A reversible Nekhoroshev theorem for persistence of invariant tori in systems with symmetry

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Abstract

We prove a variant of a theorem by Nekhoroshev on persistence of invariant tori in systems with symmetry. The new proof applies to reversible non Hamiltonian systems equivariant under the action of an Abelian group and is much simpler then the original one.

1 Introduction

In 1994 Nekhoroshev [Nek94] published a theorem ensuring the persistence of n-dimensional invariant tori in a system of n commuting Hamiltonian vector fields. In some sense this theorem can be viewed as an extension to systems with symmetry of Poincaré's theorem of persistence of periodic orbits. A complete proof of this theorem based on the ideas explained by Nekhoroshev can be found in [BG02] (see also [Gae02]). Subsequently the theorem was applied to some infinite dimensional lattices [BV02] and to PDEs [BB09] (see also [JA97]).

In the present paper we prove a new version of Nekhoroshev's theorem valid in the non Hamiltonian reversible case. More precisely, we are interested in the dynamics of a vector field which is invariant under the action of a commutative group. The result we obtain coincides with the original Nekhoroshev's theorem (obviously except for the existence of action angle variables).

We emphasize that the new proof which, following [BB09], is based on Lyapunov-Schmidt decomposition, is much simpler then those previously published. The main motivation for the present work, however is that of understanding better the role of reversibility in the construction of invariant tori (see [Sev86, ZGY11, BBP14]). Indeed sometimes reversibility plays a surprising role (we have in mind the papers [BDG10, CG15]) and we hope that the present paper, which deals with a very simple situation can help to clarify the situation.

Before stating and proving the main theorem it is worth to add a few words on the problem of persistence of invariant tori in systems with symmetry. This will also be useful in order to explain the structure that we will assume for the vector fields we study.

The main point is that the theory of persistence of invariant tori in systems with symmetry is a generalization of Poincaré theory of persistence of periodic orbits. The essential result of such a theory is that, if the Floquet multiplier 1 has multiplicity 1, then a periodic orbit persists under perturbation. However, in the case of conservative systems, the Floquet multiplier 1 has always multiplicity at least 2, and so one cannot expect periodic orbits to persist under generic perturbations (actually there are well known counterexamples, see e.g.[MZ05]). Persistence of the periodic orbit is ensured when the perturbed system is conservative or reversible.

Also in the case of Nekhoroshev's invariant tori one can define a concept of Floquet multiplier (see [BG02]), which extends also to the non Hamiltonian case. One can prove that for an n dimensional torus Λ, if the Floquet multiplier 1 has multiplicity n, then Λ persists under general perturbations [Gae02].

However, in the Hamiltonian case the Floquet multiplier 1 always has multiplicity at least $2n$, thus one cannot expect the torus to persist under generic perturbation. Nekhoroshev's theorem ensures that it persists if the perturbation is Hamiltonian. Here we extend the result to the case where the perturbation is non Hamiltonian, but reversible.

The paper also contains an appendix on the "Normal form of Hamiltonian vector fields invariant under a torus action" in which a result showing that the model problem studied in the core of the paper occurs quite generally when one is perturbing a Hamiltonian system invariant under a symplectic torus action.

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2 Statement

Consider the phase space $\mathbb{R}^n \times \mathbb{T}^n \times \mathbb{R}^k \times \mathbb{R}^k$ in which the coordinates will be denoted by $z \equiv (I, \alpha, p, q)$. Consider n commuting vector fields of the form

$$X_0^{(l)}(z) = \frac{\partial}{\partial \alpha_l} + \sum_{j=1}^{k} \Omega_j^{(l)} \left(p_j \frac{\partial}{\partial q_j} - q_j \frac{\partial}{\partial p_j} \right) , \quad l = 1, ...n , \tag{2.1}$$

where $\Omega_j^{(l)}$ are numbers playing the role of frequencies. The $X_0^{(l)}$ will play the role of unperturbed vector fields.

