Once in a lifetime: Haar Measures on locally compact groups

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1 Recalls on measure theory and topological groups

Definition 1. Let X be a set, $E \subseteq \mathcal{P}(X)$. The smallest σ-algebra containing the sets which are elements of E is denoted as $\sigma[E]$.
Let (X, τ) be a topological space. The σ-algebra $\Sigma := \sigma[\tau]$ is called the Borel σ-algebra.
If (X, τ) is a T2 topological space and μ is a measure on the Borel σ-algebra of X, then the measure space $(X, \sigma[\tau], \mu)$ is called a **Borel space**, and μ is said to be a **Borel measure**.

Definition 2. Let $(X, \sigma[\tau], \mu)$ be a Borel space. The measure μ is said to be a **regular measure** if

- $\mu(K) < +\infty$ for every compact subset $K \subseteq X$;
- It holds the **inner regularity**: for every open subset $U \subseteq X$, it is:
$$\mu(U) = \sup\{m(K) \colon K \subset U, K \text{ compact}\};$$
- It holds the **outer regularity**: for every $A \in \Sigma$ it is
$$\mu(A) = \inf\{m(U) \colon A \subseteq U, U \text{ open }\}.$$

Definition 3. A **topological group** is a topological space G with a group structure such that the multiplication $m : G \times G \to G$ and the inversion

$.^{-1} : G \to G$ are continuous maps.
A **locally compact group** is a topological group G which is locally compact and T2.

Remark 1. A topological group is T2 if and only if there is a closed point.

Definition 4. A **left/right Haar measure** on a topological group G is a regular Borel not zero measure on G such that

$$\mu(g \cdot A) = \mu(A) \ (\mu(A \cdot g) = \mu(A))$$

for every $g \in G$, for every $A \in \sigma[\tau]$.

The purpose of this seminar is to prove the following important theorem:

Let G be a locally compact group. Then there exists a unique left Haar measure on G up to multiplication by a positive real number.

A motivatory example

Let K be a number field and let $\mathcal{O}_K$ be its ring of integers. A famous theorem by Dirichlet says that there is the group isomorphism

$$\mathcal{O}_K^* \simeq \mathbb{Z}^m \times \mu_K$$

where $m \in \mathbb{N}$ and μ_K is the finite subgroup made by the roots of unity contained in K.
Let $\sigma_1, \ldots, \sigma_{m+1}$ be the embeddings of K in $\mathbb{C}$: in the theory of L-functions it is often useful to consider the homomorphism

$$\begin{aligned} \psi : &\mathcal{O}_K \to (\mathbb{R}_{>0})^{m+1} \\ &\eta \to (|\sigma_1(\eta)|, \ldots, |\sigma_{m+1}(\eta)|) \end{aligned}$$

and the functions defined over $\psi(\mathcal{O}_K)$. Suppose that one is going to integrate one of such functions on $G := (\mathbb{R}_{>0})^{m+1}$; suppose moreover that this group (which is in an obvious way a topological group) admits a measure μ which is invariant by group translations.
What if $G = G_1 \times G_2$, where $G_1$and G_2 are two topological groups which are easier to deal with? The theory of Haar measure claims that there exist μ_1 and μ_2 invariant measures on G_1 and G_2 respectively such that

$$\int_G f(x) d\mu(x) = \int_{G_1} \int_{G_2} f(ab) d\mu_1(a) d\mu_2(b).$$

2 From right to left measures

We give some useful lemmas that permit us to reduce the proof of existence and uniqueness of right Haar measures to the one for left Haar measures.

Lemma 1. *Let $f : X \to Y$ be a function, $E \subseteq \mathcal{P}(Y)$. Then $\sigma[f^{-1}(E)] = f^{-1}(\sigma[E])$.*

Lemma 2. *Let $(X, \sigma[\tau], \mu)$ be a Borel measure space, and let $f : X \to X$ be a homeomorphism. Then, for every $A \subset X$, TFAE:*

- $A \in \sigma[\tau]$;
- $f(A) \in \sigma[\tau]$;
- $f^{-1}(A) \in \sigma[\tau]$.

