

On the conservation of adiabatic invariants for a system of coupled rotators

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Abstract

We study the accuracy of the conservation of adiabatic invariants in a model of n weakly coupled rotators. Most attention is devoted to $n = 2$ and frequency $\omega = (\omega_1, \omega_2)$, with ω_2/ω_1 quadratic irrational. We apply a heuristic approximation scheme, going back to Jeans and to Landau and Teller, and perform a very accurate numerical check of the result, observing a quite remarkable agreement.

Keywords: Adiabatic invariants; Exponential estimates; Perturbation theory; Equilibrium times in gases

1. Introduction

The purpose of this paper is to discuss the accuracy of conservation of adiabatic invariants in a system with more than one independent frequencies. We shall work in connection with a simple model example, and proceed as follows: first, working at a heuristic level, we apply to our model an elementary approximation scheme, which goes back (to our knowledge) to Jeans [1] and to Landau and Teller [2], and leads to a simple expression for the change ΔI in the actions of the system. We then perform an accurate numerical test of such an approximate expression, finding very good agreement. The possibility of understanding the behavior of the system within Hamiltonian perturbation theory is also discussed.

The model we are concerned with represents a system of fast rotators, coupled by a time-dependent potential. The Hamiltonian has the form

$$H(I, \varphi, t) = \sum_{j=1}^n \frac{I_j^2}{2A} + f(t) g(\varphi), \quad I = (I_1, \dots, I_n) \in \mathbb{R}^n, \quad \varphi = (\varphi_1, \dots, \varphi_n) \in \mathbb{T}^n, \quad (1.1)$$

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where A is the moment of inertia of rotators, while f and g are analytic functions, respectively, of time and of the angles. More precisely, f is assumed to be analytic in a strip $|\operatorname{Im} t| < \tau$, and to decay to zero in an integrable way for $t \rightarrow \pm\infty$, while g is assumed to be analytic in a strip $|\operatorname{Im} \varphi_j| < \rho$, $j = 1, \dots, n$; to be definite, and to simplify our analysis, we shall make special choices of f and g , namely

$$f(t) = \sigma^2 \frac{e^{-(t/\sigma)^2}}{\tau^2 + t^2}, \quad \sigma, \tau > 0, \quad \text{and} \quad g(\varphi) = \sum_{k \in \mathbb{Z}^n} \mathcal{G}_k e^{ik \cdot \varphi},$$

the Fourier coefficients \mathcal{G}_k decaying exactly exponentially:

$$\mathcal{G}_k = \frac{1}{2} G e^{-\rho|k|}, \quad G \in \mathbb{R}, \quad (1.2)$$

with $|k| = \sum_{j=1}^n |k_j|$. The relevance of such choices will be discussed later.

We shall consider asymptotic data, for $t \rightarrow -\infty$, of the form

$$I(t) \rightarrow I^0, \quad \varphi - \omega(I^0)t \rightarrow \varphi^0, \quad (1.3)$$

where of course $\omega(I) = I/A$. We shall assume $\omega(I^0)$ to be large, more precisely we shall take

$$I^0 = \lambda A \Omega, \quad \lambda \text{ large},$$

with some fixed Ω , so that

$$\omega(I^0) = \lambda \Omega. \quad (1.4)$$

We shall study the overall change in the actions:

$$\Delta I = (\Delta I_1, \dots, \Delta I_n), \quad \Delta I_j = I_j(+\infty) - I_j^0,$$

looking for “exponential laws”, namely (in a sense to be made precise) exponential decaying of ΔI with λ .³

