C(V)_{\mathrm{dR}}$ coincides with the relative de Rham homology

$$C(V)_{\mathrm{dR}} = \mathcal{H}_{1,\mathrm{dR}}(A),$$

the connection is the so called *Gauss-Manin connection* and the filtration is given by the Hodge filtration

$$0 \longrightarrow \mathrm{R}_1\pi_*(\mathcal{O}_A)^\vee \to \mathcal{H}_{1,\mathrm{dR}}(A) \longrightarrow \pi_*(\Omega_A^1)^\vee \longrightarrow 0.$$

**Example I.** Recall that in this case $n = 0$ and $C^+(L) \subset C^+(V) = K$ is an order in the quadratic imaginary field $K$. Then

$$A_z = A_z^+ \times A_z^-,$$

where $A_z^+$ is an elliptic curve with complex multiplication by $C^+(L)$ and

$$A_z^- = A_z^+ \otimes_{C^+(L)} L.$$

**Example II.** In this case $V = M_{2\times 2}(\mathbb{Q})^{Tr=0}$, we have

$$M(\mathbb{C}) \cong Y_0(N)(\mathbb{C})$$

by $z \mapsto [E_z \to E_z']$ and

$$A_z = A_z^+ \times A_z^-$$

and

$$A_z^+ = A_z^- = E_z \times E_z'.$$

Consider $\ell \colon V \hookrightarrow \mathrm{End}(C(V))$ given by

$$v \mapsto \ell_v := \{ \text{ left multiplication by } v \text{ on } C(V)\}.$$

It is a morphism in the category of representation of $G$. Indeed, recall that $G$ acts on $C(V)$ via the inclusion into $C^+(V)^*$ and the left multiplication action of $C^+(V)^*$ on $C(V)$. The induced action of $C^+(V)^*$ on $\mathrm{End}(C(V))$ takes $\alpha \in C^+(V)^*$ and $g \in \mathrm{End}(C(V))$ to $\alpha g(\alpha^{-1}{}_{-}) \in \mathrm{End}(C(V))$ which restricted to $\ell_v$ gives $\alpha\ell_v(\alpha^{-1}(_{-})) = \ell_{\alpha v \alpha^{-1}}(_{-})$. Hence, we get a morphism of Hodge structures

$$(3) \hspace{4em} \ell_{\mathrm{Betti}} \colon \mathbb{V}_{\mathrm{Betti}} \hookrightarrow \mathrm{End}(\mathcal{H}_1(A,\mathbb{Q}))$$

and a morphism of vector bundles with connections over $M_L(\mathbb{C})$

$$(4) \hspace{4em} \ell_{\mathrm{dR}} \colon V_{\mathrm{dR}} \hookrightarrow \mathrm{End}(\mathcal{H}_{1,\mathrm{dR}}(A)).$$

## 5. THE BIG CM POINTS

We follow here [BKY]. Let $E$ be a CM field of degree $2d$ with a totally real subfield $F$ (of degree $d$). We label the embeddings $\{\sigma_0, \ldots, \sigma_{d-1}\} = \mathrm{Hom}(F, \mathbb{R})$. For every integer $0 \leq i \leq d-1$ label by $\sigma_i$ and $\overline{\sigma}_i \colon E \to \mathbb{C}$ the two conjugate embeddings of $E$ extending $\sigma_i \colon F \to \mathbb{R}$.

Let $\lambda \in F$ be an element such that $\sigma_0(\lambda) < 0$ and $\sigma_i(\lambda) > 0$ for $1 \leq i \leq d-1$. Consider the quadratic space $W = E$ of dimension 2 over $F$ with bilinear pairing

$$B_W \colon W \times W \longrightarrow F, \quad (x, y) \mapsto B_W(x, y) = \mathrm{Tr}_{E/F}(\lambda x \overline{y}).$$

It is negative definite at one place and positive definite at the others. Namely, for all embedding $\sigma_i \colon F \to \mathbb{R}$ with $i \neq 0$, writing $W_{\sigma_i} := W \otimes_F^{\sigma_i} \mathbb{R}$, the induced bilinear form $B_{\sigma_i} \colon W_{\sigma_i} \times W_{\sigma_i} \longrightarrow \mathbb{R}$ is positive definite for $i = 1, \ldots, d-1$, and it is negative definite at the remaining place $\sigma_0$. We let

$$Q_W(x) = \frac{1}{2} B_W(x, x)$$

be the associated quadratic form. The even Clifford algebra $C_F^+(W)$ over $F$ is identified with $E$.

One then lets

$$V = \mathrm{Res}_{F/\mathbb{Q}}(W).$$

That is, $V$ is simply $E$, as $\mathbb{Q}$-vector space of dimension $2d$ so that

$$V \otimes_{\mathbb{Q}} \mathbb{R} = W \otimes_F \mathbb{R} = \bigoplus_j W_{\sigma_j}$$

equipped with the form

$$B_V(x, y) = \mathrm{Tr}_{E/\mathbb{Q}}(\lambda x \overline{y}).$$

**Example.** Let $L = \mathfrak{a}$ be a fractional ideal of $E$. Let $\overline{L} := \overline{\mathfrak{a}}$ be the image of $\mathfrak{a}$ under complex conjugation. Then $B_V$ is integrally valued on $L$ if and only if $\lambda \mathfrak{a} \overline{\mathfrak{a}} \subset \mathfrak{D}_{E/\mathbb{Q}}^{-1}$. In this case $L^\vee = (\lambda \mathfrak{D}_{E/\mathbb{Q}} \overline{\mathfrak{a}})^{-1}$ and

$$(5) \qquad L^\vee / L \cong \mathcal{O}_E / \lambda \mathfrak{D}_{E/\mathbb{Q}} \mathrm{Norm}_{E/F}(\mathfrak{a})$$

**Definition 5.1.** Given a quadratic lattice $L \subset V$ we say that the prime $p$ is *good for $L$* if the following conditions hold. For every prime $\mathfrak{p}$ of $F$ over $p$ set

$$L_{\mathfrak{p}} = (L \otimes \mathbb{Z}_p) \cap V_{\mathfrak{p}}.$$

We demand that
- For every $\mathfrak{p} \mid p$ unramified in $E$, the $\mathbb{Z}_p$-lattice $L_{\mathfrak{p}}$ is $\mathcal{O}_{E,\mathfrak{p}}$-stable and self-dual for the induced $\mathbb{Z}_p$-valued quadratic form.
- For every $\mathfrak{p} \mid p$ ramified in $E$, the $\mathbb{Z}_p$-lattice $L_{\mathfrak{p}}$ is maximal for the induced $\mathbb{Z}_p$-valued quadratic form, and there exists an $\mathcal{O}_{E,\mathfrak{q}}$-stable lattice $\Lambda_{\mathfrak{p}} \subset V_{\mathfrak{p}}$ such that

$$\Lambda_{\mathfrak{p}} \subset L_{\mathfrak{p}} \subsetneq \mathfrak{d}_{E_{\mathfrak{q}}/F_{\mathfrak{p}}}^{-1} \Lambda_{\mathfrak{p}}.$$

  where $\mathfrak{q} \subset \mathcal{O}_E$ is the unique prime above $\mathfrak{p}$.

