Lefschetz Embedding Theorem

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See Lang's "Introduction to abelian and algebraic functions" for further references.

1 A brief recall on theta functions

Let V be a complex vector space of dimension n, and let $\Lambda \subset V$ be a lattice (i.e. a free abelian subgroup of rank $2n$).

Definition 1. A meromorphic function $F : V \to \mathbb{C}$ is called a **theta function for V over Λ of type (L, J)** if for every $x \in V, \lambda \in \Lambda$ is

$$F(x + \lambda) = e(L(x, \lambda) + J(\lambda))F(x) \quad (1)$$

where $L : V \times \Lambda \to \mathbb{C}$ is $\mathbb{C}$-linear in the first coordinate, $J : \Lambda \to \mathbb{C}$ and $e(x) := e^{2\pi i x}$.

The function L can be extended $\mathbb{R}$-linearly on the second coordinate. Define the function $E : V \times V \to \mathbb{C}$, $E(x, y) := L(x, y) - L(y, x)$.

Proposition 1. *The function E defined above is $\mathbb{R}$-bilinear and alternating. Moreover, $E(\Lambda, \Lambda) \subset \mathbb{Z}$ and thus E has real values.*

Definition 2. Let F be a theta function of type (L, J). The $\mathbb{C}$-bilinear function $H : V \times V \to \mathbb{C}$ such that

$$H(x, y) := E(ix, y) + iE(x, y) \quad (2)$$

is called **the Riemann form associated to F**. It is easy to verify that H is hermitian on V.

Definition 3. A theta function F of type (L, J) is said to be **non-degenerate** if its Riemann form H is positive defined

2 Frobenius decompositions

We give a useful technical lemma, due to Frobenius. From now on, if $a, b \in V$, we denote $[a, b] := \mathbb{Z}a \oplus \mathbb{Z}b$.

Lemma 1. *Let Λ be a free abelian group or rank $2n$, and let E be a not zero bilinear alternating form on Λ such that $E(\Lambda, \Lambda) \subset \mathbb{Z}$. Then there exist $e_1, v_1, \ldots, e_n, v_n \in \Lambda$ such that*

$$\Lambda = [e_1, v_1] \oplus \cdots \oplus [e_n, v_n] \tag{3}$$

where the $[e_i, v_i]$'s are pairwise E-orthogonal of rank 2 and the numbers $d_i := E(e_i, v_i)$ are positive and such that $d_1 | d_2 | \cdots | d_n$.

Proof. Let d_1 be the least positive value hold by E on $\Lambda \times \Lambda$ and let $e_1, v_1 \in \Lambda$ such that $E(e_1, v_1) = d_1$. Call $\Lambda_1 := [e_1, v_1]$; it is of rank 2 (otherwise $E(e_1, v_1) = aE(e_1, v_1) = 0$). Call Λ_2 its othogonal complement with respect to E.
First it is $\Lambda_1 \cap \Lambda_2 = \{0\}$ (very easy to prove). Then we want to show that $\Lambda = \Lambda_1 \oplus \Lambda_2$: in order to do this we prove that for every $u \in \Lambda$ there exist $a, b \in \mathbb{Z}$ such that $\tilde{u} := u - ae_1 - bv_1 \in \Lambda_2$.
Requiring this implies that

$$0 = E(\tilde{u}, e_1) = E(u, e_1) + bd_1$$

and we can solve for b, being the set $\{E(u, ce_1 + dv_1) \colon u \in \Lambda, c, d \in \mathbb{Z}\}$ an ideal of $\mathbb{Z}$ generated by d_1. Thus we get

$$a = \frac{E(u, v_1)}{d_1}, b = -\frac{E(u, e_1)}{d_1} \in \mathbb{Z}$$

and the decomposition we looked for.
Now, let d_2 be the least positive value hold by E on $\Lambda_2 \times \Lambda_2$; using the decompostions above it is easy to verify that the set of values of E is an ideal of $\mathbb{Z}$ generated by d_1 and d_2, and so $d_1 | d_2$. □

The set $\{e_1, v_1, \ldots, e_n, v_n\}$ is said to be a **Frobenius base for Λ with respect to** E.

2.1 Application to theta functions

Let $F : V \to \mathbb{C}$ be a non-degenerate theta function of type (L, J) over a lattice Λ. Let $\{e_1, v_1, \ldots, e_n, v_n\}$ be a Frobenius basis for Λ with respect to Λ.

