RECENT PROGRESS
ON THE ABRAHAM–LORENTZ–DIRAC EQUATION

Andrea CARATI, Luigi GALGANI
Università di Milano, Dipartimento di Matematica
Via Saldini 50, 20133 Milano, Italy

ABSTRACT

A short review is given of two quantum–like effects occurring in classical electrodynamics of point particles, as described by the Abraham–Lorentz–Dirac equation. They concern: 1) A characteristic nonlocal aspect of classical electrodynamics of point particles, related to an analogue of the tunnel effect, which leads to a violation of Bell's inequalities; 2) The implementation of an idea, conceived by Stueckelberg and Feynman, allowing one to describe in a classical framework the processes of pair creation and annihilation.

1. Introduction. As is well known, the Abraham–Lorentz–Dirac equation was introduced more or less a century ago in order to describe the self–interaction of a charged particle with the electromagnetic field, while its relativistic version was found by Dirac [1] in the year 1938. In its nonrelativistic version, with which we will be mostly concerned in this review, the equation has the form

$$\epsilon \dddot{\mathbf{x}} = \ddot{\mathbf{x}} - \frac{1}{m}\mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}, t) \ ;$$

here $\epsilon = \frac{2}{3}\frac{e^2}{mc^3}$, while, as usual, e and m are the particle's charge and mass, c the speed of light, $\mathbf{x}$ the particle's position, $\mathbf{F}$ an external force, including the Lorentz force due to the free evolution of the initial field.

In principle such an equation should be deduced from the Maxwell–Lorentz system, i.e. Maxwell's equations for the field having the particle as a source for charge and current, and Newtons equation for the particle, the force being that of Lorentz due to the field plus possibly an external mechanical one. This system poses no special problems for the case of a macroscopic body such as a charged stone, but the situation is completely different for a point particle or also for a small enough particle. The reason of the difficulty is the presence of the self–field, typically the Coulomb one, which diverges for a point particle; this causes the need of mass renormalization, and the existence of the absurd runaway solutions, as will be briefly recalled below. In any case, due to the fact that all known deductions of the Abraham–Lorentz–Dirac equation from the Maxwell–Lorentz system are at most heuristic, while such an equation presents the absurd runaway solutions, the scientific community essentially performed an action of psicological removal (or repression), behaving as if that equation did not exist. In consideration of this, a certain relevance should be attributed to some recent works by Bambusi, Noja and Posilicano (see [2], [3], [4]), where the equation was deduced from the Maxwell–Lorentz system with the kind of rigour presently accepted in mathematical analysis. Unfortunately, the result was obtained only for what is called the "dipole approximation", which is a linearization of the system; essentially it consists in replacing the current, namely $\dot{\mathbf{x}}\delta(\mathbf{r}-\mathbf{x}(t))$, simply by $\dot{\mathbf{x}}\delta(\mathbf{r}(t))$, where $\mathbf{r}$ denotes the generic point in space, and in retaining just the linear term in the Lorentz force.

So the problem is still open whether in the full nonlinear case, and in particular also in the relativistic case, the strange peculiarities (runaway solutions) of the Abraham–Lorentz–Dirac equation will still be present. Here we will proceed by assuming that this is the case, aiming at illustrating some relevant consequences of such a situation. If the situation will instead prove to be completely different in the nonlinear case, as is the hope even of some good friends of ours, we will gladily accept it.