Consider the torus $\Lambda = (0, \mathbb{T}^n, 0, 0)$.

Remark 2.1. *The torus Λ is invariant and linearly stable for the flow of each one of the vector fields $X_0^{(l)}$.*

Let $\mathcal{U}$ be a fixed open neighborhood of Λ. For $l = 1, ..., n$, let $X_1^{(l)} \in C^\infty(\bar{\mathcal{U}}; \mathbb{R}^{n+k})$ be functions possibly depending also on a small parameter μ, and consider the vector fields $X_\mu^{(l)} := X_0^{(l)} + \mu X_1^{(l)}$.

Denote by Φ_l^t the flows of the vector fields $X_\mu^{(l)}$; we assume that

(1) for $l = 2, ..., n$, $\forall z \in \mathcal{U}$ and μ small enough, one has $\Phi_l^{t+2\pi} = \Phi_l^t$, $\forall t \in \mathbb{R}$.

(2) $[X_\mu^{(m)}; X_\mu^{(l)}] \equiv 0$, $\forall l, m = 1, ..., n$.

Remark 2.2. *The vector fields $X_\mu^{(l)}$, $l = 2, ..., n$ generate a group action of $\mathbb{T}^{n-1}$ on the phase space; the vector field $X_\mu^{(1)}$ is then equivariant with respect to such a group action.*

Remark 2.3. *In the Appendix we will prove a theorem (see Theorem A.1) showing that the above structure is typical when one is dealing with perturbations of a Hamiltonian vector field invariant under a symplectic action of the group $\mathbb{T}^{n-1}$. Precisely, if the unperturbed vector field is integrable on a submanifold, then close to one of the invariant tori of the submanifold there exist coordinates in which the field takes the form $X_\mu^{(1)}$. The same is true for the vector fields generating the symmetry; such vector fields will also fulfill assumption (2).*

Remark 2.4. *All what we will do extends easily to the case where the unperturbed invariant torus is not linearly stable.*

Then we assume that the fields are reversible. Precisely define the involution S by

$$S(I, \alpha, p, q) := (I, -\alpha, -p, q) \ , \tag{2.2}$$

and assume

(3) For any $l = 1, ..., n$ and any μ small enough one has

$$X_\mu^{(l)}(S\zeta) = -SX_\mu^{(l)}(\zeta) \ .$$

It follows that if $z(t)$ is a solution of the equations of one of the vector fields then also $Sz(-t)$ is a solution of the same equations.

Finally we assume that

(4) One has

$$\Omega_j^{(1)} \notin \mathbb{Z} \ , \quad \forall j = 1, ..., k \ . \tag{2.3}$$

In the following statement, by invariant manifold, we will mean invariant under the flows of each one of the fields $X_\mu^{(l)}$, $l = 2, ..., n$.

Theorem 2.5. *Assume (1-4), then there exists $\mu_* > 0$ s.t., for any $|\mu| < \mu_*$ there exists a $2n$ dimensional invariant manifold $M_\mu \subset \mathcal{U}$ foliated in n dimensional invariant tori $\Lambda_{\mu,a}$, $a \in \mathcal{V} \subset \mathbb{R}^n$, where $\mathcal{V}$ is open. Furthermore the flow of $X_\mu^{(1)}$ on each one of the tori $\Lambda_{a,\mu}$ is quasiperiodic with a number of independent frequencies smaller or equal then n. The same is true for any linear combination of the fields $X_\mu^{(l)}$.*

Remark 2.6. *We emphasize that in the Hamiltonian case the vector field $X_\mu^{(1)}$, due to the commutation property, admits n integrals of motion. On the contrary in our case, such a vector field may have no integrals of motion at all.*

We find surprising the fact that the condition of reversibility is enough to substitute that of existence of n integrals of motion and to ensure the existence of a manifold of n dimensional invariant tori.

