The Second Dirichlet Coefficient of Certain L-functions

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Abstract

This work is completely based on Farmer-Koutsoliatis' article [1]. No original contribution was made by the writer of this resume.

1 Two interesting phenomenons

1.1 Newforms

Let N be a positive integer, and let $\Gamma_0(N)$ be the subgroup of $PSL(2,\mathbb{Z})$ defined as

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{Z})\colon c \equiv 0 \bmod N \right\}.$$

Let k be an even positive integer. Let $S_k(N)$ be the space of modular cusp forms of weight k and level N, i.e. the $\mathbb{C}$-vector space of holomorphic functions $f : \mathbb{H} := \{\operatorname{Im} z > 0\} \to \mathbb{C}$ such that:

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \qquad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N),$$

$$f(z) = \sum_{n=1}^{+\infty} a(n) e^{2\pi i n z}.$$

Among these cusp forms, there are some forms which can be seen as $g(dz)$ where $g \in S_k(\Gamma_0(M))$ with $M|N$ and $d|\frac{N}{M}$. The cusp forms which do not admit this representation form a subspace denoted with $S_k^{\text{new}}(N)$.

Finally, it can be shown that this subspace admits a basis of simultaneous eigenfunctions with $a(1) = 1$ for a class of specific self-adjoint operators: these cusp forms are called **newforms**. To each newform f we can associate a L-function

$$L(f, s) := 1 + \sum_{n=2}^{+\infty} \frac{a(n)}{n^{s+\frac{k-1}{2}}}.$$

It is evident that the first coefficient characterizing the newforms and the related L-functions is $a(2)$.

Consider now the following spaces:

$$S_2^{\text{new}}(11), S_4^{\text{new}}(5), S_6^{\text{new}}(3), S_8^{\text{new}}(2), S_{12}^{\text{new}}(1).$$

Each one of these spaces has dimension 1 and is thus generated by a unique newform f_k. Moreover, if we lower k or N we obtain a trivial space.
Now, if we look at their Dirichlet coefficients, something interesting shows to us: their second coefficient $a(2)$ is always negative. This also happens for the spaces $S_2^{\text{new}}(N)$ for $N \leq 21$ (when they are not trivial), for $S_3^{\text{new}}(5)$ and $S_3^{\text{new}}(6)$, and also for several $S_k^{\text{new}}(1)$ with increasing k (even if it does not happen always: see for example $S_{16}^{\text{new}}(1)$). Each one of these spaces could be considered a "border space", in some empirical sense with respect to the possible existence of non trivial newforms. So we ask ourselves:

Do L-functions of newforms which "barely exist" have the tendency to have negative second Dirichlet coefficient?

1.2 Elliptic curves

To every elliptic curve E we can associate an L-function: define the numbers

$$t_1(E) := 2$$

$$t_{p^e}(E) := p^e + 1 - |E(\mathbb{F}_{p^e})|$$

where $|E(\mathbb{F}_{p^e})|$ is the number of rational points of E in the finite field $\mathbb{F}_{p^e}$. Furthermore, consider the trivial multiplicative character 1_E, such that

$$1_E(p) = \begin{cases} 0 & \text{if } p|N \\ 1 & \text{otherwise} \end{cases}$$

Then we define the L-function associated to E as

$$L(s, E) := \sum_{n=1}^{+\infty} \frac{a_n}{n^s}$$

where the numbers a_n are such that

$$\begin{gathered} a_1 = 1 \\ a_p = t_p(E) \\ a_{p^e} = t_p(E) t_{p^{e-1}}(E) - 1_E(p) \cdot p \cdot t_{p^{e-2}}(E) \\ a_{mn} = a_m \cdot a_n \qquad \text{if } (m, n) = 1. \end{gathered}$$

We recall now the Modularity Theorem:

Theorem 1. *To every elliptic curve $E/\mathbb{Q}$ of conductor N one can associate a unique newform f in $S_2^{\text{new}}(N)$.*

The database LMFDB (`www.lmfdb.org`) gathers many information about elliptic curves over $\mathbb{Q}$: in particular, there is a complete classification of the curves with conductor less than 350.000.
John Cremona created a labeling system for the elliptic curves, called **LMFDB label**, which permits to give them an order.
As an example, consider the curve

$$E \ : \ y^2 = x^3 + x^2 + 210x + 1764.$$

This curve has the label ”672.e2”, which means:

- E has conductor 672;
- If we consider the newform f_E associated to E in the space $S_k^{\text{new}}(672)$ and we order the newforms lexicographically by the Dirichlet coefficients, then f_E appears fifth on the list (observe that we are using Modularity Theorem, and this ordering gives an order to the isogeny classes of E);
- If we put the elliptic curves in the isogeny class of E in Weierstrass form

  $$y^2 + a_1 xy + a_3 y = x^3 + a_2 + a_4 x + a_6$$

  and we order the $[a_1, a_2, a_3, a_4, a_6]$ lexicographically, then E appears as second.

