LONG TIME STABILITY IN PERTURBATIONS OF COMPLETELY RESONANT PDE'S

D. BAMBUSI*, N.N. NEKHOROSHEV**

Abstract. In this paper we will present some results concerning long time stability in nonlinear perturbations of resonant linear PDE's with discrete spectrum. In particular we will prove that if the perturbation satisfies a suitable nondegeneracy condition then there exists a periodic like trajectory, i.e. a closed curve in the phase space with the property that solutions starting close to it remain close to it for very long times. Secondly, in the special case where the average of the main part of the perturbation is integrable we will prove that if the energy is initially essentially concentrated on finitely many modes, then along the corresponding solutions all the actions are approximatively constant for very long times. Applications to nonlinear wave and Schrödinger equations on a segment will also be given.

1. Introduction

In this paper we will present some results concerning the qualitative behavior of solutions of some Hamiltonian PDE's. In particular we will concentrate on perturbations of resonant linear PDE's and give results in the spirit of the works [1,2].

We recall that, for finite dimensional systems, the corner stone of qualitative theory of near to integrable Hamiltonian systems is KAM theory which ensures that solutions corresponding to the majority of initial data lie on invariant tori. For solutions starting outside KAM tori Arnold diffusion could appear, but it has been shown that it would take very long times: One can prove that the actions of the unperturbed system are approximatively constant for times growing exponentially with an inverse power of the size of the perturbation [1,2]. In particular this is very important for applications in which the actions are usually the relevant physical quantities.

* Dipartimento di Matematica dell'Università, Via Saldini 50, 20133 Milano, Italy. email: bambusi@mat.unimi.it

** Department of Mechanics and Mathematics, Moscow State University, Moscow 119899, Russia. email: nekhoros@mech.math.msu.su

A generalization of KAM theory to PDE's has been obtained by Kuksin [3] (see also [4,5,6]), and extended by Craig and Wayne and Bourgain (see [7,8,9,10]). Such results ensure that most of the *finite dimensional* tori of the unperturbed system survive a small enough perturbation. These results form a very general theory that applies e.g. to perturbations of Klein Gordon equation [11], Korteweg de Vries equation [12], nonlinear Schrödinger equation [13] and 2 dimensional Schrödinger equation [10]. However these theorems give informations only on solutions starting on finite dimensional submanifolds of the phase space. Moreover the threshold for the perturbation under which tori of dimension N survive converges very quickly to zero as $N \to \infty$. It follows that althoght very interesting, invariant tori correspond to exceptional initial data: It is therefore very important to be able to describe solutions starting outside such tori.

In the present paper we will present a technique that allows to describe the qualitative behavior of solutions with initial data in open sets of the phase space, when the unperturbed system is linear and completely resonant, like in the case of the wave equation on a segment or the nonlinear Schrödinger equation on a segment. We will first discuss the case of a perturbation fulfilling a suitable nondegeneracy condition and we will construct some families of closed curves $\{\gamma_\epsilon\}_{0<\epsilon<\epsilon_*}$ in the phase space (here ϵ^2 is the harmonic energy evaluated on any point of the curve) which behaves as "practically stable periodic like solutions", by this we mean that solutions starting $\mathcal{O}(\epsilon^2)$ close to γ_ϵ remain $\mathcal{O}(\epsilon^2)$ close to it up to times of order $\exp(c\epsilon^{-2})$ (see [14,15]). We will also present a result ensuring existence of a Cantor family of exact periodic solutions close to the family $\{\gamma_\epsilon\}_{0<\epsilon<\epsilon_*}$ (see [16]). The abstract result will be applied to the nonlinear wave equation [17]

$$u_{tt} - u_{xx} = \pm u^3 \ . \tag{1.1}$$

We point out that the curves γ_ϵ are characterized as critical points of a functional constructed using the average of the perturbation over the linear flow.

Then we consider the (exceptional) case where the average of the perturbation is an integrable system; this happens e.g. in the case of the nonlinear Schrödinger equation

$$iu_t = u_{xx} + u\varphi(|u|^2) \ , \quad \varphi'(0) \neq 0 \ . \tag{1.2}$$

Following [18], we prove that along solutions corresponding to initial datum in which the majority of the energy is stored in a finite number of modes all the actions of the linearized system are approximatively constant for times exponentially long with the inverse of the initial energy.

We point out that all the proofs we give here are much simplified with respect to the original ones.

The paper is organized as follows: In sect 2 we give an heuristic description of our methods and proofs. In sect 3 we give a precise statement of the normal forms results on which the whole theory is based. We also give the scheme of its proof. In sect. 4 we give statements and proofs of the results concerning existence of periodic like solutions. In sect. 5 we give the results concerning the case of perturbations with integrable average, and its application to the nonlinear Schrödinger equation.

2. Heuristic description

Consider a Hamiltonian system

$$H(p,q)=h_0(p,q)+P(p,q)\ , \tag{2.1}$$

where

$$h_0(p,q):=\sum_{k\geq 1}k^d\frac{p_k^2+q_k^2}{2}\ , \tag{2.2}$$

$d\geq 1$ is an integer, and P is the perturbation which will be assumed to be an homogeneous polynomial of some degree $r\geq 3$. All what follows holds with minor changes also for the case of perturbations which are analytic in a neighborhood of the origin.

Remark that all the solutions of the linearized system $P\equiv 0$ are periodic with the same period 2π.

We are interested to small solutions, so it is useful to introduce the rescaled variables (p',q') defined by $(p,q)=\epsilon(p',q')$. Omitting primes the Hamiltonian takes the form

$$H(p,q)=h_0(p,q)+\mu P(p,q)\ , \tag{2.3}$$

with $\mu:=\epsilon^{r-2}$.

Consider action angle coordinates I_k,ϕ_k defined by

$$p_k=\sqrt{2I_k}\cos\phi_k\ ,\quad q_k=\sqrt{2I_k}\sin\phi_k\ ,$$

then h_0 takes the form $h_0(I)=\sum_{k\geq 1}k^dI_k$; it is thus natural to introduce the new actions

$$J_1:=\sum_{k\geq 1}k^dI_k\ ,\quad J_k=I_k\ ,\quad \forall\, k\geq 2$$

and the corresponding angles ψ_k; in these variables the Hamiltonian takes the form

$$H=J_1+\mu P(J,\psi)\ . \tag{2.4}$$

Remark that along the unperturbed flow all the variables are constant except the fast angle ψ_1 that rotates with angular velocity 1. Thus the idea is to average with respect to the fast angle, i.e. to look for a canonical transformation (we will prove its existence in the next section) that transform (2.4) into

$$H:=J_1+\mu+\mathcal{Z}_\mu(J,\psi_2,\psi_3,\psi_4,...)+e^{-1/\mu}\mathcal{R}_\mu(J,\psi)\ , \tag{2.5}$$

where

$$\mathcal{Z}_\mu(J,\psi_2,\psi_3,\psi_4,...):=\langle P\rangle\,(J,\psi_2,\psi_3,\psi_4,...)+\mu Z_\mu(J,\psi_2,\psi_3,\psi_4,...)$$

is the part in normal form of the Hamiltonian, $\langle P\rangle$ is the average of P with respect to ψ_1, namely

$$\langle P\rangle\,(J,\psi_2,\psi_3,\psi_4,...):=\frac{1}{2\pi}\int_0^{2\pi}P(J,\psi)d\psi_1$$

(a quantity that can be explicitly computed in concrete cases), Z_μ is a higher order correction to the normal form; finally $\mathcal{R}_\mu$ is an exponentially small remainder that depends on all the actions and all the angles.

