FAMILIES OF PERIODIC SOLUTIONS OF REVERSIBLE PDEs

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1 Introduction

Some families of periodic solutions of PDEs can be constructed using KAM theory; however a different approach leading to stronger results and simpler proofs is available. It is based on the Lyapunov–Schmidt decomposition combined with a suitable analysis of small denominators. The main advantage of this approach is elimination of the second Melnikov condition (see (**??**)) which allows to apply it to problems with periodic boundary conditions and to equations in more than one space dimension. Most of the general theory has been developed for equations that are of second order in time and we will mainly deal with this case. Moreover, we will concentrate on problems involving small denominators and only briefly report on results of a different kind.

2 An abstract theorem for nonresonant PDEs

Let $\{X_s\}$ be a scale of Hilbert spaces with norms $\|\cdot\|_s$ and scalar product $\langle\cdot;\cdot\rangle_s$. Let A be a (linear) morphism of the scale, and assume that there exists a Hilbert basis $\{\varphi_j\}_{j=1}^{\infty}$ such that

$$A\varphi_j = \omega_j^2\varphi_j \ , \quad \omega_j > 0$$

Let us fix s, consider a neighbourhood $\mathcal{U}$ of the origin in X_s and a smooth map $g : \mathcal{U} \to X_s$, having at the origin a zero of second order. We are interested in families of small amplitude periodic solutions of the equation

$$\ddot{x} + Ax = g(x) \ . \tag{2.1}$$

Example 2.1. The nonlinear wave equation with periodic boundary conditions:

$$w_{tt} - w_{xx} + V(x)w = f(x,w) \ , \tag{2.2}$$

$$w(x,t) = w(x+2\pi,t) \ , \quad w_x(x,t) = w_x(x+2\pi,t) \ , \tag{2.3}$$

where the potential V and the nonlinearity f are periodic of period 2π in x, and $f(x,w) = O(|w|^2)$. Let λ_j be the periodic eigenvalues of the Sturm Liouville operator $-\partial_{xx} + V(x)$ and assume $\lambda_j > 0\ \forall j$. Then the frequencies are $\omega_j := \sqrt{\lambda_j}$. In this case $X_s = H^s(\mathbb{T})$, and a smooth f induces a smooth operator from X_s to itself, provided that $s > 1/2$. $\square$

Example 2.2. The nonlinear plate equation in the d dimensional cube:

$$w_{tt} + \Delta\Delta w + aw = f(w)\ , \quad x \in \mathcal{Q} \tag{2.4}$$

$$w\big|_{\partial\mathcal{Q}} = \Delta w|_{\partial\mathcal{Q}} = 0\ , \tag{2.5}$$

where $a > 0$, $f(w) = O(|w|^3)$ and

$$\mathcal{Q} := \left\{x = (x_1, ..., x_d) \in \mathbb{R}^d\ :\ 0 < x_i < \pi\right\}.$$

Then the eigenfunctions of the linearised system are given by

$$\varphi_n = \sin(n_1 x_1)\sin(n_2 x_2)...\sin(n_d x_d)$$

and the corresponding frequencies are $\omega_n = \sqrt{(n_1^2 + ... + n_d^2)^2 + a}$, where $n \in \mathbb{Z}^d$ and $n_i \geq 1$. To fit the above scheme we order the basis in such a way that the frequencies are in nondecreasing order. Now $X_0 = L^2(\mathcal{Q})$, and $X_s = D((\Delta\Delta)^s) \subset H^{4s}$ endowed with the graph norm. If the nonlinearity f is smooth and odd (i.e. $f(-w) = -f(w)$), then it defines a smooth map from X_s to itself for any $s > [d/2]/4$ (see example **??**). $\square$

In the linear approximation ($g \equiv 0$) the general solution of (2.1) is the superposition of the linear normal modes, i.e. of the families of periodic solutions

$$x^{(j)}(t) = (a_j \cos(\omega_j t) + b_j \sin(\omega_j t))\varphi_j\ . \tag{2.6}$$

Fix one of the families, say $x^{(1)}$. To ensure its persistence in the nonlinear problem we make the following assumptions:

H1) (Nonresonance) For small enough $\gamma > 0$ there exists a closed set $W_\gamma \subset \mathbb{R}^+$ having ω_1 as an accumulation point both from the right and from the left, and such that for any $\omega \in W_\gamma$ one has

$$|\omega l - \omega_j| \geq \frac{\gamma}{l}\ , \quad \forall\, l \geq 1\ ,\ \forall j \geq 2 \quad \square \tag{2.7}$$