Now, consider the **rank** of the elliptic curve E, i.e. the rank r of the group $E(\mathbb{Q})$ of rational points. The famous Birch and Swinnerton-Dyer conjecture affirms that r is precisely the order of vanishing of $L(s, E)$ at $s = 1/2$.
Take now the first curves of rank 1 and consider their isogeny class: it is observed that the first 11 curves of the list have all isogeny class labeled with "a" (even if there are other isogeny classes). These curves are

37.a1, 43.a1, 53.a1, 57.a1, 58.a1, 61.a1, 65.a1, 65.a2, 77.a1, 79.a1, 82.a1.

The very same phenomenon happens when we consider curves of rank 2: the first 9 curves have all isogeny class labeled with a. This seems quite strange, because the LMFDB labeling seems not to have a connection with the rank. So we ask ourselves:

Fixed a rank r, do the first elliptic curves of rank r have isogeny class labeled with a?

2 L-functions and Explicit Formula

The answers to the previous questions will be partially answered by putting ourselves in the stronger conditioned case for L-functions: with this setting we say that

L-functions which "barely exist" tend to have negative coefficients.

2.1 L-functions

From now, with L-function we mean a Dirichlet series

$$L(s) := \sum_{n=1}^{+\infty} \frac{b(n)}{n^s} \tag{1}$$

where $s = \sigma + it \in \mathbb{C}$. We assume that the coefficients $b(n)$ are real and $b(n) \ll n^\delta$ for every $\delta > 0$.
The L-function must satisfy the functional equation

$$\Lambda(s) := Q^s \prod_{j=1}^{d} \Gamma\left(\frac{s}{2} + \mu_j\right) L(s) = \varepsilon \Lambda(1-s) \tag{2}$$

where $\varepsilon \in \{1, -1\}$, $Q > 0$ is called (somehow improperly) **conductor**, $\mu_j \geq 0$ for every j and d is the **degree**.
Moreover, the L-function can be rewritten in a certain half-plane as an **euler product**

$$L(s) = \prod_p L_p(p^{-s})^{-1} \tag{3}$$

where the L_p's are polynomials of degree d for almost every p.
Finally, the L-function must satisfy the two following conjectures:

- **Ramanujan-Petersson Conjecture**: the polynomials L_p have all the zeros on or outside the unit circle. This implies that

$$|b(p)| \leq d \tag{4}$$

for every prime p.

- **Generalized Riemann Hypothesis**: the zeros of L in the strip $0 < \sigma < 1$ have the form $\rho = \frac{1}{2} + i\gamma$.

The main technical tool we need is Weil's explicit formula:

Theorem 2. *Supppose $L(s)$ is an L-function such that $\Lambda(s)$ continues to an entire function and such that*

$$L(\sigma + it) \ll |t^A|$$

for some $A > 0$, uniformly in t for bounded σ.
Let $f(s)$ be an even function which is holomorphic in the horizontal strip $-(1/2 + \delta) < \operatorname{Im} s < 1/2 + \delta$ with $f(s) \ll \min(1, |s|^{-(1+\varepsilon)})$ in this region, and suppose that $f(x)$ is real if x is real.
Define the Fourier transform of f as

$$\hat{f}(x) := \int_{-\infty}^{+\infty} f(u) e^{-2\pi i u x} du$$

and suppose that

$$\sum_{n=1}^{\infty} \frac{c(n)}{n^{1/2}} \hat{f}\left(\frac{\log n}{2\pi}\right)$$

converges absolutely, where $c(n)$ is defined by

$$\frac{L'}{L}(s) = \sum_{n=1}^{\infty} \frac{c(n)}{n^s}.$$

Then

$$\sum_{\gamma} f(\gamma) = \frac{\hat{f}(0)}{\pi} \log Q + \frac{1}{2\pi} \sum_{j=1}^{d} l(\mu_j, f) + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{c(n)}{n^{1/2}} \hat{f}\left(\frac{\log n}{2\pi}\right) \tag{5}$$

where

$$l(\mu, f) := \mathrm{Re}\left\{\int_{\mathbb{R}} \frac{\Gamma'}{\Gamma}\left(\frac{1}{2}\left(\frac{1}{2} + it\right) + \mu\right) f(t)dt\right\} - \hat{f}(0)\log \pi$$

and the sum $\sum_\gamma$ *runs over all non trivial zeros of* $L(s)$*.*

Remark 1. The proof is contained in [2], even if the Fourier transform is defined with a different normalization. Keep in mind that without GRH the left hand side would be the sum over the non trivial zeros of L but in the right hand side there would be something more complicated than $\hat{f}$.