2.1 The case of generic perturbations

To understand the behavior of the system (2.5) let us consider first the averaged system defined by the Hamiltonian

$$H_a := J_1 + \mu \langle P \rangle (J, \psi_2, \psi_3, \psi_4, ...) ;$$

it is clear that since J_1 is an integral of motion H_a, is equivalent to the reduced system with Hamiltonian

$$\langle P \rangle (J, \psi_2, \psi_3, \psi_4, ...) , \tag{2.6}$$

where J_1 is considered as a parameter, and the dynamical variables are now

$$(J_2, J_3, J_4, ..., \psi_2, \psi_3, \psi_4, ...) .$$

In general this is a non integrable infinite dimensional system, so one can start to study some features of its dynamics, and in particular to look for its critical points. Remark in particular that extrema of $\langle P \rangle$ are stable stationary solutions of such a system. These are the solutions on which we concentrate. So, consider an extremum of $\langle P \rangle$, then we take into account the rest of the normal form Z_μ (which is also independent of ψ_1), and consider the simplified reduced system with Hamiltonian given by

$$H_s^r := \langle P \rangle (J, \psi_2, \psi_3, \psi_4, ...) + \mu Z_\mu (J, \psi_2, \psi_3, \psi_4, ...) . \tag{2.7}$$

It is clear that if the considered extremum of $\langle P \rangle$ is non degenerate, then it can be continued to a non degenerate extremum of this system. We want to see whether there exists a trace of this object in the dynamics of the complete system (2.5). To understand this point remark that, returning to the complete phase space, the extremum of (2.7) corresponds to a solution of (2.7) in which all the variables are constant except the angle ψ_1 that rotates with constant velocity, therefore critical points correspond to periodic orbits (of the simplified system). Moreover, using J_1 and the function (2.7) as Lyapunof functions one obtains that the considered periodic orbit is orbitally stable, i.e. solutions starting close to its trajectory remain forever close to it.

Finally add the remainder $\mathcal{R}_\mu$, then neither J_1 nor (2.7) are integral of motion, however, they move with an exponentially small velocity, and therefore solutions starting close to the above critical point can move away with an exponentially small velocity. This is the content of theorem 4.1 below. A second question pertains the existence of exact periodic orbits close to the considered approximative solutions. In the finite dimensional case one can use Poincaré theorem to obtain exact periodic orbits (i.e. use implicit function theorem in order to continue fixed points of the Poincaré map of the simplyfied system to fixed points of Poincaré map the complete system). We recall that in the finite dimensional case exitence of such periodic orbits is due to Bogolyubov and Mitropolsky [19]. In the case of PDE's however, this is not possible due to some small denominator problems. Nevertheless it is possible to show that periodic orbits exist, but they appear in Cantor families, instead of continuous families. This is the content of theorem 4.2 below.

There are some difficulties in order to make rigorous the above scheme. The main one is that the flow of the linearized system is not time differentiable (since we deal with partial differential equations). The consequence of this fact is that the angle ψ_1 which is the time along the solutions of the linearized system is not a smooth variable on the phase space, and therefore the system of variables (J, ψ) is not a smooth coordinate system. So one cannot introduce it to actually make proofs. A related difficulty is that one cannot use Marsden Weinstein reduction procedure to pass to the reduced system, since (again) the group action given by the linearized flow is not smooth, and therefore the reduced manifold would be non smooth. So, we will work in the original phase space.

The first point to obtain the proof is to reformulate the above construction in the original phase space space remarking that the average with respect to the angle ψ_1 is substituted by the average with respect the unperturbed flow (see (3.1) below), then the function (2.6) (with J_1 considered as a parameter) is substituted by the restriction of the average of P to the surface of constant harmonic energy (remark that J_1 is just the harmonic energy). So one has to consider the average of the perturbation with respect to the unperturbed flow and its restriction to the surface of constant harmonic energy. Compute the extrema of this function. Such extrema are the candidates for the "periodic like exponentially stable orbits" of the system. To actually prove this fact we first prove that such curves are smooth, and then we introduce a suitable system of coordinates in a tubular neighborhood of them.

2.2 The case of integrable average.

Consider the case where $\langle P \rangle$ depends only on the action variables J_k, then the simplified system would turn out to have the form

$$H_s = \langle P \rangle (J) + \mu Z_\mu(J, \psi_2, \psi_3, \psi_4, ...) \ ,$$

which is a perturbation of an integrable system! So one could try to use perturbation theory to study its dynamics. Clearly this problem is as difficult as the original one, so it is worth to consider a specific model. The simplest model in which this happens is given by the nonlinear Schroedinger equation (1.2). Using the variables (I, ϕ) one has

$$\langle P \rangle (I) = \frac{1}{2} \sum_{k \geq 1} I_k^2 - \frac{1}{2} \left(\sum_{k \geq 1} I_k \right)^2 \ ,$$

notice that $\sum_{k \geq 1} I_k$ is just the L^2 norm of u, and this is an integral of motion for the complete system. So, the Hamiltonian $\langle P \rangle$ is equivalent to

$$\frac{1}{2} \sum_{k \geq 1} I_k^2 \ , \tag{2.8}$$

a function which has very nice properties. In particular it is convex and diagonal as a function of the action variables. So the idea is to extend the usual theory of exponential stability to deal with this kind of systems.

We now explain briefly how this will be done. First consider a finite dimensional resonant torus. To this end fix an N dimensional resonant vector $\omega_j = \nu n_j$ $(j = 1, ..., N)$ where ν is a parameter and n_j are integers and introduce the translated actions

$$I'_k := I_k - \omega_k \ ;$$

in terms of these variables the Hamiltonian (of the simplified system) is given by

$$\sum_k \omega_k I'_k + \sum_{k\geq 1} \frac{(I'_k)^2}{2} + \mu Z \ ,$$

Putting it in resonant normal form one gets the system

$$\sum_k \omega_k I'_k + \sum_{k\geq 1} \frac{(I'_k)^2}{2} + \mu Z' + \mu^2 e^{-\mu_*/\mu} \mathcal{R}' \ ,$$

where Z' is in resonant normal form, i.e. it is such that

$$\left\{ Z', \sum_k \omega_k I'_k \right\} = 0 \ .$$

This has a very important consequence, namely that, up to exponentially long times one has that the change of the function $\sum_k \omega_k I'_k$ can be bounded by quantities of order μ, using conservation of energy and the fact that $\mu Z'$ is of order μ, and that the remainder is small one obtains that also the change of the quantity

$$\sum_{k\geq 1} \frac{(I'_k)^2}{2}$$

is of order μ over the considered time scales. It follows that, for solutions corresponding initial data with $\sum (I'_k)^2 = \mathcal{O}(\mu)$ all the actions remain approximatively constant for the considered times, i.e. they remain close the initial value $I_k = \omega_k$. This is true for initial data close to resonant tori. Then the idea by Lochak [20] is that, by Dirichlet theorem of approximation by rational numbers, each initial datum is close to a resonant torus, and therefore one has stability close to any finite dimensional small amplitude torus.

3. Normal form results

We state here the normal form theorem for system (2.3). To this end we have to introduce formally the phase space, a norm in it, and to state in a precise way the assumptions on the Hamiltonian.

Having fixed $s \geq 0$ consider the Hilbert space ℓ^2_s of the sequences $\{q_k\}_{k\geq 1}$ such that

$$\sum_{k\geq 1} |k|^{2s} |q_k|^2 < \infty \ ,$$

and define the phase space to be the space

$$\mathcal{P} := \ell^2_{d/2} \times \ell^2_{d/2} \ni (p, q)$$

endowed by the standard norm and by the symplectic form $\Omega = dp \wedge dq$ (which, when $d > 0$ is a weakly symplectic form [21]). Given a smooth Hamiltonian function f, we will denote by X_f the Hamiltonian vector field of f, namely the unique vector field such that $\Omega(X_f, Y) = df Y$ for all $Y \in \mathcal{P}$.