H2) (Nondegeneracy) Let $g_r(x)$ be the first non-vanishing (homogeneous) Taylor polynomial of g. Assume that $r \geq 3$ and $\beta_0 \neq 0$, where

$$\beta_0 := \begin{cases} \langle g_r(\varphi_1), \varphi_1\rangle_0 & \text{if} \quad r \quad \text{is} \quad \text{odd,} \\ \langle g_{r+1}(\varphi_1), \varphi_1\rangle_0 & \text{if} \quad r \quad \text{is} \quad \text{even.} \quad \square \end{cases} \tag{2.8}$$

Denoting $\xi_1(\omega_1 t) = \cos(\omega_1 t)\varphi_1$ one has

Theorem 2.3. *Suppose that assumptions H1,H2 hold. Then there exists a set* $\mathcal{E} \subset \mathbb{R}$ *having zero as an accumulation point, a positive* ω_*, *and a family of periodic solutions* $\{x_\epsilon(t)\}_{\epsilon\in\mathcal{E}}$ *of (2.1) with frequencies* $\{\omega^\epsilon\}_{\epsilon\in\mathcal{E}}$ *fulfilling*

$$\sup_t \|x_\epsilon(t) - \epsilon\xi_1(t\omega^\epsilon)\|_s \le C\epsilon^r \ , \quad |\omega^\epsilon - \omega_1| \le C\epsilon^{r-1} \ . \tag{2.9}$$

Moreover, the set $\mathcal{E}$ *is in one to one correspondence either with* $W_\gamma \cap [\omega_1, \omega_1+\omega_*)$ *if* $\beta_0 < 0$, *or with* $W_\gamma \cap (\omega_1 - \omega_*, \omega_1]$ *if* $\beta_0 > 0$.

Proof. We consider only the case of odd r, the general case can be obtained by a slightly different treatment of the forthcoming equation ω. We are looking for an X_s-valued function $q(t)$ which is 2π-periodic and reversible (i.e., $q(t) = q(-t)$), and for a positive ω, close to ω_1, such that $q(\omega t)$ is a solution of (2.1). They must satisfy the equation

$$L_\omega q = g(q) \ , \quad L_\omega := \omega^2 \frac{d^2}{dt^2} + A \ , \tag{2.10}$$

which will be considered as an ω-dependent functional equation in the space $\mathcal{H} \subset H^1(\mathbb{T}, X_s)$, formed by the reversible periodic functions. Equation (2.10) is studied using the Lyapunov–Schmidt decomposition, namely by decomposing it into an equation on $\mathrm{Ker} L_{\omega_1} \equiv \mathrm{span}(\xi_1)$ and an equation on its complement R. Precisely, denote by Q the projector on ξ_1 and by P the projector on R and make the Ansatz $q = \epsilon\xi_1 + \epsilon^r u$, where $u \in R$. Then (2.10) is equivalent to the system

$$\omega^2 = \omega_1^2 + \beta\epsilon^{r-1} \tag{2.11}$$

$$L_\omega u = Pg_r(\xi_1) + PG(\epsilon, u) \tag{2.12}$$

$$-\beta\xi_1 = Qg_r(\xi_1) + QG(\epsilon, u) \tag{2.13}$$

for the unknowns (ϵ, u, β). Here G contains all higher order corrections and $\omega \in W_\gamma$ is a parameter. The equations (2.11), (2.12) and (2.13) are called the ω, the P and the Q equation, respectively.

First one solves the P equation (2.12). To this end one has to invert the linear operator $L_\omega\big|_R$. Its eigenfunctions are $\cos(lt)\varphi_j$, and the corresponding eigenvalues are

$$\lambda_{jl} = -l^2\omega^2 + \omega_j^2 = (l\omega + \omega_j)(\omega_j - l\omega) \ , \ j \ge 2 \ , \ l \ge 1.$$

By (2.7), $|\lambda_{jl}| > C\gamma$. So $(L_\omega\big|_R)^{-1}$ exists and is bounded. Applying this operator to the P equation and using the implicit function theorem one obtains a smooth function $u(\epsilon)$ that depends parametrically on $\omega \in W_\gamma$ and solves the P equation.