Proposition 2. *The set* $\{e_1, e_2, \dots, e_n\}$ *is a* $\mathbb{C}$*-basis of* V*.*

Recall that, given any Z-basis of Λ, it is also a R-basis of V: thus the matrix representing E with respect to any such basis has integer coefficients, and its determinant is (in modulus) independent from the choice.

Definition 4. The **pfaffian of** E is the number $Pf(E) := \sqrt{|\det E|}$.

Corollary 1. $Pf(E)$ *is a positive integer.*

Proof. Let $\{e_1, v_1, \dots, e_n, v_n\}$ be a Frobenius basis for Λ with respect to E. Then, with respect to this basis, the matrix representing E has the form

$$\begin{pmatrix} 0 & d_1 & & & \\ -d_1 & 0 & & & \\ & & \dots & & \\ & & & 0 & d_n \\ & & & -d_n & 0 \end{pmatrix}$$

hence its determinant is equal to $d_1^2 \cdots d_n^2$. □

3 Spaces of entire theta functions

3.1 Results on dimension of spaces

Let F be an entire theta function of type (L, J). The $\mathbb{C}$-vector space of entire theta functions of the same type is denoted with $\mathrm{Th}(F)$ or $\mathrm{Th}(L, J)$.
In this section the goal is to prove the following:

Theorem 1. *The space* $\mathrm{Th}(L, J)$ *has dimension equal to* $Pf(E)$*.*

Remark 1. Let θ be a trivial theta function (i.e. a function of the form $e(q(x)+l(x)+c)$ where q is a quadratic form on V, l is $\mathbb{C}$-linear on V and $c \in \mathbb{C}$). Then the multiplication by θ provides an obvious isomorphism of $\mathbb{C}$-vector spaces

$$\mathrm{Th}(F) \simeq \mathrm{Th}(F\theta)$$

By multiplying with a suitable trivial theta function,and chosen a Frobenius basis $\{e_1, v_1, \dots, e_n, v_n\}$, we can assume that L and J satisfy the following condition $(*)$:

- $L(x + e_j) = 0$ for every $x \in V$, for every $j \in \{1, \dots, n\}$;

- $J(e_j) = 0$ for every $j \in \{1, \dots, n\}$.

We assume from now on that every theta function we are dealing with is non-degenerate. We are reduced to prove the following:

Theorem 2. *The space* $\mathrm{Th}(L, J)$*, where* (L, J) *satisfy the condition (*), has dimension equal to* $Pf(E)$*.*

Proof. Let $\{e_1, v_1, \dots, e_n, v_n\}$ be a Frobenius basis for Λ; for every $z \in V$ let $\{z_1, \dots, z_n\}$ be its coordinates with respect to the $\mathbb{C}$-basis $\{e_1, \dots, e_n\}$ of V. Call $c_j =: J(v_j)$.
Then the space $\mathrm{Th}(L, J)$ is precisely the space of entire theta functions F which satisfy

$$F(z + e_j) = F(z)$$

$$F(z + v_j) = F(z)e(z_j d_j + c_j)$$

The first condition implies that any theta function has a Fourier expansion

$$F(z) = \sum_{r \in \mathbb{Z}^n} a(r)e(r \cdot z)$$

while the second one induces the following recurrence formula on the coefficients:

$$a(r - d_j e_j) = a(r)e(r \cdot v_j - c_j).$$

Thus the values of the coefficients $a(r)$ can be formally and uniquely determined once we have fixed the values for $0 \leq r_j < d_j$ (where $j \in \{1, \dots, n\}$).
Hence we have proved the claim once one is able to show that each of this series converge. The two main tools in order to prove this are:

- Show that $r \cdot v_j = \sum r_k d_k^{-1} L(v_j, v_k)$;
- Show that, if L satisfies (*), then the imaginary part of L is negative definite on the $\mathbb{R}$-space generated by the v_i's.

□

3.2 A remark on lattices

Suppose F is a non degenerate theta function of type (L, J) for V over Λ. It could be that F is a theta function of the same type over some bigger lattice Λ'.