Remark 3.1. *We recall that in the case of weakly symplectic form X_f does not need to exist. Indeed, in our case we have $X_f = \left(-\frac{\partial f}{\partial q_k}, \frac{\partial f}{\partial p_k}\right)$, and it is clear that even if f is differentiable, X_f might not belong to the phase space $\mathcal{P}$. The typical example is given by h_0.*

Consider the complixification $\mathcal{P}^{\mathbf{C}} = \mathcal{P} \otimes \mathbf{C}$ of $\mathcal{P}$ and denote by $B(\rho) \subset \mathcal{P}^{\mathbf{C}}$ the ball of radius ρ in $\mathcal{P}^{\mathbf{C}}$ centered at 0. We will denote by $A := X_{h_0}$ the linear operator of the unperturbed system, and by e^{At} the corresponding flow. We will also denote by $\langle P \rangle$ the average of P with respect such a flow, namely

$$\langle P \rangle (z) := \frac{1}{2\pi} \int_0^{2\pi} P(e^{At}(z)) dt \ . \tag{3.1}$$

Theorem 3.2. *Consider system (2.3), assume that X_P is bounded as a map from $\mathcal{P}$ to itself; then there exists a positive μ_* and a positive C_1 such that provided $|\mu| \leq \mu_*$ there exists an analytic canonical transformation $\mathcal{T}_\mu : B(2) \to B(3)$ which puts the Hamiltonian (2.3) in the resonant normal form*

$$H \circ \mathcal{T}_\mu := h_0 + \mu \mathcal{Z}_\mu + \mu^2 e^{-\mu_*/\mu} \mathcal{R}_\mu$$

where $\mathcal{Z}_\mu$ is in resonant normal form, i.e. one has

$$\{h_0, \mathcal{Z}_\mu\} \equiv 0 \ ,$$

and is a perturbation of $\langle P \rangle$, namely

$$\mathcal{Z}_\mu = \langle P \rangle + \mu Z_\mu \ ,$$

and for $z \in B(2)$ the following estimates hold

$$\begin{gathered} \left\| X_{Z_\mu}(z) \right\| \leq C_1 \ , \quad |Z_\mu(z)| \leq C_1 \ , \\ \left\| X_{\mathcal{R}_\mu}(z) \right\| \leq C_1 \ , \quad |\mathcal{R}_\mu(z)| \leq C_1 \ , \\ \|z - \mathcal{T}_\mu(z)\| \leq \mu C_1 \ , \end{gathered} \tag{3.2}$$

moreover $\mathcal{T}_\mu$, Z_μ, $\mathcal{R}_\mu$ *depend in a* C^∞ *way on* μ.

Remark 3.3. *Both* h_0 *and* Z_μ *are integral of motion for the dynamics of the simplified system. For the dynamics of the complete system one has (for* $z \in B(2)$*),*

$$\left|\frac{dh_0}{dt}\right| = \mu^2 e^{-\mu_*/\mu} |\left\{h_0, \mathcal{R}_\mu\right\}| \leq \mu^2 e^{-\mu_*/\mu} \left\|dh_0\right\| \left\|X_{\mathcal{R}_\mu}\right\| \leq C_2 \mu^2 e^{-\mu_*/\mu}$$

where the last inequality follows from the analyticity of h_0. *Similar inequalities hold for the time derivative of* Z_μ *and of* $\langle P\rangle$.

The rest of this section will be occupied by the scheme of the proof of theorem 3.2. The uninterested reader can simply skip it and pass to the next section.

Definition. *The Hamiltonian system*

$$H_a := h_0 + \mu \langle P\rangle$$

will be called the averaged system.

Definition. *The Hamiltonian system*

$$H_s := h_0 + \mu \langle P\rangle + \mu^2 Z_\mu$$

will be called the simplified system.

Scheme of the proof. The idea of the proof is to construct the normalizing transformation as the composition of a sequence of canonical transformations $\mathcal{T}_1, ..., \mathcal{T}_N$, each of which is the time one flow of a suitable Hamiltonian vector field. Each transformation increases by one the order of the non normalized part of the Hamiltonian. At each step we estimate the size of the remainder (i.e. of the non normalized part of the Hamiltononian), after N steps such an estimate will turn out to be given by

$$C_3 \left(\mu C_4 N\right)^N \ ,$$

Choosing $N = [1/eC_4\mu]$ we obtain the exponentially small remainder of theorem 3.2. We explain the first step in detail, while we refer to [18] and [14] for the rest of the proof.

Given an analytic function $\chi_1 : B(\rho) \to \mathbb{R}$, with an analytic Hamiltonian vector field, we consider the differential equation

$$\dot{z} = \mu X_{\chi_1}(z) \ ,$$

and denote by $\mathcal{T}_1$ the corresponding time one flow. We compute $H \circ \mathcal{T}_1$, to this end remark that we have

$$\mathcal{T}_1(z) = z + \mu X_{\chi_1}(z) + \mathcal{O}(\mu^2) \ ,$$

so that

$$H \circ \mathcal{T}_1 = h_0 + \mu \left[\{\chi_1, h_0\} + P\right] + \text{higher order terms} \ , \tag{3.3}$$

where explicitly we have

$$\text{higher order terms} = \left(h_0 \circ \mathcal{T}_1 - h_0 - \mu \left\{\chi_1, h_0\right\}\right) + \mu \left(P \circ \mathcal{T}_1 - P\right) \ . \tag{3.4}$$

So, we construct χ_1 in such a way that $\{\chi_1, h_0\} + P$ is reduced to normal form, i.e. to $\langle P\rangle$. Hence, χ_1 has to satisfy the homological equation

$$\{\chi_1, h_0\} + P = \langle P\rangle \ . \tag{3.5}$$

It is easily seen that the wonted χ_1 is given by

$$\chi_1(z) := \frac{1}{T}\int_0^T t\tilde{P}(e^{At}(z))dt \ , \tag{3.6}$$

where $\tilde{P} := P - \langle P\rangle$, and $T = 2\pi$. Indeed one has

$$\begin{aligned} -\left\{\chi_1, h_0\right\}(z) &= \left.\frac{d}{ds}\right|_{s=0} \chi_1\left(e^{As}(z)\right) \\ = \frac{1}{T}\int_0^T t \left.\frac{d}{ds}\right|_{s=0} \tilde{P}\left(e^{A(t+s)}(z)\right) dt &= \frac{1}{T}\int_0^T t\frac{d}{dt}\tilde{P}\left(e^{At}(z)\right) dt \\ &= \left.\frac{1}{T}t\tilde{P}\left(e^{At}(z)\right)\right|_0^T - \frac{1}{T}\int_0^T \tilde{P}\left(e^{At}(z)\right) dt = \tilde{P} \ . \end{aligned} \tag{3.7}$$

Remark 3.4. *The above computation is correct for z in the domain of of A. However such a domain is dense, and moreover both sides of (3.7) are continuous, therefore the equality holds for all z such that $\mathcal{T}_1(z)$ is defined.*

So, we use χ_1 to generate the first canonical transformation $\mathcal{T}_1$. In order to conclude the first step we have to estimate the domain of definition of $H \circ \mathcal{T}_1$ and the size of the higher order terms in (3.3). This will be done in a few lemmas.

