Cauchy inequality in Banach spaces

D. Bambusi

28.05.07

Let $\mathcal{P}$ and $\mathcal{F}$ be reflexive real or complex Banach spaces, and let

$$\widetilde{F} : \underbrace{\mathcal{F} \times \mathcal{F} \times ... \times \mathcal{F}}_{r-\text{times}} \to \mathcal{P}$$

be a multilinear function.

Definition 0.1. A multilinear function is said to be *bounded* if there exists a constant C such that

$$\left\| \widetilde{F}(a^{(1)}, ..., a^{(r)}) \right\| \leq C \quad \forall a^{(1)}, ..., a^{(r)} \text{ with } \left\| a^{(i)} \right\| \leq 1 \ . \tag{0.1}$$

The best constant such that (0.1) holds is called *the norm of* $\widetilde{F}$ and is denoted by $\left\| \widetilde{F} \right\|$

Remark 0.2. Let $\widetilde{F}$ be a bounded multilinear function, then one has

$$\left\| \widetilde{F}(a^{(1)}, ..., a^{(r)}) \right\| \leq \left\| \widetilde{F} \right\| \left\| a^{(1)} \right\| ... \left\| a^{(r)} \right\| \ , \quad \forall a^{(i)} \in \mathcal{F} \ . \tag{0.2}$$

Proposition 0.3. *A multilinear function $\widetilde{F}$ is continuous (and therefore C^∞) if and only if it is bounded.*

The easy proof is left to the reader.

Definition 0.4. A function $F : \mathcal{F} \to \mathcal{P}$ is called a polynomial if there exists a multilinear function $\widetilde{F}$ such that

$$\widetilde{F}(a, ..., a) = F(a) \tag{0.3}$$

for all a in the domain of definition of F. The degree of a polynomial is its degree of homogeneity.

Definition 0.5. A polynomial F of degree r is said to be *bounded* if there exists a constant such that

$$\|F(a)\| \leq C \|a\|^r \ . \tag{0.4}$$

The best constant such that (0.4) is fulfilled is called the norm of F.

Proposition 0.6. *Let F be a polynomial of degree r, then, for $a^{(i)} \in \mathcal{F}$, the corresponding multilinear function is given by the polarization formula*

$$\widetilde{F}(a^{(1)}, ..., a^{(r)}) = \frac{1}{r!2^r} \sum_{\epsilon_j = \pm 1} \epsilon_1 ... \epsilon_r F(a^{(0)} + \epsilon_1 a^{(1)} + ... + \epsilon_r a^{(r)}) \ . \tag{0.5}$$

Remark 0.7. A polynomial is bounded if and only if the corresponding multilinear form is bounded.

Remark 0.8. A polynomial is continuous (and therefore C^∞) if and only if it is bounded.

Remark 0.9. A multilinear function on a real Banach space extends naturally by multilinearity to a multilinear function from the complexification of $\mathcal{F}$ to the complexification of $\mathcal{P}$. The same is true for a polynomial (just extend the corresponding multilinear form).

Let $f \in C^\infty(\mathcal{U}, \mathcal{P})$, with $\mathcal{U} \subset \mathcal{F}$ open and convex. It is well known that $\forall a_0 \in \mathcal{U}$ and $\forall r \geq 0$, one has

$$f(a) = \sum_{l=0}^{r} F_l(a - a_0) + \mathcal{R}_r(a, a_0) \ , \tag{0.6}$$

where F_l is a homogeneous polynomial of degree l and $\mathcal{R}_r$ has a zero of order $r+1$ at a_0. The polynomial $l!F_l$ is called the l-th differential of f at a_0 and will be denoted by $\mathrm{d}^l f(a_0)$. The expansion

$$f(a) = a_0 + F_1(a - a_0) + F_2(a - a_0) + ... \tag{0.7}$$

is called the Taylor expansion of f at a_0.

By Lagrange formula the remainder is actually estimated by

$$\|\mathcal{R}_r(a, a_0)\| \leq \frac{1}{(r+1)!} C_{r+1} \|a - a_0\|^{r+1} \tag{0.8}$$

where

$$C_{r+1} = \sup_{t \in [0,1]} \left\| \mathrm{d}^{r+1} f((1-t)a + t a_0) \right\| \tag{0.9}$$

From now on we will work in complex Banach spaces.

Definition 0.10. A function $f \in C^\infty(\mathcal{U}, \mathcal{P})$ is said to be *analytic* if $\forall a_0 \in \mathcal{U}$ there exist $R > 0$ such that the Taylor expansion of f at a_0 is convergent in the open ball $B(a_0, R)$ of center a_0 and radius R, uniformly in any compact set contained in $B(a_0, R)$.

The vector space of functions analytic on $\mathcal{U}$ with values in $\mathcal{P}$ will be denoted by $\mathcal{H}(\mathcal{U}, \mathcal{P})$

Remark 0.11. Let $f \in \mathcal{H}(\mathcal{U}, \mathcal{P})$, and let $a \in \mathcal{U}$. Fix also $b \in \mathcal{F}$ and a functional L in the dual of $\mathcal{P}$. It follows that the complex valued function $Lf(a + zb)$ of the complex variable z is analytic in a neighborhood of the origin.

Theorem 0.12. *(Cauchy inequality.) Let $f \in \mathcal{H}(\mathcal{U}, \mathcal{P})$. Let $a_0 \in \mathcal{U}$ and let $R > 0$ be such that $\overline{B(a_0, R)} \subset \mathcal{U}$ then one has*

$$\|\mathrm{d}f(a_0)\| \leq \frac{1}{R} \sup_{a \in \partial B(a_0,R)} \|f(a)\| \ . \tag{0.10}$$

Proof. Fix $h \in \mathcal{F}$ with $\|h\| = 1$, and an element L of the dual of $\mathcal{P}$ and consider the function $g(z) := Lf(a_0 + zh)$, which is analytic $|z| \leq R$. One has

$$L\mathrm{d}f(a_0)h = \left.\frac{d}{dz}\right|_{z=0} g(z) \ .$$

Using Cauchy inequality on g one has

$$\begin{aligned} |L\mathrm{d}f(a_0)h| &\leq \sup_{|z|=R} \frac{1}{R} |Lf(a_0 + zh)| \leq \|L\| \frac{1}{R} \sup_{|z|=R} \|f(a_0 + zh)\| \\ &\leq \|L\| \frac{1}{R} \sup_{a \in \partial B(a_0,R)} \|f(a)\| \ . \end{aligned} \tag{0.11}$$

But, since $\mathcal{P}$ is reflexive one has, for $b \in \mathcal{P}$,

$$\sup_{L \neq 0} \frac{|Lb|}{\|L\|} = \|b\| \ ,$$

and thus, taking the supremum of (0.11) over L one gets

$$\|\mathrm{d}f(a_0)h\| \leq \frac{1}{R} \sup_{a \in \partial B(a_0,R)} \|f(a)\| \ . \tag{0.12}$$

Since this equation holds for any h with $\|h\| = 1$ the thesis follows. □