All but finitely many primes are good: Choose any $\mathcal{O}_E$-stable lattice $\Lambda \subset L$. Then, for all but finitely many primes $p$, $\Lambda_{\mathbb{Z}_p} = L_{\mathbb{Z}_p}$ will be self-dual and hence good. We let $D_{L,\mathrm{bad}}$ be the product of all primes that are *not* good for $L$.

For every $j = 0, \ldots, d-1$ we have $C^+(W_{\sigma_j}) \cong C_F^+(W) \otimes_F^{\sigma_j} \mathbb{R}$. There are maps

$$\prod_j C^+(W_{\sigma_j}) \longrightarrow \otimes_j C^+(W_{\sigma_j}) \longrightarrow C^+(\bigoplus_j W_{\sigma_j}) = C^+(V \otimes_{\mathbb{Q}} \mathbb{R}).$$

(The tensor product is over $\mathbb{R}$ or more generally any field over which $F$ splits completely.) The first map is multiplicative and multi-linear and the second map is a ring homomorphism. The ring $C_F^+(W)$ has a basis over $F$ given by $\{1, e_1 e_2\}$. A general element has the form $a + be_1 e_2$; via the isomorphism $C_F^+(W) \otimes_{\mathbb{Q}} \mathbb{R} \cong \prod_j C^+(W_{\sigma_j})$ and the maps above, the image of this element in $C^+(\bigoplus_j W_{\sigma_j})$ is $\prod_j (\sigma_j(a) + \sigma_j(b) e_1^j e_2^j)$ (product in the Clifford algebra). In particular, $a \in F$ is mapped to $\mathrm{Norm}_{F/\mathbb{Q}}(a)$ and $e_1 e_2$ to $\prod_j e_1^j e_2^j$.

Passing to GSpin, making use of $\mathrm{GSpin}_F(W) = C_F^+(W)^*$, we get a homomorphism of groups over $\mathbb{R}$ (or any field splitting $F$):

$$\text{(6)} \qquad\qquad \prod_j \mathrm{GSpin}_{\mathbb{R}}(W_{\sigma_j}) \longrightarrow \mathrm{GSpin}_{\mathbb{R}}(\bigoplus_j W_{\sigma_j}).$$

The homomorphism (6) descends to a homomorphism of algebraic groups over $\mathbb{Q}$

$$\text{(7)} \qquad\qquad g \colon \mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GSpin}_F(W)) \longrightarrow \mathrm{GSpin}(V)$$

(cf. [BKY, §2]). Let $T$ be the image in $\mathrm{GSpin}(V)$ of this homomorphism; it is a torus and there is an exact sequence (loc. cit.)

$$1 \longrightarrow T_F^{\mathrm{Nm}=1} \longrightarrow T_E \overset{g}{\longrightarrow} T \longrightarrow 1.$$

Here $T_E = \mathrm{Res}_{E/\mathbb{Q}}(\mathbb{G}_{m,E}) = \mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GSpin}_F(W))$ and $T_F^{\mathrm{Nm}=1}$ is the subgroup of norm 1 elements of $T_F = \mathrm{Res}_{F/\mathbb{Q}}(\mathbb{G}_{m,F})$.

Note that there is a unique, up to isomorphism, rank 2 non-split torus over $\mathbb{R}$ and, thus, there is an isomorphism

$$\text{(8)} \qquad\qquad h \colon \mathbb{S} \cong \mathrm{GSpin}(W_{\sigma_0}) \subset \mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GSpin}_F(W))(\mathbb{R}).$$

This gives a point $z_0 \in D_{\mathbb{R}}$. Fixing a quadratic lattice $L \subset V$ we get the Shimura variety $M_L(\mathbb{C})$ as explained in §3.4. Taking a compact open subgroup $K_T \subset K_L \cap T(\mathbb{A}_f)$ we can define the Shimura variety

$$Y_L(\mathbb{C}) = T(\mathbb{Q}) \backslash \{z_0\} \times T(\mathbb{A}_f)/K_T,$$

which consists of finitely many points, and a homomorphism

$$Y_L(\mathbb{C}) \to M_L(\mathbb{C}) = G(\mathbb{Q}) \backslash D \times G(\mathbb{A}_f)/K_L,$$

whose image is called the *big CM cycle* associated to $(E, \sigma_0, \lambda)$ (the specific subgroup $K_T$ we will consider depends only on $L$). It is defined over its reflex field $E$ and we let $\mathcal{Y}_L$ be the normalization of $\mathrm{Spec}(\mathcal{O}_E)$ in $Y_L$.

*Remark* 5.2. The world "big" suggests the existence of *small CM cycles* as well. This is the case and they are constructed starting from quadratic imaginary extensions of $\mathbb{Q}$ instead of a CM field extension of degree $n+2$. There are interesting conjectures in this setting as well in the spirit of Kudla's program, elaborated by Bruinier and Yang in [BY]. These conjectures have been proven under some mild assumptions in [AGHMP1].

5.1. **The total reflex algebra.** Write

$$(9) \qquad V_{\mathbb{C}} = E \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{\rho \in \mathrm{Hom}(E,\mathbb{C})} \mathbb{C} \cdot e_{\rho}.$$

where the $e_{\rho}$'s are orthogonal idempotents of the algebra $E \otimes_{\mathbb{Q}} \mathbb{C}$. We recall the following results from [Fi].

**Lemma 5.3.** *Define for $\rho \in \mathrm{Hom}(E,\mathbb{C})$ an element of $C^+(V_{\mathbb{C}})$,*

$$\delta_{\rho} = \frac{1}{\rho(\lambda)} e_{\rho} e_{\bar{\rho}}.$$

*There are $2d$ such elements $\delta_{\rho}$. They all commute. Furthermore $\delta_{\rho}^2 = \delta_{\rho}$ and $\delta_{\rho}\delta_{\bar{\rho}} = 0$.*

Denote by $\mathrm{CM}(E)$ the set of CM types on $E$.

**Lemma 5.4.** *For every $\phi \in \mathrm{CM}(E)$ define the following element of $C^+(V_{\mathbb{C}})$:*

$$\Delta_{\phi} := \prod_{\rho \in \mathrm{CM}(E)} \delta_{\rho}$$

(1) *There are $2^d$ such elements $\Delta_{\phi}$. They all commute.*
(2) $\Delta_{\phi}^2 = \Delta_{\phi}$.
(3) $\Delta_{\phi_1}\Delta_{\phi_2} = 0$ *if* $\phi_1 \neq \phi_2$.
(4) *for* $\alpha \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$, $\alpha(\Delta_{\phi}) = \Delta_{\alpha\phi}$.
(5) $\sum_{\phi \in \Phi} \Delta_{\phi} = 1$.

In particular the $\overline{\mathbb{Q}}$-span of the $\Delta_{\Phi}$'s in $C^+(V) \otimes \overline{\mathbb{Q}}$ is an étale subalgebra, corresponding to the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-set $\mathrm{CM}(E)$. Taking $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariants one realizes $E^{\sharp}$ as a $\mathbb{Q}$-subalgebra of $C^+(V)$.

