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**Dedication.** *It occurred to the five of us to come in contact with Asim Barut in the very short period from June to September 1994, between a conference in Erice and “his” conference in Edirne, where he was so kind to invite all of us and for which the present notes are written down. It was a joyful experience to share with him the sensation that classical electrodynamics had not been fully exploited, and the hope that a full appreciation of it might prove relevant even for the foundations of quantum mechanics. Certainly there will be other people among his pupils that will take up his heritage, but we very willingly acknowledge that, although working along already established lines, a great support comes to us from the sensation of continuing also his work.*

## INTRODUCTION

By classical electron theory we mean what is in principle a very simple thing, namely the Maxwell–Lorentz system, which consists of Maxwell equations with sources due to a point particle, and the relativistic Newton equation for the particle, with Lorentz force due to the electromagnetic field. The unknowns are then the fields  $\mathbf{E}(\mathbf{x}, t)$ ,  $\mathbf{B}(\mathbf{x}, t)$  and the particle motion  $\mathbf{q}(t)$ , governed by the equations

$$\begin{aligned} \operatorname{div} \mathbf{E} &= \rho & \operatorname{div} \mathbf{B} &= 0 \\ \operatorname{rot} \mathbf{E} &= -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} & \operatorname{rot} \mathbf{B} &= \frac{1}{c} \left( \mathbf{j} + \frac{\partial \mathbf{E}}{\partial t} \right) \\ \frac{d}{dt} \left( \frac{m_0}{\sqrt{1 - \dot{\mathbf{q}}^2/c^2}} \dot{\mathbf{q}} \right) &= e \left( \mathbf{E}(\mathbf{q}, t) + \frac{1}{c} \dot{\mathbf{q}} \times \mathbf{B}(\mathbf{q}, t) \right) \\ \rho(\mathbf{x}, t) &= e \delta(\mathbf{x} - \mathbf{q}(t)) & \mathbf{j}(\mathbf{x}, t) &= \frac{e}{c} \dot{\mathbf{q}} \delta(\mathbf{x} - \mathbf{q}(t)) \end{aligned}$$

from which the continuity equation  $\frac{\partial \rho}{\partial t} + \operatorname{div} \mathbf{j} = 0$  follows; here  $c$  is the speed of light while  $e$  is the particle’s charge and  $m_0$  its (bare!) mass. The intent would be to study such a system as a dynamical system in the standard sense, namely by looking at the Cauchy problem, also taking into proper account (but we are not able to do it, at present) the nonlinearities, which appear for example in the definitions of the charge density  $\rho$  and of the current density  $\mathbf{j}$ , and in the Lorentz force. The main mathematical problem comes from the presence of the delta function in the definition of the current, which leads to a singularity in the Lorentz force at the right hand side of the equation governing the particle’s motion (think of the Coulomb force in the static case). This problem can be dealt with by suitable regularizations; a standard procedure, well familiar from quantum field theory, consists in introducing cutoffs

in the Fourier transforms of the fields, and studying the limit when the cutoffs are removed. In this connection we like to quote one of our favorite authors, namely E. Nelson,<sup>[1]</sup> who, in his book “Quantum Fluctuations” (page 65) says: “*With suitable ultraviolet and infrared cutoffs, this is a dynamical system of finitely many degrees of freedom, and we have global existence and uniqueness... . Is it an exaggeration to say that nothing whatever is known about the behavior of the system as the cutoffs are removed, and there is not one single theorem that has been proved ?*”. Though incredible, this is just the actual situation: nothing whatever seemed to be known rigorously for the motion of a point particle in interaction with the electromagnetic field, when the latter is not assumed to be assigned in advance.

We believe that we were finally able to provide at least some preliminary results concerning the limit in which the cutoffs are removed, on which we will report below. The results might at first sight appear to be almost trivial, because we essentially confirmed the validity of the famous Abraham–Lorentz (or AL) equation for the particle in the so-called dipole approximation. But on the other hand this required a strong conceptual effort, because we had to become convinced of a deep fact, which paradoxically was well known, but on the other hand was essentially removed by the scientific community (see however [2]). Namely, the fact that classical electrodynamics, when extended to microscopic bodies, is radically different from the macroscopic one, due to the fact that it requires negative bare masses, and so leads for generic initial data to absurd runaway solutions; and these can be removed by some prescription *à la* Dirac, which leads to a conceptually different theory, exhibiting nonlocal aspects. By the way, as foreseen by Nelson too, this turns out to have strong implications on the relations between classical physics and quantum mechanics.

These facts will be illustrated, though in a rather sketchy way, in the present notes. In this introduction we would like, however, to mention some authorities, in support of the significance of our studies. Indeed, if it turns out that microscopic classical electrodynamics has so many and great complications, why at all to insist on it and not just abandon it ? Here the quotation we like most is taken from the beautiful chapter of the Feynman lectures devoted to the electromagnetic mass (page 28.10), namely: “*....it might be a waste of time to straighten out the classical theory, because it could turn out that in quantum electrodynamics the difficulties will disappear or may be resolved in some other fashion. But the difficulties do not disappear in quantum electrodynamics.... The Maxwell theory still has the difficulties after the quantum mechanics modifications are made.*”. Essentially the same opinion is expressed in some works of Dirac and Haag; in particular, in the introduction to his famous work on the selfinteraction of the electron<sup>[3]</sup> Haag says: “*This often discussed subject will be here reconsidered in light of the difficulties of quantum field theory*”. For other relevant contributions see [4], [5] (see also [6]).

## RIGOROUS RESULTS FOR THE LINEARIZED MAXWELL–LORENTZ SYSTEM

We give here a short review of some results recently found by two of us (see [7]), which somehow constitute the culmination of a long line of research (see [8], [9] and also [10] from a numerical point of view, and [11], [12] from an analytical point of view; however, in all these works the bare mass was kept positive, because it took much time to understand the necessity of negative bare masses for microscopic particles). Other aspects of the problem were discussed in the very recent work [13]; finally, in a

work still in preparation<sup>[14]</sup> for the first time the closed equation was found which is obeyed by the field in the limit in which the cutoffs are removed.