Let us stress that a possible motivation for such a study comes from statistical physics, namely from the problem of understanding the rate of the approach to equilibrium in a classical gas of fast rotating (or slow translating) molecules. In the simplest situation, one can consider a planar model, and look at the two-body collisions as at the basic process leading eventually to equipartition of energy among all degrees of freedom. A possible Hamiltonian for the two-body collision is

$$H = \sum_{j=1}^2 \frac{I_j^2}{2A} + \frac{\|p\|^2}{2\mu} + V(r, \varphi_1, \varphi_2), \quad (1.5)$$

where $r \in \mathbb{R}^2$ is the vector joining the centers of mass of the molecules, μ the reduced mass, and $p \in \mathbb{R}^2$ is conjugate to r , $\|p\|^2 = p_1^2 + p_2^2$. One should clearly assume $V \rightarrow 0$ for $|r| \rightarrow \infty$, and thus (for unbounded motions) for $t \rightarrow \pm\infty$. It is then clear that Hamiltonian (1.1) mimics, so to speak, Hamiltonian (1.5), namely the explicit dependence on time replaces there the translational degrees of freedom. The problem gets simpler (in particular for numerical computations), but remains nontrivial, and the behavior of the actions is expected to be essentially the same. Let us recall that, if ΔI is small, then the energy exchange per collision is also small, and correspondingly the time scale for equilibrium in the gas gets larger. As a matter of fact, this is precisely the scenario

³ Such a problem is clearly equivalent, via a trivial rescaling, to a problem of weakly interacting rotators, with Hamiltonian $\tilde{H} = \sum_j (I_j^2/2A) + \lambda^{-2} f(t/\lambda) g(\varphi)$, and asymptotic data $I(t) \rightarrow J^0$, $\varphi(t) - \Omega t \rightarrow \varphi^0$.

proposed by Boltzmann [3] and Jeans [1] to explain classically, as a nonequilibrium phenomenon, the nowadays called “quantum freezing” of fast degrees of freedom (see, for a discussion of the problem [4,5]).

As a result of the above-mentioned heuristic approximation, that we shall call the Jeans–Landau–Teller (JLT) approximation – actually a simple “computation along an unperturbed orbit” – one gets ΔI in the form of a Fourier series in the asymptotic phases φ^0 , say

$$\Delta I = \sum_{k \in \mathbb{Z}^n} \mathcal{I}_k e^{ik \cdot \varphi^0}, \quad (1.6)$$

with separate exponential laws for the different Fourier components: namely, for the above choices of f and g ,

$$\mathcal{I}_k \simeq \text{const. } k e^{-\lambda \tau |k \cdot \Omega| - \rho |k|}. \quad (1.7)$$

Similar exponential laws appeared, in the very recent literature, in connection with different problems, both at rigorous [6–9] and at heuristic [10] level.⁴ Analyzing (1.6) and (1.7), and in particular understanding the behavior of ΔI for large λ , is not easy, and crucially depends on n and on the arithmetic properties of Ω . We will not perform a general analysis of them, and will restrict ourselves to the easier (but nontrivial) case $n = 2$ and $\Omega = (1, \theta)$, with θ quadratic irrational; to be definite we shall take $\theta = \sqrt{2}$. In such a case we shall deduce (heuristically) from (1.7) a more transparent expression for the change in the actions, specifically for the quantity

$$\overline{\Delta I} = \max_{\varphi^0 \in \mathbb{T}^2} \|\Delta I(\varphi^0)\|. \quad (1.8)$$

$\|\cdot\|$ denoting the Euclidean norm. According to such an expression, $\log \overline{\Delta I}$ turns out to be practically a convex piecewise linear monotonically decreasing function of λ , with slopes determined by the arithmetic properties of θ , whose asymptotic behavior (in a sense to be made precise) is $\sim \sqrt{\lambda}$. Such a result is confronted with very precise numerical computations of $\overline{\Delta I}$, for $\overline{\Delta I}$ in a rather wide range (more than 20 orders of magnitude), and a quite remarkable agreement is found.