In order to simplify the notations we will write

$$\left\|X_h\right\|_\rho := \sup_{z\in B(\rho)} \left\|X_h\right\| \ , \quad |h|_\rho := \sup_{z\in B(\rho)} |h|$$

The estimate of χ_1 is contained in the following lemma

Lemma 3.5. *Consider the solutions $\langle P\rangle$ and χ_1 of the homological equation (3.5), then the following estimates hold*

$$\begin{gathered} \left\|X_{\langle P\rangle}\right\|_\rho \leq \left\|X_P\right\|_\rho \ , |\langle P\rangle|_\rho \leq |P|_\rho \ , \\ \left\|X_{\chi_1}\right\|_\rho \leq T\left\|X_P\right\|_\rho \ , |\chi_1|_\rho \leq T|P|_\rho \ . \end{gathered}$$

Proof. Remark that by our assumption e^{At} is a canonical transformation, namely one has

$$\Omega\left(e^{At}X, e^{At}Y\right) = \Omega(X, Y) \ , \quad \forall X, Y \in \mathcal{P} \ .$$

from this one also has

$$X_{\chi_1}(z) = \frac{1}{T}\int_0^T te^{-At} X_{\tilde{P}}(e^{At}(z))dt \ , \tag{3.8}$$

which, remaking that e^{At} is a unitary operator, immediately gives the wonted estimates. □

We estimate now the domain of definition of $\mathcal{T}_1$ as well as the difference with respect to the identical transformation.

Lemma 3.6. *Let χ_1 be analytic in $B(\rho)$ together with its symplectic gradient, fix $\delta<\rho$. Assume $\mu\left\|X_{\chi_1}\right\|_\rho<\delta$ and consider the time t flow $\mathcal{T}_1^t$ of $X_{\mu\chi_1}=\mu X_{\chi_1}$. Then, for $|t|\leq 1$, one has*

$$|\mathcal{T}_1^t - \mathbb{1}|_{\rho-\delta} \leq \mu \left\|X_\chi\right\|_\rho \ . \tag{3.9}$$

Proof. It is just an application of the equality

$$z(t)-z(0)=\int_0^t \frac{dz}{dt}(s)ds = \int_0^t \mu X_{\chi_1}(z(s))ds \ .$$

□

From this lemma it also follows that, provided μ is small enough, in our case the transformation is well defined and analytic for $z\in B(3-\delta)$ ($\delta<3$).

We have now to estimate the higher order terms of eq. (3.4). Using Lagrange mean value theorem, Cauchy estimate of derivatives, and the above lemma it is immediate to prove that

$$|P\circ\mathcal{T}_1 - P|_{\rho-\delta} \leq \mu\frac{2}{\delta}\left\|X_{\chi_1}\right\|_\rho |P|_\rho \ . \tag{3.10}$$

It is slightly more difficult to estimate the vector field of $P\circ\mathcal{T}_1-P$. To this end the following lemma is useful.

Lemma 3.7. *Consider χ_1 as above and let $h: B(\rho)\to \mathbf{C}$ be an analytic function with vector field analytic in $B(\rho)$, fix $0<\delta<\rho$ assume $\left\|X_{\chi_1}\right\|_\rho\leq\delta/3$, then, for $|t|\leq 1$, one has*

$$\left\|X_{h\circ\mathcal{T}_1^t}\right\|_{\rho-\delta} \leq \left(1+\mu\frac{3}{\delta}\left\|X_\chi\right\|_\rho\right)\left\|X_h\right\|_\rho$$

Proof. First remark that, since $\mathcal{T}_1^t$ is a canonical transformation one has

$$X_{h\circ\mathcal{T}_1^t}(z) = d\mathcal{T}_1^{-t}(\mathcal{T}_1^t(z))X_h(\mathcal{T}_1^t(z)) \ , \tag{3.11}$$

from which

$$X_{h\circ\mathcal{T}_1^t}(z) = \left(d\mathcal{T}_1^{-t}(\mathcal{T}_1^t(z)) - \mathbb{1}\right)X_h(\mathcal{T}_1^t(z)) + X_h(\mathcal{T}_1^t(z)) \ .$$

To estimate the first term fix $\delta_1:=\delta/3$; we have

$$\begin{aligned}\sup_{z\in B(\rho-3\delta_1)}\left\|d\mathcal{T}_1^{-t}(\mathcal{T}_1^t(z)) - \mathbb{1}\right\| &\leq \sup_{z\in B(\rho-2\delta_1)}\left\|d\mathcal{T}_1^{-t}(z)-\mathbb{1}\right\| \\ &\leq \frac{1}{\delta_1}\sup_{z\in B(\rho-\delta_1)}\left\|\mathcal{T}_1^{-t}(z)-z\right\| \leq \frac{1}{\delta_1}\mu\left\|X_{\chi_1}\right\|_\rho \ .\end{aligned} \tag{3.12}$$

Going back to δ and adding the trivial estimate of the second term one has the thesis. $\square$

Using this lemma we have the wonted estimate, indeed, one has

$$X_{h\circ\mathcal{T}_1^t}(z)-X_h(z)=\left(d\mathcal{T}_1^{-t}(\mathcal{T}_1^t(z))-\ \mathbb{1}\right)X_h(\mathcal{T}_1^t(z))+\left[X_h(\mathcal{T}_1^t(z))-X_h(z)\right]\ .$$

The norm of the square bracket is estimated in the same way as (3.10). The other term was already estimated in (3.12), so we have the estimate of the vector field of the first term of (3.4):

$$\left\|X_{P\circ\mathcal{T}_1-P}\right\|_{\rho-\delta}\leq\mu\frac{5}{\delta}\left\|X_P\right\|_\rho\left\|X_{\chi_1}\right\|_\rho\ . \tag{3.13}$$

In order to estimate the other term of (3.4) we take advantage of the homological equation.

Lemma 3.8. *Denote*

$$\ell:=h_0\circ\mathcal{T}_1-h_0-\mu\left\{\chi_1,h_0\right\}\ ,$$

then, for any $\delta<3$, the Hamiltonian vector field of ℓ is analytic on $B(3-\delta)$, and satisfies

$$|\ell|_{3-\delta}\leq\mu^2\frac{4}{\delta}\left|P\right|_3\left\|X_{\chi_1}\right\|_3\ ,\quad\left\|X_\ell\right\|_{3-\delta}\leq\mu^2\frac{10}{\delta}\left\|X_{\chi_1}\right\|_3\left\|X_P\right\|_3 . \tag{3.14}$$

Proof. One has

$$h_0(\mathcal{T}_1(z))-h_0(z)=\int_0^1\frac{d}{dt}h_0(\mathcal{T}_1^t(z))dt=\int_0^1\tilde{P}(\mathcal{T}_1^t(z))dt\ ,$$

where we used the homological equation to calculate $\{h_0,\chi_1\}$. Frome this one has

$$\ell(z)=\int_0^1(\tilde{P}(T^t(z))-\tilde{P}(z))dt\ ;$$

Using (3.10) and $|\tilde{P}|\leq 2|P|$ one gets the first of equations (3.14). The second is obtained similarly. $\square$

Collecting these results one has that the vector field of the norm of the vector field of the higher order terms is controlled in $B(3-\delta)$ by

$$\frac{C_3}{\delta}\mu^2\left\|X_P\right\|_3$$

with a suitable C_3 independent of μ and of δ. Then one has to iterate the construction. This requires some care since the Hamiltonian contains now the terms $\langle P\rangle$ which is also transformed, but the question is only technical, and we skip it referring to [18] for more details.

4. Periodic like trajectories.

4.1 Statements

We construct here periodic like solutions of system (2.1) under the assumption that P fulfills a suitable nondegeneracy assumption.

Denote by $\mathcal{S}_E$ the surface of constant harmonic energy equal to E:

$$\mathcal{S}_E := \left\{ z \in \mathcal{P} \ : \ \sum_{k\geq 1} \omega_k \frac{p_k^2 + q_k^2}{2} = E \right\} ,$$

and by $\mathcal{F} := \langle P \rangle \big|_{\mathcal{S}_1}$ the restriction of $\langle P \rangle$ to $\mathcal{S}_1$.