The aim is thus to have information on the dynamics when the regularization is removed. In fact, the preliminary results illustrated here are restricted to the projection of the solution on the particle variables, i.e. concern only the motion of the particle, which is induced by the solution of the full system. However, the main drawback is that such results are obtained not for the original Maxwell–Lorentz system, but only for its linearization about the equilibrium point defined by the particle at rest at the origin and vanishing field (a mechanical linear restoring force acting on the particle is also included); in particular, within such an approximation (often called the “dipole approximation”) the system is nonrelativistic; finally (but this is an unessential technical point), the regularization is performed not by imposing cutoffs on the fields, but by considering, as we use to say, a “fat” particle, i.e. by substituting the  $\delta$  function appearing in the charge and current densities by a smooth charge distribution (or form factor)  $\rho$ . Because of this, the limit in which the cutoffs are removed will be called here the “point limit”.

We chose to work in the Coulomb gauge. Let us recall that in such a gauge the dynamically relevant parts of the fields depend just on the vector potential  $\mathbf{A} = \mathbf{A}(\mathbf{x}, t)$  (subject to the constraint  $\text{div} \mathbf{A} = 0$ ), and that the complete Maxwell–Lorentz system takes the form

$$\begin{aligned} \frac{1}{c^2} \ddot{\mathbf{A}} - \Delta \mathbf{A} &= \frac{4\pi e}{c} \Pi(\dot{\mathbf{q}} \delta_{\mathbf{q}}) , \\ \frac{d}{dt} \left( \frac{m_0 \dot{\mathbf{q}}}{\sqrt{1 - (\dot{\mathbf{q}}/c)^2}} \right) &= -\frac{e}{c} \dot{\mathbf{A}}(\mathbf{q}, t) + \frac{e}{c} \dot{\mathbf{q}} \times \text{rot} \mathbf{A}(\mathbf{q}, t) - \alpha \mathbf{q} , \end{aligned} \quad (1)$$

where  $\alpha > 0$  is a constant characterizing the external linear force, while  $\delta_{\mathbf{q}}$  is the delta function translated by  $\mathbf{q}$  with respect to the  $\delta$  function centered at the origin, i.e.  $\delta_{\mathbf{q}}(\mathbf{x}) := \delta(\mathbf{x} - \mathbf{q})$ . Finally  $\Pi$  is the projector on the subspace of vector fields with vanishing divergence, i.e.  $\Pi(\mathbf{j})$  is the so called transversal part of the current  $\mathbf{j}$ , often denoted by  $\mathbf{j}_{\text{tr}}$ . Now we take the so called dipole approximation, i.e. linearize the system about the equilibrium point  $\mathbf{q} = \dot{\mathbf{q}} = \mathbf{A} = \dot{\mathbf{A}} = 0$ , and regularize it by substituting the delta function by a smooth normalized (in  $L^1$ ) charge distribution  $\rho$ . So we obtain the system

$$\begin{aligned} \frac{1}{c^2} \ddot{\mathbf{A}} - \Delta \mathbf{A} &= \frac{4\pi e}{c} \Pi(\dot{\mathbf{q}} \rho) , \\ m_0 \ddot{\mathbf{q}} &= -\frac{e}{c} \int_{\mathbb{R}^3} \rho(\mathbf{x}) \dot{\mathbf{A}}(\mathbf{x}, t) d^3 \mathbf{x} - \alpha \mathbf{q} , \end{aligned} \quad (2)$$

which is the one actually studied; the appropriate configuration space  $_0$  is

$$_0 := \S_*(\mathbb{R}^3, \mathbb{R}^3) \oplus \mathbb{R}^3 \ni (\mathbf{A}, \mathbf{q}) ,$$

where  $\S_*$  denotes the subset of the vector fields belonging to the Schwartz space  $\S$  ( $C^\infty$  functions decaying at infinity faster than any power) having vanishing divergence.

Concerning the form factor  $\rho$  one assumes that it is  $C^\infty$ , decays at least exponentially fast at infinity, is spherically symmetric, and its Fourier transform  $\hat{\rho}$ , defined by

$$\rho(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \hat{\rho}(\mathbf{k}) e^{i\mathbf{k}\mathbf{x}} d^3 \mathbf{k} ,$$

is everywhere non-vanishing (*i.e.*  $\rho(\mathbf{k}) \neq 0, \forall \mathbf{k} \in \mathbb{R}^3$ ); finally, in order to simplify the discussion of the point limit one assumes that  $\rho$  has the form

$$\rho_R(x) := \frac{1}{R^3} \left( \frac{x}{R} \right), \quad (3)$$

where  $\rho$  is a positive, normalized (in  $L^1$ ) function, and  $R > 0$  a parameter characterizing the “radius” of the particle. For any  $R > 0$  the Cauchy problem for system 2 is well posed in the phase space  $\mathfrak{o} \times \mathfrak{o}$ , and the problem is to discuss the limit of the particle’s motion as the “radius”  $R$  of the charge distribution tends to zero.