Proof:
The proof is straightforward from lemma 1. □

Proposition 1. *Let G be a topological group and μ be a Haar measure on G. Define $\mu^{'}(A) := \mu(A^{-1})$.*
Then μ is a left/right Haar measure if and only if $\mu^{'}$ is a right/left Haar measure.

Proof:

1) Let us show that $\mu^{'}$ is a Borel measure of G.
The inversion $.^{-1} : G \to G$ is a homeomorphism of G. Then by lemma 2 $\mu^{'}$ is defined on the Borel sets of G, with $\mu^{'} \geq 0$ and $\mu^{'}(\emptyset) = 0$.
Moreover, suppose $\{A_n : n \in \mathbb{N}\}$ be a collection of disjoint and measurable sets. Then $\{A_n^{-1} : n \in \mathbb{N}\}$ is a collection of disjoint and measurable sets and so

$$\mu^{'}(\cup_{n\in\mathbb{N}} A_n) = \mu((\cup_{n\in\mathbb{N}} A_n)^{-1}) = \mu(\cup_{n\in\mathbb{N}} A_n^{-1}) = \sum_{n\in\mathbb{N}} \mu(A_n^{-1}) = \sum_{n\in\mathbb{N}} \mu^{'}(A_n).$$

2) Let us show that $\mu^{'}$ is regular.
If K is compact, then K^{-1} is a compact and so

$$\mu^{'}(K) = \mu(K^{-1}) < +\infty.$$

Let $U \subseteq G$ be an open subset. It is $K^{-1} \subseteq U$ compact if and only if $K \subseteq U^{-1}$ compact. Then:

$$\begin{aligned} \mu^{'}(U) = \mu(U^{-1}) = & \sup\{\mu(K) \colon K \subset U^{-1}, K \text{ compact}\} = \\ & \sup\{\mu(K^{-1}) \colon K \subset U, K \text{ compact}\} = \\ & \sup\{\mu^{'}(K) \colon K \subset U, K \text{ compact}\}. \end{aligned}$$

This proves the inner regularity, and the proof for the outer regularity is similar.

3) $\mu^{'}$ is a right Haar measure.
This is obvious because

$$\mu^{'}(Ag) = \mu((Ag)^{-1}) = \mu(g^{-1}A^{-1}) = \mu(A^{-1}) = \mu^{'}(A).$$

The converse proof (from right to left measure) is completely similar, and thus we have reduced ourselves to work only with left Haar measures. □

3 Existence of the Haar measure

Let G be a locally compact group. We divide the proof of the existence of the Haar measure in several steps.

Step 1: define the values μ_U.
Let $\mathcal{K}$ be the collection of compact subsets of G and let $\mathcal{U}$ be the collection of open neighborhoods of 1_G.
For every $K \in \mathcal{K}$ and $V \subseteq G$ a subset with non empty interior. The collection $\{gV^o \colon g \in G\}$ is an open covering of K, and so there exist $g_1, .., g_n \in G$ such that

$$K \subseteq \cup_{k=1}^{n} g_k V^o. \tag{1}$$

Let us define $(K : V)$ as the minimum non negative integer such that (1) holds.
Now, being G locally compact, there exists $K_0 \in \mathcal{K}$ with non empty interior. For every $U \in \mathcal{U}$ let us define the function

$$\begin{aligned} \mu_U \colon & \mathcal{K} \to \mathbb{R} \\ & K \to \mu_U(K) := \frac{(K : U)}{(K_0 : U)}. \end{aligned}$$

(It is well defined being $K_0^o \neq \emptyset$).

We show that for every $K \in \mathcal{K}$ it is $0 \leq \mu_U(K) \leq (K : K_0)$.
It is obviously $\mu_U \geq 0$. At the same time $\mu_U(K) \leq (K : K_0)$ if and only if $(K : U) \geq (K : K_0)(K_0 : U)$.
Suppose $(K : K_0) = n$ and $(K_0 : U) = m$. Then there exist $g_1, \dots, g_n \in G$ such that $K \subseteq \cup_{i=1}^n g_i K_0^o$ and there exist $h_1, \dots, h_m \in G$ such that $K_0 \subseteq \cup_{j=1}^m h_j U$.
Then $K \subseteq \cup_{i=1}^n \cup_{j=1}^m g_i h_j U$ and so $(K : U) \leq nm$.