The deduction of the exponential law (1.7), as well as its analysis for n and Ω as above, are performed in Section 2. Section 3 is instead devoted to numerical results. Section 4 contains some concluding comments, including a discussion on the possibility of recovering the behavior of the system by the usual methods of classical Hamiltonian perturbation theory. Finally, Appendix A is devoted to the asymptotic expression of $\overline{\Delta I}$. This paper can be considered to be a (nontrivial) generalization of the heuristic and numeric results reported in [11], concerning the case of a single frequency.

2. JLT approximation

2.1. Computation along unperturbed orbits

Let us consider Hamiltonian (1.1) with f and g as described above; to simplify the notation, we use the (totally irrelevant) constants σ and G , respectively, as units of time and energy, that is, from now on we set $\sigma = G = 1$. For large λ , the coupling $f(t)g(\varphi)$ plays the role of a perturbation of the kinetic part, or unperturbed Hamiltonian, $\sum_j I_j^2/(2A)$. The unperturbed motion corresponding to the asymptotic conditions (1.3) is then

$$\widehat{I}(t) = I^0, \quad \widehat{\varphi}(t) = \varphi^0 + \lambda \Omega t,$$

⁴ In particular, the very recent preprints [8,9], that we received during the preparation of our manuscript, have some significant superposition with our work, more precisely with the theoretical analysis performed in Section 2.2.

and the naive first-order approximation that we propose here, is defined by ⁵

$$\dot{I} = -f(t) \frac{\partial g}{\partial \varphi}(I^0, \varphi^0 + \lambda \Omega t), \quad \Delta I = - \int_{-\infty}^{\infty} f(t) \frac{\partial g}{\partial \varphi}(I^0, \varphi^0 + \lambda \Omega t) dt.$$

For the above choice of g , one then immediately gets the Fourier series (1.6) with

$$\mathcal{I}_k = -\frac{i}{2} k e^{-\rho|k|} \mathcal{F}_k, \quad \mathcal{F}_k = \int_{-\infty}^{\infty} f(t) e^{i\lambda k \cdot \Omega t} dt. \quad (2.1)$$

Note that $\mathcal{F}_{-k} = \mathcal{F}_k$ (as follows from the parity of f), and correspondingly

$$\mathcal{I}_{-k} = -\mathcal{I}_k.$$

For the above choice of f the integral in (2.1) can be explicitly computed, and one finds ⁶

$$\mathcal{F}_k = \frac{\pi e^{\tau^2}}{2\tau} [e^{-\lambda \tau k \cdot \Omega} (1 - \Phi(\tau - \frac{1}{2} \lambda k \cdot \Omega)) + e^{\lambda \tau k \cdot \Omega} (1 - \Phi(\tau + \frac{1}{2} \lambda k \cdot \Omega))],$$

where Φ is the probability integral

$$\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\xi^2} d\xi.$$

The function Φ converges to ± 1 very rapidly for $x \rightarrow \pm\infty$ [one has indeed $\Phi(x) - 1 = O(e^{-x^2})$ for large positive x], so for fixed τ , and $\lambda|k \cdot \Omega|$ large, one gets

$$\mathcal{F}_k \simeq C e^{-\lambda \tau |k \cdot \Omega|}, \quad C = \pi e^{\tau^2} / \tau.$$

In conclusion, one deduces (1.7), more precisely

$$\mathcal{I}_k \simeq -\frac{i}{2} k C e^{-\lambda \tau |k \cdot \Omega| - \rho|k|}. \quad (2.2)$$

This is what we call the JLT approximation, or first-order approximation, for the problem at hand.