The first result is obtained for the special class of initial data such that the particle is at rest in some position  $q_0 \neq 0$ , and the field vanishes. One has the

**Proposition.** *Having fixed  $m_0 > 0$  and  $R > 0$  denote by  $\mathbf{q}^{(R)}(t)$  the solution of 2 with form factor 3, corresponding to initial data  $\mathbf{A}_0 = \dot{\mathbf{A}}_0 = \dot{\mathbf{q}}_0 = 0$ , and  $\mathbf{q}_0 \neq 0$ . Assume that  $\mathbf{q}^{(R)}(\cdot)$  converges weakly to a distribution  $\mathbf{q}(\cdot)$  as  $R \rightarrow 0$ . Then there exists a constant vector  $\bar{\mathbf{q}}$  such that  $\mathbf{q}(t) \equiv \bar{\mathbf{q}}$ .*

This means that, for fixed positive bare mass, the limit dynamics of the point particle is trivial, if it exists. This result would be obvious for physicists of the old generation, and should be obvious even for undergraduates that are familiar with the Feynman lectures on physics, where he explains that there is an electromagnetic contribution to mass, which diverges in the point limit;<sup>1</sup> in fact, the above proposition just proves that a point particle is unaffected by the presence of a force  $(-\alpha \mathbf{q}_0)$  no matter how large it is, so that it behaves as if its actual mass were infinite. From the mathematical point of view this is seen as follows. The proof of all results discussed here is based on the use of a representation formula for the solution of the Cauchy problem of 2 through normal modes, and it turns out that in such a formula the bare mass  $m_0$  appears always summed to the quantity

$$m_{\text{em}} := \frac{32}{3} \pi^2 \frac{e^2}{c^2} \int_0^\infty |\hat{\rho}(k)|^2 dk = \frac{1}{R} \left[ \frac{32}{3} \pi^2 \frac{e^2}{c^2} \int_0^\infty |\hat{\rho}(k)|^2 dk \right], \quad (4)$$

which is just the electromagnetic mass corresponding to the given charge distribution; and this is in agreement with the expectation that the particle should behave as if its experimental mass  $m$  were the sum of  $m_0$  and  $m_{\text{em}}$ :  $m = m_0 + m_{\text{em}}$ . Notice that  $m_{\text{em}} = m_{\text{em}}(R) \rightarrow +\infty$  as  $R \rightarrow 0$ , so that the effective mass also  $\rightarrow \infty$  as  $R \rightarrow 0$  if the bare mass  $m_0$  is kept fixed.

So, in order to obtain meaningful results, one has to renormalize mass, *i.e.* to consider the bare mass  $m_0$  as a function of  $R$ , by putting

$$m_0(R) := m - m_{\text{em}}(R), \quad (5)$$

where  $m$  is a fixed phenomenological parameter to be identified with the physical mass of the particle, and which does not appear at all in the original Maxwell–Lorentz

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<sup>1</sup> This is seen in the simplest way (see the quoted chapter of Feynman’s book) as follows. Consider a “fat” particle, in the form of a sphere of radius  $R$ , in uniform rectilinear motion with velocity  $v$ . Compute by the familiar formulae of retarded potentials the fields  $E$  and  $B$  “created” by the particle, and the corresponding Poynting vector, proportional to  $E \times B$ , and integrate in the domain outside the particle, thus obtaining the total momentum  $\mathbf{p}_{\text{em}}$  of the electromagnetic field dragged along by the particle. In the nonrelativistic approximation one immediately finds

$$\mathbf{p}_{\text{em}} = \frac{2}{3} m_{\text{em}} v,$$

in terms of the electromagnetic mass  $m_{\text{em}}$  defined by  $m_{\text{em}} = \frac{e^2}{Rc^2}$ .

system. This point should be particularly emphasized. If one considers the original regularized Maxwell–Lorentz system, and takes the limit  $R \rightarrow 0$  with the prescription 5, one is in fact defining a new system, which can be said to be just suggested by the original one. This is particularly evident, if one remarks that, according to 5 there exists a threshold radius  $\bar{R}$  such that

$$m_0(\bar{R}) = 0, \quad m_0(R) < 0 \quad \text{if } R < \bar{R},$$

and  $m_0(R) \rightarrow -\infty$  as  $R \rightarrow 0$ . The critical radius  $\bar{R}$  is called the “classical electron radius” Concerning the behaviour of the system in the limit  $R \rightarrow 0$  one has the

**Proposition.** *Consider the Cauchy problem for system 2 with form factor  $\rho_R$  given by 3,  $m_0$  given by 5, and initial data  $\mathbf{A}_0 = \dot{\mathbf{A}}_0 = \dot{\mathbf{q}}_0 = 0$ , and  $\mathbf{q}_0 \neq 0$ ; let  $\mathbf{q}^{(R)}(t)$  be the corresponding particle’s motion. Then, for any  $T > 0$ , as  $R \rightarrow 0$  the function  $\mathbf{q}^{(R)}(.)$  converges in  $C^1([-T, T], \mathbb{R}^3)$  to a non constant function.*

So, the particular solution corresponding to the above initial data has a point limit which is nontrivial, provided mass is renormalized.

Consider now the case where the initial particle’s velocity too is different from zero; this is a nontrivial generalization, because it requires that one be familiar with the notion of the field “adapted” to the given initial velocity. This is illustrated by the following result which shows that, with  $\dot{\mathbf{q}}_0 \neq 0$ , if one takes a vanishing initial field the trajectory of the particle turns out to have no sensible point limit. Precisely one has the

**Proposition.** *Consider the Cauchy problem for system 2 with form factor  $\rho_R$  given by 3,  $m_0$  given by 5, and initial data  $\mathbf{A}_0 = \dot{\mathbf{A}}_0 = \mathbf{q}_0 = 0$ , and  $\dot{\mathbf{q}}_0 \neq 0$ ; let  $\mathbf{q}^{(R)}(t)$  be the corresponding particle’s motion. Then, for any  $T > 0$ , as  $R \rightarrow 0$  one has*

$$|\mathbf{q}^{(R)}(t)| \rightarrow \infty, \quad \forall t \in [-T, T] \setminus ,$$

where  $\setminus$  is a finite (possibly empty) set.

It is not difficult to prove that the same happens also if one takes as initial data for the field any regular function (i.e. without singularities).