By a variational argument one can show that there exists an extremum z_m of $\mathcal{F}$ (see [17]). It is then immediate to realize that the whole curve $\gamma := \cup_t e^{At}(z_m)$ is constituted by extrema of $\mathcal{F}$.* We assume now that such an extremum is non degenerate in the transversal direction, precisely our assumption is the following

(ND) Let Π be a hyperplane in $T_{z_m}\mathcal{S}_1$ transversal to γ, denote by $d^2\mathcal{F}_{z_m}$ the second differential of $\mathcal{F}$ at z_m (which is a symmetric bilinear form on $T_{z_m}\mathcal{S}_1$), then we assume that $d^2\mathcal{F}_{z_m}$ (which is semidefinite) is weakly nondegenerate on Π, i.e. that

$$[d^2\mathcal{F}_{z_m}(X,X) = 0 \ , \quad X \in \Pi] \Longrightarrow X = 0 \ .$$

In order to ensure that γ is smooth we also assume

(S1) For any integer n the vector field X_P is a smooth map from $D(A^n)$ to itself. $D(A^n)$ is endowed by the graph norm.

(S2) The differential of $\langle P \rangle$ does not vanish at z_m, namely $d\langle P \rangle (z_m) \neq 0$.

Finally we will denote $\gamma_\epsilon := \cup_t e^{At}\epsilon z_m$ the rescaled curve with harmonic energy ϵ^2.

Theorem 4.1. *Under the above assumptions there exist constants $\epsilon_\sharp$, C_5, C_6 with the following properties: fix $\epsilon < \epsilon_\sharp$, consider a solution $z(t)$ of the Hamilton equation of (2.1), with initial datum satisfying*

$$d(\gamma_\epsilon, z(0)) \leq C_5 \epsilon^r \ , \tag{4.1}$$

then one has

$$d(\gamma_\epsilon, z(t)) \leq C_6 \epsilon^r \ , \tag{4.2}$$

for all times t with $|t| \leq \frac{C}{\epsilon^{r-1}} \exp\left(\frac{\epsilon_*}{\epsilon}\right)^{r-1}$

Concerning existence of exact periodic solutions we have the following

Theorem 4.2. *Assume also that P depend only on q, then there exists a family $\{z_\epsilon(t)\}_{\epsilon\in\mathcal{E}}$ of periodic solutions of (2.1), with the following properties:*

1) The set $\mathcal{E} \subset \mathbb{R}^+$ is uncountable and has zero as an accumulation point;

---

* we will prove below that γ is a C^∞ curve

2) z_ϵ *is periodic with period* $2\pi/\omega_\epsilon$*, the frequency* ω_ϵ *being close to 1, i.e.*

$$|1-\omega_\epsilon| \le C_7 \epsilon^{r-1} \ ; \tag{4.3}$$

3) z_ϵ *is close to* γ_ϵ*, precisely*

$$\sup_{t\in \mathbb{R}} d\left(z_\epsilon(t), \gamma_\epsilon\right) \le C_8\epsilon^2 \ . \tag{4.4}$$

Here we will not discuss the proof of theorem 4.2, which is based on Lyapunov Schmidt decomposition. We just refer to the original work [16] (for related results see [22,23]). We also recall that periodic solutions of nonlinear string type equations can be found by variational methods (see e.g.[24,25,26]), however these periodical solutions typically do not form "regular" families.

We come to applications, to this end we remark that the smoothness assumptions on the perturbation are typically satisfied in semilinear equations. The assumption that has to be verified in each model is (ND). In the paper [17] it has been proved that (ND) is satisfied in the case of equation (1.1). In this case the only extremum of $\mathcal{F}$ is given by

$$u(x) = V_m \mathrm{sn}(\mathtt{w}x, \mathtt{m}) \ , \quad p(x) \equiv 0 \ ,$$

with V_m, $\mathtt{w}$ and $\mathtt{m}$ suitable constants, and sn the Jacobi elliptic sine. Therefore the curve γ is the phase space trajectory of the solution of the linearized system with such an initial datum. Theorems 4.1 and 4.2 ensure that solutions starting close to it remain close to it for very long times, and moreover that close to γ_ϵ there is a family of true periodic solutions.

4.2 Proof of theorem 4.1

For the proof it is convenient to work with rescaled variables, so we will consider system (2.3).

We first give some definitions and notations

Definition. *Let* $z(t)$ *be a non constant periodic solution of the linear system with Hamiltonian* h_0*. If there exists* $C \neq 0$ *such that* $z(Ct)$ *is a solution of the equations of motion of the averaged (or of the simplified) system, then* $z(Ct)$ *is called a linear–like periodic solution of the averaged (or of the simplified) system.*

We will denote by $\mathcal{Z}$ the part in normal form of the Hamiltonian, namely

$$\mathcal{Z} := \langle P\rangle + \mu Z_\mu$$

.

Remark 4.3. *The curve* γ *is a linear–like periodic solution of the averaged system.*

Remark 4.4. *By the theory of Lagrange multipliers one has that* z_m *fulfills the following (equivalent) equations.*

$$\begin{aligned} \lambda dh_0(z_m) &= d\left\langle P\right\rangle(z_m) \ , \\ \lambda A z_m &= X_{\langle P\rangle}(z_m) \ , \\ \lambda \nabla h_0(z_m) &= \nabla\langle P\rangle(z_m) \end{aligned} \tag{4.5}$$

where we denoted by ∇ the gradient with respect to the metric of $\mathcal{P}$. Moreover by (S2) the Lagrange multiplier λ does not vanish.

Remark 4.5. *By the homogeneity assumption on P, one has that $\langle P\rangle$ is a homogeneous polynomial of degree r; it follows that, if z_m is an extremum of $\mathcal{F}$, then $\sqrt{E}z_m$ is an extremum of the restriction of $\langle P\rangle$ to $\mathcal{S}_E$.*

Remark 4.6. *By our assumptions $X_{\langle P\rangle}$ is smooth, therefore $X_{\langle P\rangle}(z_m) \in \mathcal{P}$ and also (using (4.5)) $Az_m \in \mathcal{P}$, it follows that z_m is in the domain of A, and therefore the curve γ is at least of class C^1. By assumption (S1) it immediately follows that γ is of class C^∞.*

For definiteness in what follows we will assume that z_m is a minimum of $\mathcal{F}$.

Lemma 4.7. *Assumption (ND) is equivalent to the assumption that there exists a positive constant C_9 such that*

$$d^2\mathcal{F}_{z_m}(X,X) \geq C_9 \left\|X\right\|^2 \ , \quad \forall X \in \Pi \cap T_{z_m}\mathcal{S}_1 \ .$$

Proof. First remark that by the theory of Lagrange multipliers one has

$$d^2\mathcal{F}_{z_m}(X,X) = -\lambda d^2 h_0(X,X) - d^2 \left\langle P\right\rangle_{z_m}(X,X) \ , \tag{4.6}$$

where λ is the value of the Lagrange multiplier. Moreover using the metric of $\mathcal{P}$ in order to represent the second differentials at right hand side of (4.6)with linear operators one easily sees that the r.h.s. of (4.6) is equal to

$$-\lambda\langle X,X\rangle + \langle X,BX\rangle \ ,$$

with a suitable operator B, which by the regularity of X_P, is compact. It follows that the operator $\lambda\,\mathbb{1} + B$ is selfadjoint and Frehdolm and therefore 0 is an isolated point of its spectrum. The same holds for its restriction to $\Pi \cap T_{z_m}\mathcal{S}_1$. But such a restriction does not have the zero eigenvalue, and therefore the constant C_9 is just the smallest eigenvalue of $-\lambda\,\mathbb{1} + B$. □