Using (6), we also obtain a multiplicative homomorphism, called the *complete reflex norm*, that factors through $T$:

$$(10) \qquad T_E \longrightarrow T_{E^{\sharp}} \hookrightarrow \mathrm{GSpin}(V), \qquad \sum_{\rho} x_{\rho} e_{\rho} \mapsto \prod_{j} (x_{\rho_j} \delta_{\rho_j} + x_{\bar{\rho}_j} \delta_{\bar{\rho}_j}).$$

In particular the inclusion $T \subset G$ factors via $T_{E^{\sharp}}$. We denote by $V^{\sharp}$ the $\mathbb{Q}$-vector space $E^{\sharp}$, viewed as a $\mathbb{Q}$-representation of $T$. The induced map $T_E \to \mathrm{GL}(V^{\sharp})$ is given by sending $\alpha \in E^*$ to the automorphism $V^{\sharp} \to V^{\sharp}$ sending $\Delta_{\Phi}$ to $\prod_{\rho \in \Phi} \rho(\alpha) \Delta_{\Phi}$.

We further get a homomorphism:

$$(11) \qquad \ell \colon V \to \mathrm{End}(V^{\sharp})$$

defined by sending $e_{\rho} \in V_{\mathbb{C}} = E_{\mathbb{C}}$ to the $\mathbb{C}$-linear endomorphism of $V_{\mathbb{C}}^{\sharp} = \oplus_{\Phi \in \mathrm{CM}(E)} \mathbb{C}\Delta_{\Phi}$ given by $e_{\rho} \mapsto \sum_{\rho \in \Phi \in \mathrm{CM}(E)} \Delta_{\Phi} \otimes \Delta_{\Phi \cup \{\bar{\rho}\} \setminus \{\rho\}}^{\vee}$.

**Corollary 5.5.** *Let $h \colon \mathbb{S} \xrightarrow{\cong} \mathrm{GSpin}(W_{\sigma_0}) \subset \mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GSpin}_F(W))(\mathbb{R}) = (E \otimes_{\mathbb{Q}} \mathbb{R})^{\times}$ be as in (8). There is a unique way to choose $h$ so that $g \circ h(i)$, where $g$ is the homomorphism (7), is the element*

$$\varphi(i e_{\rho_0} - i e_{\bar{\rho}_0} + \sum_{\rho \notin \{\rho_0, \bar{\rho}_0\}} e_{\rho}).$$

*That is,*

$$g \circ h(i) = \sum_{\phi \in \Phi} \epsilon(\phi) \cdot i \cdot \Delta_\phi, \qquad \epsilon(\phi) = \begin{cases} 1 & \rho_0 \in \phi \\ -1 & \rho_0 \notin \phi. \end{cases}$$

*In the following, we shall denote $g \circ h \colon \mathbb{S} \longrightarrow \mathrm{GSpin}(V_{\mathbb{R}})$ simply by $h$.*

5.2. **Extra structure on the Big CM cycle.** We start with

**Lemma 5.6.** *The action of $E = C^+(W)$ on $W = C^-(W)$ gives an action of $E$ on $V$, as $V$ is equal to $W$ only viewed as a rational vector space. This action will be denoted $\beta \cdot v$ for $\beta \in E$ and $v \in V$. On the other hand, through the homomorphisms*

$$E^\times \xrightarrow{g} \mathrm{GSpin}(V) \xrightarrow{\pi} \mathrm{SO}(V),$$

*we get another action of $E^\times$ on $V$ that we call $\rho(\beta)v, \beta \in E, v \in V$  ($\rho = \pi \circ g$). The actions are related as follows:*

$$\rho(\beta)v = (\beta\bar\beta^{-1}) \cdot v.$$

*In particular the homomorphism $\ell$ of (11) is a homomorphism as representations of $T$.*

*Proof.* The two actions of $E$ on $W$ are related by the formula $\rho(\beta)w = (\beta\bar\beta^{-1}) \cdot w$; indeed, $\rho(\beta)w = \beta w \beta^{-1} = (\beta\bar\beta^{-1})w = (\beta\bar\beta^{-1}) \cdot w$. The lemma now follows applying restriction of scalars and the commutativity of the following diagram:

$$
\begin{array}{ccccc}
C^+(W) \otimes_{\mathbb{Q}} \mathbb{R} & \xrightarrow{\cong} & \prod_j C^+(W_j) & \longrightarrow & \mathrm{GSpin}(V) \\
 & & \downarrow & & \downarrow \\
 & & \prod_j \mathrm{SO}(W_j) & \longrightarrow & \mathrm{SO}(V).
\end{array}
$$

$\square$

**Proposition 5.7.** *The endomorphism ring of $V$ as a rational Hodge structure is precisely $E$ with the "dot action". Moreover, $E$, viewed in $\mathrm{End}(V)$ via the "dot action" is the $\mathbb{Q}$-linear span of the image of $T(\mathbb{Q})$.*

*Proof.* Let $T_E$ be the $\mathbb{Q}$-torus associated to $E$ and let $\gamma \colon T_E \longrightarrow T_E$ be the morphism given by $\alpha \mapsto \alpha/\bar\alpha$ on $\mathbb{C}$-points. This morphism is defined over $\mathbb{Q}$ and its image, by Lemma 5.6, is nothing but the image of $T(\mathbb{Q})$ in $\mathrm{GL}(V)$. One can check that $E$ is the $\mathbb{Q}$-span of the elements of $T(\mathbb{Q})$, namely $\{\alpha/\bar\alpha : \alpha \in E^\times\}$.            $\square$

Notice that $V$ has the extra structure of $E$-vector space that defines endomorphisms of $V_{\mathrm{Betti}}|_{Y(\mathbb{C})}$. In fact, for any $z \in Y(\mathbb{C})$ the $E$ action on $V_{\mathrm{Betti},\mathbb{Q},z}$ induces a decomposition

$$V_{\mathrm{Betti},\mathbb{Q},z} \otimes_{\mathbb{Q}} \mathbb{C} = V_{\mathrm{dR},z} = \oplus_{i=0}^d V_{\mathrm{dR},z}(\sigma_i) \oplus V_{\mathrm{dR},z}(\bar\sigma_i),$$

where $V_{\mathrm{dR},z}(\sigma)$ is the 1-dimensional $\mathbb{C}$-vector space on which $E$ acts via $\sigma \colon E \to \mathbb{C}$. Then

$$\mathrm{Fil}^1 V_{\mathrm{dR},z} = V_{\mathrm{dR},z}(\sigma_0), \mathrm{Gr}^{-1} V_{\mathrm{dR},z} = V_{\mathrm{dR},z}(\bar\sigma_0)$$

and

$$\mathrm{Gr}^0 V_{\mathrm{dR},z} = \oplus_{i=1}^d V_{\mathrm{dR},z}(\sigma_i) \oplus V_{\mathrm{dR},z}(\bar\sigma_i).$$

We summarize our findings. Write $V := E$, considered as $\mathbb{Q}$-vector space. Then:

- there is a morphism $T_E \to \mathrm{GSpin}(V)$, factoring through $T$, where $\alpha \in T_E(\mathbb{Q}) = E^\times$ acts on $V = E$ through multiplication by $\alpha\bar{\alpha}^{-1}$;
- we can realize $E^\sharp$ as a $\mathbb{Q}$-subalgebra of $C^+(V)$ such that the homomorphism $T \to \mathrm{GSpin}(V)$ factors via the subtorus $T_{E^\sharp} \subset \mathrm{GSpin}(V)$.
- there is a morphism $h\colon \mathbb{S} \to T$ inducing a Hodge structure on $V$ where

$$\mathrm{Fil}^1 V_\mathbb{C} = E \otimes_E^{\sigma_0} \mathbb{C}, \mathrm{Gr}^{-1} V_\mathbb{C} = E \otimes_E^{\bar{\sigma}_0} \mathbb{C}, \mathrm{Gr}^0 V_\mathbb{C} = \oplus_{i=1}^{d-1} (E \otimes_E^{\sigma_i} \mathbb{C} \oplus E \otimes_E^{\bar{\sigma}_i} \mathbb{C})$$

and a Hodge structure of type $(-1, 0)$ and $(0, -1)$ on $V^\sharp$ where, writing $V_\mathbb{C}^\sharp = E^\sharp \otimes_\mathbb{Q} \mathbb{C} = \oplus_{\Phi \in \mathrm{CM}(E)} \mathbb{C}\Delta_\Phi$ as a sum of idempotents,

$$(V_\mathbb{C}^\sharp)^{(-1,0)} = \oplus_{\sigma_0 \in \Phi \in \mathrm{CM}(E)} \mathbb{C}\Delta_\Phi, \quad (V_\mathbb{C}^\sharp)^{(0,-1)} = \oplus_{\bar{\sigma}_0 \in \Phi \in \mathrm{CM}(E)} \mathbb{C}\Delta_\Phi.$$

- there is an embedding $j\colon V \to C^+(V)$ such that multiplication on $C^+(V)$ induces a $T$-equivariant morphism $\ell\colon V \to \mathrm{End}(V^\sharp)$.

We now consider a second level structure $K_0 \subset T(\mathbb{A}_f)$ defining a CM cycle

$$Y_0(\mathbb{C}) = T(\mathbb{Q})\backslash\{z_0\} \times T(\mathbb{A}_f)/K_0.$$

We define $K_0 = \prod_p K_{0,p}$ where $K_{0,p} \subset T(\mathbb{Q}_p)$ sits in an exact sequence

$$1 \to \mathbb{Z}_p^\times \to K_{0,p} \to \frac{(\mathbb{Z}_p \otimes_\mathbb{Z} \mathcal{O}_E)^\times}{(\mathbb{Z}_p \otimes_\mathbb{Z} \mathcal{O}_F)^\times} \to 1$$

via the exact sequence

$$1 \to \mathbb{G}_m \to T \to T_E/T_F \to 1$$

defined by the natural projection $T = T_E/T_F^{\mathrm{Nm}=1} \to T_E/T_F$. Then $Y_0$ is defined over $E$ and we let $\mathcal{Y}_0$ be the normalization of $\mathrm{Spec}(\mathcal{O}_E)$ in $Y_0$.

Consider the integral structures $L_0 := \mathcal{O}_E \subset E = V$ and $L_0^\sharp := \mathcal{O}_{E^\sharp} \subset E^\sharp = V^\sharp$. Then $K_0$ preserves their profinite completions and the Hodge structure $V^\sharp$ of type $(-1, 0)$ and $(0, -1)$ defines an abelian variety $A^\sharp$ over $Y_0$ as follows: given $z := (z_0, g) \in Y_0(\mathbb{C}) := T(\mathbb{Q})\backslash\{z_0\} \times T(\mathbb{A}_f)/K_0$ then $A_z^\sharp := V_\mathbb{R}^\sharp/(V^\sharp \cap g\hat{L}_0^\sharp g^{-1})$. It extends to an abelian scheme $\mathcal{A}^\sharp$ over $\mathcal{Y}_0$. We let $\mathbb{H}^\sharp$ be its first de Rham homology group. It is a filtered $\mathcal{O}_{\mathcal{Y}_0}$-module. Thanks to the choice of the compact open subgroup $K_0$ we have

**Proposition 5.8.** *$\mathcal{Y}_0$ is étale over $\mathrm{Spec}(\mathcal{O}_E)$. Furthermore the inclusion $\ell\colon V \to \mathrm{End}(V^\sharp)$ defines a strict morphism of filtered $\mathcal{O}_{\mathcal{Y}_0}$-modules*

$$\ell\colon \mathbb{V}_{0,\mathrm{dR}} \to \mathrm{End}(\mathbb{H}^\sharp)$$

*and the $\mathcal{O}_E$-action on $L_0$ extends to a $\mathcal{O}_E$-action on $\mathbb{V}_{0,\mathrm{dR}}$.*

*Proof.* The first claim follows from Shimura's reciprocity laws. The second claim follows from Kisin's theory. See [AGHMP2, Prop. 3.5.5] $\qquad\square$

In particular we define the invertible $\mathcal{O}_{\mathcal{Y}_0}$-module $\omega_0 := \mathrm{Fil}^1 \mathbb{V}_{0,\mathrm{dR}}$. We endow it with a metric at infinity induced by the standard Hermitian metric $Q_0$ on $V_\mathbb{C} = E \otimes_\mathbb{Q} \mathbb{C}$, $x \mapsto Q_0(x) := \mathrm{Tr}_{F/\mathbb{Q}}(x\bar{x})$.

## 6. Integral models

In order to proceed with the computation of the intersections numbers we want, we need models for $M_L(\mathbb{C})$ and the CM cycle $Y_L(\mathbb{C})$ over $\mathbb{Z}$.

6.1. **Integral models of** GSpin**-Shimura varieties.** We know from work of Shimura and Deligne [Del] that $M_L(\mathbb{C})$ is the complex analytic space associated to a quasi-projective variety $M_L$ over a number field $K$ called the *reflex field*. In the case of GSpin, for $n \geq 1$, the reflex field is $\mathbb{Q}$. For $n = 0$ it is the quadratic imaginary field $K = C^+(V)$.

We assume next that $n \geq 1$. Let $\Delta_L$ be the discriminant $\Delta_L := [L^\vee : L]$ where $L^\vee$ is the $\mathbb{Z}$-dual of $L$ and the inclusion $L \subset L^\vee$ is defined using the bilinear form $[\_, \_]$. By work of Vasiu [Va] and Kisin (see in particular [Kis]) the scheme $M_L$ has a canonical integral model $\mathcal{M}_{L, \mathbb{Z}[2^{-1}|\Delta_L|^{-1}]}$, smooth over $\mathbb{Z}[2^{-1}|\Delta_L|^{-1}]$. Also $V_{\mathrm{dR}}$ has a model $\mathbb{V}_{\mathrm{dR}}$ which is a locally free $\mathcal{O}_{\mathcal{M}_{L, \mathbb{Z}[2^{-1}|\Delta_L|^{-1}]}}$-module, endowed with a descending filtration $\mathrm{Fil}^\bullet \mathbb{V}_{\mathrm{dR}}$ by locally free submodules, and an integrable connection satisfying Griffiths' transversality. For the purpose of computing some arithmetic intersection we wish to have a model

$$\mathcal{M} \longrightarrow \mathrm{Spec}(\mathbb{Z})$$

to which some of the extra structures described above extend as well.