This result should be not completely astonishing, because it is very well known (and was also recalled above) that, in the case of a uniform motion, a particle drags with it a field which in the case of a point particle has a singularity at the particle’s position. So, it seems natural to study the particular class of initial data such that a particle with a non vanishing velocity is accompanied by such a “proper or adapted field”.<sup>2</sup> In order to give a precise statement we recall that<sup>[15,11]</sup> in the non–linear Maxwell–Lorentz system a free particle can move uniformly with velocity  $\mathbf{v}$ , only if accompanied by a field  $\mathbf{X}$ , which is defined as the solution of equation

$$\Delta \mathbf{X} - \frac{1}{c^2} \sum_{i,l=1}^3 v_i v_l \frac{\partial^2}{\partial x_i \partial x_l} \mathbf{X} = -4\pi \frac{e}{c} \Pi(\rho \mathbf{v})$$

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<sup>2</sup> Within our group, the awareness of this elementary fact was obtained through the work [8], where it was shown that such a field produces a Lorentz force vanishing exactly at the instantaneous particle position; by the way, it is just this fact that allows dynamically for the existence of uniform motions (a circumstance that Abraham used to qualify as “consistency of electrodynamics with the inertia principle”). Moreover, numerical integrations with a positive bare mass and a cutoff on the field showed that, if the initial data are nonvanishing particle velocity and vanishing field, then the solution of the complete system is such that the proper field of the particle tends to be created. As this fact was discovered by Lia Forti, in our jargon we use to call the proper field “il campo della Lia”, i.e. Lia’s field.

vanishing at infinity. In the dipole approximation, i.e. after a linearization in the velocity and in the field, such an equation reduces to

$$\Delta \mathbf{X} = -4\pi \frac{e}{c} \Pi(\rho \mathbf{v}) . \quad (6)$$

We denote by  $\mathbf{X}_\mathbf{v}$  the unique solution of equation 6 vanishing at infinity, and study the point limit of the solutions of the Cauchy problem corresponding to initial data of the form  $\dot{\mathbf{q}}_0 \neq 0, \mathbf{A}_0 = \mathbf{X}_{\dot{\mathbf{q}}_0}$ . Initial data of the form  $(\mathbf{q}_0, \dot{\mathbf{q}}_0, \mathbf{X}_{\dot{\mathbf{q}}_0}, 0)$  will be called of “congruent type”. One has then the

**Proposition.** *Consider the Cauchy problem for system 2 with form factor  $\rho_R$  given by 3,  $m_0$  given by 5, and initial data  $(\mathbf{q}_0, \dot{\mathbf{q}}_0, \mathbf{A}_0, \dot{\mathbf{A}}_0) = (\mathbf{q}_0, \dot{\mathbf{q}}_0, \mathbf{X}_{\dot{\mathbf{q}}_0}, 0)$ ; let  $\mathbf{q}^{(R)}(t)$  be the corresponding particle’s motion. Then, for any  $T > 0$ , as  $R \rightarrow 0$  the function  $\mathbf{q}^{(R)}(\cdot)$  converges in  $C^1([-T, T], \mathbb{R}^3)$  to a non constant function.*

Analogously, for the case of general initial data for the field, one has the following

**Theorem.** *Consider the Cauchy problem for system 2 with form factor  $\rho_R$  given by 3,  $m_0$  given by 5, and initial data  $(\mathbf{q}_0, \dot{\mathbf{q}}_0, \mathbf{A}_0, \dot{\mathbf{A}}_0) = (\mathbf{q}_0, \dot{\mathbf{q}}_0, \mathbf{X}_{\dot{\mathbf{q}}_0} + \mathbf{A}'_0, \dot{\mathbf{A}}_0)$  with  $(\mathbf{A}'_0, \dot{\mathbf{A}}_0) \in \mathfrak{s}_*(\mathbb{R}^3, \mathbb{R}^3) \times \mathfrak{s}_*(\mathbb{R}^3, \mathbb{R}^3)$ ; let  $\mathbf{q}^{(R)}(t)$  be the corresponding particle’s motion. Then, for any  $T > 0$ , as  $R \rightarrow 0$  the function  $\mathbf{q}^{(R)}(\cdot)$  converges in  $C^0([-T, T], \mathbb{R}^3)$ . Moreover, the limiting particle’s motion depends continuously on*

$$(\mathbf{q}_0, \dot{\mathbf{q}}_0, \mathbf{A}'_0, \dot{\mathbf{A}}_0) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathfrak{s}_*(\mathbb{R}^3, \mathbb{R}^3) \times \mathfrak{s}_*(\mathbb{R}^3, \mathbb{R}^3) .$$

So the above theorem shows that the dynamics of a point particle is well defined in the point limit, at least for initial fields which are regular modifications of congruent fields. Moreover, the Cauchy problem is well posed in the sense of Hadamard. By the way this existence result could be extended to the case of initial fields  $(\mathbf{A}'_0, \dot{\mathbf{A}}_0)$  which are only  $C^0$ , and decay at infinity faster than  $r^{-3/2}$ .

In the case of congruent initial data it is possible to calculate explicitly the point limit of the solution of the Maxwell-Lorentz system. Let us introduce some notations: define

$$\omega_0^2 := \frac{\alpha}{m} , \quad \tau := \frac{2}{3} \frac{e^2}{mc^3} ,$$

consider the equation

$$\tau \nu^3 - \nu^2 - \omega_0^2 = 0 , \quad (7)$$

and denote by  $\nu_r, \nu_+ = \nu_3 + i\nu_2, \nu_- = \nu_3 - i\nu_2$  its three solutions ( $\nu_2, \nu_3 > 0$ ). One has then the