Remark 2.7. *The above theorem (and its proof) extends in a straightforward way to the case of infinite dimensional systems of the kind of those studied in [BV02]. An extension to the case with small denominators as the ones studied in [BB09] is also possible.*

3 Proof

The first and main step of the proof consists in determining some parameters $\omega, \epsilon_2, .., \epsilon_n$ and a function $\zeta(\tau)$ periodic of period 2π which is a solution of the equations

$$\omega \frac{d\zeta}{d\tau} = X_\mu^{(1)} + \sum_{l=2}^{n} \epsilon_l X_\mu^{(l)} \tag{3.1}$$

(so that $z(t) := \zeta(\omega t)$ is periodic with period $2\pi/\omega$ in the time t).

We will show that for all μ small enough there exists an n-dimensional manifold of periodic solutions. Then Theorem 2.5 follows by considering the manifold obtained applying the flow of the vector fields $X_\mu^{(l)}$ $l = 2, .., n$ to the manifold of such solutions. The details will be given at the end of the section.

In order to determine the functions $\zeta(\tau)$ we will interpret the equation (3.1) as a functional equation in a suitable function space and use the Lyapunov Schmidt decomposition and the implicit function theorem in order to solve it. We now look for a 2π periodic function

$$u(\tau) \equiv (I(\tau), \psi(\tau), p(\tau), q(\tau)) \ ,$$

such that $\zeta(\tau) := (0, e_1\tau, 0, 0) + u(\tau)$ is a solution of (3.1) with e_1 the unit vector of the first axis. Furthermore we will take reversible solution, namely solutions fulfilling $\zeta(\tau) = S\zeta(-\tau)$.

We rewrite the equations for u, ω, ϵ in the form

$$F(u, \omega, \epsilon, \mu) = 0 \ , \tag{3.2}$$

where $F \equiv (F_I, F_\psi, F_p, F_q)$ and the components are defined by

$$F_{I_1} := \omega \frac{\partial \psi_1}{\partial \tau} + (\omega - 1) - \mu X^{(1)}_{1,\psi_1} - \sum_{l=2}^{n} \mu \epsilon_l X^{(l)}_{1,\psi_1} \tag{3.3}$$

$$F_{I_j} := \omega \frac{\partial \psi_j}{\partial \tau} - \epsilon_j - \mu X^{(1)}_{1,\psi_j} - \sum_{l=2}^{n} \mu \epsilon_l X^{(l)}_{1,\psi_j} \tag{3.4}$$

$$F_{\psi_j} := \omega \frac{\partial I_j}{\partial \tau} - \mu X^{(1)}_{1,I_j} - \sum_{l=2}^{n} \mu \epsilon_l X^{(l)}_{1,I_j} \tag{3.5}$$

$$F_{p_j} := \omega \frac{\partial q_j}{\partial \tau} - \Omega^1_j p_j - \mu X^{(1)}_{1,q_j} - \sum_{l=2}^{n} \mu \epsilon_l X^{(l)}_{1,q_j} \tag{3.6}$$

$$F_{q_j} := \omega \frac{\partial p_j}{\partial \tau} + \Omega^1_j q_j - \mu X^{(1)}_{1,p_j} - \sum_{l=2}^{n} \mu \epsilon_l X^{(l)}_{1,p_j} \tag{3.7}$$

(the inversion of the order of the variables is useful in the following).

We now fix the function space in which we will look for solutions. Let $H^s(\mathbb{T}^1; \mathbb{R}^{2N})$ be the space of the periodic functions of τ taking value in $\mathbb{R}^{2N}$, having s weak derivatives of class L^2. Then we work in the subspace composed by reversible functions (namely by those the functions $u \in H^s$ such that $u(\tau) = Su(-\tau)$). This space will be denote by $\mathcal{H}^s$. We fix $s \geq 2$.

First we have the following simple lemma:

Lemma 3.1. *There exists a neighborhood of* $(0, 1, 0, 0)$ *such that the function* $F : \mathcal{H}^s \times \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} \to \mathcal{H}^{s-1}$ *defined by* (3.3)-(3.7) *is of class* C^∞.

