Some Basic Results on Existence and Uniqueness in Semilinear equations*

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1 Introduction

To start with remark that the standard existence and uniqueness theorem for smooth vector fields holds also for infinite dimensional systems.

Precisely, let $\mathcal{P}$ be a Banach space then the following theorem holds

Theorem 1.1. *Let $X \in C^\infty(\mathcal{U}, \mathcal{P})$, with $\mathcal{U}$ open in $\mathcal{P}$, be a vector field, and let $\mathcal{V}$ be an open subset of $\mathcal{P}$ such that $\bar{\mathcal{V}} \subset \mathcal{U}$. Then there exists $\bar{t} > 0$ and, $\forall t$, $|t| \leq \bar{t}$ there exists a map $\Phi^t \in C^\infty(\mathcal{V}, \mathcal{U})$ s.t. $\forall x \in \mathcal{V}$ one has*

$$\frac{d}{dt}\Phi^t(x) = X(\Phi^t(x)) \ .$$

In the case of PDEs however the vector fields are not smooth, so one has to develop a more general theory, whose starting point is the theory of linear equations. The corner stones of the theory of linear equations are the Stone theorem and the theorem of Hille–Yoshida. Here we are going to prove a much simpler result adapted to the situation we are studying.

2 Linear theory

In the space $\mathcal{P}_s := \ell^2_s \times \ell^2_s$ $(s \geq 0)$ consider a quadratic Hamiltonian of the form

$$H_0(p, q) := \sum_{j\geq 1} \omega_j \frac{p_j^2 + q_j^2}{2} \tag{2.1}$$

with

$$\omega_j = aj^d + \nu_j \ , \quad |\nu_j| \leq Cj^\delta \ , \quad \delta < d \tag{2.2}$$

*this is a very preliminary version: in order to point out error or suggestion please write to bambusi@mat.unimi.it and consider the corresponding Hamilton equations that can be written in the form

$$\begin{cases} \dot{p}_j = -\omega_j q_j \\ \dot{q}_j = \omega_j p_j \end{cases} \iff \quad \dot{x} = Ax \tag{2.3}$$

where $x = (p, q)$ and $A : D(A) \to \mathcal{P}_s$ is the linear operator

$$\begin{pmatrix} p_j \\ q_j \end{pmatrix} \mapsto \begin{pmatrix} -\omega_j q_j \\ \omega_j p_j \end{pmatrix} \tag{2.4}$$

whose domain $D(A)$ clearly coincides with $\mathcal{P}_{s+d}$.

Remark 2.1. If $d > 0$ then A is an unbounded operator and therefore the standard existence and uniqueness theorem 1.1 does not apply to equation (2.3).

Using the first expression of (2.3) it is immediate to write its formal solution, namely

$$\begin{aligned} p_j(t) &= p_j^0 \cos(\omega_j t) - q_j^0 \sin(\omega_j t) \\ q_j(t) &= p_j^0 \sin(\omega_j t) + q_j^0 \cos(\omega_j t) \end{aligned}$$

Correspondingly define the family of linear operators

$$\begin{aligned} \mathcal{U}(t) : \mathcal{P}_s \quad &\to \quad \mathcal{P}_s \\ \begin{pmatrix} p_j \\ q_j \end{pmatrix} \quad &\mapsto \quad \begin{pmatrix} p_j \cos(\omega_j t) - q_j \sin(\omega_j t) \\ p_j \sin(\omega_j t) + q_j \cos(\omega_j t) \end{pmatrix} \end{aligned} \tag{2.5}$$

Remark 2.2. For any $t \in \mathbb{R}$ the operator $\mathcal{U}(t)$ is unitary:

$$\|\mathcal{U}(t)x\|_{\mathcal{P}_s} = \|x\|_{\mathcal{P}_s} \ ,$$

in particular it is globally defined, i.e. $D(\mathcal{U}(t)) = \mathcal{P}_s$.

The idea is that, for fixed $x_0 \in \mathcal{P}_s$ the curve

$$\gamma_{x_0}(t) := \mathcal{U}(t)x_0 \tag{2.6}$$

should be the solution of the equation (2.3), however, the situation is slightly complicated by the fact that equation (2.3) is defined only for $x \in D(A)$.