2.2. Interpreting the JLT approximation

Now we restrict ourselves to $n = 2$ and $\Omega = (1, \theta)$, $\theta = \sqrt{2}$; in fact, we shall profit of the very simple continued fraction expansion $\sqrt{2} = [1; 2, 2, 2, \dots]$, but a little reflection shows that everything extends easily, with minor

⁵ For the physically more realistic Hamiltonian (1.5), the corresponding approximation would be as follows: one preliminarily replaces the coupling $V(r, \varphi_1, \varphi_2)$ with its average on the angles, denoted $v(r)$; if $\hat{r}(t)$ is the motion corresponding to the Hamiltonian $\sum_j |p_j^2|/(2\mu) + v(r)$, then one writes $\Delta I = - \int_{-\infty}^{\infty} \frac{\partial V}{\partial \varphi}(\hat{r}(t), \varphi^0 + \lambda \Omega t) dt$. This is, in essence, what Jeans and Landau, and Teller proposed (although in connection with a problem with a single frequency; see [11] for comments).

⁶ See for example, I.S. Gradshteyn and I.M. Ryzhik, Table of Integrals, Series, and Products (Academic Press, New York, 1980) p. 497 no. 3.954-2.

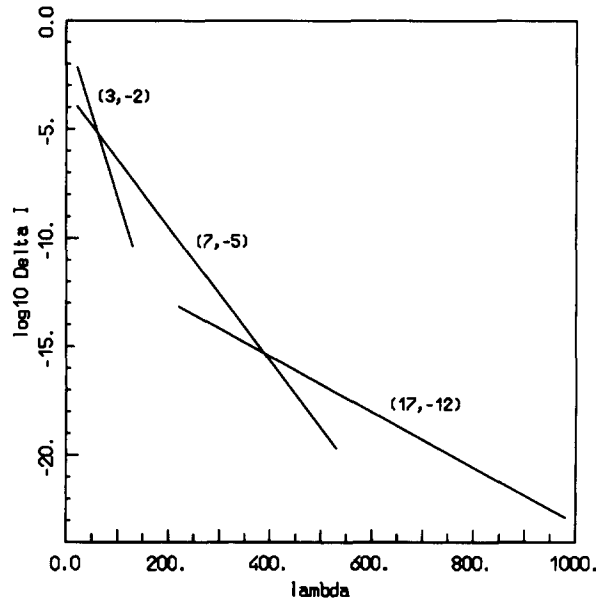


Fig. 1. The amplitudes $\|I_k\|$ vs. λ , for k in the resonant sequence, according to the JLT approximation.

modification, to any θ with periodic continued fraction. Moreover, we profit of the symmetry $\mathcal{I}_{-k} = -\mathcal{I}_k$ to write

$$\Delta I \simeq \sum_{k \in \mathcal{K}} \tilde{\mathcal{I}}_k \sin(k \cdot \varphi^0), \quad \tilde{\mathcal{I}}_k = 2i\mathcal{I}_k = kC e^{-\lambda|k \cdot \Omega| \tau - \rho|k|},$$

the sum now extending on “one half” of \mathbb{Z}^2 , namely on the set

$$\mathcal{K} = \{(k_1, k_2) \in \mathbb{Z}^2: k_1 > 0 \text{ or } k_1 = 0 \text{ and } k_2 > 0\}.$$

It is quite evident from (2.2) that, for large λ , only those harmonics with nearly resonant k (the “small denominators”) do significantly contribute to the Fourier series. From the theory of continued fractions, one immediately obtains, for the above choice $\theta = \sqrt{2}$, the following “resonant sequence”:

$$k = (1, 0), (1, -1), (3, -2), (7, -5), (17, -12), (41, -29), (99, -70), \dots$$

(the rule is that (k_1, k_2) is followed by $(k_1 - 2k_2, -k_1 + k_2)$); a little reflection shows that one can confidently restrict the attention to k in such a sequence (this is also confirmed numerically, see later). However, and this is crucial, even inside the resonant sequence the amplitudes of the different harmonics, say their Euclidean norms $\|\tilde{\mathcal{I}}_k\| = \|k\| C e^{-\lambda|k \cdot \Omega| \tau - \rho|k|}$, turn out to be quite different, and generically, for a given (large) λ , one single harmonic dominates the Fourier series (exceptionally, two of them give comparable contributions). It can be useful, to understand this point, to plot $\|\tilde{\mathcal{I}}_k\|$ vs. λ in log scale. One clearly gets straight lines,

