A few basic results on Sobolev spaces*

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24.05.07

In this section we give some definitions and properties of the Sobolev spaces of square integrable functions with square integrable derivatives of suitable order.

On $C^\infty(\mathbb{T})$ consider the the following seminorm

$$\|u\|_{H^s}^2 := \|u\|_{L^2}^2 + \|\partial_x^s u\|_{L^2}^2 \tag{0.1}$$

Definition 0.1. The Sobolev space $H^s(\mathbb{T})$ is the closure of $C^\infty(\mathbb{T})$ in the norm (0.1).

Remark 0.2. Consider the Fourier basis $\hat{e}_j$ (cf. eq.(**??**)), and the map

$$H^s(\mathbb{T}) \ni u \mapsto \{u_j\}_{j\in\mathbb{Z}} \in \ell_s^2$$

with u_j defined by $u = \sum_j u_j \hat{e}_j$. By the very definition of H^s such a map is an isometric isomorphism.

Remark 0.3. $\forall r \leq s$ one has

$$\|\partial_x^r u\|_{L^2} \leq \|u\|_{H^s} \tag{0.2}$$

Indeed, by expanding in Fourier series one has

$$\|\partial_x^r u\|_{L^2}^2 = \sum_j |j|^{2r} |u_j|^2 < \|u\|_{H^s}^2 \ .$$

Theorem 0.4. *Assume $s \geq 1$, then $\forall r < s-1/2$ one has that H^s is continuously embedded in $C^r(\mathbb{T})$, i.e. any function $u \in H^s(\mathbb{T})$ is r times differentiable and there exists a constant $C_{r,s}$ s.t.*

$$\sup_{x\in\mathbb{T}} |\partial_x^r u(x)| \leq C_{r,s} \|u\|_{H^s} \ . \tag{0.3}$$

Proof. First we prove that (0.3) holds for C^∞ functions. The general result follows by density. Writing $u = \sum_j u_j \hat{e}_j$, one has

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*this is a very preliminary version: in order to point out error or suggestion please write to bambusi@mat.unimi.it

$$\begin{aligned} |\partial_x^r u(x)| &\leq \sum_j \frac{1}{\sqrt{\pi}} |j|^r |u_j| = \sum_j \frac{1}{\sqrt{\pi}} \frac{|j|^r}{\sqrt{1+|j|^{2s}}} \sqrt{1+|j|^{2s}} |u_j| \\ &\leq \frac{1}{\sqrt{\pi}} \sqrt{\sum_j \frac{|j|^{2r}}{1+|j|^{2s}}} \sqrt{\sum_j (1+|j|^{2s})|u_j|^2} = \frac{1}{\sqrt{\pi}} C_{r,s} \left\| u \right\|_{H^s} \ . \end{aligned}$$

Take now a general $u \in H^s$ and a sequence $u^{(n)} \to^{H^s} u$. By (0.3) the sequence $u^{(n)}$ converges to u also in C^r and thus u is r times differentiable, moreover (0.3) holds by continuity of the norm. ☐

Remark 0.5. Equation (0.3) is a particular case of the Gagliardo Nirenberg inequality, which holds for much more functions spaces. For the general result see e.g. [**?**].

Theorem 0.6. $\forall s \geq 1$ *the bilinear map*

$$\begin{array}{rcl} H^s \times H^s & \to & H^s \\ (u,v) & \mapsto & uv \end{array}$$

(point product) is analytic. Equivalently there exists a constant C_s *s.t.*

$$\left\| uv \right\|_{H^s} \leq C_s \left\| u \right\|_{H^s} \left\| v \right\|_{H^s} \tag{0.4}$$

Proof. We are going to prove that $\forall r \leq s$ one has

$$\left\| \partial_x^r (uv) \right\|_{L^2} \leq C \left\| u \right\|_{H^s} \left\| v \right\|_{H^s} \ , \tag{0.5}$$

which implies the result. One has

$$\partial_x^r (uv) = \sum_{j=0}^{r} \binom{r}{j} (\partial_x^j u)(\partial_x^{r-j} u) \ . \tag{0.6}$$

Consider a single monomial of (0.6) with $j \neq r$, one has

$$\begin{aligned} \left\| (\partial_x^j u)(\partial_x^{r-j} u) \right\|_{L^2}^s &= \int_{\mathbb{T}} (\partial_x^j u)^2 (\partial_x^{r-j} u)^2 \\ \leq \sup_{x \in \mathbb{T}} \left| \partial_x^j u(x) \right| \int_{\mathbb{T}} (\partial_x^{r-j} u)^2 &\leq C_{j,s} \left\| u \right\|_{H^s}^2 \left\| \partial_x^{r-j} v \right\|_{L^2}^2 \\ &\leq C_{j,s} \left\| u \right\|_{H^s}^2 \left\| v \right\|_{H^s}^2 \end{aligned}$$

For the term $j = r$ the same result is immediately obtained by interchanging the role of u and v. Summing up one gets the thesis. ☐

Corollary 0.7. *For any* $s \geq 1$ *and any integer* k *the map* $H^s \ni u \mapsto u^k \in H^s$ *is analytic.*

Proof. By theorem (0.6) one has that $u \mapsto u^2$ fulfills the inequality

$$\left\|u^2\right\|_{H^s} \leq C \left\|u\right\|_{H^s}^2 \ ,$$

which shows analyticity of the map for $k = 2$. Iterating (i.e. writing $u^k = u^{k-1}u$) one gets the result. $\square$