If $L$ is maximal among the quadratic lattices of $V$ and is self dual at 2, Madapusi Pera [MP] and Kim-Madapusi Pera [KMP] constructed such a canonical integral model which has singular fibers at the primes dividing $\Delta_L$. Unfortunately at primes whose square divides $\Delta_L$ this model is not well behaved: for our purposes the CM cycle $Y_L$ will have a model $\mathcal{Y}_L$, finite over $\mathrm{Spec}(\mathcal{O}_E)$ but the morphism $Y_L \to M_L$ on the generic fiber does *not* in general extend to a morphism from $\mathcal{Y}$ to the Madapusi Pera model. In [AGHMP2, §4.4] we proceeded differently. Let $p$ a prime dividing $\Delta_L$ and let $L \subset L^\diamond$ be an isometric embedding of quadratic lattices with signature $(n, 2)$ and $(n^\diamond, 2)$ respectively. We take $L^\diamond$ that is self-dual at $p$. By functoriality we will have a morphism of Shimura varieties $M_L \to M_{L^\diamond}$ and $M_{L^\diamond}$ will admit an integral model $\mathcal{M}_{L^\diamond, \mathbb{Z}[\Delta_{L^\diamond}^{-1}]}$ smooth over $\mathrm{Spec}(\mathbb{Z}[\Delta_{L^\diamond}^{-1}])$. We define $\mathcal{M}_{L, \mathbb{Z}[\Delta_{L^\diamond}^{-1}]}$ to be the normalization of $\mathcal{M}_{L^\diamond, \mathbb{Z}[\Delta_{L^\diamond}^{-1}]}$ in $M_L$. The restriction of the tautological bundle on $\mathcal{M}_{L^\diamond, \mathbb{Z}[\Delta_{L^\diamond}^{-1}]}$ defines a line bundle on $\omega_{L^\diamond}$ on $\mathcal{M}_{L, \mathbb{Z}[\Delta_{L^\diamond}^{-1}]}$. We have the following proposition proven in [AGHMP2, Prop. 4.4.1].

**Proposition 6.1.** *The models $\mathcal{M}_{L, \mathbb{Z}[\Delta_{L^\diamond}^{-1}]}$ and the line bundles $\omega_{L^\diamond}$ glue for varying embeddings $L \subset L^\diamond$ to a normal model $\mathcal{M}_L$ over $\mathrm{Spec}(\mathbb{Z})$ and a line bundle $\omega$ that agree with the construction on $\mathcal{M}_{L, \mathbb{Z}[2^{-1}|\Delta_L|^{-1}]}$ provided by Kisin and Vasiu.*

Via the uniformization map

$$\mathcal{D} \to M_L(\mathbb{C}), \quad z \mapsto (z, g)$$

the fiber of $\omega$ at $z \in \mathcal{D}$ is the isotropic line $\mathbb{C}z = V_{\mathbb{C}}^{(1,-1)} \subset V_{\mathbb{C}}$. We then endow $\omega$ with the metric $||z||^2 := -[z, \overline{z}]$, called the Petersson metric, and we obtain a metrized line bundle in the sense of Ch. VII, §1.3,

$$\widehat{\omega} \in \widehat{\mathrm{Pic}}(\mathcal{M}_L).$$

6.2. **Integral models of big CM cycles.** Consider now the big CM cycles $\mathcal{Y}_L$ (associated to the compact open subgroup $K_T := K_L \cap K_0$) and $\mathcal{Y}_0$ constructed in the previous sections. The definition $\mathcal{M}_L$ is made so that the morphism $Y_L(\mathbb{C}) \to M_L(\mathbb{C})$ extends to a morphism $\mathcal{Y}_L \to \mathcal{M}_L$. As $K_T \subset K_0$ we get the following

diagram

$$
\begin{array}{ccc}
\mathcal{Y}_L & \longrightarrow & \mathcal{M}_L \\
\downarrow & & \\
\mathcal{Y}_0. & &
\end{array}
$$

In particular we have two metrized line bundles on $\mathcal{Y}_L$: the pull back $\widehat{\omega}_{\mathcal{Y}_L}$ of the metrized line bundle $\widehat{\omega}$ over $\mathcal{M}_L$ and the pull back $\widehat{\omega}_{0,\mathcal{Y}_L}$ of the metrized line bundle $\widehat{\omega}_0$ over $\mathcal{Y}_0$.

For the first we will be able to compute the arithmetic degree using work of Bruinier, Kudla and Yang. The second is related to the Faltings' height of the abelian variety $A^\sharp$ which has an action of $\mathcal{O}_{E^\sharp}$. Let us start with this connection. Recall from Proposition 5.8 that we have a strict morphism of filtered $\mathcal{O}_{\mathcal{Y}_0}$-module

$$
\ell \colon \mathbb{V}_{0,\mathrm{dR}} \to \mathrm{End}(\mathbb{H}^\sharp),
$$

where $\mathbb{H}^\sharp$ is the de Rham homology of $\mathcal{A}^\sharp$ (the extension of $A^\sharp$ to an abelian scheme over $\mathcal{Y}_0$). Then $\omega_0 = \mathrm{Fil}^1 \mathbb{V}_{0,\mathrm{dR}}$ maps to the endomorphisms of $\mathbb{H}^\sharp$ sending $\mathrm{Gr}^{-1}\mathbb{H}^\sharp \to \mathrm{Fil}^0\mathbb{H}^\sharp$, i.e., upon taking determinants we have a map of $\mathcal{O}_{\mathcal{Y}_0}$-modules

$$
\omega_0^{2^{d-1}} \otimes_{\mathcal{O}_{\mathcal{Y}_0}} \det\mathrm{Gr}^{-1}\mathbb{H}^\sharp \longrightarrow \det\mathrm{Fil}^0\mathbb{H}^\sharp.
$$

By definition of the Hodge filtration of $\mathbb{H}^\sharp$ we have $(\det\mathrm{Gr}^{-1}\mathbb{H}^\sharp)^{-1} = \omega^\sharp$, the Hodge bundle of $\mathcal{A}^\sharp$. With this notation we have the following result proven in [AGHMP2, Thm 9.4.2].