$$\log \|\tilde{\mathcal{I}}_k\| = -\alpha_k \lambda - \beta_k, \quad \alpha_k = \tau|k \cdot \Omega|, \quad \beta_k = \rho|k| - \log \|k\| - \log C, \quad (2.3)$$

and for larger k in the resonant sequence the slope α_k gets smaller, while the constant β_k gets larger (see Appendix A for quantitative estimates on α_k and β_k). The result is reported in Fig. 1. As is evident, apart from narrow intervals around the intersections of nearby lines, one Fourier component is *much* larger than the others.

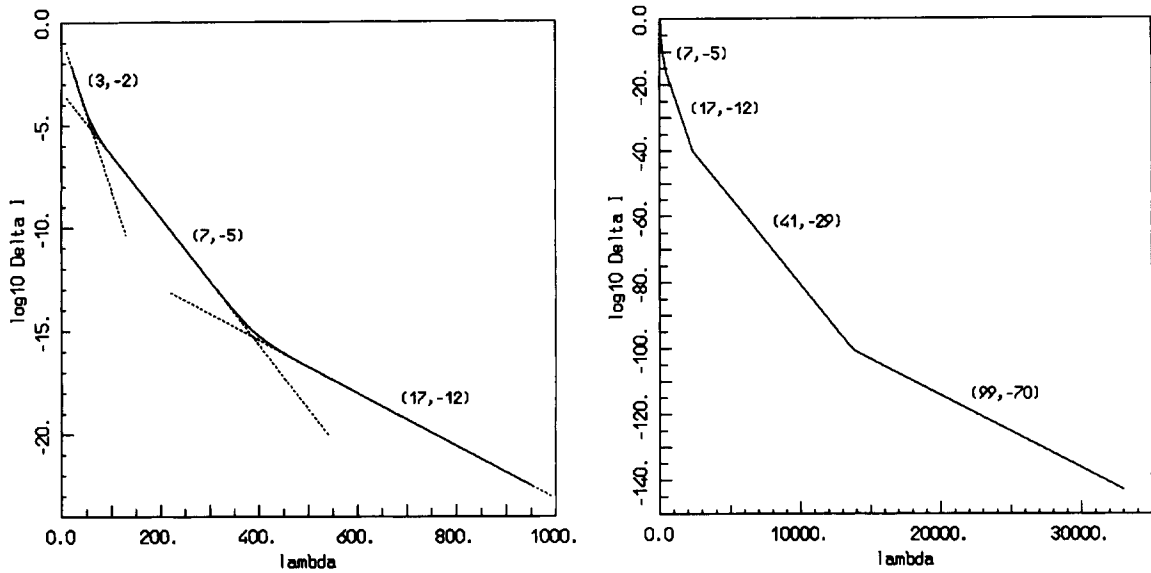


Fig. 2. A numerical plot of (2.5).

To be precise, we can make reference to the quantity $\overline{\Delta I}$ defined in (1.8) (taking the maximum avoids the spurious fact that for special values of φ^0 the dominating harmonic could vanish; moreover, $\overline{\Delta I}$ turns out to be a nice quantity to be compared with numerical results). The above considerations can be summarized by saying that

$$\overline{\Delta I} \simeq e^{-\alpha_k \lambda - \beta_k} \quad \text{in the interval where } k \text{ dominates,} \quad (2.4)$$

and also

$$\overline{\Delta I} \simeq \sum_k e^{-\alpha_k \lambda - \beta_k} \quad \text{in the union of the above intervals,} \quad (2.5)$$