Lemma 4.8. *Provided $|E-1|$ and μ are small enough the function $\mathcal{Z}\big|_{\mathcal{S}_E}$ has a nondegenerate (in the transversal direction) extremum $z_{\mu,E}$. Moreover $z_{\mu,E}$ can be chosen to depend in a C^∞ way on μ and E.*

Proof. The proof is obtained by implicit function theorem. Consider the hyperplane Π transversal to γ at z_m. Since h_0 is a submersion, there exists a coordinate system (E,ξ) in Π, defined in a neighborhood of z_m such that $h_0(E,\xi) = E$ and $z(1,0) = z_m$. Consider now the function $\mathcal{Z}\big|_{\Pi}(z) = \mathcal{Z}(E,\xi,\mu)$ (where we explicitly wrote the dependence of $\mathcal{Z}$ on μ). As a function of ξ such a function has a nondegenerate minimum at zero when $E=1$ and $\mu = 0$. By applying the implicit function theorem to the differential with respect to ξ of such a function one immediately obtains that as (E,μ) vary in a neighborhood of $(1,0)$ the function $\mathcal{Z}\big|_{\Pi}$ has a nondegenerate minimum (in the ξ directions) which depends differentiably on these parameters. □

Corollary 4.9. *The curve $\gamma_{E,\mu} := \cup_{t\in \mathbb{R}} e^{At} z_{\mu,E}$ is a nondegenerate linear–like periodic solution of the simplified system and is of class C^∞.*

Define now the Hilbert space $\mathcal{H} := \Pi \cap T_{z_{\mu,1}}\mathcal{S}_1$, with naturally induced scalar product.

Lemma 4.10. *There exists a coordinate system $(E,\phi,x) \in \mathbb{R}\times\mathbb{R}\times\mathcal{H}$, defined in a neighborhood of $z_{\mu,1}$, such that*

$$h_0(E,\phi,x) = E \ , \quad z(E,\phi,0) \in \gamma_{\mu,E} \ . \tag{4.7}$$

Proof. Take a coordinate system (E,ϕ,ξ) such that the first of (4.7) holds, and $(1,0,0) = z_{\mu,1}$ with $\frac{\partial}{\partial\phi}$ tangent to the curve at $z_{\mu,1}$. In such coordinates the curve $\gamma_{\mu,E}$ have equation $(E,\phi,\xi(E,\phi))$: indeed, ϕ can be chosen as a parameter along the curve by the condition on $\frac{\partial}{\partial\phi}$; we define the wonted coordinates as $(E,\phi,x) := (E,\phi,\xi-\xi(E,\phi))$. □

Introduce the function (defined in a neighborhood of γ)

$$\mathcal{W}(z) := \mathcal{Z}(z) - \mathcal{Z}(\gamma_{h_0(z),\mu}) \ ;$$

then the following corollary holds

Corollary 4.11. *There exists a neighborhood $\mathcal{U}$ of γ and constants C_{10}, C_{11} such that, for any $z\in\mathcal{U}$ one has*

$$C_{10}\sqrt{\mathcal{W}(z)} \le d(z,\gamma_{\mu,h_0(z)}) \le C_{11}\sqrt{\mathcal{W}(z)} \ . \tag{4.8}$$

Proof. First remark that γ is close to $\gamma_{\mu,1}$, and therefore (having fixed μ), it is possible to cover a whole neighborhood of γ by a finite number of coordinate systems of the form constructed in lemma 4.10. Fix now a point z of such a neighborhood, one has

$$d(z,\gamma_E) \le C_{12}\left\|x\right\|_{\mathcal{H}} \le C'_{11}\sqrt{\mathcal{W}(z)} \ ,$$

where the first inequality is a general property of the coordinate system, and the second follows from the fact that $\mathcal{Z}$ has a nondegenerate minimum on $\gamma_{\mu,E}$. The constant C'_{11} clearly depends only on the coordinate system, and therefore in order to obtain the thesis just take the minimum over all the considered coordinate systems. The other inequality of (4.8) is simpler and we omit its proof. □

Proof of theorem 4.1. First remark that from remark 3.3 one has

$$\left|\frac{d\mathcal{W}}{dt}\right| \le \left|\frac{d\mathcal{Z}}{dt}\right| + C_{13}\left\|d\mathcal{Z}\right\|\left|\frac{dh_0}{dt}\right| \le C_{14}\mu^2 e^{-\mu_*/\mu} \ .$$

Consider first the case where the initial datum $z(0)$ lies on $\mathcal{S}_1$, we find

$$\begin{aligned} C_{10}\sqrt{\mathcal{W}(z(0))} \le d(z(0),\gamma_{1,\mu}) &\le d(z(0),\mathcal{T}_\mu^{-1}(\gamma_{1,0})) + d(\mathcal{T}_\mu^{-1}(\gamma_{1,0}),\gamma_{1,0}) + d(\gamma_{1,0},\gamma_{1,\mu}) \\ &\le C_{13}\mu + \mu + \mu \end{aligned}$$

where the last inequality follows from the fact that $\gamma_{1,0} \equiv \gamma$, from the differentiable dependence of γ on μ, from inequality (3.2), and the remark that the condition on the initial datum (4.1) is written in the original coordinates (before the normalizing transformation $\mathcal{T}_\mu$). From this it follows that, within the considered times, one has

$$\mathcal{W}(z(t)) \leq \mathcal{W}(z(0)) + |\mathcal{W}(z(t)) - \mathcal{W}(z(0))| \leq C_{13}\mu^2 \ ;$$

we thus have

$$d(z(t), \gamma) \leq d(z(t), \gamma_{E,\mu}) + d(\gamma_{E,\mu}, \gamma) \leq C_{15}[\mu + (\mu + \mu^2)] \ ,$$

where we used lemma 4.11, and the fact that $\gamma_{E,\mu}$ depends smoothly on E, μ and moreover that we have (by 3.3) $|h_0(z(t)) - h_0(z(0))| \leq C_{16}\mu^2$. Going back to the original coordinates one has that the result holds for initial data on $\mathcal{S}_1$. To extend to the general case remark that, rescaling suitably the variables then one can obtain $z_0 \in \mathcal{S}_1$, however with this rescaling the curve to which refers condition (4.1) lie on a surface μ close to $\mathcal{S}_1$. Since the curve depends smoothly on the energy level it follows that z_0 is also μ close to the curve on his level surface, and therefore the theorem holds. $\square$

5. The case of the (perturbed) nonlinear Schrödinger equation

We proceed now to use our theory to study the dynamics of the Nonlinear Schrödinger equation (1.2), which is the prototype of the equations for which $\langle P \rangle$ turns out to be integrable.

The aim of this section is to present the scheme of the proof of a long time stability result for small solutions of (1.2) that was proved in [18]. To give a precise statement consider the Fourier expansion of the solution

$$u(x, t) = \sum_{k \geq 1} u_k(t) \sqrt{\frac{2}{\pi}} \sin(kx) \ ,$$

and denote by $I_k := |u_k|^2/2$, the (square of the) amplitude of the $k - th$ Fourier mode, and by

$$\epsilon := \left(\frac{1}{2} \int_0^\pi |u_x^{(0)}(x)|^2 \right)^{1/2}$$

the H^1 norm of the initial datum $u^{(0)}$.