**Theorem.** *The point limit of the particle’s motion corresponding to the solution of the Cauchy problem for the linearized Maxwell-Lorentz system 2 with initial data of congruent type  $(\mathbf{q}_0, \dot{\mathbf{q}}_0, \mathbf{A}_0, \dot{\mathbf{A}}_0) = (\mathbf{q}_0, \dot{\mathbf{q}}_0, \mathbf{X}_{\dot{\mathbf{q}}_0}, 0)$  is given by*

$$\mathbf{q}(t) = \begin{cases} e^{-\nu_3 t} [\mathbf{1}^+ \cos(\nu_2 t) + \mathbf{2}^+ \sin(\nu_2 t)] + \mathbf{3}^+ e^{\nu_r t} , & \text{if } t > 0 \\ e^{\nu_3 t} [\mathbf{1}^- \cos(\nu_2 t) + \mathbf{2}^- \sin(\nu_2 t)] + \mathbf{3}^- e^{-\nu_r t} , & \text{if } t < 0 \end{cases} , \quad (8)$$

where  $\mathbf{1}^\pm, \mathbf{2}^\pm, \mathbf{3}^\pm$  are real vector constants depending on the initial data and on  $e, m, \omega_0$ . Moreover one has the asymptotics

$$\begin{cases} \nu_r = \frac{\omega_0}{\epsilon} + O(\epsilon) \\ \nu_2 = \omega_0 + O(\epsilon^2) \\ \nu_3 = \omega_0 \epsilon / 2 + O(\epsilon^2) \end{cases} , \quad \begin{cases} \mathbf{1}^\pm = \mathbf{q}_0 + O(\epsilon^2) \\ \mathbf{2}^\pm = \frac{\dot{\mathbf{q}}_0}{\omega_0} + O(\epsilon^2) \\ \mathbf{3}^\pm = O(\epsilon) \end{cases} \quad (9)$$

for  $\epsilon \rightarrow 0$ , in terms of the dimensionless parameter  $\epsilon := \omega_0 \tau$ .

We can now compare 8 with the solutions of the celebrated Abraham–Lorentz (AL) equation

$$m\tau\ddot{\mathbf{q}} = m\ddot{\mathbf{q}} + \alpha\mathbf{q} , \quad (10)$$

which was known since a century, but the deduction of which from the Maxwell–Lorentz system should be considered as a heuristic one (see [16] or [17]). In this connection one has the

**Theorem.** *The point limit 8 of the particle’s motion in the Maxwell–Lorentz system is also a solution of the problem*

$$\begin{aligned} -m\tau\ddot{\mathbf{q}} &= m\ddot{\mathbf{q}} + \alpha\mathbf{q} , & t < 0 , \\ m\tau\ddot{\mathbf{q}} &= m\ddot{\mathbf{q}} + \alpha\mathbf{q} , & t > 0 . \end{aligned}$$

So, for initial data of congruent type, the particle’s motion satisfies exactly the AL equation for positive times, and the corresponding one with the substitution  $\tau \rightarrow -\tau$  for negative times. Notice that the fact that different equations occur for negative and positive times is not a particular feature of the present model; a classical case where this happens is that of the Boltzmann equation, and a simple and enlightening mechanical model where this phenomenon occurs was given by Lamb at the beginning of the century.

## A HEURISTIC APPROACH TO THE ABRAHAM–LORENTZ EQUATION

It was shown above that the motion of the particle in the point limit for the linearized Maxwell–Lorentz system with the prescription of mass renormalization, for a special class of initial data satisfies exactly the Abraham–Lorentz equation 10. Now, the rigorous deduction reported above is based on an explicit representation formula using of the normal modes of the complete system, which does not allow one to really understand what is going on. So we intend to devote the present section to the illustration of a heuristic deduction, which allows to see by eyes what is occurring (see [18]). A rigorous treatment along similar lines can be found in [13], where it was also shown how the initial acceleration for the AL equation is defined by the initial field for the complete system.

In order to study the linearized Maxwell–Lorentz system 2, it is convenient to introduce the space Fourier transform of the vector potential by

$$\mathbf{A}(\mathbf{x}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \hat{\mathbf{A}}(\mathbf{k}, t) e^{i\mathbf{k} \cdot \mathbf{x}} d^3\mathbf{k} ,$$

so that the system becomes

$$\begin{aligned} m_0 \ddot{\mathbf{q}}(t) &= -\frac{e}{c} \int \hat{\rho}^*(\mathbf{k}) \frac{\partial \hat{\mathbf{A}}}{\partial t}(\mathbf{k}, t) d^3k - \alpha\mathbf{q} + \mathbf{F}_{\text{ext}}(\mathbf{q}) \\ \frac{d^2}{dt^2} \hat{\mathbf{A}}(\mathbf{k}, t) + k^2 c^2 \hat{\mathbf{A}}(\mathbf{k}, t) &= 4\pi e c \hat{\rho}(\mathbf{k}) \left[ \dot{\mathbf{q}}(t) - \frac{(\dot{\mathbf{q}}(t) \cdot \mathbf{k}) \mathbf{k}}{k^2} \right] , \end{aligned} \quad (11)$$

where  $*$  denotes complex conjugate, and a generic external force  $\mathbf{F}_{\text{ext}}$  additional to the linear one  $-\alpha\mathbf{q}$  has also been introduced. By the variation of constants formula, the second equation can also be written in the integral form

$$\hat{\mathbf{A}}(\mathbf{k}, t) = \hat{\mathbf{A}}_{\text{hom}}(\mathbf{k}, t) + \frac{e\hat{\rho}(\mathbf{k})}{k} \int_0^t \left[ \dot{\mathbf{r}}(t') - \frac{(\dot{\mathbf{r}}(t') \cdot \mathbf{k}) \mathbf{k}}{k^2} \right] \sin[kc(t - t')] dt' ,$$

where  $\mathbf{A}_{\text{hom}}$  is the solution of the homogeneous problem. By substitution in the equation for the particle, exchanging the order of the integrations and performing the integrations on the angular variables (by exploiting the spherical symmetry of the form factor) one then obtains an integro-differential equation for the particle, namely

$$m_0 \ddot{\mathbf{q}}(t) = - \frac{32\pi^2}{3} e^2 \int_0^t dt' \dot{\mathbf{q}}(t') \int_0^\infty dk k^2 |\tilde{\rho}(k)|^2 \cos[kc(t-t')] \\ + \mathbf{F}_{\text{hom}} - \alpha \mathbf{q}(t) + \mathbf{F}_{\text{ext}}(\mathbf{q}) ,$$

where  $\mathbf{F}_{\text{hom}}$  represents the Lorentz force on the particle due to the free evolution of the initial field; in particular,  $\mathbf{F}_{\text{hom}}$  vanishes if the initial field does. From now on we will simply denote  $\mathbf{F}_{\text{ext}} + \mathbf{F}_{\text{hom}}$  by  $\mathbf{F}_{\text{ext}}$ .