Before entering into a more detailed discussion about the Abraham–Lorentz–Dirac equation, we presently recall a heuristic argument illustrating the source of the difficulties related to mass renormalization, following the presentation of Feynman's handbook (chapter 28 of the second volume). Everyone knows that a particle in uniform motion drags along with it an electromagnetic field. To this corresponds an electromagnetic momentum, whose intensity p_{em} is immediately computed in the nonrelativistic approximation; one finds $p_{\mathrm{em}} = m_{\mathrm{em}}v$, where v is the particle's velocity and the factor m_{em} (called the electromagnetic mass) is given by

$$m_{\mathrm{em}} = \frac{2}{3}\frac{e^2}{ac^2}$$

if a is the particle's radius. So, if we denote by m_0 the "bare" or "mechanical" mass, namely the one appearing in Newton's equation of motion, the particle should behave as if it were endowed with an experimentally detectable mass m given by

$$m = m_0 + m_{\mathrm{em}} \ .$$

Consequently, in an attempt at defining a limit system for vanishing radius from the Maxwell–Lorentz system, one is naturally led to take the value of m as a given phenomenological parameter. This in turn has the effect that the bare mass m_0 has to be considered as a function of the particle's radius a; in particular m_0 turns out to vanish at the so–called "classical radius" $a_0 = \frac{2}{3}\frac{e^2}{mc^2}$, becoming negative below it, and tending to $-\infty$ as $a \to 0$.

Such a heuristic argument is confirmed in the rigorous deductions recalled above, where it is shown that the classical renormalization procedure is necessary if the limit system has to be nontrivial at all, i.e. if in the limit system the particle and the field have to proceed with some mutual interaction. Particularly elegant from a mathematical point of view is the deduction given in [3], where also the limit equation for the field was given for the first time. The following comment might be in order. Many people appear to have difficulties in understanding how a third order equation occurs at all for the particle. The point is that one should have in mind the complete (infinitely dimensional) system particle + field, with the corresponding Cauchy problem involving initial data both for the particle and the field (see especially [5]). So somehow it is just by chance that a closed equation for the particle is met at all, and in any case it is clear that in dealing with the Abraham–Lorentz–Dirac equation a rule should be found to determine the Cauchy datum for the particle's acceleration in terms of the Cauchy data for the field. This was done for the first time in the paper [4] (see also [6]). By the way, another relevant point that came out of the rigorous deductions recalled above is that, in order to get the limit, in the Cauchy problem for the system one has to put a constraint among the Cauchy data of the particle and of the field, inasmuch as the field should have the correct singularity corresponding to the presence of the point particle.

Finally, let us recall what essentially are the celebrated runaway solutions. To go to the heart of the problem, consider the simplest case, namely that of the free particle, with $\mathbf{F} = 0$. In such a case the Abraham–Lorentz–Dirac equation is a closed equation for the acceleration $\mathbf{a}$, of the form $\epsilon\dot{\mathbf{a}} = \mathbf{a}$, with general soilution $\mathbf{a}(t) = \mathbf{a}_0 \exp(t/\epsilon)$. So one sees that for generic initial data the free particle accelerates exponentially fast, which is absurd. The only nonabsurd solutions (uniform motions) are obtained if one takes as initial acceleration $\mathbf{a}_0 = 0$.

2. The Bopp–Haag phenomenon, the analogue of the tunnel effect, the nonlocality of classical electrodynamics of point particles, and a violation of Bell's inequalities. Our studies on the Abraham–Lorentz–Dirac equation started with a very simple remark that psicologically was very inportant for us. The problem is the character of the series expansions (in the parameter ϵ) which define the solutions of the equation. Looking at the form of the equation, it should be almost obvious that such series should be asymptotic and in general divergent, because this is typical of situations concerned with singular perturbation theory, where the order itself of the equation is lowered (here from three to two) when the expansion parameter is set equal to zero (think even of the algebraic equation $\epsilon x^2 + ax + b = 0$). And indeed, in the present case the proof just takes a few lines (see [7]). The point that particularly impressed us is that a scientist that is usually considered to be one of the main authorities in the field conjectured instead explicitly the contrary, namely that such series expansions might in general be analytic, and this even in a paper dedicated to Dirac [8]. This fact gave us the impression that some relevant work might still be performed in the field.