**Theorem 6.2.** *The following holds*

$$
\frac{1}{2^d}\sum_\Phi h_{(E,\Phi)}^{\mathrm{Falt}} = \frac{1}{4}\frac{\widehat{\deg}(\widehat{\omega}_0)}{\deg_{\mathbb{C}}(Y_0)} + \frac{1}{4}\log|D_F| - \frac{1}{2}d\cdot\log(2\pi).
$$

*Proof.* We first use (1) to relate $\sum_\Phi h_{(E,\Phi)}^{\mathrm{Falt}}$ with $h_{(E^\sharp,\Phi^\sharp)}^{\mathrm{Falt}}$ via the formula $2d h_{(E^\sharp,\Phi^\sharp)}^{\mathrm{Falt}} = \sum_\Phi h_{(E,\Phi)}^{\mathrm{Falt}}$. Second, by definition the degree of the metrized Hodge bundle $\widehat{\Omega}^\sharp$ of $\mathcal{A}^\sharp$ satisfies

$$
\frac{\deg\widehat{\Omega}^\sharp}{\deg Y_0} = 2d h_{(E^\sharp,\Phi^\sharp)}^{\mathrm{Falt}}.
$$

Finally we study the inclusion of invertible sheaves:

$$
\omega_0^{2^{d-1}} \subset (\det\mathrm{Gr}^{-1}\mathbb{H}^\sharp)^{-2} \otimes_{\mathcal{O}_{\mathcal{Y}_0}} \det\mathbb{H}^\sharp \cong (\omega^\sharp)^2 \otimes_{\mathcal{O}_{\mathcal{Y}_0}} \det\mathbb{H}^\sharp.
$$

One proves that the metrics on the two sides coincide; here $\det\mathbb{H}^\sharp$ is endowed with the standard metric given by integration over $A^\sharp(\mathbb{C})$ of top degree $C^\infty$ de Rham classes. Its arithmetic degree is computed in [AGHMP2, Lemma 9.4.3] and coincides with

$$
\widehat{\deg}\det\mathbb{H}^\sharp = 2^d d\cdot\deg Y_0\cdot\log(2\pi).
$$

We are left to study the cokernel of the displayed inclusion. A delicate algebra computation, see [AGHMP2, Prop. 9.4.1], shows that the difference of these line bundles has degree $2^{d-1}\deg Y_0\log|D_F|$ and the claim follows. $\qquad\square$

Next we compare the metrized line bundles $\widehat{\omega}_{\mathcal{Y}_L}$ and $\widehat{\omega}_{0,\mathcal{Y}_L}$.

**Proposition 6.3.** *We have $\deg(\widehat{\omega}_{\mathcal{Y}_L}) \sim_L \deg(\widehat{\omega}_{0,\mathcal{Y}_L}) + \log|D_F|$ where $\sim_L$ means equal up to rational linear combinations of log of primes dividing $D_{L,\mathrm{bad}}$; see Definition 5.1.*

*Proof.* Over $\mathbb{Q}$ the two sheaves coincide as they are associated to the same Hodge structure, namely $V$. Via this identification the metric on $\omega$ is the metric on $\omega_0$ times $\sigma_0(\lambda)$ (recall that the quadratic form on $V$ is defined by $x \mapsto \mathrm{Tr}_{F/\mathbb{Q}}(\lambda x \bar{x})$). We have only to check that over $\mathcal{Y}_L[D_{L,\mathrm{bad}}^{-1}]$ we have $\omega = \lambda \mathfrak{D}_{F/\mathbb{Q}}^{-1} \otimes_{\mathcal{O}_F} \omega_0$. Using Kisin's correspondence one is reduced to prove this statement for the lattices $L$ and $L_0$ of $V$ and the claim follows by local calculations (see the proof of [AGHMP2, Prop. 9.5.1] using Definition 5.1). $\qquad\square$

## 7. The Bruinier, Kudla, Yang conjecture

Thanks to Theorem 6.2 and Proposition 6.3 and the $\mathbb{Q}$-linear independence of log of primes, in order to conclude the proof of the averaged version of Colmez's conjecture we need to prove the following

**Theorem 7.1.** *The degree of $\widehat{\omega}_{\mathcal{Y}_L}$ satisfies*

$$\frac{\deg \widehat{\omega}_{\mathcal{Y}_L}}{\deg Y_L} \sim_L -\frac{2\Lambda'(\chi,0)}{\Lambda(\chi,0)} - d\log(4\pi e^\gamma).$$

*Moreover, $\cap_L D_{L,\mathrm{bad}} = \emptyset$.*

The assertion that $\cap_L D_{L,\mathrm{bad}} = \emptyset$ can be turned via (5) into a class field theory question that we do no discuss here; we refer to [AGHMP2, Prop. 9.5.2] for details.

The main idea is then to realize $\widehat{\omega}$ as a combination of arithmetic divisors using Borcherds theory and results of Bruinier, Kudla, Yang that compute the contribution at infinity of the intersection of these arithmetic divisors with $\mathcal{Y}_L$. Bruinier, Kudla, Yang provided also a conjecture for the contribution at finite places that we verify in [AGHMP2] in sufficiently many cases to get the result.

7.1. **Special divisors.** In this section we will show how, given an element $\lambda \in V$ with $Q(\lambda) > 0$, we can construct a divisor in $M_L(\mathbb{C})$ as a Shimura subvariety. These will give the special divisors mentioned in the introduction. The fact that we have such a large supply of easily constructed divisors, and in general of cycles of higher codimension obtained by intersecting such divisors, makes the theory of GSpin-Shimura varieties extremely rich.

Given $\lambda$ as above, set $V_\lambda := \lambda^\perp \subset V$. This is a dimension $(n-1)+2$ subspace of $V$ and $Q_\lambda := Q|_{V_\lambda}$ is a quadratic form of signature $(n-1,2)$. Then we get a subgroup

$$G_\lambda = \mathrm{GSpin}(V_\lambda, Q_\lambda) \subset \mathrm{GSpin}(V) = G.$$

The symmetric space $D_\lambda$ for $G_\lambda$ is identified with

$$D_\lambda = \left\{ [z] \in D_\mathbb{C} \subset V_\mathbb{C} \backslash \{0\} : z \in V_{\lambda,\mathbb{C}} = \lambda^\perp \right\} / \mathbb{C}^*.$$

Let $L_\lambda := L \cap V_\lambda$ so we have $K_\lambda \subset G_\lambda(\mathbb{A}_f)$ and we get a GSpin-Shimura variety

$$M_\lambda(\mathbb{C}) = G_\lambda(\mathbb{Q}) \backslash D_\lambda \times G_\lambda(\mathbb{A}_f) / K_\lambda$$

together with a finite analytic map

$$M_\lambda(\mathbb{C}) \to M_L(\mathbb{C}) = G(\mathbb{Q}) \backslash D \times G(\mathbb{A}_f) / K_L.$$

Notice that such map is in general not an injection but the image of this map consists of divisors of $M_L(\mathbb{C})$. The discrepancy between $M_\lambda(\mathbb{C})$ and its image in $M_L(\mathbb{C})$ makes the intersection theory of special divisors more involved. We will ignore this issue here for sake of simplicity and pretend that we can identify $M_\lambda(\mathbb{C})$ and its image. We refer to [AGHMP2, Def. 4.5.6] for the correct treatment using

stacks. Also, in order to have a good theory of the integral models of such divisors, following Borcherds we group together all the $M_\lambda(\mathbb{C})$'s considering the $\lambda$'s with equal norm $Q(\lambda) = m$ and equal class $\mu \in L^\vee/L$:

**Definition 7.2.** For any $m \in \mathbb{N}_{>0}$ and every $\mu \in L^\vee/L$, let

$$Z(m,\mu)(\mathbb{C}) := \amalg_{g \in G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_L} \Gamma_g \backslash \left( \amalg_{\substack{\lambda \in \mu_g + L_g \\ Q(\lambda)=m}} D_\lambda \right).$$

where $\Gamma_g = G(\mathbb{Q}) \cap gK_Lg^{-1}$, $L_g \subset V$ is the lattice $V \cap (g\widehat{L}g^{-1})$, and $\mu_g \in L_g^\vee/L_g$ is the class of $g\mu g^{-1}$.