Step 2: We show the candidate for the Haar measure.
Define the space $X := \prod_{K \in \mathcal{K}} [0, (K : K_0)]$. Each factor is compact and T2 with the Euclidean topology, and so the space X is compact and T2 (by Tychonoff's theorem).
Now, for every $V \in \mathcal{U}$, define the closed sets

$$\mathcal{C}(V) := \overline{\{\mu_U \colon U \in \mathcal{U} \colon U \subseteq V\}}.$$

We are thinking the functions μ_U as elements in X, as a consequence of the inequality proved in Step 1.
Observe that if $V_1, .., V_n \in \mathcal{U}$, then $\mu_{\cap_{i=1}^n V_i} \in \cap_{i=1}^n \mathcal{C}(V_i)$. Thus the family $\{\mathcal{C}(V)\}$ has the finite intersection property, abd so there exists $\mu \in \cap_{V \in \mathcal{U}} \mathcal{C}(V)$.
μ is our candidate for being a left Haar measure on G.

Remark 2. The existence of X is due to the Axiom of Choice: this fact is fundamental in Weil's proof but was not appreciated at his time. This is why Cartan provided another proof for the existence of the Haar measure.

Step 3: We prove the following properties of μ:

A) $\mu(K_1) \leq \mu(K_2)$ if $K_1 \subseteq K_2$;

B) $\mu(K_1 \cup K_2) \leq \mu(K_1) + \mu(K_2)$;

C) $\mu(K_1 \cup K_2) = \mu(K_1) + \mu(K_2)$ if $K_1 \cap K_2 = \emptyset$.

In order to prove this we show that similar properties hold for every μ_U.

A) For every $U \in \mathcal{U}$ we have that $\mu_U(K_1) \leq \mu_U(K_2)$ if $K_1 \subseteq K_2$.
This is clear because $K_1 \subseteq K_2 \subseteq \cup_{i=1}^{(K_2:U)} g_i U$.

Now, define the map $\psi : X \to \mathbb{R}$ which sends $f \in X$ to $f(K_2) - f(K_1)$ (where for every $K \in \mathcal{K}$ we define $f(K)$ as the image of f under the projection of X onto $[0, (K : K_0)]$).

The map ψ is continuous, being a composition of continuous functions:

for every $K \in \mathcal{K}$ the map

$$\begin{aligned} &X \to [0, (K : K_0)] \to \mathbb{R} \\ &f \to f(K) \to f(K) \end{aligned}$$

is continuous, and then, for every $K_1 \subseteq K_2$ fixed, we have that

$$\begin{aligned} &X \to \mathbb{R} \times \mathbb{R} \to \mathbb{R} \\ &f \to (f(K_2), f(K_1)) \to f(K_2) - f(K_1) \end{aligned}$$

is continuous.

The map ψ is non negative on every μ_U: thus it is non negative on every $\mathcal{C}(V)$ (because of the continuity), and so it is in non negative on μ, i.e. $\mu(K_2) - \mu(K_1)$.

B) As before it easy to show that for every $U \in \mathcal{U}$ we have $\mu_U(K_1 \cup K_2) \leq \mu_U(K_1) + \mu_U(K_2)$ (same argument with open coverings).
As before we have that the map $\psi : X \to \mathbb{R}$ which sends f to $f(K_1 \cup K_2) - f(K_1) - f(K_2)$ is continuous and not negative on every $\mathcal{C}(V)$, therefore on μ.

C) To show the third property one has to prove before that $\mu_U(K_1 \cup K_2) = \mu_U(K_1) + \mu_U(K_2)$ if $K_1 U^{-1} \cap K_2 U^{-1} = \emptyset$.
The proof of this fact is the following: let $n = (K_1 \cup K_2 : U)$ and $K_1 \cup K_2 \subseteq \cup_{i=1}^n g_i U$. If some $g_i U$ intersects both K_1 and K_2, then $g_k \in K_1 U^{-1} \cap K_2 U^{-1}$, which is a contradiction.
Then there is $m < n$ not negative integer such that (up to new labeling of the indexes) $K_1 \subseteq \cup_{i=1}^m g_i U$ and $K_2 \subseteq \cup_{i=m+1}^n g_i U$, and so $(K_1 : U) + (K_2 : U) \leq (K_1 + K_2 : U)$ and the equality follows from B).