The next result came out indirectly from a suggestion of an anonymous referee of the paper mentioned above, that we would like to thank personally some time, if we only could know his name. By an analogy with the situation occurring in the semiclassical regime, the referee suggested that the asymptotic character of the series might be particularly relevent in connection with the scattering of a particle by a potential barrier. So we started the study of the one–dimensional scattering of a particle by a potential barrier, according to the Abraham–Lorentz–Dirac equation, and this led us to understand a significative qualitative property of the equation, that was somehow known already (as we discovered later) to Bopp[5] and Haag[9], but did not receive the emphasis it would in our opinion deserve. We now illustrate it briefly; more details can be found in the original paper[10].

The problem is the following. In the one–dimensional case, the relevant phase space for the Abraham–Lorentz–Dirac equation is three–dimensional, with coordinates x, $v = \dot{x}$, $a = \dot{v}$, namely particle's position, velocity and acceleration; moreover, generic initial data lead to abusrd runaway solutions, with the acceleration exploding exponentially fast with increasing time. So one could be tempted to simply throw the Abraham–Lorentz–Dirac equaition away. A different possibility was however suggested already by Dirac in the year 1938, namely that of restricting the phase space to the invariant manifold (if not empty) constituted of motions (i.e. solutions of the equation) having nonrunaway character; for example, in the typical case of scattering from a barrier, one requires $a(t) \to 0$ as $t \to +\infty$. Such a manifold will be called here the Dirac or the physical manifold.

Notice that the Dirac prescription should be considered as a constitutive part of the limit theory itself defining classical electrodynamics of point particles in the framework of the complete Maxwell– Lorentz system. Namely, classical electrodynamics of point particles with mass renormalization and elimination of the runaway solutions through Dirac's prescription is a new theory, logically distinct from macroscopic classical electrodynamics. In particular, Dirac's condition confers to the theory a peculiar nonlocal aspect, being a prescription on what will occur at $t = +\infty$, so that one is in presence of a differential system to be studied as a Sturm–Liouville type problem.

From the mathematical point of view the problem is then as follows. At an initial time one fixes the mechanical state (position and velocity), and the initial data should be completed by assigning freely the initial acceleration. Then one asks whether, for a fixed mechanical state, there exists an initial acceleration leading to a nonrunawy solution (i.e. with $a(t) \to 0$ as $t \to +\infty$). The problem was defined exactly in these terms by Hale and Stokes[11] who, for a quite general class of forces, could prove existence, but could not prove in general uniqueness (indeed, existence was proved by topological fixed point methods, for which nonuniqueness is rather the rule than an exception). Geometrically, nonuniqueness can be described by saying that the physical or Dirac manifold is folded above the mechanical plane (x, v).

The result we found[10], first numerically and then analytically, by methods of the qualitative theory of differential equations, is that the Dirac manifold is folded, and even with infinitely many folds, for all potential barriers with a sufficiently sharp maximum. More precisely, for an initial position sufficiently far away from the barrier, the nonuniqueness phenomenon occurs only for initial energies inside a small strip situated about the top of the barrier. Moreover, for any positive integer n there exists a substrip admitting $2n$ nonrunaway solutions, n of which cross the barrier while the remaining n ones are reflected. So a knowledge of the mechanical state is not sufficient to predict whether the particle will cross the barrier or will be reflected from it, because this depends on the particular value of the remaining variable, which in fact acts as a hidden variable. Such a qualification of hidden seems to be appropriate, because it is found that the different allowed values of the variable all tend to a unique value (actually, the value zero) exponentially fast as the initial particle's position recedes from the barrier, so that the variable is actually uncontrollable, and a statistical description for it is needed. In this sense we are here in presence of a classical analogue of the tunnel effect. We in fact also investigated whether such a classical analogue might prove to be physically meaningful, but it seems to us that a factor of about a thousand in the parameter ϵ is lacking in order to explain for example the alpha decay. Finally. as recalled above, it turns out that the value of the particle's acceleration is in fact uniquely defined by the electromagnetic field in the complete Maxwell–Lorentz system, so that the hidden variable in our model is indeed a variable of a nonmechanical type.