Recall that $M_L(\mathbb{C}) = \amalg_{g \in G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_L} \Gamma_g \backslash D_\mathbb{C}$ so that we have a natural morphism

$$Z(m,\mu)(\mathbb{C}) \to M_L(\mathbb{C})$$

whose image is the union of the images of various $M_\lambda(\mathbb{C})$.

The image of $Z(m,\mu)(\mathbb{C})$ singles out points $z$ of $M_L(\mathbb{C})$ where the $\mathbb{Z}$-Hodge structure $\mathbb{V}_{\mathrm{Betti},z}^\vee$ acquires a Hodge $(0,0)$-class $\lambda_z$ of norm $Q_{\mathrm{Betti}}(\lambda_z) = m$ and whose class in $\mathbb{V}_{\mathrm{Betti},z}^\vee/\mathbb{V}_{\mathrm{Betti},z}$, which can be canonically identified with $L^\vee/L$, is $\mu \in L^\vee/L$.

We can give an intrinsic characterization of the image of $Z(m,\mu)(\mathbb{C})$ in $M_L(\mathbb{C})$ as follows; take $(z,g) \in M_L(\mathbb{C})$ and consider the corresponding element $\ell_{\mathrm{Betti}}(\lambda) \in \mathrm{End}(\mathrm{H}_{1,\mathrm{Betti}}(A_{(z,g)}))$ (using (3)). Then,

**Proposition 7.3.** *The point $(z,g)$ is in the image of $\mathcal{D}_\lambda$ if and only if $\ell_{\mathrm{Betti}}(\lambda)$ arises as the Betti realization of a rational endomorphism $\ell_\lambda \in \mathrm{End}(A_{(z,g)})$.*

*Proof.* The point $(z,g)$ lies in $D_\lambda$ if and only if

$$\lambda \in z^\perp = \mathrm{Fil}^0\mathbb{V}_{\mathrm{dR},z} \subset \mathbb{V}_{\mathrm{dR},z}$$

if and only if the element $\ell_{\mathrm{dR}}(\lambda) \in \mathrm{End}(\mathrm{H}_{1,\mathrm{dR}}(A_{(z,g)}))$ of (4) lies in $\mathrm{Fil}^0\mathrm{End}(\mathrm{H}_{1,\mathrm{dR}}(A_{(z,g)}))$, i.e., $\ell_{\mathrm{dR}}(\lambda)$ preserves the Hodge filtration of $\mathrm{H}_{1,\mathrm{dR}}(A_{(z,g)})$. This is equivalent to require that the element $\ell_{\mathrm{Betti}}(\lambda)$ defines an endomorphism of $A_{(z,g)}$. $\qquad\square$

As before we take $\lambda \in V$ be an element with $Q(\lambda) > 0$. The next lemma shows that the images of $Y_L(\mathbb{C})$ and $M_\lambda(\mathbb{C})$ in $M_L(\mathbb{C})$ do not intersect. This will imply that for the associated arithmetic objects, i.e., the associated objects over $\mathbb{Z}$, we have *proper intersection*. This is not at all the case for the small CM points of Remark 5.2 where we have *improper intersection*; see [AGHMP1] for a discussion.

**Lemma 7.4.** *The intersection of the images $Y_L(\mathbb{C})$ and $M_\lambda(\mathbb{C})$ in $M_L(\mathbb{C})$ is empty.*

*Proof.* Assume that we have an element $z \in D_\lambda$ whose image on $(z,g) \in M_L(\mathbb{C})$ lies in the image of $Y_L(\mathbb{C})$. This is equivalent to saying that $\lambda \in V_{\mathrm{Betti},\mathbb{Q},z}$ is such that $\lambda \in (\mathbb{C}z \oplus \mathbb{C}\bar{z})^\perp$. But $V_{\mathrm{Betti},\mathbb{Q},z} = V = E$ as a $\mathbb{Q}$-vector space and we get that

$$V = E \cdot \lambda \subset (\mathbb{C}z \oplus \mathbb{C}\bar{z})^\perp$$

as the pairing on $V$ is $E$-hermitian. This is clearly a contradiction as $(\mathbb{C}z \oplus \mathbb{C}\bar{z})^\perp \oplus (\mathbb{C}z \oplus \mathbb{C}\bar{z}) = V_\mathbb{C}$. $\qquad\square$

7.2. **Integral models of special divisors.** We need to define models of the special divisors $Z(m, \mu)$ over $\mathbb{Q}$ and even integral models $\mathcal{Z}(m, \mu)$ over $\mathbb{Z}$ in order to compute arithmetic intersection theory. Clearly we can not use complex uniformization as we did in the previous section. Proposition 7.3 leads us to an intrinsic definition that makes perfect sense over $\mathbb{Z}$ and provides us with the definition we want. We set $\mathcal{Z}(m, \mu)$ to be the scheme representing the functor of pairs $(\rho, f)$ where $\rho \colon S \to \mathcal{M}_L$ and $f \in \mathrm{End}^0(\mathcal{A} \times_{\mathcal{M}_L} S)$ is a rational endomorphism of the Kuga-Satake abelian scheme $\mathcal{A}_S := \mathcal{A} \times_{\mathcal{M}_L} S$ such that

    i. $f \circ f = [m]$ (the multiplication by $m$ map on $\mathcal{A}_S$);

    ii. the endomorphism defined by $f$ on the de Rham homology of $\mathcal{A}_S$, on the Tate module of $\mathcal{A}_S$ and on the crystal defined by $\mathcal{A}_S$ is in the image of the de Rham realization, étale or crystalline realization of $L^\vee$ of class $\mu \in L^\vee/L$.

We explain what me mean in (ii). Over $\mathbb{Z}[\Delta_L^{-1}]$ the lattice $L \subset V$ defines a *motive* $\mathbb{V}$ over $\mathcal{M}_L$, namely a $\mathbb{Z}$-variation of Hodge structures over $M_L(\mathbb{C})$, a filtered vector bundle with connection over $\mathcal{M}_L$, a lisse $\ell$-adic étale sheaf over $\mathcal{M}_L[\ell^{-1}]$, a crystal over $\mathcal{M}_L \otimes \mathbb{F}_p$. The map $\ell$ defines an embedding of motives $\mathbb{V} \subset \mathrm{End}(\mathbb{H})$ where $\mathbb{H}$ is the motive associated to the abelian scheme $\mathcal{A}$: its de Rham homology, its $\ell$-adic Tate module, its Dieudonné module. In (ii) we ask that the realization of $f$ lies in the image of $\mathbb{V}$.

To extend this notion to the whole of $\mathbb{Z}$ one works with auxiliary lattices $L \subset L^\diamond$, that are self dual at a given prime $p$ and demands this condition working on $\mathcal{M}_{L^\diamond}$. A result analogous to Proposition 6.1 implies that we get a well-posed definition, independent of the auxiliary choice of $L^\diamond$.