Theorem 5.1. *Fix $N \geq 1$, then there exist constants ϵ_*, C_{17}, C_{18}, C_{19} such that, if the initial datum $u^{(0)}$ is such that*

$$\epsilon < \epsilon_* \ , \qquad \sum_{k \geq N+1} |u_k^{(0)}|^4 \leq C_{17} \epsilon^4 \epsilon^{2-1/N} \ ,$$

then along the corresponding solution of (1.2) one has

$$\sum_{k\geq 1} |I_k(t) - I_k(0)|^2 \leq C_{18}\epsilon^{4+1/N}$$

for the times t fulfilling

$$|t| \leq C_{19} \exp\left(\frac{\epsilon_*}{\epsilon}\right)^{1/N} .$$

To start with we recall that (1.2) is an Hamiltonian equation with Hamiltonian of the form (2.1) with the canonical variables (p, q) defined as the real and the imaginary part of the Fourier coefficients of u, namely

$$u_k = p_k + iq_k$$

the parameter $d = 2$ and the perturbation given by

$$P(p, q) = \int_0^\pi \Phi(u(x))dx$$

with $\Phi' = -\varphi$. Moreover an important feature of (1.2) is that it has an exact integral of motion given by the L^2 norm of the solution, namely

$$N(u) := \int_0^\pi \frac{|u|^2}{2} dx = \sum_{k\geq 1} \frac{p_k^2 + q_k^2}{2} .$$

Expanding Φ in Taylor series we get

$$P(u) = \alpha_0 N(u) + \alpha_1 \int_0^\pi \frac{|u|^4}{4} dx + ...$$

First of all we want to apply to our system the normal form lemma 3.2, however we need a slightly improved form in order to take into account the presence of an exact integral of motion and the fact that the perturbation has terms of different degree. Such an improved version (which is conceptually identical to theorem 3.2) was proven in [18]; in order to state the result obtained by its application we first introduce the rescaled variables (p', q') by $(p, q) = (\epsilon p', \epsilon q')$ and omit primes. Thus we have

Proposition 5.2. *Consider the Nonlinear Schrödinger equation*

$$iu_t = u_{xx} + u\varphi(\epsilon^2|u|^2) , \tag{5.1}$$

with φ analytic in a neighborhood of zero; there exists a positive $\epsilon_\sharp$ such that, for any $\epsilon < \epsilon_\sharp$ there exists an analytic canonical transformation $\mathcal{T}_\epsilon^{(0)} : B(2) \to B(3)$, which leaves invariant

N, (namely such that $N \circ \mathcal{T}_\epsilon^{(0)} = N$), and puts the Hamiltonian of the system in resonant normal form, namely such that

$$H_{(1)} := H \circ \mathcal{T}_\epsilon^{(0)} = h_0 - \epsilon^2 \frac{\alpha_1}{\pi} \langle P_0 \rangle + \epsilon^4 Z_\epsilon + \epsilon^4 e^{-(\epsilon_\sharp/\epsilon)^2} \mathcal{R}_\epsilon + \alpha_0 N \tag{5.2}$$

where $\langle P_0 \rangle$ is the average of the quartic term of P and is explicitly given by

$$\langle P_0 \rangle (u) := \frac{1}{2} \sum_{k \geq 1} \left(\frac{p_k^2 + q_k^2}{2} \right)^2 - \frac{1}{2} \Big(N(u) \Big)^2 , \tag{5.3}$$

Z_ϵ is in normal form, i.e. it satisfies $\{h_0, Z_\epsilon\} = 0$ and commutes with N, namely $\{N, Z_\epsilon\} = 0$; moreover Z_ϵ and $\mathcal{R}_\epsilon$ are analytic on $B(2)$ together with their symplectic gradients, and the estimates (3.2) are satisfied.

So $\langle P_0 \rangle$ is integrable, and moreover, since N is an integral of motion, $\langle P_0 \rangle$ is equivalent to $\sum_{k \geq 1} I_k^2/2$ (rescaling the variables in and assuming for simplicity $\alpha_1 < 0$), where we used the notation

$$I_k := \frac{p_k^2 + q_k^2}{2} \ .$$

As explained in the heuristic section 2.2, the system can now be considered as a perturbation of the integrable system $\sum_{k \geq 1} I_k^2/2$. So our next step is to fix a finite dimensional resonant torus of this system and to make a resonant normal form in a neighborhood of this torus. So, fix a positive ν, and a sequence of integers $\{n_j\}_{j=1,...,N}$, with a fixed N. Consider the torus with actions $I_k = \nu n_k$, introduce the notations

$$I'_k := \frac{p_k^2 + q_k^2}{2} - \nu n_k \ , \quad k = 1, ..., N$$
$$I'_k := I_k \ , \quad k \geq N$$

divide the Hamiltonian by ϵ^2, then it takes the form

$$H^{(2)} := \sum_{k=1}^{N} \nu n_k I'_k + \sum_{k \geq 1} \frac{I_k^{'2}}{2} + \epsilon^2 Z_\epsilon + \epsilon^2 e^{-(\epsilon_\sharp/\epsilon)^2} \mathcal{R}_\epsilon + \left[\frac{\alpha_0}{\epsilon^2} N - \frac{1}{2} \Big(N \Big)^2 + \frac{h_0}{\epsilon^2} \right]$$

which, provided $\sum_{k \geq 1} \frac{I_k^{'2}}{2}$ is small, is a perturbation of the resonant system $\sum_{k=1}^{N} \nu n_k I'_k$ (remark that the square bracket contains approximate or exact constant of motions).

We state now a normal form result close to the considered resonant torus. Its proof (which goes again along the lines of the proof of theorem 3.2) was first given in [18], a simplified proof can be found in [27] (which develops a method of [28]). Define the domain

$$\mathcal{G}_{r,\rho} := \left\{ z \in B(\rho) \ : \ \sum_{k \geq 1} \frac{I_k^{'2}}{2} \leq r^2 \right\} .$$

Lemma 5.3. *Define*

$$\delta := \frac{1}{\nu}\left(r + \frac{\epsilon^2}{r}\right) \ ; \tag{5.4}$$

there exists δ_* (independent of ϵ, r, ν) such that, if $\delta < \delta_*$ then there exists an analytic canonical transformation $\mathcal{T}_\epsilon^{(2)} : \mathcal{G}_{r,3/2} \to \mathcal{G}_{2r,2}$ which puts $H^{(2)}$ in resonant normal form, namely such that

$$\begin{aligned} H^{(2)} \circ \mathcal{T}_\epsilon^{(2)} = \sum_{k=1}^{N} \nu n_k I'_k + \sum_{k\geq 1} \frac{I'^2_k}{2} + \epsilon^2 Z'_\epsilon + \epsilon^2 e^{-(\epsilon_\sharp/\epsilon)^2} \mathcal{R}'_\epsilon \\ + \delta^2 e^{-(\delta_*/\delta)} \mathcal{R}'' + \left[\frac{\alpha_0}{\epsilon^2} N - \frac{1}{2}\left(N\right)^2 + \frac{h_0}{\epsilon^2}\right] \end{aligned} \tag{5.5}$$

where Z'_ϵ is in normal form, i.e.

$$\left\{ Z'_\epsilon, \sum_{k=1}^{N} \nu n_k I'_k \right\} = \{Z'_\epsilon, h_0\} = 0$$

moreover $\mathcal{T}^{(2)}$ leaves invariant both N and h_0, each term of the Hamiltonian commutes with N, and all the functions and their vector fields (except the one of h_0) are bounded in the considered domain, uniformly with respect to all the relevant constants, and moreover

$$\left\| \mathcal{T}_\epsilon^{(2)} - \mathbb{1} \right\| \leq \delta$$

.

Using h_0 and $\sum_{k\geq 1} \frac{I'^2_k}{2}$ as Lyapunov function one immediately has the following stability result for initial data starting close to resonant invariant tori.