We need two topological lemmas in order to proceed:

Lemma 3. *Let X be a T2 space, K_1 and K_2 compact and disjoint subspaces. Then there exists U_1 and U_2 open subsets such that $K_i \subseteq U_i$ and $U_1 \cap U_2 = \emptyset$.*

Lemma 4. *Let G be a topological group and $K \subseteq G$ a compact subset. Let $U \subseteq G$ be an open subset such that $K \subseteq U$. Then there exists an open neighborhood V of 1_G such that $KV \subseteq U$.*

Now let us prove the fact about μ: let $K_1, K_2 \in \mathcal{K}$ such that $K_1 \cap K_2 = \emptyset$. By lemma 3 there exist U_1 and U_2 open sets such that $K_i \subseteq U_i$.
By lemma 4 there exist V_1 and V_2 open neighborhoods of 1_G such that $K_i V_i \subseteq U_i$.
Define $V := V_1 \cap V_2$. Then $K_i V \subseteq K_i V_i \subseteq U_i$ and so $K_1 V \cap K_2 V = \emptyset$.
Now, if $U \in \mathcal{U}$ such that $U \subseteq V^{-1}$: then $K_i U^{-1} \subseteq K_i V$ and thus $K_1 U^{-1} \cap K_2 U^{-1} = \emptyset$, and so for every such U we have $\mu_U(K_1 \cup K_2) = \mu_U(K_1) + \mu_U(K_2)$.
Thus the continuous map from X to R which sends f to $f_U(K_1 \cup K_2) - f_U(K_1) + f_U(K_2)$ is zero on $\mathcal{C}(V)$ and so on μ.

Step 4: We extend μ over every subset of G.
We begin with a definition on the open sets: if $U \subseteq G$ is open, define

$$\tilde{\mu}(U) := \sup\{\mu(K) \colon K \subseteq U, K \in \mathcal{K}\}.$$

We prove that for every K compact and open subset in G we have $\tilde{\mu}(K) = \mu(K)$.
Being $K \subseteq K$, it is clearly $\mu(K) \leq \tilde{\mu}(K)$.
At the same time, for every compact subset J of K, we have $\mu(J) \leq \mu(K)$ and so $\tilde{\mu}(K) \leq \mu(K)$.
From now on, we denote $\tilde{\mu}$ as μ and we show that if $U_1 \subseteq U_2$ are open sets, then $\mu(U_1) \leq \mu(U_2)$: this is clear because

$$\mu(U_1) = \sup\{\mu(K) \colon K \subseteq U_1, K \in \mathcal{K}\} \leq \sup\{\mu(K) \colon K \subseteq U_2, K \in \mathcal{K}\} = \mu(U_2).$$

Now, for every set $A \subset G$, define

$$\mu(A) := \inf\{\mu(U) \colon A \subseteq U, U \text{ open }\}.$$

As before, we can show that this definition coincides with the one given for open and compact sets, and that $A_1 \subseteq A_2$ implies $\mu(A_1) \leq \mu(A_2)$.

Step 5: We show that μ is an outer measure over G.
We need μ to be not negative, has value 0 on the emptyset and holds the countable sub-additivity.

- In order to prove that $\mu \geq 0$ we can restrict ourselves to prove it on compact subsets.
  Fix $K \in \mathcal{K}$: then the map $\psi : X \to \mathbb{R}$ which sends f to $f(K)$ is continuous and positive on every $\mathcal{C}(V)$, and so on μ.

- Being $(\emptyset : U) = 0$ for every $U \in \mathcal{U}$, we have $\mu_U(\emptyset) = 0$ and so (arguing as before) $\mu(\emptyset) = 0$.