Then the number of such lattices is finite: in fact, let $u \in \Lambda^{'}$ and $\{e_1, v_1, \ldots, e_n, v_n\}$ be a Frobenius basis for Λ. We can write

$$u = \sum_j a_j e_j + \sum_j b_j v_j$$

where the coefficients a_j, b_j are real numbers. It is clearly

$$E(u, e_j) = -b_j d_j \text{ and } E(u, v_j) = a_j d_j.$$

From the previous remark on $\Lambda^{'}$, these values must be integers and hence a_j and b_j can take only a finite number of values mod $\mathbb{Z}$, which means that there exists only a finite number of such $\Lambda^{'}$'s.

Theorem 3. *Every entire non-degenerate theta function of type (L, J) for V over Λ, with the possible exception of those lying in a finite union of subspaces of dimension $< Pf(E)$, are not theta functions over a lattice strictly larger than Λ.*

Proof. Suppose F is a theta function for V over a bigger lattice $\Lambda^{'}$: then the Pfaffian of E with respect to $\Lambda^{'}$ is equal to

$$\frac{Pf(E)}{(\Lambda' : \Lambda)} < Pf(E).$$

Hence the space of these functions has lower dimension, and from the previous remark there is only a finite number of such subspaces, associated to the lattices $\Lambda^{'s}$. $\square$

4 The Main Theorem

Let F be a entire non degenerate theta function for V over Λ. The space $\mathrm{Th}(F)$ is finite dimensional and admits a basis $\{F_0, \ldots, F_m\}$. Then we can define (or at least try to define) a map

$$\mathcal{F} : V/\Lambda \to \mathbb{P}^m$$

$$x \to (F_0(x) : \cdots : F_m(x))$$

at least in the points where the functions F_i do not vanish at the same time: in those points is well defined towards the projective space (due to the Theta periodicity).
Before we start, we need a useful lemma (which gives the connection of this topic with the divisors).

Lemma 2. *If F is an entire non degenerate theta function , then the kernel of the map*

$$\varphi_F : a \to Cl(F(\cdot - a)/F)$$

is finite.

Theorem 4 (Lefschetz)**.** *Let F be an entire non degenerate theta function. Then the map $\mathcal{F}$ induced by the base of $\mathrm{Th}(F^3)$ is everywhere defined and is an analytic embedding of V/Λ into a projective space.*

Proof. The proof is divided into 3 major steps.

Step 1: the function is defined on every point of V.
This is true if and only if, given a basis $\{F_0, \ldots, F_m\}$ of $\mathrm{Th}(F^3)$, these functions do not all vanish on x; it is enough to show that there exists a theta function of the same type of F^3 which does not vanish on x.
Given $a, b \in V$ consider the theta function

$$F(x-a)F(x-b)F(x+a+b).$$

It is entire and has the same type of F^3; moreover we can find suitable a and b such that this function does not vanish on x (this follows from the fact that F is non degenerate).

Step 2: the function is a homomeorphism with the image The map $\mathcal{F}$ is clearly continuous and closed (being V/Λ compact). The only difficult thing to show is that it is injective, which means that two points x and y with the same image must differ by an element in Λ.
Assume $\mathcal{F}(x) = \mathcal{F}(y)$: then there exists $\gamma \in \mathbb{C}^*$ such that for every b, z and for every $\theta \in \mathrm{Th}(F^3)$ we have

$$\theta(x-z)\theta(x-b)\theta(x+z+b) = \gamma\theta(y-z)\theta(y-b)\theta(y+z+b)$$

We can select θ which is not a theta function with respect to any lattice strictly larger than Λ.
Let $v := x - y$ and $\Lambda^{'} := \Lambda \oplus \mathbb{Z}v$: we shall prove that θ is a theta function over $\Lambda^{'}$, so that $v \in \Lambda$.
Given any point z_0, we can find an element b such that

$$\theta(x-b)\theta(x+z_0+b)\theta(y-b)\theta(y+z_0+b) \neq 0$$

and by the continuity this inequality holds in a neighborhood of z_0: this means that in such a neighborhood there is a holomorphic function g_0 such that

$$\theta(x-z)=\theta(y-z)g_0(z).$$

We can glue together this functions (taking all the points z_0) in order to obtain an entire function g without zeros such that

$$\theta(x-z)=\theta(y-z)g(z).$$

Recall that $v=x-y$, so that (with a change of variables) the previous formula becomes