Corollary 5.4. *Consider the system (5.1); there exists a (small) constant C_{20} such that, given an initial datum z_0 fulfilling $\|z_0\| \leq 1$ and*

$$\sum_{k\geq 1} |I_k(0) - \nu n_k|^2 \leq C_{20} r^2 \tag{5.6}$$

then along the corresponding solution one has

$$\sum_{k\geq 1} |I_k(t) - \nu n_k|^2 \leq r^2$$

for all times t with

$$|t| \leq C_{21} e^{(\delta_*/\delta)}$$

Following Lochak we now use Dirichlet approximation theorem for rational numbers in order to show that any initial datum fulfilling

$$\sum_{k\geq N+1} I_k(0)^2 \leq \epsilon^{2-\frac{1}{N}} \tag{5.7}$$

is close enough to a resonant torus in order to apply the above corollary 5.4. This gives the

Proof of theorem 5.1. We continue to work in rescaled variables; so, having fixed an initial datum fulfilling (5.7), denote by $\omega_1 := I_1(0), ..., \omega_N := I_N(0)$ its N first actions, and by

$$\omega := \max_i \omega_i$$

the maximum of these quantities. Remark that since in rescaled variables the H^1 norm of the initial datum is 1 then ω can be estimated from below uniformly for all the considered initial data. By Dirichlet theorem, having fixed any $Q \geq 1$, there exist integers n_i and $q \leq Q$ such that

$$\left|\omega_k - \frac{\omega n_i}{q}\right| \leq \frac{\omega}{qQ^{1/(N-1)}}.$$

It follows that for the considered initial datum one has

$$\sum_{k\geq 1} |I_k(0) - \nu n_k|^2 \leq \epsilon^{2-\frac{1}{N}} + N\left(\frac{\omega}{qQ^{1/(N-1)}}\right)^2 \tag{5.8}$$

with $\nu = \omega/q$.

The idea is now to choose r in such a way that (5.8) implies (5.6), to insert in the definition of δ which thus turns out to be a function of Q and ϵ, and finally to choose Q as a function of ϵ in order to minimize δ.

We will see that the correct estimate is such that the second term of (5.8) is larger than than the first one, so we define

$$r := \frac{2\sqrt{N}\omega}{\sqrt{C_{20}}}\frac{1}{qQ^{1/(N-1)}}$$

it follows that

$$\delta \leq C_{22}\left(\frac{1}{Q^{1/(N-1)}} + \epsilon^2 q^2 Q^{\frac{1}{N-1}}\right) \leq C_{22}\left(\frac{1}{Q^{1/(N-1)}} + \epsilon^2 Q^{2+\frac{1}{N-1}}\right) .$$

We impose now that the two terms are equal, and this gives $\frac{1}{Q^{1/(N-1)}} = \epsilon^{1/2N}$. Inserting in the above formulas we get the result. $\square$

References

[1] N.N. Nekhoroshev: *Exponential estimate of the stability time of near integrable Hamiltonian systems.* Russ. Math. Surveys. **32** (6) 1-65 (1977)

[2] N.N. Nekhoroshev: *Exponential estimate of the stability time of near integrable Hamiltonian systems, II.* Trudy Sem. Petrovs. (5) 5-50 (1979), in Russian.

[3] S.B. Kuksin: Nearly Integrable Infinite-Dimensional Hamiltonian Systems, Lect. Notes Math. 1556 (Springer 1994)

[4] S.B. Kuksin: *Hamiltonian Perturbations of Infinite-dimensional Linear Systems with Imaginary Spectrum.* Funktsional. Anal. i Prilozhen. **21**, (3), 22-37 (1987) = Funct. Anal. Appl. **21** (1987).

[5] C.E. Wayne: *Periodic and Quasi-periodic Solutions of Nonlinear Wave Equation via KAM Theory.* Commun. Math. Phys. **127**, 479-528 (1990).

[6] J. Pöschel: *A KAM–Theorem for some Partial Differential Equations* Ann. Scuola Norm. Sup. Pisa Cl. Sci. **23**, 119–148 (1996).

[7] W. Craig, C.E. Wayne: *Newton's Method and Periodic Solutions of Nonlinear Wave Equations* Comm. Pure and Appl. Math. **46** 1409-1501 (1993).

[8] J. Bourgain: *Construction of periodic solutions of nonlinear wave equations in higher dimension.* GAFA **5**, 629-639 (1995)

[9] J. Bourgain: *Construction of approximative and almost periodic solutions of perturbed linear Schrödinger and wave equation* GAFA **6** 201–230 (1995).

[10] J. Bourgain: *Quasi-periodic solutions of Hamiltonian perturbations of 2D linear Schrödinger equations.* Ann. of Math. **148**, 363–439, (1998).

[11] A.I. Bobenko, S.B. Kuksin: *The nonlinear Klein-Gordon equation on an interval as a perturbed Sine–Gordon equation.* Comment. Math. Helvetici. **70**, 63–112 (1995).

[12] S.B. Kuksin: *A KAM-theorem for equations of the Korteweg-de Vries type.* Rev. Math. Math. Phys. **10** ii+64 pp. (1998).

[13] S.B. Kuksin, J. Pöschel: *Invariant Cantor manifolds of quasi-periodic oscillations for a nonlinear Schrödinger equation* Ann. of Math. **142** 149–179 (1995).

[14] D. Bambusi, N.N. Nekhoroshev: *A property of exponential stability in nonlinear wave equation near the fundamental linear mode.* Physica D **122**, 73-104 (1998).

[15] N.N. Nekhoroshev: *Exponential stability of the approximate fundamental mode of a nonlinear wave equation.* (Russian) Funktsional. Anal. i Prilozhen. **33**, 80–83, (1999). Translation in Funct. Anal. Appl. **33**, 69–71, (1999).

[16] D. Bambusi, S. Paleari: *Families of periodic solutions in perturbation of resonant PDE's* J. Nonlinear Sci. **11**, 69-87, (2001).

[17] S. Paleari, D. Bambusi, S. Cacciatori: *Normal form and exponential stability for some nonlinear string equations.* ZAMP, (to appear).

[18] D. Bambusi: *Nekhoroshev theorem for small amplitude solutions in nonlinear Schrödinger equations.* Math. Z. **130**, 345–387, (1999).

[19] N.N. Bogoljubov, Ju.A. Mitropoiskii, Asymptotic methods in the theory of nonlinear oscillations. Gordon and Breach, New York 1961 (transl. of Russian ed. 1955).

[20] P. Lochak: *Canonical Perturbation Theory via Simultaneous Approximation.* Uspekhi Math. Nauk, **47** (6)59–140, 1992. English translation: Russ. Math. Surv. **47** 57–133 (1992).

[21] J.E. Marsden: Lectures on Geometric Methods in Mathematical Physics. SIAM (Philadelphia, 1981).

[22] D. Bambusi: *On Lyapunov center theorem for some nonlinear PDE's: a simple proof.* Ann. Scuola Norm. Sup. Pisa Cl. Sci., **29**, 823-837 (2000).

[23] J. Bourgain: *Periodic solutions of nonlinear wave equations.* in *Harmonic analysis and partial differential equations (Chicago, IL, 1996)*, 69–97, Chicago Lectures in Math., Univ. Chicago Press, Chicago, IL, 1999.

[24] P.H. Rabinowitz: *Free Vibrations for a Semilinear Wave Equation* Comm. Pure Appl. Math. **31**, 31-68 (1978).

[25] P.H. Rabinowitz: *Periodic Solutions of Nonlinear Hyperbolic Partial Differential Equations* Comm. Pure Appl. Math. **20** 145–205 (1967)

[26] H. Brezis: *Periodic Solutions of Nonlinear Vibrating String and Duality Principle* Bull. AMS **8**, (3) 409-426 (1983)

[27] J. Pöschel: *On Nekhoroshev estimates for a nonlinear Schrödinger equation and a theorem by Bambusi* Nonlinearity **12**, 1587-1600, (1999).

[28] L. Niederman: *Nonlinear stability around an elliptic equilibrium point in an Hamiltonian system.* Preprint 1997.