After completing the article we discovered that many years before already Bopp [5] and Haag [9] had found a similar phenomenon, in a particularly simple case (one–dimensional step) which allowed them to find the solutions by elementary methods. However, what they found in their particular cases is that there existed just two, and not an arbitrary number of nonrunaway solutions for a given mechanical state (in a suitable domain). This led some people to suggest (see the quotation in [12]) that only one (chosen is some way) of the two solutions should be retained as physically meaningful; but this seems to be impracticable in the present case, where the interpretation of the acceleration (or of the electromagnetic field) as a hidden variable requiring a probabilistic description seems to be almost compulsory.

So we have here a situation which allows to produce a simple highly nontrivial model of a hidden variable theory (see [13]). In fact, we have a charged particle moving on a line in presence of a potential barrier, and it turns out that the particle's initial state is in general not uniquely defined by its mechanical part (position and velocity), because it requires the further assignement of a hidden variable (the particle's acceleration, or rather the corresponding electromagnetic field in the complete Maxwell–Lorentz system). Such a variable is properly speaking a hidden one, because it is essentially uncontrollable (the different allowed values converge to a unique value (zero) exponentially fast on recedes from the barrier), but its most striking property is its characteristic nonlocal character. Indeed, the domain of definition of the hidden variable turns out to depend not only on the initial mechanical state, but also by the height of the barrier (because the initial energies where nonuniqueness occurs are about the top of the potential barrier), no matter how far it is situated (say in Tokyo) from the initial particle's position (say in Milan).

We believe that in such a way we managed to make somehow trivial what E. Nelson had conjectured [14,15]. Indeed he suggested that characteristic nonlocal properties might occur in classical field theories when regularizing cutoffs are removed. From our point of view, the regularizing cutoff is nothing but the form factor of the particle in the Maxwell–Lorent system, which is eliminated in the point particle limit. The taking of the limit requires mass renormalization; this leads to a lack of positivity of energy, which in turn produces the generic runaway solutions; and this is cared by Dirac's prescription, which is essentially nonlocal. Such a nonlocality property (usually described as "acausality") was considered as a blame by most people, and we now discover instead that it is exactly what was to be expected, according to Nelson's suggestion. Moreover, we know very well that some nonlocality property should be present in any good description of nature, as Bell and Aspect seem to have shown, and so this blame for the Abraham–Lorentz–Dirac equation appears now to be welcome.

Having this in mind, it was then a simple matter to determine a specific model violating Bell's inequalities. The simple idea was to let the barrier act as the measuring instrument in an Einstein–Podolsky-Rosen type experiment. In the case of a single particle moving on a line, the particle's initial mechanical state is taken in the range where nonuniqueness is guaranteed, so that the initial acceleration is the hidden variable for which a probability distribution has to be assigned. The dicothomic measurement consists in observing whether the particle crosses the barrier (result $+1$) or is reflected from it (result -1), the result depending on the precise initial value of the hidden variable. In turn, it is assumed that the measuring instrument (i.e. the barrier) can be prepared in some (typically three) different settings (i.e. heights), and things can be adjusted in such a way that, for a given initial mechanical state in a certain open set, the domain of definition of the hidden variable depends on the setting of the measuring instrument.

Thus, following the general idea of any Einstein–Podolsky–Rosen type experiment, we consider two charged particles which, having first interacted in some irrelevant way, then proceed along a straight line in opposite directions, each independently of the other; finally, they are subjected to separate measurements of a dicothomic quantity, by observing whether they cross or not two far away potential barriers, adjusted in one of three possible settings as described above. The two hidden variables are here the values of the accelerations a_1, a_2 of the two particles at a time (say T, to use the same letter of Eintein–Podolsky–Rosen) when they enter the two regions of independent motions; obviously, such values correspond to some precise values of the accelerations at any previous time, when the particles were located in the interaction region, and the probability distribution for such values is just in one to one correspondence with the probability distribution at time T. The relevant point is that the latter distribution depends on the settings of both measuring instruments, because even the dimensionality of the probability space (i.e. the number of possible accelerations) depends on it. This is a characteristic nonlocal property of the hidden variables which is by definition excluded for Bell's type hidden variables. So it is obvious that one can find probability distributions of the hidden variables leading to a violation of Bell's inequalities. A specific example is reported in the paper [13], where reference was made to Bell's inequalities in the form discussed by Nelson. As shown there, the violation occurs for initial probability distributions which are unfactorized, i.e. are not products of independent probability distributions for the two particles; indeed, it occurs that factorized states correspond to trivial situations where each particle either certainly crosses the barrier or certainly is reflected from it.