Inserting $u(\epsilon)$ in the Q equation one determines the parameter β as a function of ϵ. In particular one has $\beta(\epsilon) = C\beta_0$+higher order corrections, where $C > 0$. Inserting $\beta(\epsilon)$ in the ω equation one gets an equation for ϵ (remember that ω is fixed), which is a perturbation of the equation $\omega^2 - \omega_1^2 = C\beta_0\epsilon^{r-1}$. By the nondegeneracy this can be reduced to a fixed point equation for ϵ^{r-1} which is solvable by the contraction mapping principle. □

Remark 2.4. The theorem holds also in the case $r = 2$, but in this case the nondegeneracy condition takes a more complicated form. □

Theorem 2.3 was proved in [Bam00]. The technique of the Lyapunov–Schmidt decomposition was used for the first time to construct families of periodic solutions in PDEs by Craig and Wayne [CW93] who considered the model problem of the wave equation with periodic boundary conditions (see example 2.1); we will report on this work in Section 4.

Example 2.5. Consider the nonlinear wave equation with periodic boundary conditions (see example 2.1). Let ω_1 be such that $\omega_1 \neq \omega_j$ for each $j \neq 1$. Decompose V into its average a and a part $\tilde{V}$ of zero average, then condition H1 is satisfied if a belongs to an uncountable set which is dense in a neighbourhood of the origin (for the proof see Lemma 3.1 of [BP02]). Condition H2 can be expressed in terms of the eigenfunctions of the Sturm–Liouville operator. If it holds, then Theorem 2.3 applies and ensures persistence of the corresponding family of periodic orbits. Note that, in a difference with the case of Dirichlet boundary conditions (see example **??**), the nonlinearity does not need to have some particular parity. □

Example 2.6. Consider the nonlinear plate equation (see example 2.2). In the case $d = 1$ all the frequencies are simple and the assumption H1 is satisfied if a is chosen in a subset of $\mathbb{R}^+$ having full measure. In the case $d > 1$, all the frequencies are multiple except the smallest one. Taking for ω_1 the smallest frequency, H1 is fulfilled if a belongs to a dense uncountable subset of [0,1/4]. H2 holds trivially provided the Taylor expansion of f at zero does not vanish identically (remember that $f(-w) = f(w)$). Then Theorem 2.3 ensures persistence of the corresponding family of periodic orbits (for details see [BP02]). □

3 The resonant case

It is possible to generalise the above theorem to the case when the frequencies satisfy some resonance relations. We will consider only the Lagrangian case, when $g = -\nabla H$.

Fix a frequency ω_1 of the linearised system. We replace the assumption H1 by the following one:

H1R) For any small enough γ there exists a closed set $W_\gamma \subset \mathbb{R}^+$ having ω_1 as an accumulation point both from the right and from the left, and such that for any $\omega \in W_\gamma$ one has

$$\text{either} \qquad |\omega l - \omega_j| \geq \frac{\gamma}{l}\ , \qquad \text{or} \qquad l\omega_1 - \omega_j = 0. \tag{3.1}$$

□

To pass to the nondegeneracy assumption, we define the resonant set as

$$\mathcal{I}_R := \{k \geq 1 \ : \ \exists l \geq 1 \ : \ l\omega_1 - \omega_k = 0\}\ , \tag{3.2}$$

consider the linear space generated by $\{\varphi_k\}_{k\in\mathcal{I}_R}$, and denote by $\mathcal{N}$ its closure in the graph norm of $D(A)$. Note that all solutions of the linearised system with initial datum in $\mathcal{N}$ and vanishing initial velocity are periodic of period $2\pi/\omega_1$. Let H_r be the first non vanishing Taylor coefficient of H. For $x \in \mathcal{N}$ define the average of H_r by

$$\langle H_r\rangle(x) := \frac{\omega_1}{2\pi}\int_0^{2\pi/\omega_1} H_r(\cos(At)x)dt \ .$$

Consider the hypersurface $\mathcal{S} \subset \mathcal{N}$ of the points $x \in \mathcal{N}$ such that $\langle x; Ax\rangle_0 = 1$.

H2R) There exists a nondegenerate critical point x_0 of the functional $\langle H_r\rangle\big|_{\mathcal{S}}$. The corresponding Lagrange multiplier β_0 does not vanish. □

Denote by $\xi_0(\omega_1 t)$ the solution of the linearised system with initial datum x_0 and vanishing initial velocity.