Theorem 2.3. *The operator family $\mathcal{U}(t)$ has the following properties*

1) *it is a group of continuous linear operators i.e.* $\mathcal{U}(t + s) = \mathcal{U}(t)\mathcal{U}(s)$, $\mathcal{U}(0) = \mathrm{I}$

2) *For any fixed t, $\mathcal{U}(t)$ commutes with A*

3) *$\forall x \in \mathcal{P}_s$ the curve $\gamma_x(t)$ is continuous as a curve in $\mathcal{P}_s$, i.e.*

$$\lim_{t \to t_1} \gamma_x(t) = \gamma_x(t_1)$$

4) *$\forall x \in D(A)$ the curve $\gamma_x(t)$ is differentiable and one has*

$$\frac{d}{dt}\gamma_x(t) = A\gamma_x(t) \iff \frac{d}{dt}\mathcal{U}(t)x = A\mathcal{U}(t)x \tag{2.7}$$

5) Let r be a positive integer, and let $D(A^r) = \mathcal{P}_{s+dr}$ be the domain of A^r then $\forall x \in D(A^r)$ the curve $\gamma_x(t)$ is of class C^r

6) $\forall r \geq 0$ one has $\mathcal{U}(t)D(A^r) = D(A^r)$.

The easy proof is omitted.

Remark 2.4. By equation (2.7), for $x \in D(A)$ the curve $\gamma_x(t)$ is really the solution of the equation (2.3). On the contrary, if $x \notin D(A)$ then $\gamma_x(t)$ is only continuous, and thus it does not really fulfill the equation (2.3). However, $\forall \bar{t} < \infty$, $\gamma_x(t)$ is the limit in $C([-\bar{t}, \bar{t}], \mathcal{P}_s)$ of true solutions.

Definition 2.5. $\mathcal{U}(t)$ is called the group generate by A, and $\forall x \in \mathcal{P}_s$ the curve $\gamma_x(t)$ is called the solution of equation (2.3) with initial datum x. Often $\mathcal{U}(t)$ is called the flow generated by A.

We will always use the notation

$$\mathrm{e}^{At} := \mathcal{U}(t) \tag{2.8}$$

A direct application of this theory allows to prove existence and uniqueness for the linearization of the systems described in chapter **??**, i.e. the wave equation, the Schrödinger equation and the transport equation with periodic boundary conditions. In particular one has that given an initial datum in H^s the solution remains H^s for all times.

Theorem 2.3 is a particular case of the Stone theorem that ensures the same conclusions for the equation

$$\dot{x} = Ax$$

where x is the element of an abstract Hilbert space and A is an abstract anti-selfadjoint operator. In the abstract case one gives the following definition

Definition 2.6. A linear operator A is said to generated a group of unitary operators if the conclusions of theorem 2.3 hold for the equation $\dot{x} = Ax$.

A further interesting example of linear equation is given by the transport equation

$$u_t = u_x \ , \quad x \in \mathbb{R} \tag{2.9}$$

in $L^2(\mathbb{R})$. In this framework one can write (2.9) in the form

$$\dot{u} = Au \ , \quad A \equiv \partial_x \ , \quad D(A) := H^1(\mathbb{R}) \ . \tag{2.10}$$

The formal solution of (2.9) is obviously

$$u(x,t) = u_0(x+t) \ , \tag{2.11}$$

where u_0 is the initial datum. A formula which is meaningful for all $u \in L^2$. However it is clear that if $u_0 \notin H^1$ then (2.11) is not time differentiable. Thus in this case one defines $\mathcal{U}(t)u \equiv \mathrm{e}^{At}u$ by the formula

$$\left[\mathrm{e}^{At}u\right](x) := u(x+t)$$

and the same conclusions of theorem 2.3 hold.

2.1 Dirichlet boundary conditions

The situation is different in the case of Dirichlet boundary conditions. To illustrate the behaviour of linear equations in such a case consider the wave equation. Define q_j by

$$u(x) = \sum_{j\geq 1} q_j \frac{\sin(jx)}{\sqrt{\pi}}$$

then the sequence q_j is in ℓ^2_s if and only if $u \in H^s$ **and the following boundary conditions are satisfied**

$$\partial^{2k} u(0) = \partial^{2k} u(\pi) = 0 \ , \quad \forall k \leq s/2 \ . \tag{2.12}$$

Thus in particular one has the following important corollary

Corollary 2.7. *Consider the Cauchy problem for the wave equation, namely*

$$u_{tt} - u_{xx} + m^2 u = 0$$
$$u(x,0) = u_0(x) \quad u_t(x,0) = v_0(x)$$

Assume $u_0 \in H^s$ *and* $v_0 \in H^{s-1}$ *and*

$$u_0^{2k}(0) = u_0^{2k}(\pi) = 0 \ , \quad \forall k \leq \frac{s}{2} \tag{2.13}$$
$$v_0^{2k}(0) = v_0^{2k}(\pi) = 0 \ , \quad \forall k \leq \frac{s-1}{2}$$

then the solution $u(t)$*,* $v(t)$ *is of class* $H^s \times H^{s-1}$ *as a function of* x*.*

One can see that if (2.13) are violated then the solution does not have the same smoothness of the initial datum.