In order to apply the method of Lyapunov-Schmidt decomposition to solve equation (3.2), we first have to study the operator $L := d_u F(0, 1, 0, 0)$. A simple computation shows that

$$L \begin{bmatrix} I \\ \psi \\ p_j \\ q_j \end{bmatrix} = \begin{bmatrix} \frac{\partial \psi}{\partial \tau} \\ \frac{\partial I}{\partial \tau} \\ \frac{\partial q_j}{\partial \tau} - \Omega^{(1)}_j p_j \\ \frac{\partial p_j}{\partial \tau} + \Omega^{(1)}_j q_j \end{bmatrix} . \tag{3.8}$$

In particular one gets that, under the nonresonance condition (4) one has that $Lu = 0$ implies $I_j(\tau) = a_j$, $\psi_j(\tau) = \phi_j$ (independent of τ). But *the reversibility condition* implies $\phi_j = 0$. Thus one has

$$K := KerL = (a, 0, 0, 0) .$$

One also has that the range R of L coincides with the space of the functions $(I, \psi, p, q) \in \mathcal{H}^{s-1}$, with $I(\tau)$ having zero average. Denote by P the projector on R and by Q the projector on K. We write $u = a + w$ with $a \in K$ (identified with $\mathbb{R}^n$) and $w \in R \cap \mathcal{H}^s$ (function in $\mathcal{H}^s$ with zero average).

Applying as usual P and Q to eq. (3.2) one gets

$$PF(a+w, \omega, \epsilon, \mu) = 0 \ , \tag{3.9}$$

$$QF(a+w, \omega, \epsilon, \mu) = 0 \ , \tag{3.10}$$

and of course $d_w PF(0,1,0,0) = L\big|_R$ which is an isomorphism between $R \cap \mathcal{H}^s$ and R.

Thus applying the implicit function theorem to equation (3.9) one gets the following lemma:

Lemma 3.2. *There exists a neighborhood* $\mathcal{U}$ *of* $(0,1,0,0) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}$ *and a* $C^\infty(\mathcal{U}, R)$ *map* $(a, \omega, \epsilon, \mu) \mapsto w(a, \omega, \epsilon, \mu)$ *which fulfills* $w(0,1,0,0) = 0$ *and solves* (3.9)*.*

Inserting into equation (3.10) and exploiting the explicit form of the function F one gets the equations

$$0 = (\omega - 1) - \mu Q X^{(1)}_{1,\psi_1}(a + w(a, \omega, \epsilon, \mu)) - \sum_{l=2}^{n} \mu \epsilon_l Q X^{(l)}_{1,\psi_1}(a + w(a, \omega, \epsilon, \mu)) \tag{3.11}$$

$$0 = \epsilon_j - \mu Q X^{(1)}_{1,\psi_j}(a + w(a, \omega, \epsilon, \mu)) - \sum_{l=2}^{n} \mu \epsilon_l Q X^{(l)}_{1,\psi_j}(a + w(a, \omega, \epsilon, \mu)) \ . \tag{3.12}$$

By implicit function theorem one immediately gets the following lemma

Lemma 3.3. *There exists a neighborhood* $\mathcal{V}$ *of the origin in* $\mathbb{R}^n \times \mathbb{R}$ *and a* C^∞ *map* $\mathcal{V} \ni (a, \mu) \mapsto (\omega, \epsilon)$ *which solves* (3.11)*,* (3.12)*, furthermore, one has* $\omega(0,0) = 1$ *and* $\epsilon(0,0) = 0$*.*

As a consequence one gets the existence of the wanted periodic solutions which is given by the following lemma

Lemma 3.4. *There exists a smooth map*

$$\mathcal{V} \ni (a, \mu) \mapsto (u, \omega, \epsilon) \in \mathcal{H}^s \times \mathbb{R} \times \mathbb{R}^{n-1}$$

such that $\zeta_{a,\mu}(\tau) := u_{a,\mu}(\tau) + (0, e_1\tau, 0, 0)$ *is a periodic solution of* (3.1) *with period* 2π*.*

