Some elements of Hamiltonian formalism for PDEs

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1 The Hamiltonian formalism for PDEs

1.1 The gradient of a functional

Definition 1.1. *Consider a function $f \in C^\infty(\mathcal{U}_s, \mathbb{R})$, $\mathcal{U}_s \subset H^s(\mathbb{T})$ open, $s \geq 0$ a fixed parameter and $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ is the 1 dimensional torus. We will denote by $\nabla f(u)$ the gradient of f with respect to the L^2 metric, namely the unique function such that*

$$\langle \nabla f(u), h \rangle_{L^2} = \mathrm{d}f(u)h \ , \quad \forall h \in H^s \tag{1}$$

where

$$\langle u, v \rangle_{L^2} := \int_{-\pi}^{\pi} u(x)v(x)\mathrm{d}x \tag{2}$$

*is the L^2 scalar product and $\mathrm{d}f(u)$ is the differential of f at u. The gradient is a smooth map from H^s to H^{-s} (see e.g. [**?**]).*

Example 1.2. Consider the function

$$f(u) := \int_{-\pi}^{\pi} \frac{u_x^2}{2}\mathrm{d}x \ , \tag{3}$$

which is differentiable as a function from $H^s \to \mathbb{R}$ for any $s \geq 1$. One has

$$\mathrm{d}f(u)h = \int_{-\pi}^{\pi} u_x h_x \mathrm{d}x = \int_{-\pi}^{\pi} -u_{xx} h \mathrm{d}x = \langle -u_{xx}, h \rangle_{L^2} \tag{4}$$

and therefore in this case one has $\nabla f(u) = -u_{xx}$.

Example 1.3. Let $\mathcal{F} : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function and define

$$f(u) = \int_{-\pi}^{\pi} \mathcal{F}(u, u_x)\mathrm{d}x \tag{5}$$

then the gradient of f coincides with the so called functional derivative of $\mathcal{F}$:

$$\nabla f \equiv \frac{\delta \mathcal{F}}{\delta u} := \frac{\partial \mathcal{F}}{\partial u} - \frac{\partial}{\partial x}\frac{\partial \mathcal{F}}{\partial u_x} \ . \tag{6}$$

1.2 Lagrangian and Hamiltonian formalism for the wave equation

Until subsection 1.4 we will work at a formal level, without specifying the function spaces and the domains.

Definition 1.4. *Let $L(u, \dot{u})$ be a Lagrangian function, then the corresponding Lagrange equations are given*

$$\nabla_u L - \frac{\mathrm{d}}{\mathrm{d}t}\nabla_{\dot{u}} \ L = 0 \tag{7}$$

where $\nabla_u L$ is the gradient with respect to u only, and similarly $\nabla_{\dot{u}}$ is the gradient with respect to $\dot{u}$.

Example 1.5. Consider the Lagrangian

$$L(u, \dot{u}) := \int_{-\pi}^{\pi} \left(\frac{\dot{u}^2}{2} - \frac{u_x^2}{2} - \mu^2 \frac{u^2}{2} - F(u) \right) \mathrm{d}x \ . \tag{8}$$

then the corresponding Lagrange equations are given by (**??**) with $f = -F'$.

Given a Lagrangian system with Lagrangian function L one defines the corresponding Hamiltonian system as follows.

Definition 1.6. *Consider the momentum $v := \nabla_{\dot{u}} L$ conjugated to u; assume that L is convex with respect to $\dot{u}$, then the Hamiltonian function associated to L is defined by*

$$H(v, u) := \left[\langle v; \dot{u} \rangle_{L^2} - L(u, \dot{u}) \right]_{\dot{u} = \dot{u}(u,v)} \ . \tag{9}$$

Definition 1.7. *Let $H(v, u)$ be a Hamiltonian function, then the corresponding Hamilton equations are given by*

$$\dot{v} = -\nabla_u H \ , \quad \dot{u} = \nabla_v H \ . \tag{10}$$

As in the finite dimensional case one has that the Lagrange equations are equivalent to the Hamilton equation of H.