An equivalent interesting form for the particle equation is obtained by performing two integrations by parts on the variable  $t'$ . Indeed this leads to

$$(m_0 + m_{\text{em}}) \ddot{\mathbf{q}}(t) = -\alpha \mathbf{q} + \mathbf{F}_{\text{ext}} + \\ + \frac{32\pi^2 e^2}{3c^3} \int_0^t dt' \ddot{\mathbf{q}}(t') \int_0^\infty dk |\tilde{f}(k)|^2 \cos[kc(t-t')] + \\ - \frac{32\pi^2 e^2}{3c^3} \int_0^\infty |\tilde{f}(k)|^2 \left\{ \dot{\mathbf{q}}(0) k \sin(kct) - \frac{\ddot{\mathbf{q}}(0)}{c} \cos(kct) \right\} dk$$

where it appears that the bare mass  $m_0$  occurs only summed to the electromagnetic mass  $m_{\text{em}}$  defined by 4.

We now concentrate our attention on a particularly convenient form factor, namely the sharp one corresponding to a cutoff  $\mathcal{K}$  in momentum space, defined by

$$\tilde{\rho}(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \theta(\mathcal{K} - k)$$

where  $\theta$  is the usual step function; this is somehow equivalent to considering an extended particle of spatial dimension (or “radius”)  $R$  with  $R \simeq \mathcal{K}^{-1}$ . Correspondingly, we affix an index ( $\mathcal{K}$ ) to the quantities of interest, and we will be looking for the limit as  $\mathcal{K} \rightarrow \infty$ . In particular, for the electromagnetic mass one has then

$$m_{\text{em}}(\mathcal{K}) = \frac{4}{3} \frac{e^2}{\pi c^2} \mathcal{K} .$$

Introduce now time Fourier transforms, so that one has for example, for the particle acceleration  $\mathbf{a}^{(\mathcal{K})}(\mathbf{t}) \equiv \ddot{\mathbf{q}}^{(\mathcal{K})}(\mathbf{t})$ ,

$$\mathbf{a}^{(\mathcal{K})}(t) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \tilde{\mathbf{a}}^{(\mathcal{K})}(\omega) e^{i\omega t} d\omega ,$$

while the time Fourier transform of the external force is given by

$$\hat{\mathbf{F}}_{\text{ext}}^{(\mathcal{K})}(\omega) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \mathbf{F}_{\text{ext}}(\mathbf{q}^{(\mathcal{K})}(t), t) e^{i\omega t} dt ,$$

and turns out to be a functional of the particle motion  $\mathbf{q}^{(\mathcal{K})}(t)$ .



By computing some trivial integrals, the equation for the particle then takes the form

$$\tilde{\mathbf{a}}^{(\mathcal{K})}(\omega) = \frac{\tilde{\mathbf{F}}_{\text{ext}}^{(\mathcal{K})}(\omega)}{m_0 + m_{\text{em}}(\mathcal{K}) + \frac{2e^2}{3\pi c^3} \omega \log \frac{\omega - c\mathcal{K}}{\omega + c\mathcal{K}} + \frac{\alpha}{\omega^2}} \quad (12)$$

By the way, it is of interest to remark that for  $\omega = 0$  equation 12 reduces to

$$\tilde{\mathbf{a}}^{(\mathcal{K})}(0) = \frac{\tilde{\mathbf{F}}_{\text{ext}}^{(\mathcal{K})}(0) - \alpha \tilde{\mathbf{q}}(0)}{m}$$

with  $m = m_0 + m_{\text{em}}$ ; this relation is consistent with a kind of “correspondence principle” because it is analogous to the mechanical equation  $m\ddot{\mathbf{q}} = \mathbf{F}_{\text{ext}}$  for the integrated quantities  $\tilde{\mathbf{a}}^{(\mathcal{K})}(0) = \int_{-\infty}^{+\infty} \mathbf{a}^{(\mathcal{K})}(t) dt$  and  $\tilde{\mathbf{F}}_{\text{ext}}^{(\mathcal{K})}(0) = \int_{-\infty}^{+\infty} \mathbf{F}_{\text{ext}}(\mathbf{q}^{(\mathcal{K})}(t)) dt$ .

Equation 12 for the Fourier transform of the particle acceleration allows in a rather simple way, through a control of its poles, to obtain an information on the function  $\ddot{\mathbf{q}}(t)$  itself. Clearly there are poles due to the particular form chosen for the external force, but the most relevant ones are those that are independent of it, namely those corresponding to the zeroes of the denominator, i.e. the zeroes of the function

$$\zeta^{(\mathcal{K})}(\omega) = m_0 + m_{\text{em}}(\mathcal{K}) + \frac{2e^2}{3\pi c^3} \omega \log \frac{\omega - c\mathcal{K}}{\omega + c\mathcal{K}} + \frac{\alpha}{\omega^2}.$$

One easily proves that:

- \* if  $m_0 > 0$ , then the function  $\zeta(\omega)$  has two real zeroes  $\pm\bar{\omega}$ , with  $\bar{\omega} > c\mathcal{K}$ ;
- \* if  $m_0 < 0$ , then the function  $\zeta(\omega)$  has two conjugate imaginary zeroes.