the sum extending over the resonant sequence (or also over \mathcal{K} , with negligible difference). In essence: since, in each interval, a single term in the Fourier series dominates, the maximum of $\|\Delta I\|$ on φ^0 can be replaced by the sum of the maxima of the single harmonics with negligible error. Near the crossover points (i.e., the intersection points of two nearby lines) expression (2.5) could fail: indeed, around such points $\overline{\Delta I}$ is (approximately) the sum of two dominant contributions, which are nearly equal, and could partially cancel. But this is in a sense exceptional, and typically one should expect that (2.5) is valid there too, although being possibly less accurate (in any case, of course, the RHS of (2.5) is there an upper bound to the LHS). As is remarkable, (2.5) provides a single simple analytic expression, valid for all (large) λ .

Expression (2.5) for $\overline{\Delta I}$ is plotted in Fig. 2 with two different choices of the scale of λ ; the curve refers to $\tau = \rho = 1$, and the sum is restricted to k in the resonant sequence, up to $k = (577, -408)$. For comparison, in the left part of Fig. 2 the resonant straight lines are also drawn (dashed lines). By the way, one can numerically check that the restriction to k in the resonant sequence is definitely reliable; for example, including in the sum all $k \in \mathcal{K}$ up to $|k| = 1000$ gives an absolutely negligible difference: for λ larger than, say, 10, the difference is fairly covered by the width of the lines in Fig. 2.

As is remarkable, the curve, although analytic, closely resembles a broken line, in particular for large λ . A simple analysis, reported in Appendix A and close to Refs. [9,10], shows that, asymptotically for large λ , $\overline{\Delta I}$ behaves like

$$A \frac{e^{\tau^2}}{\rho \tau} (1 + O(\lambda^{-1/2})) \sqrt{c \rho \lambda \tau} e^{-\sqrt{c \rho \lambda \tau}} \quad (2.6)$$

with

$$A = \frac{1}{2} \sqrt{3} \pi (\sqrt{2} - 1), \quad c = 2 + \sqrt{2}.$$

More precisely (see Appendix A) the logarithm of (2.6) is a convex curve, which provides an upper bound to $\overline{\Delta I}$ and is tangent to the broken line once in each interval.

Before passing to the numerical results, we also remark that, besides the norm $\|\Delta I\|$, the JLT approximation also determines the direction of ΔI , which according to (2.1) should be parallel to the dominating k : so, denoting $\gamma = \Delta I_1 / \Delta I_2$, one expects

$$\gamma \simeq \frac{k_1}{k_2} \quad \text{in the interval where } k \text{ dominates.} \quad (2.7)$$

3. Numerical results

We now test numerically the two formulas (2.5) and (2.7). Computations are performed in quadruple precision (approximately 33 significant digits) using a symplectic algorithm, the common “leap-frog”. As shown in [12] (see also [11,13]), the use of a symplectic algorithm, although elementary, in a scattering problem, allows one to measure reliably quite small quantities.