- Suppose $\{U_n : n \in \mathbb{N}\}$ is a countable collection of open subsets of G. Let K be a compact subset of $\cup_{n\in\mathbb{N}} U_n$: there exists $U_{i_1}, \dots, U_{i_n}$ such that $K \subseteq \cup_{j=1}^{n} U_{i_j}$. We exploit the following lemma:

  Lemma 5. *Let X be a T2 space, let K be a compact subset of X, and let U_1 and U_2 be open subsets of X such that $K \subseteq U_1 \cup U_2$. Then, there are compact sets K_1 and K_2 of X such that $K_1 \subseteq U_1, K_2 \subseteq U_2$ and $K = K_1 \cup K_2$.*

  By several applications of lemma 5 we find $K_1, \dots, K_n$ compact subsets of G such that $K_j \subseteq U_j$ for every $j \in \{1, \dots, n\}$ and $K = K_1 \cup \ldots \cup K_n$. Then:
  $$\mu(K) = \sum_{j=1}^{n} \mu(K_j) \leq \sum_{j=1}^{n} \mu(U_j) \leq \sum_{n\in\mathbb{N}} \mu(U_n).$$
  Taking the supremum of the values of μ on the compact sets in $\cup_{n\in\mathbb{N}} U_n$ we have done the job on the open sets.

- Let $\{A_n : n \in \mathbb{N}\}$ be a collection of subsets of G. The claim is obviously true if $\sum_{n\in\mathbb{N}} \mu(A_n) = \infty$.
  Otherwise, let $\varepsilon > 0$ and for every $n \in \mathbb{N}$ take U_n open set such that $A_n \subseteq U_n$ and $\mu(U_n) \leq \mu(A_n) + \varepsilon 2^{-n}$. Then:
  $$\mu(\cup_{n\in\mathbb{N}} A_n) \leq \mu(\cup_{n\in\mathbb{N}} U_n) \leq \sum_{n\in\mathbb{N}} \mu(U_n) \leq \sum_{n\in\mathbb{N}} \mu(A_n) + \varepsilon \sum_{n\in\mathbb{N}} 2^{-n} = \sum_{n\in\mathbb{N}} \mu(A_n) + \varepsilon 2^{-1}.$$
  Being ε an arbitrary value, we have the truth of the claim.

Step 6: We show that the Caratheodory measurable sets for μ contain the Borel sets of G.

A Caratheodory measurable set is a set A such that for every set E it holds

$$\mu(E) = \mu(A \cap E) + \mu(A \cap E^c). \tag{2}$$

Suppose $U \subset G$ is open, and let $E \subseteq G$ such that $\mu(E) < +\infty$ (otherwise (2)) is trivially true).
Let $\varepsilon > 0$ and $V \subseteq G$ be an open set such that $E \subseteq V$ and $\mu(V) \leq \mu(E) + \varepsilon$.
Let $K \subseteq V \cap U$ be a compact set such that $\mu(V \cap U) - \varepsilon \leq \mu(K)$ and let $L \subseteq V \cap K^c$ be such that $\mu(V \cap K^c) - \varepsilon \leq \mu(L)$.
Being $K \subset U$ and $V \cap U^c \subseteq V \cap K^c$ we have

$$\mu(V \cap U^c) - \varepsilon \leq \mu(V \cap K^c) - \varepsilon \leq \mu(L).$$

Then we obtain

$$\mu(E \cap U) + \mu(E \cap U^c) - 2\varepsilon \leq \mu(V \cap U) - \varepsilon + \mu(V \cap U^c) - \varepsilon \leq \mu(K) + \mu(L) = \\ \mu(K \cup L) \leq \mu((V \cap U) \cap (V \cap K^c)) \leq \mu(V) \leq \mu(E) + \varepsilon$$

and so

$$\mu(E \cap U) + \mu(E \cap U^c) \leq \mu(E) + 3\varepsilon.$$

Step 7: We show that μ is a left Haar measure.

The measure μ is not zero: in fact for every $U \in \mathcal{U}$ we have $\mu_U(K_0) = 1$, and so $\mu(K_0) = 1$ (it is the usual procedure shown in the previous points).
The measure μ is regular: this follows immediately from the definition of μ in Step 4.
Let us show that μ is translation invariant. Fix $g \in G$. We have $K \subseteq \cup_{i=1}^n g_i U$ if and only if $gK \subseteq \cup_{i=1}^n g g_i U$. Then $(K : U) = (gK : U)$ for every $U \in \mathcal{U}$ and so $\mu_U(K) = \mu_U(gK)$.
Thus the map $\psi : X \to \mathbb{R}$ which sends f to $f(K) - f(gK)$ is continuos and zero on every $\mathcal{C}(V)$ and so on μ.