Theorem 3.1. *[BP01] Suppose the assumptions H1R, H2R hold. Then there exists a family of periodic solutions* $\{x_\epsilon(t)\}_{\epsilon\in\mathcal{E}}$ *of (2.1) with frequencies* ω^ϵ*, satisfying*

$$\sup_t \|x_\epsilon(t) - \epsilon\xi_0(t\omega^\epsilon)\|_s \leq C\epsilon^r \ , \quad |\omega^\epsilon - \omega_1| \leq C\epsilon^{r-1} \ . \tag{3.3}$$

The set $\mathcal{E}$ *has the same properties as in the nonresonant case.*

The proof is obtained by proceeding as in the nonresonant case. The only difference is that in this case the kernel of L_{ω_1} is no longer one dimensional, but is isomorphic to $\mathcal{N}$ (the isomorphism being given by the map $x \mapsto \cos(At/\omega_1)x$). So the Q equation can be transformed into an equation in $\mathcal{N}$. The latter turns out to be a perturbation of the equation for the critical points of $\langle H_r\rangle\big|_{\mathcal{S}}$, and the nondegeneracy condition H2R allows to solve it by the implicit function theorem.

Applying the above theorem, one can construct countably many families of periodic solutions of the ϕ^4-model

$$w_{tt} - w_{xx} = \pm w^3 + \text{higher order terms}$$

with Dirichlet boundary conditions, and also higher frequency periodic solutions of the nonlinear plate equation of example 2.2 (see [BP01, BP02], see also [LS88, Bou99b]).

In general it is difficult to check condition H2R. In the case of Hamiltonian systems with $n < \infty$ degrees of freedom, topological arguments allow to avoid it. Indeed, the Weinstein–Moser theorem (see [Wei73, Mos76]) ensures that close to a minimum of the energy on each surface of a constant energy there exist at least n periodic orbit. In general they do not form regular families. A corresponding result for PDEs is not available at present. However there exists an *ad hoc* variational result for the wave equation

$$w_{tt} - w_{xx} = \pm w^p + \text{higher order terms} \ , \quad p \geq 2 \ , \tag{3.4}$$

which ensures that, having fixed $j \geq 1$, there exists a sequence of periodic orbits accumulating at zero, whose frequencies accumulate at j (which plays here the role of the $j-$th linear frequency). Corresponding theorem is due to Berti and Bolle [BB03].

Periodic solutions in the nonlinear wave equation

$$w_{tt} - w_{xx} + f(x, w) = 0 \ , \quad u(0, t) = u(\pi, t) = 0 \tag{3.5}$$

where constructed for the first time by Rabinowitz [Rab78] using global variational methods and a Lyapunov–Schmidt decomposition. Rabinowitz proved that, under suitable assumptions on f, equation (3.5) has at least one periodic solution with period $T = 2\pi p/q$, for any choice of the integers p and q. Note that, when the period T is commensurable with 2π, the operator $L_\omega|_R$ has a compact inverse, i.e. there are no small denominators. The work [Rab78] was followed by a series of papers, simplifying the proof and sharpening the result (see [Bre83] and references therein). In particular, we mention the paper [BCN80] by Brezis, Coron and Nirenberg, where existence of periodic orbits is proved by a particularly simple method: the authors write a variational principle, dual to the usual one, and look for its critical points, using the mountain pass lemma. It is remarkable that in this approach the Q equation becomes trivial.

4 Weakening the nonresonance condition

The main limitation of the results presented in Sections 2 and 3 rests in the nonresonance conditions H1 and H1R. Indeed, such conditions are fulfilled with large probability (in a suitable parameter space) when $\omega_j \sim j^\nu$ with $\nu > 1$; when $\nu = 1$ the nonresonance conditions are satisfied typically on uncountable sets of zero measure, but when $\nu < 1$ they are satisfied only exceptionally (as in the plate equation). As a consequence the results of Sections 2 and 3 are not applicable to general equations in more than one space dimensions. Furthermore, the method of Lyapunov–Schmidt decomposition can be extended to the case of reversible systems of first order in time, but the approach of Section 2 is no more applicable.