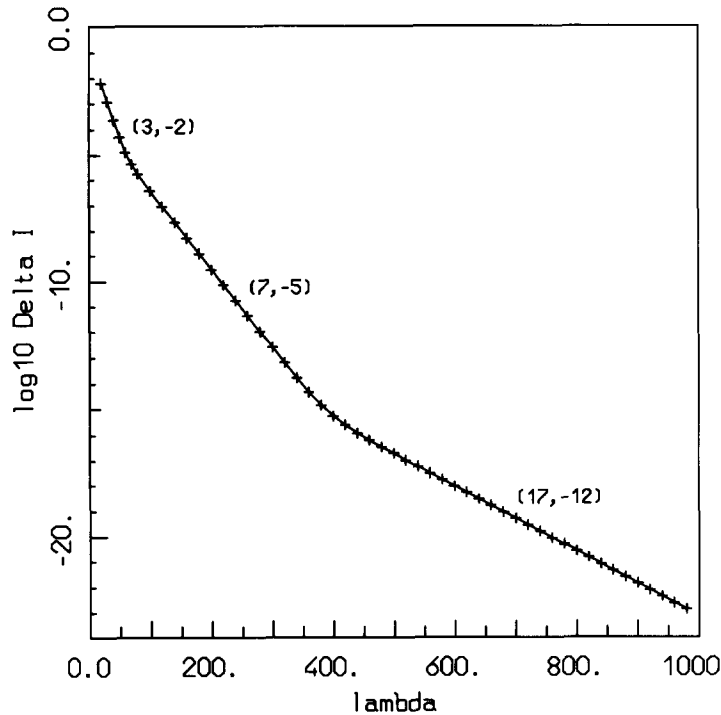


Fig. 3. $\overline{\Delta I}$ vs. λ for $\tau = 1$ and $\rho = 1$; crosses represent the experimental data, while the continuous line is the JLT approximation, more precisely the nearly broken line (2.5).

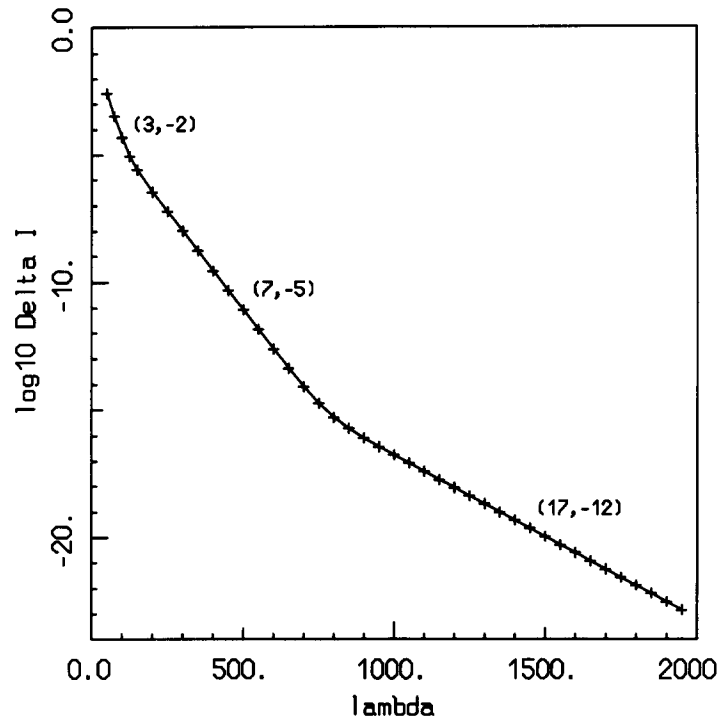


Fig. 4. The same as Fig. 3 for $\tau = 0.5$ and $\rho = 1$.

Numerical experiments are performed in the most obvious way: one fixes the parameters τ and ρ , and truncates the Fourier series for g to $|k| \leq K$, with sufficiently large K ; since the computer time is proportional to the number of Fourier components, most computations were performed using only integer vectors k in the resonant sequence, up to $K = 70$ (the first six harmonics of the sequence), but occasionally, as a control, 18 nonresonant harmonics were added, with absolutely negligible difference. Concerning time, we took of course a finite integration interval $-T \leq t \leq T$; it turns out that $T = 8$ is enough in order for $f(T)\mathcal{G}_k$ to be negligible also for small k . For each choice of the parameters τ and ρ , we varied λ from small values to values as large as possible, compatibly with computer time (unfortunately, the time step needs to be taken inversely proportional to λ). For each value of λ , in order to obtain the maximum $\overline{\Delta I}$ on the phases, we considered 50 different initial data with different values of φ_1^0 , φ_2^0 , namely 10 values of φ_1^0 equally spaced in $(0, 2\pi)$, and for each of them five values of φ_2^0 equally spaced in $(0, \pi)$. The time step was, typically, $0.08/\lambda$.