4 Uniqueness of the Haar measure

The aim of this section is to prove that, given a Haar measure on a locally compact group G, it is possible to describe all the others Haar measures on G.

Theorem 1. *Let G be a locally compact group and let μ and $\mu^{'}$ be two Haar measures on G. Then there exists $a > 0$ such that $\mu = a\mu^{'}$.*

As in the previous section, we divide the proof in several steps. We begin providing a lemma:

Lemma 6. *Let (X, Σ, μ) be a measure space, let $A \subseteq X$ be a measurable subset and $f : X \to \mathbb{R}$ a measurable function.*
If $A = \{x \in X \colon f(x) > 0\}$ and $\mu(A) > 0$, then there exists $a > 0$ such that $\mu(\{x \in A \colon f(x) \geq a\}) > 0$.

Step 1: We show that for every $f \in C_c(G)$ such that $f \geq 0$ is not the zero function, then $\int_G f d\mu > 0$.
Let $U = f^{-1}(0, +\infty)$, which is a not empty open set (by hypothesis). Let K be a compact set such that $\mu(K) > 0$ (we know it exists from the construction given in the previous section). Let $g_1, \ldots, g_n \in G$ be such that $K \subseteq \cup_{i=1}^{n} g_i U$. Then:

$$0 < \mu(K) \leq \sum_{i=1}^{n} \mu(g_i U) = n\mu(U)$$

and so $\mu(U) > 0$.
Now we apply lemma 6: there exists $a > 0$ such that $V = \{g \in G \colon f(g \geq a)\}$ has positive measure: then

$$\int_G f d\mu \geq \int_V f d\mu \geq a\mu(V) > 0.$$

Remark 3. These considerations are valid also for the measure $\mu^{'}$.

Step 2: We define a function that connects μ and $\mu^{'}$.
Let $f \in C_c(G)$ such that $f \geq 0$ and is not the null function. Let $g \in C_c(G)$. Define:

$$h(x, y) := \frac{g(x)f(yx)}{\int_G f(tx)d\mu^{'}(t)}.$$

The function h is well defined thanks to the previous step. We show now that it is a continuous function with compact support; we need the following lemma.

Lemma 7. *Let G be a locally compact group, and $C_c(G)$ the setof continuous functions $f : G \to \mathbb{R}$ with compact support. Then for every $\varepsilon > 0$ there exists an open neighborhood U of 1_G such that $|f(x) - f(y)| < \varepsilon$ for any $y \in xU$.*

Now, come back to the function h. It is enough to prove that the function $I(x) := \int_G f(tx)d\mu^{'}(t)$ is continuous.
Let $K := \operatorname{supp} f$. Let $x_0 \in G$ and let U be an open neighborhood of x_0 with compact closure. The set $K \times \overline{U}^{-1}$ is compact in $G \times G$ and so $K\overline{U}^{-1}$ is compact in G.
Now, let $\varepsilon > 0$ and choose $\delta > 0$ such that $\delta \cdot \mu^{'}(K\overline{U}^{-1}) < \varepsilon$ (we can do this because tha Haar measure is finite on every compact subset).
By lemma 7 there exists an open neighborhood V of 1_G such that for every $y \in xV$ it is $|f(x) - f(y)| < \delta$. Then, if $x \in U \cap x_o V$ we have $tx \in tx_0 V$ and so:

$$|I(x) - I(x_0)| \leq \int_G |f(tx) - f(tx_0)| d\mu^{'}(t) \leq \delta\mu^{'}(K\overline{U}^{-1}) < \varepsilon.$$

We introduce the last technical tool needed for the proof of uniqueness.