3 Semilinear equations

We will develop the theory in an abstract real Hilbert space $\mathcal{P}$. In $\mathcal{P}$, consider the differential equation

$$\dot{x} = Ax + f(x) \tag{3.1}$$

Definition 3.1. The equation (3.1) is called semilinear if A is a linear operator generating a group, and f is a smooth (at least Lipschitz) map from some open set $\mathcal{U} \subset \mathcal{P}$ to $\mathcal{P}$.

The starting point of the theory of semilinear equation is the formula of variation of arbitrary constants. Indeed working formally one has that a solution of (3.1) with initial datum x_0 also fulfills the integral equation

$$x(t) = \mathrm{e}^{At} x_0 + \int_0^t \mathrm{e}^{A(t-s)} f(x(s)) ds \ . \tag{3.2}$$

Now, while eq. (3.1) is meaningful only when $x \in D(A)$, eq. (3.2) is meaningful also when $x \notin D(A)$. The idea is to use the contraction mapping principle in order to construct a solution of (3.2) and to define the solution of the integral equation to be the solution also of the corresponding differential equation (3.1).

Definition 3.2. A *mild solution* of (3.1) is a solution of the integral equation (3.2).

Theorem 3.3. *Assume that*

- *A generates a group*
- *$f \in C^k(\mathcal{U}, \mathcal{P})$ for some $k \geq 1$, and $\mathcal{U} \subset \mathcal{P}$ open*
- *there exists $\bar{r} > 0$ s.t. $f(D(A^r)) \subset D(A^r)$ $\forall r \leq \bar{r}$*

Let $\mathcal{V} \subset \mathcal{U}$ be open such that $\bar{\mathcal{V}} \subset \mathcal{U}$, then there exists $\bar{t} > 0$ and, $\forall t$ with $|t| \leq \bar{t}$ a unique C^k map $\Phi^t : \mathcal{V} \to \mathcal{U}$ with the following properties

1) $\forall x_0 \in \mathcal{V}$, $x(t) := \Phi^t(x_0)$ is a mild solution of (3.1),

2) $\Phi^t(D(A^r)) \subset D(A^r)$ $\forall r \leq \bar{r}$

3) Assume $k \geq \bar{r}$ then $\forall x_0 \in \mathcal{V} \cap D(A^r)$, $r \leq \bar{r}$ the curve

$$\gamma_{x_0}(t) := \Phi^t(x_0)$$

is of class C^r

4) If $\bar{r} \geq 1$ and $x_0 \in D(A)$, then $\gamma_{x_0}(t)$ fulfills (3.1).

Sketch of the proof. Fix $x_0 \in \mathcal{U}$ and $\bar{t} > 0$, remark that, provided $\bar{t}$ is small enough then there exists a solution $x(.) \in C^0([-\bar{t}, \bar{t}], \mathcal{P})$ of (3.2). This is an immediate consequence of the fact that the map

$$C^0([-\bar{t}, \bar{t}], \mathcal{P}) \ni x(.) \mapsto \mathrm{e}^{At} x_0 + \int_0^t \mathrm{e}^{A(t-s)} f(x(s)) ds \in C^0([-\bar{t}, \bar{t}], \mathcal{P})$$

is a contraction of a small ball centered at the constant function $x(t) \equiv x_0$. Thus one simply defines Φ^t in the obvious way. Smoothness is proved in the usual way. The invariance of the domain of the powers of A can be obtained for example by repeating the argument in the space $D(A^r)$ endowed by the graph norm.