Remark 3.5. *By standard regularization argument for the solutions of ordinary differential equations, one has that actually* u *depends on the rescaled time* τ *in a* C^∞ *way.*

End of the proof of Theorem 2.5. Denote by $\Phi_j^{t_j}$ the flow of the vector field $X_\mu^{(l)}$, $l = 2, ..n$, then one has that for any vector $(t_2, ..., t_n)$, $\Phi_2^{t_2} \circ ... \circ \Phi_n^{t_n} \zeta_{a,\mu}(\tau)$ with $\zeta_{a,\mu}$ given by Lemma 3.4is also a periodic solution of (3.1). It follows that

$$\Lambda_{a,\mu} := \bigcup_{\tau, t_2, ..., t_n} \Phi_2^{t_2} \circ ... \circ \Phi_n^{t_n} \zeta_{a,\mu}(\tau) \tag{3.13}$$

is an n-dimensional torus which is invariant under the dynamics of each one of the fields. $\square$

A Normal form of Hamiltonian vector fields invariant under a torus action

On a $2N$ dimensional symplectic manifold M consider a symplectic torus action g:

$$\mathbb{T}^{n-1} \times M \ni (\beta, x) \mapsto g^{\beta}(x) \in M \ ;$$

admitting a momentum map. For $l = 2, ..., n$, denote by $X^{(l)}(x) := \left[\partial g^{\beta}(x)/\partial\beta_l\right]_{\beta=0}$ the vector fields generating the symmetry group and by $H^{(l)}$ the corresponding Hamiltonian functions. On M consider a further Hamiltonian system with Hamiltonian function $H^{(1)}$ invariant under the above group action. Denote by $X^{(1)}$ the corresponding Hamiltonian vector field. We assume that there exists a $2n$-dimensional submanifold $\mathcal{T}^{2n} \subset M$ invariant under the flow of each one of the fields $X^{(l)}$, $l = 1, ..., n$, and that the vector fields $X^{(l)}$ are independent on $\mathcal{T}^{2n}$. Then each of the Hamiltonian systems is integrable on $\mathcal{T}^{2n}$, which is thus foliated in invariant tori. We fix one of such tori, denoted by Λ, and consider just the intersection of $\mathcal{T}^{2n}$ with a small open neighborhood of such a torus. We still denote by $\mathcal{T}^{2n}$ the restricted manifold.

Theorem A.1. *There exist an open neighborhood $\mathcal{U}$ of $\mathcal{T}^{2n}$, open neighborhoods of the origin $\mathcal{I} \subset \mathbb{R}^n$, $\mathcal{W} \subset \mathbb{R}^{2k}$, $k := N - n$, and a coordinate system*

$$\mathcal{I} \times \mathbb{T}^n \times \mathcal{W} \ni (I, \alpha, y) \mapsto x(I, \alpha, y) \in \mathcal{U} \subset M \ ,$$

with the following properties: $\Lambda = x(0, \mathbb{T}^n, 0)$, $\mathcal{T}^{2n} = x(\mathcal{I}, \mathbb{T}^n, 0)$ and there exist n commuting linear operators $A^{(l)} : \mathbb{R}^{2k} \to \mathbb{R}^{2k}$, skew symmetric with respect to the standard symplectic form, s.t. one has

$$X^{(l)} = \frac{\partial}{\partial\alpha_l} + \sum_{i,j} [A^{(l)}]^i_j y^j \frac{\partial}{\partial y^i} + O(|y|^2 + |I||y|) \ , \quad l = 2, ..., n \ . \tag{A.1}$$