Lemma 8. *Let G be a topological group and let μ be a Haar measure on G. Then for every $f \in L^1(G)$, for every $x \in G$ it holds:*

$$\int_G f(xg)d\mu(g) = \int_G f(g)d\mu(g).$$

Step 3: We present a number connected to μ and $\mu^{'}$ but which does not depend on them.
Let f, g and h as introduced in Step 2. By several applications of lemma 8 and Fubini's theorem (which permits us to exchange the order of integration) we have:

$$\int_G \Big[\int_G h(x,y)d\mu^{'}(y)\Big]d\mu(x) = \int_G \Big[\int_G h(x,y)d\mu(x)\Big]d\mu^{'}(y) =$$
$$\int_G \Big[\int_G h(y^{-1}x,y)d\mu(x)\Big]d\mu^{'}(y) = \int_G \Big[\int_G h(y^{-1}x,y)d\mu^{'}(y)\Big]d\mu(x) =$$
$$\int_G \Big[\int_G h(y^{-1},xy)d\mu^{'}(y)\Big]d\mu(x).$$

Then one obtains:

$$\int_G g(x)d\mu(x) = \int_G g(x)\Big[\frac{\int_G f(yx)d\mu^{'}(y)}{\int_G f(tx)d\mu^{'}(t)}\Big]d\mu(x) =$$
$$\int_G \Big[\int_G \frac{g(x)f(yx)}{\int_G f(tx)d\mu^{'}(t)}d\mu^{'}(y)\Big]d\mu(x) =$$
$$\int_G \Big[\int_G h(x,y)d\mu^{'}(y)\Big]d\mu(x) = \int_G \Big[\int_G h(y^{-1},xy)d\mu^{'}(y)\Big]d\mu(x) =$$
$$\int_G \Big[\int_G \frac{g(y^{-1})f(x)}{\int_G f(ty^{-1})d\mu^{'}(t)}d\mu^{'}(y)\Big]d\mu(x) =$$
$$\Big(\int_G f(x)d\mu(x)\Big)\Big(\int_G \frac{g(y^{-1})}{\int_G f(ty^{-1}d\mu^{'}(t))}d\mu^{'}(y)\Big)$$

and so one sees that the quotient $C := \int_G gd\mu / \int_G fd\mu$ is independent from the left Haar measure chosen, which means:

$$\frac{\int_G gd\mu}{\int_G fd\mu} = \frac{\int_G gd\mu^{'}}{\int_G fd\mu^{'}}.$$

Call $a := \int_G fd\mu^{'} / \int_G fd\mu$; it is well defined by the choice of f and from this we get

$$\int_G gd\mu^{'} = a\int_G gd\mu$$

for every $g \in C_c(G)$.

Step 4: We finally show that $\mu^{'} = a\mu$.
For every $f \in C_c(G)$ define $\phi(f) := \int_G fd\mu$ and $\psi(f) := \frac{1}{a}\int_G fd\mu^{'}$.
Then ϕ and ψ are linear functionals on the space $C_c(G)$ and the difference (which is associated to linear combination of measures $\mu - \mu^{'}/a$) $\phi - \psi$ is the zero functional.
By Riesz's representation theorem (see Rudin or Cohn) we conclude that $\mu - \mu^{'}/a$ is in fact the zero measure, and thus we have

$$\mu^{'} = a\mu.$$

5 Some example

Example 1. $(\mathbb{R}, +)$ with the euclidean topology admits as Haar measure the Lebesgue measure: its translation invariance comes from the fact that the volume of pluri-interval is translation invariant.

Example 2. Let $(\mathbb{R}_{>0}, *)$ with the group structure given by the usual meultiplication and the Euclidean topology.
For every Borel set $E \subseteq \mathbb{R}_{>0}$ the Haar measure is defined as

$$\mu(E) = \int_E \frac{dm(x)}{x}$$

where m is the Lebesgue measure on $\mathbb{R}$.
In fact, for every $a > 0$ one has

$$\mu(aE) = \int_{aE} \frac{dm(x)}{x} = \int_E \frac{a \cdot dm(y)}{ay} = \int_E \frac{dm(y)}{y}.$$