This result allows one to deduce several interesting consequences. The first one is that if one goes to the point limit  $\mathcal{K} \rightarrow \infty$  while keeping the bare mass positive, then the system presents oscillations of increasingly high frequencies, diverging with the cutoff. The second remark concerns the case in which the point limit is taken while performing a coherent renormalization procedure, i.e. one considers, for any cutoff  $\mathcal{K}$ , a corresponding bare mass  $m_0(\mathcal{K})$  such that  $m_0(\mathcal{K}) + m_{\text{em}}(\mathcal{K}) = m$  where  $m$  is a fixed positive mass; in particular this requires to take  $m_0 < 0$  if  $\mathcal{K} > \bar{k}$  where  $\bar{k} \simeq \bar{R}^{-1}$ ,  $\bar{R} = e^2/mc^2$  being the so called “classical radius”. In such a case one has that the real poles escape to infinity, reaching it for  $\mathcal{K} = \bar{k}$ ; then they reappear on the imaginary axis, tending to  $\pm i\tau^{-1}$ , where  $\tau$  is the familiar parameter defined by  $\tau = e^2/mc^3$ . In particular this has the fundamental consequence that for large enough cutoff the solutions for generic initial data have runaway character, i.e. diverge exponentially fast with time  $t$  as  $t \rightarrow +\infty$ , due to the presence of the pole in the lower half plane, and also for  $t \rightarrow -\infty$ , due to the other pole in the upper half plane.

In the literature it appears that the existence of oscillatory motions for “fat” particles with positive bare mass was put into evidence particularly by Bohm and Weinstein<sup>[19]</sup>, after the classical works of Schott.<sup>[20]</sup> The idea of such authors was to find a relation between the bare mass  $m_0$  and the form factor in order to have exactly periodic (and thus non radiating) solutions for the complete system; moreover they were stressing that such oscillations “*do not constitute a form of instability, as does the self-acceleration of the Dirac classical electron*”. What we have shown here for the case of the sharp cutoff is that the presence of nonradiating oscillatory motions for “fat” particles (i.e. for particles with  $R > \bar{R}$ ) is just a premonition of the appearance of runaway motions for “thin” particles (i.e. for particles with  $R < \bar{R}$ ), and in particular also in the point limit. The classical radius  $\bar{R}$  appears thus as a *critical radius* or a threshold: for “thin” particles of radius  $R < \bar{R}$  the

Maxwell–Lorentz system presents in general absurd runaway solutions which are not present for larger radii. In fact for  $R \gg \bar{R}$  the motions are just the smooth ones to which we are accustomed in macroscopic electrodynamics, and the transition to the absurd microscopic electrodynamics occurs with the premonition of the high frequency nonradiating oscillations of Schott and Bohm–Weinstein.

So, microscopic classical electrodynamics is qualitatively completely different from the macroscopic one, and one could be tempted to decide to throw it out, because it could be “*a waste of time to try to straighten it*”. Another possible attitude is to take the Maxwell–Lorentz equations seriously even for “thin” particles, and to try to give sense to them by restricting the attention to a special class of initial data, as was suggested by Dirac and will be recalled in the next section. This is the attitude we are trying to pursue.

As a final comment we indicate how equation 12 allows one to understand qualitatively that the particle motion  $\mathbf{q}(t)$  in the point limit has to satisfy approximately the Abraham–Lorentz equation for times larger than the characteristic time  $\tau$ . Indeed, defining

$$\tilde{\mathbf{a}}(\omega) = \lim_{\kappa \rightarrow +\infty} \tilde{\mathbf{a}}^{(\kappa)}(\omega) ,$$

one finds

$$\tilde{\mathbf{a}}(\omega) = \frac{\tilde{\mathbf{F}}_{\text{ext}}(\omega)}{m + im\tau\omega \text{Sign}[\text{Im}(\omega)] + \frac{\alpha}{\omega^2}} . \quad (13)$$

So the function  $\tilde{\mathbf{a}}(\omega)$  has a cut on the real axis and two poles of the first order at the points  $\pm i\tau^{-1}$ . Thus, for positive times the upper pole gives an exponentially small contribution which can be neglected after a convenient time. In such a limit equation 13 reduces to

$$(m - i\omega m\tau + \frac{\alpha}{\omega^2})\tilde{\mathbf{a}}(\omega) = \tilde{\mathbf{F}}_{\text{ext}}(\omega) ,$$

which, by performing the inverse Fourier transformation, gives the Abraham–Lorentz equation

$$m\ddot{\mathbf{q}} - m\tau\ddot{\mathbf{q}} = \mathbf{F}_{\text{ext}}(\mathbf{q}) - \alpha\mathbf{q} .$$

In an analogous way, for negative sufficiently large times one gets approximately a similar equation with  $-\tau$  instead of  $+\tau$ , namely

$$m\ddot{\mathbf{q}} + m\tau\ddot{\mathbf{q}} = \mathbf{F}_{\text{ext}}(\mathbf{q}) - \alpha\mathbf{q} ,$$

in agreement with the time reversal symmetry of the full problem.

We do not have time to report here on some numerical computations (see [18]) which illustrate very well the role of the electromagnetic mass as a function of the cutoff.

## QUANTUM–LIKE ASPECTS OF CLASSICAL ELECTRON THEORY

In his paper of the year 1938, where the relativistic version of the Abraham–Lorentz equation was introduced, Dirac<sup>[21]</sup> pointed out that generic solutions of such an equation are absurd, presenting the so called runaway character. Apparently, the scientific community has not yet really digested this fact, which in our opinion is of great importance; so we concentrate now on it and on its qualitative implications.

