THEORETICAL RECALLS ON EXPLICIT FORMULAS

FRANCESCO BATTISTONI

1. Explicit formulae

Let K be a number field of degree n, signature (r_1, r_2) and discriminant d_K.
Define the *Dedekind Zeta function* of K as

$$\zeta_K(s) := \sum_{I \subset \mathcal{O}_K} \frac{1}{\mathrm{N}(I)^s}\ , \quad \operatorname{Re} s > 1$$

where $\mathrm{N}(I)$ is the norm of the ideal I, i.e. the cardinality of the quotient ring $\mathcal{O}_K/I$.
The Dedekind Zeta function has the following properties (see [1] for the proof):

- The series defining $\zeta_K(s)$ converges absolutely for every $\operatorname{Re} s > 1$, and in this half plane $\zeta_K(s)$ can be expressed as an Euler product

$$\zeta_K(s) = \prod_{\mathcal{P} \subset \mathcal{O}_K} \left(1 - \frac{1}{\mathrm{N}(\mathcal{P})^s}\right)^{-1}$$

  where the product ranges on the non zero prime ideals of $\mathcal{O}_K$.
- The function $\zeta_K(s)$ can be meromorphically extended over the complex plane and admits an infinite number of zeros ρ such that $0 < \operatorname{Re}\rho < 1$ (these are called *non trivial zeros* of ζ_K).

Now, let us consider a function $F : \mathbb{R} \to \mathbb{R}$ which satisfies the following properties:

- F is even and $F(0) = 1$.
- There exists $\varepsilon > 0$ such that the function $G(x) := F(x) \cdot \exp\left(\left(\frac{1}{2} + \varepsilon\right) x\right)$ is summable and of bounded variation over $\mathbb{R}$.
- The function $G(x)$ satisfies in every point the *mean condition*

$$G(x) = \frac{1}{2}\left(G(x^+) + G(x^-)\right).$$

- The function $(F(0) - F(x))/x$ is of bounded variation over $\mathbb{R}$.

Define now the *transform* of F as the function of complex variable

$$\Phi(s) := \int_{-\infty}^{+\infty} F(x) \cdot \exp\left(\left(s - \frac{1}{2}\right) x\right) dx.$$

The conditions set above on F imply that Φ is a holomorphic function in the strip $-a < \operatorname{Re} s < 1 + a$, for a suitable choice of $a > 0$.
All these properties, combined together, permit to recover a theorem of Weil.

Theorem 1. *Let K be a number field of degree n and signature (r_1, r_2). Let F be a function which satisfies the conditions above and let Φ be its transform.*

Then we have the equality

$$\log|d_K| = r_1\frac{\pi}{2} + n(\gamma+\log 8\pi) - r_1\int_0^{+\infty}\frac{1-F(x)}{2\cosh(x/2)}dx$$
$$- n\int_0^{+\infty}\frac{1-F(x)}{2\sinh(x/2)}dx - 4\int_0^{+\infty}F(x)\cosh(x/2)dx$$
$$+ 2\sum_{\mathcal{P}\subset\mathcal{O}_K}\sum_{m=1}^{\infty}\frac{\log \mathrm{N}(\mathcal{P})}{\mathrm{N}(\mathcal{P})^{m/2}}F(m\log\mathrm{N}(\mathcal{P})) + \sum_{\rho}\Phi(\rho).$$

where γ is Euler's constant and $\sum_\rho$ ranges over the set of non trivial zeros of ζ_K.

See [3] for all the technical details.

From now on, we do not want to consider the contribution of the sum over the non trivial zeros. This can be done if one requires $\Phi(s)$ to be positive in the strip $0 < \operatorname{Re} s < 1$: more in detail, we assume

$$F(x) := \frac{f(x)}{\cosh(x/2)} \tag{1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is such that:

- $f(x)$ is even, $f(0) = 1$ and $\int_0^{+\infty} f(x)dx$ converges.
- The function $F(x) = f(x)/\cosh(x/2)$ is of bounded variation on $\mathbb{R}$ and satisfies the mean condition.
- The function $(1-f(x))/x$ is of bounded variation on $\mathbb{R}$.
- The Fourier transform of f is a positive function.

Theorem 2. *Let K be a number field of degree n and signature (r_1, r_2). Let f be a function as the one introduced in (1) and let $y > 0$. Then we have the inequality*

$$\frac{1}{n}\log|d_K| \geq \gamma + \log 4\pi + \frac{r_1}{n} - \int_0^{\infty}(1-f(x\sqrt{y}))\left(\frac{1}{\sinh x} + \frac{r_1}{n}\frac{1}{2\cosh^2(x/2)}\right)dx$$
$$- \frac{4}{n}\int_0^{\infty} f(x\sqrt{y})dx + \frac{4}{n}\sum_{\mathcal{P}\subset\mathcal{O}_K}\sum_{m=1}^{\infty}\frac{\log\mathrm{N}(\mathcal{P})}{1+(\mathrm{N}(\mathcal{P}))^m}f(m\log\mathrm{N}(\mathcal{P})\sqrt{y}).$$

(See again [3] for the details).

Remark 1. This inequality involves the logarithm of the root discriminant of the field K, where the root discriminant is defined as $|d_K|^{1/n}$; therefore a slight increment on the right side produces a sensible increment on $|d_K|$.

Up to now, the best known choice for $f(x)$ (as described in [3] and [2]) was given by Luc Tartar, who chose

$$f(x) := \left(\frac{3}{x^3}(\sin x - x\cos x)\right)^2 \tag{2}$$

which is the square of the Fourier transform of the function

$$u(x) := \begin{cases} 1 - x^2 & |x| \leq 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Further estimates on the integral terms lead to an explicit lower bound for the logarithm of the root discriminant.

Theorem 3. *Let K be a number field of degree n, signature (r_1, r_2) and discriminant d_K. If $f(x)$ is Tartar's function (2), then we have the inequality*

$$\frac{1}{n}\log|d_K| \geq \gamma + \log 4\pi - L_1(y) - \frac{12\pi}{5n\sqrt{y}} + \frac{4}{n}\sum_{\mathcal{P}\subset\mathcal{O}_K}\sum_{m=1}^{\infty}\frac{\log \mathrm{N}(\mathcal{P})}{1+(\mathrm{N}(\mathcal{P}))^m} f(m\log \mathrm{N}(\mathcal{P})\sqrt{y}) \tag{3}$$

where

$$L_1(y) := L(y) + \frac{1}{3}L\left(\frac{y}{3^2}\right) + \frac{1}{5}L\left(\frac{y}{5^2}\right) + \cdots + \frac{r_1}{n}\left[L(y) - L\left(\frac{y}{2^2}\right) + L\left(\frac{y}{3^2}\right)\cdots\right]$$

and

$$L(y) := -\frac{3}{20y^2} + \frac{33}{10y} + 2 + \left(\frac{3}{80y^3} + \frac{3}{4y^2}\right)\left(\log(1+4y) - \frac{1}{\sqrt{y}}\arctan(2\sqrt{y})\right).$$

(Again, all the details are in [3]).

References

[1] S. Lang. *Algebraic number theory*, volume 110 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1994.

[2] A. M. Odlyzko. Bounds for discriminants and related estimates for class numbers, regulators and zeros of zeta functions: a survey of recent results. *Sém. Théor. Nombres Bordeaux (2)*, 2(1):119–141, 1990.

[3] Georges Poitou. Sur les petits discriminants. In *Séminaire Delange-Pisot-Poitou, 18e année: (1976/77), Théorie des nombres, Fasc. 1 (French)*, pages Exp. No. 6, 18. Secrétariat Math., Paris, 1977.