Furthermore there exists a function $\tilde{H}^{(1)}(H^{(1)}, ..., H^{(n)})$, whose Hamiltonian vector field has the form

$$X_{\tilde{H}^{(1)}} = \omega \frac{\partial}{\partial\alpha_1} + \sum_{i,j} [A^{(1)}]^i_j \frac{\partial}{\partial y^i} y^j + O(|y|^2 + |I||y|) \ . \tag{A.2}$$

Finally there exists a basis in $\mathbb{R}^{2k}$ s.t.

$$\sum_{i,j} [A^{(l)}]^i_j y^j \frac{\partial}{\partial y^i} = \sum_{j=1}^{k} \Omega^{(l)}_j \left(p_j \frac{\partial}{\partial q_j} - q_j \frac{\partial}{\partial p_j} \right) \ ; \quad l = 2, ..., n \ ; \tag{A.3}$$

if the torus Λ is linearly stable (A.3) *holds also for $A^{(1)}$.*

Remark A.2. *Dividing $\tilde{H}^{(1)}$ by ω and rescaling the variables according to $y = \mu\tilde{y}$, $I = \mu^2\tilde{I}$, one has that in the new variables the considered vector fields take the form $X_\mu^{(l)}$ with $X_0^{(l)}$ as in* (2.1)*. In Sect. 2, the function $X_1^{(1)}$ also contains the non Hamiltonian perturbation to the original vector field.*

Proof. First, consider the restriction of the Hamiltonians to $\mathcal{T}^{2n}$; by the procedure of Arnold Liouville theorem, $\mathcal{T}^{2n}$ is foliated into n dimensional tori, then one can construct linear combinations of the considered Hamiltonains which generate the fundamental loops of the tori in a neighborhood of Λ. Since the flows of $H^{(l)}$, $l = 2,..,n$ are periodic and they are independent, they already generated $n-1$ independent loops. The last loop is then generated by a linear combination of all the Hamiltonians with coefficients which depend on the Hamiltonians themselves. We call such a linear combination $\tilde{H}^{(1)}$. Then we construct action angle variables using such loops. We shift them in order to have that Λ correspond to $I = 0$. It follows that, on $\mathcal{T}^{2n}$ the Hamiltonians have the form

$$H^{(l)}\big|_{\mathcal{T}^{2n}} = h^{(l)}(I_l) = I_l \ , \ l = 2, ..., n$$

where the last equality is due to the fact that all the solutions of the Hamilton equations have period 2π. For $\tilde{H}^{(1)}$ it simply equals $h^{(1)}(I_1)$.

To find a coordinate system in a neighborhood of $\mathcal{T}^{2n}$ we use ideas of Floquet theory (following [Kuk92]). Denote by $\mathtt{p}(I,\alpha)$ the point of $\mathcal{T}^{2n}$ with action angle coordinates (I,α). Denote $T_{\mathtt{p}(0,0)}\mathcal{T}^{2n} \subset T_{\mathtt{p}(0,0)}M$ the tangent space to $\mathcal{T}^{2n}$ at $\mathtt{p}(0,0)$, and by $Y := T^{\perp}_{\mathtt{p}(0,0)}\mathcal{T}^{2n}$ its symplectic orthogonal.

Denote (as before) by Φ_1^t, Φ_l^t the flows of $\tilde{H}^{(1)}$ and $H^{(l)}$ $l = 2, ..., n$ respectively, and by

$$\Phi_{l*}^t(\mathtt{p}(0, \alpha_1, ..., \alpha_{l-1}, 0, \alpha_{l+1}, ..., \alpha_n)) \equiv \Phi_{l*}^t(\alpha),$$

the corresponding tangent maps. Let T^1 be the period of $\Phi_1^t|_\Lambda$, and let $T^l = 2\pi$, $l = 2, ..., n$. Since Φ_{l*}^t is symplectic, one has that