The runaway solutions were already mentioned in the previous section. But the simplest example in which they occur is the nonrelativistic Abraham–Lorentz

equation for the free particle; indeed, in such a case the equation is a closed one for the acceleration  $\mathbf{a}(t)$ , namely (for  $t > 0$ )  $\tau \dot{\mathbf{a}} = \mathbf{a}$ . The general solution is then  $\mathbf{a}(t) = \mathbf{a}_0 e^{t/\tau}$ , where  $\mathbf{a}_0$  is the initial acceleration. So the solution for the free particle explodes (i.e. diverges exponentially) for positive times, unless one takes the initial datum  $\mathbf{a}_0 = 0$ , which gives the “physical solution”  $\mathbf{a}(t) = 0$ . It is rather easy to understand by qualitative arguments that runaway solutions occur generically for the AL equation with an external force (see [22] and also [23]). Moreover, as should be clear from the discussion of the previous section, the generic runaway character is a property of the complete system and not just of the particle equation. Thus, for generic initial data microscopic classical electrodynamics is absurd.

One has then the following general mathematical problem. Given an initial “mechanical state”  $(\mathbf{q}_0, \mathbf{v}_0)$ , does there exist an initial acceleration  $\mathbf{a}_0$  (in the case of the AL equation) or an initial field (in the case of the complete system) such that the corresponding solution does not have runaway character? For example, in the case of a scattering problem, the nonrunaway character could be defined by the “final condition”  $\mathbf{a}(t) \rightarrow 0$  as  $t \rightarrow \infty$ . In geometrical terms (considering for example of the AL equation) one asks whether there exists in the complete phase space  $(\mathbf{q}, \mathbf{v}, \mathbf{a})$  a surface constituted of trajectories not having runaway character. Moreover, one has the problem whether such a surface can be expressed in the explicit form  $\mathbf{a} = \mathbf{a}(\mathbf{q}, \mathbf{v})$  (one speaks in such a case of “uniqueness” of the runaway solutions, in the sense that the nonrunaway solution is then uniquely determined by the mechanical state). The idea of Dirac was to consider electrodynamics of point particles as defined only for initial data restricted to such a “physical surface”.

From the mathematical point of view, the problem of the existence of the physical surface in the case of the AL equation was solved positively by Hale and Stokes<sup>[24]</sup> in the year 1962, for external forces of quite general a type. But in general there is no uniqueness. In fact this was already known long ago to Bopp<sup>[25]</sup> and Haag,<sup>[3]</sup> who gave an example with two solutions; but this was apparently forgotten, being not even mentioned in the most popular handbook on electron theory.<sup>[26]</sup> However, this nonuniqueness property was encountered again quite recently,<sup>[22]</sup> and even understood in terms of familiar concepts of the theory of dynamical systems. The case dealt with is the one-dimensional scattering of a particle by a barrier, and the nonuniqueness is described as follows: there exists an interval of energies for which there are any number of distinct nonrunaway solutions, that turn out to be divided into two classes. The first class corresponds to solutions reflected by the barrier, and the second one to solutions transmitted beyond the barrier (for the same initial mechanical data!); so one is here in presence of a classical effect qualitatively analogous to the tunnel effect.

Now, is this fact, namely the occurring of a phenomenon qualitatively analogous to a quantum one already within the framework of classical electron theory, just an accidental fact? In our opinion it is not so. The deep reason is that classical electron theory in the sense of the “physical solutions” *à la* Dirac described above is not at all the standard classical electron theory. As we tried to show, classical electrodynamics loses sense for point particles (or more precisely already below the so called “classical radius”), and so one could just abandon it. Another possibility is to try to keep it in some extended sense, for example just by taking the Dirac’s point of view of restricting it to the “physical surface”. But notice that such a prescription is a kind of nonlocal one, because it is a prescription on the “final time”, which mathematically leads to a kind of Sturm–Liouville problem. So it is not astonishing to find that such a new classical electrodynamics presents peculiar properties with respect to the standard (or macroscopic) one, which deals instead with “fat” particles, having dimensions larger

the so-called classical radius.

We quote in passing two examples on which some of us are presently working. First,<sup>[27]</sup> one can show that the microscopic classical electrodynamics *à la* Dirac discussed here leads to violations of the Bell inequalities (think of the initial acceleration as the hidden parameter, and take into account the nonlocality – or passive locality in the sense of Nelson – implied by the Dirac prescription). Second,<sup>[28]</sup> in the problem of the scattering of electrons by an uncharged conducting sphere one finds diffraction patterns corresponding to electron wavelengths of the order of one hundredth of the de Broglie wavelength.

This second fact might appear astonishing at first sight, but is in fact quite obvious. Indeed, for what concerns the wavelike properties of the classical electron, they are just due to the wave properties of the field which is intrinsically related to the particle (think of the complete Maxwell–Lorentz system). For what concerns the appearance of an action of the order of one hundredth of Planck’s constant, this is just due to the fact that in classical electron theory one has available the action  $e^2/c$ , which is just equal to  $(1/137)\hbar$ .

Now, the fact that the action  $e^2/c$  naturally enters in problems of classical electrodynamics is very well illustrated by the following example, where reference is made to elementary formulas that can be found in the Landau handbook but, as far as we know, were never understood previously in this way.<sup>[29]</sup> The problem is the scattering of an electron by a nucleus, with atomic number  $Z$ . In the Landau handbook one finds the formula for the emitted energy  $\Delta E$  (computed in first order approximation by Larmor’s formula along the trajectory of the purely mechanical approximation); the formula is a complicated one depending on  $Z$ . One also finds a formula for the emitted spectrum, which was known since always (particularly by Kramers) to decay exponentially at the high frequencies, so that there is a corresponding cut-off frequency  $\bar{\omega}$ ; the formula for  $\bar{\omega}$  is also a complicated one depending on  $Z$ . But for the ratio  $\Delta E/\bar{\omega}$ , which is an action, one finds the simple formula  $\Delta E/\bar{\omega} = \frac{e^2}{c}(v/c)^2$ , which is independent of  $Z$  and exhibits indeed  $e^2/c$ , namely essentially a hundredth of  $\hbar$ .

In conclusion, we hope we were able to show that classical electron theory can be extended to point (or microscopic) particles in an interesting way, leading to a theory which presents some aspects reminiscent of quantum aspects.

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