An elementary computation shows that for the wave equation one has $v = \dot{u}$ and

$$H(v, u) = \int_{-\pi}^{\pi} \left(\frac{v^2 + u_x^2 + \mu^2 u^2}{2} + F(u) \right) \mathrm{d}x \tag{11}$$

1.3 Canonical coordinates

Consider a Lagrangian system and let $\mathbf{e}_k$ be an orthonormal basis of L^2, write $u = \sum_k q_k \mathbf{e}_k$ and $\dot{u} = \sum_k \dot{q}_k \mathbf{e}_k$, then one has the following proposition

Proposition 1.8. The Lagrange equations (7) are equivalent to

$$\frac{\partial L}{\partial q_k} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_k} = 0 \tag{12}$$

Proof. Taking the scalar product of (7) with $\mathbf{e}_k$ one gets

$$\langle \mathbf{e}_k; \nabla_u L \rangle_{L^2} - \frac{\mathrm{d}}{\mathrm{d}t}\langle \mathbf{e}_k; \nabla_{\dot{u}} L \rangle_{L^2} = 0$$

but one has $\langle \mathbf{e}_k; \nabla_u L \rangle = \frac{\partial L}{\partial q_k}$ and similarly for the other term. Thus the thesis follows. $\square$

This proposition shows that, once a basis has been introduced, the Lagrange equations have the same form as in the finite dimensional case.

In the Hamiltonian case exactly the same result holds. Precisely, denoting $v := \sum_k p_k \mathbf{e}_k$ one has

Proposition 1.9. The Hamilton equations of a Hamiltonian function H, are equivalent to

$$\dot{p}_k = -\frac{\partial H}{\partial q_k} \ , \quad \dot{q}_k = \frac{\partial H}{\partial p_k} \ . \tag{13}$$

In the case of the nonlinear wave equation, in order to get a convenient form of the equations, one can choose the Fourier basis. Such a basis is defined by

$$\hat{e}_k := \begin{cases} \frac{1}{\sqrt{\pi}} \cos kx & k > 0 \\ \frac{1}{\sqrt{2\pi}} & k = 0 \\ \frac{1}{\sqrt{\pi}} \sin -kx & k < 0 \end{cases} \tag{14}$$

Thus the Hamiltonian (11) takes the form

$$H(p,q) = \sum_{k \in \mathbb{Z}} \frac{p_k^2 + \omega_k^2 q_k^2}{2} + \int_{-\pi}^{\pi} F\left(\sum_k q_k \hat{e}_k(x) \right) \mathrm{d}x \ , \tag{15}$$

where $\omega_k^2 := k^2 + \mu^2$. For the forthcoming developments it is worth to rescale the variables by defining

$$p_k' := \frac{p_k}{\sqrt{\omega_k}} \ , \quad q_k' := \sqrt{\omega_k} q_k \ , \tag{16}$$

so that, omitting primes, the Hamiltonian takes the form

$$H(p,q) = \sum_k \omega_k \frac{p_k^2 + q_k^2}{2} + H_P(p,q) \tag{17}$$

where H_P has a zero of order higher than 2. *In the following we will always study systems of the form* (17). Moreover, possibly by relabeling the variables and the frequencies it is possible to reduce to the case where k varies in $\mathbb{N} \equiv \{1, 2, 3...\}$. This is what we will assume in developing the abstract theory.

Example 1.10. An example of a different nature in which the Hamiltonian takes the form (17) is the nonlinear Schrödinger equation

$$-\mathrm{i}\dot{\psi} = \psi_{xx} + f(|\psi|^2)\psi \ , \tag{18}$$

where f is a smooth function. Eq. (18) has the conserved energy functional

$$H(\psi, \bar{\psi}) := \int_{-\pi}^{\pi} \left(|\psi|^2 + F(|\psi|^2) \right) \mathrm{d}x \ , \tag{19}$$

where F is such that $F' = f$. Introduce canonical coordinates (p_k, q_k) by

$$\psi = \sum_{k \in \mathbb{Z}} \frac{p_k + \mathrm{i}q_k}{\sqrt{2}} \hat{e}_k \ , \tag{20}$$

then the energy takes the form (17) with $\omega_k = k^2$ and the NLS is equivalent to the corresponding Hamilton equations.

Example 1.11. Consider the Kortweg de Vries equation

$$u_t + u_{xxx} + uu_x = 0 \ , \tag{21}$$

in the space of functions with zero mean value. The conserved energy is given by

$$H(u) = \int_{-\pi}^{\pi} \left(\frac{u_x^2}{2} + \frac{u^3}{6} \right) \mathrm{d}x \ , \tag{22}$$

which again is also the Hamiltonian of the system. Canonical coordinates are here introduced by

$$u(x) = \sum_{k>0} \sqrt{k}(p_k \hat{e}_k + q_k \hat{e}_{-k}) \ , \tag{23}$$

in which the Hamiltonian takes the form (17) with $\omega_k = k^3$.