It is proven in [AGHMP2, Prop. 4.5.8] that the models $\mathcal{Z}(m, \mu)$ have good properties, namely they do not have vertical components that would create troubles in computing arithmetic intersections:

**Proposition 7.5.** *If $V$ has dimension $\geq 5$ then the $\mathcal{Z}(m, \mu)$'s are flat over $\mathbb{Z}[1/2]$ and even over $\mathbb{Z}$ if $L$ is self dual at $2$.*

7.3. **Special divisors and $\widehat{\omega}$.** We now come to the main result expressing $\widehat{\omega}$ as a combination of arithmetic special divisors. Thanks to Borcherds' theory in fact the special divisors $\mathcal{Z}(m, \mu)$ are endowed with natural Green functions $\Phi_{m,\mu}$ (see [BY, Thm. 4.2] using [BF]). And we can consider the pair $\widehat{\mathcal{Z}}(m, \mu) := (\mathcal{Z}(m, \mu), \Phi_{m,\mu}) \in \widehat{CH}^1(\mathcal{M}_L)$. In the following we will freely use the equivalence $\widehat{\mathrm{Pic}}(\mathcal{M}_L) \cong \widehat{CH}^1(\mathcal{M}_L)$, between metrized line bundles and arithmetic divisors (modulo principal arithmetic divisors). We have the following fundamental result, see [AGHMP2, Thm. 4.8.1]:

**Theorem 7.6.** *Suppose that $n \geq 3$. There are finitely many integers $c(-m, \mu)$ for $m \geq 0$ and $\mu \in L^\vee/L$ with $c(0, 0) \neq 0$ and there exists a rational section $\Psi$ of $\omega^{\otimes c(0,0)}$, defined over $\mathbb{Q}$, such that*

$$
\begin{aligned}
\widehat{\omega}^{\otimes c(0,0)} &= \widehat{\mathrm{div}}(\Psi) \\
&= \sum_{m,\mu} c(-m, \mu) \widehat{\mathcal{Z}}(m, \mu) - c_f(0,0) \cdot (0, \log(4\pi e^\gamma)) + \widehat{\mathcal{E}},
\end{aligned}
$$

*where $(0, \log(4\pi e^\gamma))$ denotes the trivial divisor endowed with the constant Green function $\log(4\pi e^\gamma)$ and $\widehat{\mathcal{E}} = (\mathcal{E}, 0)$ is a vertical divisor with the trivial Green function*

*that has a decomposition*

$$\mathcal{E} = \sum_{p \mid D_L} \mathcal{E}_p,$$

*such that $\mathcal{E}_p$ is supported on the special fiber $\mathcal{M}_{L,\mathbb{F}_p}$, and:*

- *If $p$ is odd and $p^2 \nmid D_L$ then $\mathcal{E}_p = 0$;*
- *If $n \geq 5$ then $\mathcal{E} = \mathcal{E}_2$ is supported on $\mathcal{M}_{L,\mathbb{F}_2}$.*
- *If $n \geq 5$ and $L_{(2)}$ is self-dual, then $\mathcal{E} = 0$.*

We apply this result first to the lattice $L$, coming from the CM field $E$, and then also for some auxiliary lattice $L \subset L^\diamond$, self dual at a given bad prime $p$. In this case some care is needed to assure that the special divisors on $\mathcal{M}_{L^\diamond}$ do *not* contain the big CM cycle, i.e., that we have proper intersection. In particular $\mathcal{E}_p = 0$ and we will compute the contribution at $p$ of $\deg(\widehat{\omega}_{\mathcal{Y}_L})$ using the expression of $\widehat{\omega}$ in terms of the divisors $\widehat{\mathcal{Z}}(m, \mu)$ provided by the theorem above.

### 7.4. Arithmetic intersection and special values.
Consider the divisor $\widehat{\mathcal{Z}} := \sum_{m,\mu} c(-m, \mu) \widehat{\mathcal{Z}}(m, \mu)$ of Theorem 7.6. Write $[\mathcal{Y}_L : \widehat{\mathcal{Z}}]$ for the arithmetic degree of the base change of $\widehat{\mathcal{Z}}$ to $\mathcal{Y}_L$. The main result that finishes the proof of the averaged version of Colmez's conjecture is the following:

**Theorem 7.7.** *We have*

$$\frac{[\mathcal{Y}_L : \widehat{\mathcal{Z}}]}{\deg(Y)} \sim_L -2 \frac{\Lambda'(0, \chi) \cdot c(0,0)}{\Lambda(0, \chi)}$$

*(recall that $\sim_L$ means equality up to a $\mathbb{Q}$-linear combinations of $\{\log(p) : p \mid D_{L,\text{bad}}\}$).*

The first ingredient for the proof is the computation of the contribution at infinity of the degree, provided by the following theorem which is the main result of [BKY]. Thanks to Lemma 7.4 this is computed as the value of the Green function for the arithmetic divisor $\widehat{\mathcal{Z}}$ at the $\mathbb{C}$-points of $\mathcal{Y}_L$:

**Theorem 7.8.** *Let $\Phi$ be the Green function associated to the arithmetic divisor $\widehat{\mathcal{Z}}$ of the previous theorem.*

$$\frac{\Phi(\mathcal{Y}_L^\infty)}{2\deg(Y_L)} = \sum_{\substack{\mu \in L^\vee/L \\ m \geq 0}} \frac{a(m, \mu) \cdot c(-m, \mu)}{\Lambda(0, \chi)},$$

*where $\mathcal{Y}^\infty = Y_L \times_{\mathbb{Q}} \mathbb{C}$ and $\Phi(\mathcal{Y}^\infty)$ is the weighted sum of the values of $\Phi$*

$$\Phi(\mathcal{Y}^\infty) = \sum_{y \in \mathcal{Y}^\infty(\mathbb{C})} \frac{\Phi(y)}{|\text{Aut}(y)|}.$$

Here the $a(m, \mu)/\Lambda(0, \chi)$'s are the coefficients of the formal $q$-expansion of the restriction of the derivative of a suitable weight 1 Hilbert modular Eisenstein series introduced by Kudla [Ku]. In order to deduce Theorem 7.7 from the result of [BKY] above one needs to:

1. compute the coefficients $a(m, \mu)$. Typically they are expressed as orbital integrals and one wants to get explicit quantities;
2. prove that the *finite* part of the intersection is $\mathcal{Z}(m, \mu)$ along $\mathcal{Y}_L$ is $a(m, \mu)/\Lambda(0, \chi)$.

Both calculations need to be done up to $\mathbb{Q}$-linear combinations of $\log(p)$'s for $p \mid D_{L,\mathrm{bad}}$. Computation (1) is due to Kudla and Yang [KY] (and [AGHMP2, §6] for the contribution at the prime $p = 2$). Claim (2) is proven in §7 of [AGHMP2]. It is what Bruinier, Kudla and Yang conjecture to be true in [BKY], even in a more general setting than what is needed and proven in Theorem 7.7. It is a remarkable instance of Kudla's program relating generating series of (arithmetic) intersection numbers and automorphic forms.

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