Example 3. Let $G \subseteq \mathbb{R}^n$ be an open set which is a Lie group (i.e. a differentiable manifold with a group structure such that $m : G \times G \to G$ and $.^{-1} : G \to G$ are smooth maps). For every $g \in G$ let $L(g)$ be the Jacobian of the left multiplication by g. Then for every Borel set $E \subseteq G$ the Haar measure on G is defined as

$$\mu(E) := \int_E \frac{dm(x)}{L(x)}.$$

If $G = \mathrm{GL}(n, \mathbb{R})$ then the measure is defined as

$$\mu(E) := \int_E \frac{dm(x)}{|\det x|^n}.$$

Example 4. Let $\Lambda \subset \mathbb{R}^n$ be a lattice (i.e. $\Lambda = \mathbb{Z}\omega_1 \oplus \cdots \oplus \mathbb{Z}\omega_n$ where the ω_i's are $\mathbb{R}$-linearly independent). The quotient $G := \mathbb{R}^n/\Lambda$ is a compact Hausdorff space (with the quotient topology from the Euclidean space $\mathbb{R}^n$) and has a natural group structure induce by the projection map $\pi : \mathbb{R}^n \to G$: this implies that G admits a finite Haar measure μ.
Let $V := \{\sum_{i=1}^n x_i\omega_i \colon x_i \in [0, 1)\}$ be the fundamental domain of the lattice. The map π induces a bijection between V and G, and from this fact we have

a correspondence between the Lebesgue measure m on $\mathbb{R}^n$ and μ.
If $E \subseteq G$ is a Borel set, we can define

$$\mu(E) := m(\pi^{-1}(E) \cap V).$$

If $A \subseteq \mathbb{R}^n$ is a Borel set, we can define

$$m(A) = \sum_{\lambda \in \Lambda} \mu(\pi(A \cap (V + \lambda))).$$

Example 5. Let G_1 and G_2 be locally compact topological groups with Haar measures μ_1 and μ_2 respectively. Then the locally compact group $G_1 \times G_2$ admits as Haar measure the completion of the product measure $\mu_1 \times \mu_2$ (where "completion" means that if $(\mu_1 \times \mu_2)(E) = 0$ and $A \subseteq E$, then $(\mu_1 \times \mu_2)(A) = 0$).
Being $\mathbb{C}^* = S^1 \times (\mathbb{R}_{>0})$, we obtain that the Haar measure on $\mathbb{C}^*$ is induced by the measure on S^1 (constructed in example 4) and the measure on $(\mathbb{R}_{>0})$ provided in example 2.

Example 6. Let p be a prime number. If $a/b \in \mathbb{Q}^*$, we can write

$$\frac{a}{b} = p^k \cdot \frac{r}{s}$$

where r and s are both coprime with p and $k \in \mathbb{Z}$. Let $v_p : \mathbb{Q} \to \mathbb{Z}$ be the function that sends each non zero rational number to the integer k defined before.
The function v_p induces the absolute value on $\mathbb{Q}$

$$\begin{aligned} |\cdot|_p :&\mathbb{Q} \to \mathbb{R} \\ &\frac{a}{b} \to e^{-v_p(a/b)}. \end{aligned}$$

The completion of $\mathbb{Q}$ with respect to the metric induced by $|\cdot|_p$ is the field of p-adic numbers $\mathbb{Q}_p$, and the absolute value $|\cdot|_p$ admits a natural extension on this field.
The subring

$$\mathbb{Z}_p := \{x \in \mathbb{Q}_p \colon |x|_p \leq 1\}$$

is called the ring of p-adic integers and admits a structure of topological group (setting the subgroups $p\mathbb{Z}_p$ as a fundamental system of neighborhoods of 0). In this way $\mathbb{Z}_p$ is a compact Hausdorff space, and admits a Haar measure μ such that

$$\mu(p^n\mathbb{Z}_p) = p^{-n}.$$

Example 7. Let $\overline{Q}$ be the algebraic closure of $\mathbb{Q}$. The absolute Galois group $\mathrm{Gal}(\overline{Q}|\mathbb{Q})$ is isomorphic as a topological group to $\lim_{\leftarrow} \mathrm{Gal}(E|\mathbb{Q})$ where the inverse limit is made over the finite Galois extensions of $\mathbb{Q}$: this group is Hausdorff and compact (being a closed subspace of the product $\prod \mathrm{Gal}(E|\mathbb{Q})$). Thus the absolute Galois group admits a Haar measure μ, and it can be shown that

$$\mu(\mathrm{Gal}(\overline{\mathbb{Q}}|E)) = \frac{1}{[E:\mathbb{Q}]}$$

for every finite extension $\mathbb{Q} \subseteq E$.