$$\theta(z+v)=\theta(z)h(z)$$

where h is entire and without zeros. But from the theta relation of θ we get that h is in fact a trivial theta function (one shows that is theta, and being entire without zeros it must be trivial). More precisely, it is

$$h(z)=Ce(\lambda(z))$$

where λ is $\mathbb{C}$-linear.
Now, if (L,J) is the type of θ, the function h has type $(0,L(v,.))$ and for every $u\in\Lambda$ we have $\lambda(u)-L(v,u)\in\mathbb{Z}$. But

$$\lambda(u)-L(v,u)=\lambda(u)-L(u,v)+E(v,u).$$

Hence $\lambda(u)-L(u,v)$ is real valued (being λ and L $\mathbb{R}$-linear, and the elements of Λ generate V). But these functions are also $\mathbb{C}$-linear, hence

$$\lambda(z)=L(z,v).$$

This implies that the function θ is a theta function over Λ', and therefore the proof is ended.

Step 3: the function is an embedding.
We have to show that the differential operators are injective in any point. Let $x\in V/\Lambda$ and let $\theta\in \mathrm{Th}(F^3)$ such that $\theta(x)\neq 0$. The image of x via the map $\mathcal{F}$ is equal to

$$\Big(\frac{F_0(x)}{G(x)}:\dots:\frac{F_M(x)}{G(x)}\Big)$$

hence it is enough to show that,for every $v \in V \setminus \{0\}$, there is some $i \in \{0, \ldots, m\}$ such that

$$d\left(F_i/\theta\right)_x v \neq 0$$

and this is done if there is some $G \in \mathrm{Th}(F^3)$ such that

$$d\left(G/\theta\right)_x v \neq 0.$$

Suppose this were not true and for every $G \in \mathrm{Th}(F^3)$ one gets $d(G/\theta)(x)v = 0$. Let us take a basis for V such that $v = (1, 0, \ldots, 0)$. Assuming $G(x) \neq 0$, from the relation

$$d(G/\theta)_x = \frac{\theta(x)dG_x - G(x)d\theta_x}{\theta(x)^2}$$

and the hypothesis above that

$$\frac{dG_x}{G(x)}v = \frac{d\theta_x}{\theta(x)}v = \alpha$$

where α is a fixed number. By the choice of the basis for V we have

$$dG_x v = \frac{\partial G}{\partial z_1}(x) \text{ and } \frac{dG_x}{G(x)}v = \frac{1}{G(x)}\frac{\partial G}{\partial z_1}(x).$$

Define the functions $G(z) := F(z-a)F(z-b)F(z+a+b)$ (with $a, b \in V$ such that $G(x) \neq 0$) and

$$f(z) := \frac{1}{F(z)}\frac{\partial F}{\partial z_1}$$

(wherever it is defined). Combining the last three lines we get

$$f(x-a) + f(x-b) + f(x+a+b) = \alpha$$

and this relation holds for arbitrary a, b (outside the exceptional set where the denominators vanish). Now the function

$$z \to f(x-z) + f(x-b) + f(x+z+b)$$

is constant, and differentiating with respect to each variable z_j we get

$$\frac{\partial f}{\partial z_j}(x-z) = \frac{\partial f}{\partial z_j}(x+z+b).$$

The right hand side of this equation implies that these partial derivatives are constant for some open set containing z. Whence it follows that in an open set of z where f is defined, we have

$$\frac{1}{F(z)}\frac{\partial F}{\partial z_1} = \alpha_1 z_1 + \cdots + \alpha_n z_n + \beta$$

with constants $\alpha_1, \ldots, \alpha_n, \beta$.
Define finally the function

$$q(z) := \frac{1}{2}\alpha_1 z_1^2 + \alpha_2 z_1 z_2 + \cdots + \alpha_n z_1 z_n + \beta z_1.$$

Then the first partial derivate f the function

$$\tilde{F}(z) := F(z)e^{-q(z)}$$

is equal to 0 in some open set, and so it is zero everywhere. This implies that $\tilde{F}$ depends only on $n-1$ variables. But being $\tilde{F}$ equivalent to F, we have a contradiction, because of lemma 2 (the kernel of the map $a \to \mathrm{Cl}(F(\cdot - a)/F)$ must be finite, and this is not the case!). $\square$