3. Pair creation and annihilation. Another interesting result was a confirmation of the possibility of describing pair creation and annihilation, which are usually considered to be characteristic features of quantum electrodynamics. The idea is not at all new because, although generally completely unknown in the scientific community, goes back to Stueckelberg [16] and to Feynman [17] himself, as we learned from the review article [12] (see also [18] and [19]). Clearly, here one should make reference to the relativistic version of the Abraham–Lorentz–Dirac equation, provided by Dirac in the year 1938, which reads

$$\ddot{x}_\mu = \frac{e}{mc}F_{\mu\nu}\dot{x}^\nu + \frac{2e^2}{3mc^3}(\dddot{x}_\mu + \frac{\ddot{x}_\nu\ddot{x}^\nu}{c^2}\dot{x}_\mu) \ ,$$

where $F_{\mu\nu}$ represents an external electromagnetic field tensor, while, as usual, x_μ is the position four–vector and the dot denotes derivative with respect to proper time.

The main idea is the following one. In special relativity the motion of a particle is described in space–time by a time–like path, the orientation of which is arbitrary. Usually the orientation is taken with arc–length (or proper time) increasing with time, but the choice is immaterial because, for time–like paths, time can only increase or only decrease. On the other hand paths are necessarily time–like if velocity has to be smaller than the speed of light. But the mentioned authors had the genial idea of conceiving of paths presenting angular points, for example, in the simplest case, paths just composed of two time–like "branches" joining at an angle. In such a case the particle, climbing on a branch with incresing time, would attain in a finite time the speed of light at the angular point, and then would proceed with decreasing time along the other branch. Equivalently one might speak of an antiparticle climbing the second branch with increasing time, and the consideration of both branches would correspond to the description of a process of pair annihilation. Processes of pair creation are similarly described, as also all other types of analogous processes familiar from quantum electrodynamics. By the way, people familiar with Feynman's original papers in quantum electrodynamics should understand how a description of the type illustrated above could have come at all to his mind.

However, Feynman was not able to exhibit paths of the type described above within the framework of classical electrodynamics, and he could only provide an example by making reference to a nonlocal version of classical electrodynamics introduced by Bopp (whose description can also be found in chapter 28 of Feynman's handbook). In any case, the example dealt with by Feynman consists of a particle interacting with a steep potential barrier. In the paper [20] the analogous problem was dealt with in the framework of pure classical electrodynamics of a point particle, described by the relativistic Abraham–Lorentz–Dirac equation, in the presence of an external potential presenting a singularity. It was possible to show that there exists a solution corresponding to a particle arriving in a finite time at the singularity and attaining there the speed of light; moreover such a solution can be analytically continued beyond the singularity, constituting globally a path just of the type conceived by Stueckelberg and Feynman. So this shows that their genial idea is implementable in pure classical electrodynamics.

This seems to be a qualitatively interesting result, which we hope to be able to generalize in the future, for example by showing that phenomena of pair creation and annihilation and the like occur in classical electrodynamics even without the need of considering singularities due to the introduction of external potentials.

3. Conclusions. We do not have time here to illustrate several further ideas on which we and our friends are presently working, all concerned with quantum–like effects occurring in classical electrodynamics. We hope that the two examples described above are however sufficient to show that qualitatively interesting quantum–like effects, somehow unexpected, do indeed occur in classical electrodynamics of point particles.

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