In order to avoid such limitations one would like to be able to work with the weaker nonresonance condition "there exists a $\tau > 0$ such that $|l\omega - \omega_j| \geq \gamma/l^\tau$". This was done by Craig and Wayne [CW93] who used the Nash–Moser theorem to solve the P equation. The application of the Nash Moser theorem requires to construct and estimate the inverse of the linear operator describing the linearisation of the P equation at an approximate solution. This is the main difficulty of Craig–Wayne's approach. To overcome it they use the techniques by Fröhlich and Spencer [FS83], performing a careful analysis of small denominators (cf. Section **??**). The method by Craig and Wayne was extended by Bourgain in order to construct periodic (and also quasi periodic) solutions in higher dimensional equations. The resulting method seems very general, but at present a theorem “ready for application” is not available. We present here the result obtained by Bourgain by applying this method to the nonlinear wave equation

$$w_{tt} - \Delta w + aw + w^3 = 0 \tag{4.1}$$

on $\mathbb{T}^d$. Fix a multiindex $n \in \mathbb{Z}^d$ different from zero, and let

$$\xi_n(\omega_n t, x) := \cos(n \cdot x + \omega_n t) \ , \quad \omega_n := \sqrt{n_1^2 + ... + n_d^2 + a},$$

be the corresponding symmetric reversible solution.

Theorem 4.1. *[Bou95]. If a belongs to a certain subset of $\mathbb{R}^+$ of full measure, then there exists a Cantor set $\mathcal{E}$ of positive measure, accumulating at zero, and a family of periodic solutions $\{w_\epsilon(t,x)\}_{\epsilon \in \mathcal{E}}$ of (4.1) with frequencies ω^ϵ, satisfying*

$$|\epsilon \xi_n(\omega^\epsilon t, x) - w_\epsilon(t,x)| \leq C\epsilon^3 \ , \quad |\omega_n - \omega^\epsilon| \leq C\epsilon^2 \ .$$

In the case $d = 1$, the result was proved in [CW93]; subsequently, still in the case $d = 1$, Kuksin introduced a simpler technique to find the “large measure result” of theorem 4.1 (see in [Bou99a] pp. 90–94).

The Craig–Wayne–Bourgain method also allows to deal with first order in time equations. For example, it was applied to the Schrödinger equation in one [CW94] or two space dimensions [Bou98].

5 The water wave problem

A particular problem that has attracted the attention of many researchers since the very beginning of the theory of PDEs is that of existence of standing water waves. The first rigorous proof of their existence was obtained only recently by Plotnikov and Toland [PT01]; we present here their result.

Consider a perfect fluid lying above a horizontal bottom, and confined between two parallel vertical walls. The fluid is subject to gravity, and atmospheric pressure acts at the free surface. This is a dynamical system governed by the Euler equations supplemented by appropriate boundary conditions. It was pointed out by Zakharov that this system is Hamiltonian (see [Zak68]). The corresponding Hamiltonian function is the energy of the fluid, and conjugated variables are given by the wave profile and the velocity potential at the free surface.

In the linear approximation the general solution is given by the superposition of the normal modes. The problem is to continue the normal modes to families of periodic solutions of the nonlinear system (the standing waves). Fix one of the normal modes, and denote by $\eta(t, x_1)$ the corresponding profile of the free surface (x_1 being the horizontal variable). Then it is possible to choose the depth h, the width l of the region occupied by the fluid and the gravitational constant g in such a way that the period of the solution is normalised to 2π and the linear frequencies fulfill a suitable nonresonance condition. Denote by (g_0, l_0, h_0) a choice of the parameters realizing such conditions, then one has

Theorem 5.1. *[PT01] There exists an infinite set $\mathcal{E} \subset \mathbb{R}$ having zero as an accumulation point and, for any $\epsilon \in \mathcal{E}$, there exist g_ϵ, l_ϵ and a standing wave solution of the water wave problem with gravity g_ϵ in a box of width l_ϵ. Moreover, denoting by η_ϵ the corresponding profile of the free surface, one has*

$$|\eta_\epsilon(t, x_1) - \epsilon^2 \eta(t, x_1)| < C\epsilon^3 \ , \quad |g_\epsilon - g_0| + |l_\epsilon - l_0| \leq C\epsilon \ .$$

The main difficulties in proving this result are as follows: firstly, the linear frequencies behave as $\omega_n \sim n^{1/2}$, so the nonresonance conditions that can be satisfied are quite weak. Secondly, the mathematical formulation of the problem involves an unbounded nonlinear and non-local operator. To overcome these difficulties, Plotnikov and Toland use the Lagrangian description of the fluid motion and apply the Lyapunov–Schmidt approach to handle the resulting nonlinear problem. The P equation now is solved by means of the Nash–Moser theorem. The required invertibility of the linearised operator is obtained in two steps: first it is reduced to a suitable canonical form, and next this canonical form (which is essentially a perturbation of an operator involving derivatives and Hilbert transform) is studied in detail.

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