Remark 2. L-functions of rational elliptic curves and newforms are likely to be in this set of "desiderable L-functions": in fact, we have the entire continuation, the autoduality given by $\varepsilon \in \mathbb{R}$ and the Ramanujan bound (they are L-function of degree 2). We only miss Riemann Hypothesis.

2.2 The key application

We use Weil's explicit formula to prove the following theorem, which states formally the ideas introduced in the first section.

Theorem 3. *Fix non-negative numbers* $\mu_1, \ldots, \mu_d$ *in the expression (2). Then there exist real numbers* $0 < Q_0 < Q_1$ *such that :*

- *If* $0 < Q < Q_0$*, then there do not exist any* L*-functions with functional equation (2) satisfying Ramanujan bound (4) and GRH.*
- *If* $Q_0 < Q < Q_1$*, then any* L*-function satisfying those three conditions must have* $a(2) < 0$*.*

Proof. In the explicit formula, use the test function

$$f(x) := \frac{1}{2\pi} \frac{\sin^2(x/2)}{(x/2)^2}$$

which satisfies

$$\hat{f}(x) := \begin{cases} 1 - 2\pi|x| & \text{if } -\frac{1}{2\pi} < x < \frac{1}{2\pi} \\ 0 & \text{otherwise .} \end{cases}$$

The function f chosen is such that f is non-negative, $\hat{f}$ vanishes in $\log(n)/2\pi$ for $n \geq 3$ but $\hat{f}(\log(2)/2\pi) > 0$.
Substituting this in the explicit formula (5) we obtain

$$\sum_{\gamma} f(\gamma) = \frac{1}{\pi}\log Q + \frac{1}{2\pi}\sum_{j=1}^{d} l(\mu_j, f) + \frac{1-\log 2}{\sqrt{2}}c(2) + \frac{1}{\pi}. \tag{6}$$

The left hand side is non-negative and the sum over $l(\mu_j, d)$ is constant once we have fixed the μ_j's and f.
The $c(n)$'s are the coefficients of the logarithmic derivative of L: thus, for every prime p, we have

$$c(p) = -b(p)\log p$$

and by Ramunajan's bound the last term is bounded.
Thus, if Q is very near to 0, the right hand side is negative, which is a contradiction. That proves the existence of the number Q_0.
Suppose now that Q is slightly larger than Q_0: then for the right hand side to be positive it must be $(1-\log 2)c(2)/\sqrt{2}$, which is equivalent to $a(2)$ being negative. This provs the existence of Q_1. □

2.3 Back to the phenomena

Let $F \in S_k^{\text{new}}(N)$ be a newform and let L be its L-function. The parameters Q and μ for L are given by $Q = N/\pi$ and $\{\mu_1, \mu_2\} = \left\{\frac{k-1}{4}, \frac{k+1}{4}\right\}$. Applying Theorem 3 with k fixed, we see that if N is sufficiently small then $S_k^{new}(N)$ must be empty and, for N slightly larger, then $a(2) < 0$.

We can also explain why many spaces of newforms of weight 2 have $a(2) < 0$. This is because the parameters μ_1 and μ_2 with $k = 2$ are smaller than 1 and so the terms $l(\mu, f)$ in (6) tend to be negative. Thus the term $Q = \pi N$ must be larger than before in order to have a non-negative sum in the right hand side of (6).
Moreover suppose that $a(2)$ gives a contribute of $-\delta$; one needs $\log Q$ to increase by δ before it is possible for $a(2)$ to be positive. But the larger Q becomes, the slower $\log Q$ increases and so, if k is small, we need very large Q to possibly produce $a(2) < 0$, and the range of such $Q's$ is wider.

Finally, for the elliptic curves, suppose Birch and Swinnerton-Dyer conjecture holds, i.e. the L function of an elliptic curve of rank r has a zero of order r in $1/2$. This translates to r zeros with $\gamma = 0$ in the explicit formula (5). Those terms add a large positive contribution to the left side of (6). Thus, the lower bound on possible levels $N = \pi Q$ for a rank r elliptic curve is an increasing function of r and those Q which are slightly larger than the minimum give a negative value for $a(2)$. Finally, since the isogenies classes of elliptic curves are ordered lexicographically by the L-functions coefficients, those with a negative $a(2)$ are more likely to be listed first, receiving the label "a".

Remark 3. The first statement in Theorem 3 seems to have been obtained in [3] without using Riemann Hypothesis.

References

[1] David W. Farmer and Sally Koutsoliotas. The second Dirichlet coefficient starts out negative. *Ramanujan J.*, 41(1-3):335–343, 2016.

[2] Henryk Iwaniec and Emmanuel Kowalski. *Analytic number theory*, volume 53 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2004.

[3] J. Kaczorowski and A. Perelli. Lower bounds for the conductor of L-functions. *Acta Arith.*, 155(2):185–199, 2012.