$$\Phi_{l*}^{T_l}(0) : Y \to Y \ .$$

Define now the Linear operators $A^{(l)} : Y \to Y$ by

$$e^{A^{(l)}2\pi} = \Phi_{l*}^{T_l}(0) \ ; \tag{A.4}$$

then the wanted system of coordinates is defined by the map

$$\begin{aligned}&\mathcal{I} \times \mathbb{T}^n \times Y \to M\\ &\quad (I,\alpha,y) \mapsto x(I,\alpha,y) := \mathtt{p}(I,\alpha) + \Phi_{1*}^{\alpha_1 T_1/2\pi}\Phi_{2*}^{\alpha_2}...\Phi_{n*}^{\alpha_n} e^{-A^{(1)}\alpha_1}...e^{-A^{(n)}\alpha_n} y \ .\end{aligned}$$

In order to simplify the notation we write

$$\Phi_{1*}^{\alpha_1 T_1/2\pi}\Phi_{2*}^{\alpha_2}...\Phi_{n*}^{\alpha_n} e^{-A^{(1)}\alpha_1}...e^{-A^{(n)}\alpha_n} =: \Phi_*^{\alpha} e^{-\alpha A} \ .$$

To write the form of the vector field $X^{(l)}$ in this system of coordinates we write the corresponding differential equations. In order to simplify the notation we consider $l \neq 1$, the case $l = 1$ can be dealt with similarly. One has

$$\begin{aligned}\dot{x}(I,\alpha,y) &= \frac{\partial \mathtt{p}}{\partial I}\dot{I} + \frac{\partial \mathtt{p}}{\partial \alpha}\dot{\alpha} + \dot{\alpha}\frac{\partial}{\partial \alpha}(\Phi_*^\alpha e^{-\alpha A})y + \Phi_*^\alpha e^{-\alpha A}\dot{y} \\ &= X^{(l)}(x(I,\alpha,y)) = X^{(l)}(\mathtt{p}(I,\alpha)) + dX^{(l)}(\mathtt{p}(I,\alpha))\Phi_*^\alpha e^{-\alpha A} y + O(|y|^2)\ . \quad \text{(A.5)}\end{aligned}$$

Up to corrections of order y^2 one has

$$\frac{\partial \mathtt{p}}{\partial I}\dot{I} + \frac{\partial \mathtt{p}}{\partial \alpha}\dot{\alpha} = X^{(l)}(\mathtt{p}(I,\alpha)) \implies \begin{cases} \dot{I} = 0 \\ \dot{\alpha}_j = \delta_{lj} \end{cases}$$

and thus

$$\begin{aligned}\dot{\alpha}\frac{\partial}{\partial \alpha}(\Phi_*^\alpha e^{-\alpha A})y = \frac{\partial}{\partial \alpha_l}(\Phi_*^\alpha e^{-\alpha A})y &= dX^{(l)}(\mathtt{p}(0,\alpha))\Phi_*^\alpha e^{-\alpha A}y - \Phi_*^\alpha e^{-\alpha A}A^{(l)}y \\ &= -\Phi_*^\alpha e^{-\alpha A}\dot{y} + dX^{(l)}(\mathtt{p}(I,\alpha))\Phi_*^\alpha e^{-\alpha A}y \end{aligned} \quad \text{(A.6)}$$

where we used the fact that the Φ_{l*}'s commute, that the same holds for the $A^{(l)}$'s. Furthermore, the second equality follows from the fact that Φ_*^α is the evolution operator of the linearization of the equations at $\mathtt{p}(0,\alpha)$, and the third equality follows from (A.5). From (A.6) one has

$$\dot{y} = A^{(l)}y + O(|y|^2 + |y||I|)\ .$$

Finally we have to prove (A.3). First remark that the operators $A^{(l)}$ are skew symmetric with respect to the symplectic form. For $l = 2, ..., n$ they are diagonalizable and have purely imaginary spectrum, due to the assumption that the flows Φ_l are periodic with period 2π. Then, by standard theory of linear symplectic operators there exists a basis in which they take the form (A.3) (see [Gio]). In the case where the torus is linearly stable, the same holds for $A^{(1)}$. □

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