Example 3.4. In $H^s(\mathbb{T})$, $s \geq 1$, consider the nonlinear wave equation

$$u_{tt} - u_{xx} + m^2 u = \pm u^p \tag{3.3}$$

with $p \in \mathbb{N}$, $p \geq 2$. One can rewrite (3.3) in the form

$$\begin{matrix} \dot{v} = \partial_{xx} u \pm u^p \\ \dot{u} = v \end{matrix} \iff \frac{d}{dt}\begin{pmatrix} v \\ u \end{pmatrix} = \begin{pmatrix} \partial_{xx} u \\ v \end{pmatrix} \pm \begin{pmatrix} u^p \\ 0 \end{pmatrix}$$

which has the form (3.1) provided one defines $x := (v, u) \in H^{s-1} \times H^s \equiv \mathcal{P}$ and

$$A \begin{pmatrix} v \\ u \end{pmatrix} := \begin{pmatrix} \partial_{xx} u \\ v \end{pmatrix} \ , \quad f \begin{pmatrix} v \\ u \end{pmatrix} := \pm \begin{pmatrix} u^p \\ 0 \end{pmatrix}$$

The theory of the previous section shows that A generates a group. By Sobolev embedding theorem one has that, for $s \geq 1$

$$\left\| f \begin{pmatrix} v \\ u \end{pmatrix} \right\| = \|u^p\|_{H^{s-1}} \leq \|u^p\|_{H^s} \leq C_s \|u\|^p_{H^s} = C_s \|x\|^p$$

which shows that f is a bounded polynomial and therefore it is an analytic function.

Thus $\forall (v_0, u_0) \in H^{s-1} \times H^s$ equation (3.3) has a unique mild solution with the same smoothness as the initial datum.

4 Integrals of motion and global existence

Consider a semilinear equation of the form (3.1) and let $H : \mathcal{U} \to \mathbb{R}$ be a smooth function, then one can define the Lie derivative

$$\pounds_X H := \mathrm{d}HX : D(A) \to \mathbb{R} \tag{4.1}$$

Proposition 4.1. *Assume that the hypotheses of theorem 3.3 are verified, with $k \geq \bar{r} \geq 1$, and that*

$$\pounds_X H(x) = 0 \ , \quad \forall x \in D(A)$$

then along any mild solution of eq. (3.1) *one has*

$$H(x(t)) = H(x_0) \ .$$

Proof. Take a sequence $x_0^{(n)} \in D(A)$ s.t. $x_0^{(n)} \to x_0$ and denote by $x^{(n)}(t)$ the corresponding solutions. Then $x^{(n)}(t)$ is time differentiable by theorem 3.3 item 3, and, by the classical argument, one has

$$H(x^{(n)}(t)) = H(x_0^{(n)}) \ .$$

As $n \to \infty$ $x^{(n)}(t) \to x(t)$ and $x_0^{(n)} \to x_0$, thus the thesis follows. □

By standard argument one can use a priori estimates in order to prove that local solutions are global. As an example consider again the wave equation (3.3), whose Hamiltonian function is (cf. (**??**))

$$H(v, u) = \int_{-\pi}^{\pi} \left(\frac{v^2}{2} + \frac{u_x^2}{2} + \frac{m^2 u^2}{2} \mp \frac{u^{p+1}}{p+1} \right) \mathrm{d}x \tag{4.2}$$

By the previous proposition this is a conserved quantity along the flow, which exists locally in time in any of the spaces $\mathcal{P}_s := H^{s-1} \times H^s$, $s \geq 1$. We use the conservation of H in order to show global existence in $\mathcal{P}_1$.

Theorem 4.2. *If p is odd and in eq.* (3.3) *one has the sign minus, then any mild solution with initial datum in $H^0 \times H^1$ is global in time. The same conclusion holds also if p is even and/or the sign at r.h.s. is plus, but the norm of the initial datum is small.*

Proof. Denote

$$\|(v,u)\|^2 := H_0(v,u) \equiv \int_{-\pi}^{\pi} \left(\frac{v^2}{2} + \frac{u_x^2}{2} + \frac{m^2u^2}{2} \right) \mathrm{d}x$$

and

$$P(v,u) = \mp \int_{-\pi}^{\pi} \left(\frac{u^{p+1}}{p+1} \right) \mathrm{d}x$$

and consider first the case odd p and minus sign in the equation(which means plus in the energy), so that one has $P(x) > 0\ \forall x$, then one has

$$H_0(x(t)) \leq H_0(x(t)) + P(x(t)) = H(x_0)$$

which in turn is finite since P is bounded, Thus one has that $\|x(t)\|$ is bounded and therefore the existence time is infinite.

We come to the general case. First remark that by Sobolev embedding theorem one has

$$|P(x)| \leq C H_0(x)^{(p+1)/2}$$

and thus one has, at least for small x_0

$$H(x_0) = H(x(t)) \geq H_0(x(t))[1 - C H_0^{(p-1)/2}]$$

which implies the thesis. □