Remark 1.12. It is also interesting to study some of these equations with Dirichlet boundary conditions (DBC) typically on $[0, \pi]$. This will always be done by identifying the space of the functions fulfilling DBC with the space of the function fulfilling periodic boundary conditions on $[-\pi, \pi]$ which are skew symmetric. Similarly, Neumann boundary conditions will be treated by identifying the corresponding functions with periodic even functions. In some cases (e.g. in equation (**??**) with DBC and an f which does not have particular symmetries) the equations do not extend naturally to the space of skew symmetric and this has some interesting consequences (see [**?**, **?**]).

1.4 Basic elements of Hamiltonian formalism for PDEs

A suitable topology in the phase space is given by a Sobolev like topology.

For any $s \in \mathbb{R}$, define the Hilbert space ℓ^2_s of the sequences $x \equiv \{x_k\}_{k \geq 1}$ with $x_k \in \mathbb{R}$ such that

$$\|x\|_s^2 := \sum_k |k|^{2s} |x_k|^2 < \infty \tag{24}$$

and the phase spaces $\mathcal{P}_s := \ell^2_s \oplus \ell^2_s \equiv z \ni (p,q) \equiv (\{p_k\},\{q_k\})$. In $\mathcal{P}_s$ we will sometimes use the scalar product

$$\langle (p,q),(p^1,q^1)\rangle_s := \langle p,p^1\rangle_{\ell^2_s} + \langle q,q^1\rangle_{\ell^2_s} \ . \tag{25}$$

In the following we will always assume that

$$|\omega_k| \le C|k|^d \tag{26}$$

for some d.

Remark 1.13. Defining the operator $A_0 : D(A_0) \to \mathcal{P}_s$ by $A_0(p,q) = (\omega_k p_k, \omega_k q_k)$ one can write $H_0 = \frac{1}{2}\langle A_0 z; z\rangle_0$, $D(A_0) \supset \mathcal{P}_{s+d}$.

Given a smooth Hamiltonian function $\chi : \mathcal{P}_s \supset \mathcal{U}_s \to \mathbb{R}$, $\mathcal{U}_s$ being an open neighborhood of the origin, we define the corresponding Hamiltonian vector field $X_\chi : \mathcal{U}_s \mapsto \mathcal{P}_{-s}$ by

$$X_\chi \equiv \left(-\frac{\partial \chi}{\partial q_k}, \frac{\partial \chi}{\partial p_k} \right) \ . \tag{27}$$

Remark 1.14. Corresponding to a function χ as above we will denote by $\nabla\chi$ its gradient with respect to the $\ell^2 \equiv \ell^2_0$ metric. Defining the operator J by $J(p,q) := (-q,p)$ one has $X_\chi = J\nabla\chi$.

Definition 1.15. *The Poisson Bracket of two smooth functions χ_1, χ_2 is formally defined by*

$$\{\chi_1;\chi_2\} := d\chi_1 X_{\chi_2} \equiv \langle \nabla\chi_1; J\nabla\chi_2\rangle_0 \tag{28}$$

Remark 1.16. As the example $\chi_1 = \sum_k kq_k$, $\chi_2 := \sum_k kp_k$ shows, there are cases where the Poisson Bracket of two functions is not defined.

For this reason a crucial role is played by the functions whose vector field is smooth.

Definition 1.17. *A function $\chi \in C^\infty(\mathcal{U}_s,\mathcal{P}_s)$, $\mathcal{U}_s \subset \mathcal{P}_s$ open, is said to be of class* Gen_s*, if the corresponding Hamiltonian vector field X_χ is a smooth map from $\mathcal{U}_s \to \mathcal{P}_s$. In this case we will write $\chi \in \mathrm{Gen}_s$*

Proposition 1.18. Let $\chi_1 \in \mathrm{Gen}_s$. If $\chi_2 \in C^\infty(\mathcal{U}_s,\mathbb{R})$ then $\{\chi_1,\chi_2\} \in C^\infty(\mathcal{U}_s,\mathbb{R})$. If $\chi_2 \in \mathrm{Gen}_s$ then $\{\chi_1,\chi_2\} \in \mathrm{Gen}_s$.

Definition 1.19. *A smooth coordinate transformation $\mathcal{T} : \mathcal{P}_s \supset \mathcal{U}_s \to \mathcal{P}_s$ is said to be canonical if for any Hamiltonian function H one has $X_{H\circ\mathcal{T}} = \mathcal{T}^* X_H \equiv \mathrm{d}\mathcal{T}^{-1} X_H \circ \mathcal{T}$, i.e. it transforms the Hamilton equations of H into the Hamilton equations of $H \circ \mathcal{T}$.*

Proposition 1.20. Let $\chi_1 \in \mathrm{Gen}_s$, and let $\Phi^t_{\chi_1}$ be the corresponding time t flow (which exists by standard theory). Then for $\Phi^t_{\chi_1}$ is a canonical transformation.