The results for $\overline{\Delta I}$ are reported in Figs. 3–6 for different choices of τ and ρ . The crosses there represent the numerical data, while the continuous line is the theoretical expectation, namely the nearly broken line (2.5) already reported in Fig. 2. As one can see, the agreement is quite good in all situations. In fact, a very careful inspection shows that in some cases (Fig. 6, around $\omega = 1000$), immediately after the crossover, the numerical data are slightly above the theoretical curve, and flatten on it only later, let us say with some delay. We did not further investigate this point.

A more quantitative comparison between the JLT approximation and the numerical results is performed by computing numerically, by a least-squares fit, the values α_k and β_k (in the interval where k dominates), to be compared with the theoretical values given by (2.3). The computed values of γ (which are practically constant

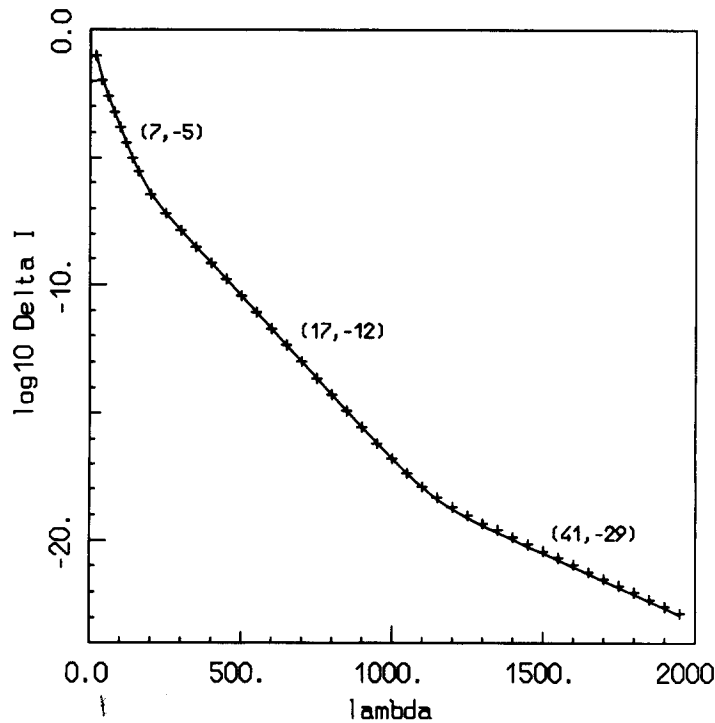


Fig. 5. The same as Fig. 3 for $\tau = 1$ and $\rho = 0.5$.

for λ in the middle of each dominating interval) are also compared with the theoretical values given by (2.7). The results are summarized in Table 1; α , β and γ there represent the theoretical values, while α' , β' and γ' denote the corresponding numerical quantities. The relative errors are typically less than 1%, and in some cases, in particular for γ , they are *much* smaller. The worst results are obtained for β : the relative error arrives, in two cases, to 2–3%.

4. Concluding remarks

4.1

First of all, let us discuss here how far one can generalize the heuristic results we obtained in connection with our model (1.1). Concerning the choice of f , its special form was certainly useful to compute everything explicitly, but the only really important features, to get (1.7), are that f has a finite analyticity strip of size τ , and decays in an integrable way for $|\operatorname{Re} t| \rightarrow \infty$. If f is an entire function, then the first-order approximation (2.1) is still expected to work, but (1.7) and (2.5), as well as the asymptotic behavior (2.6), are expected to change, namely the decay of $\overline{\Delta I}$ with λ will be faster. Let us remark however that, as far as Hamiltonian (1.1) is a mathematical model for a collision problem, and the explicit time dependence of the perturbation is intended to mimic the motion of the center of mass in a Hamiltonian problem like (1.5), singularities typically occur.⁷

⁷ Even in one dimension, if the potential is an integer function like e^{-r} , the solution $r(t)$ has poles at finite distance from the real axis, see for example [14,15].

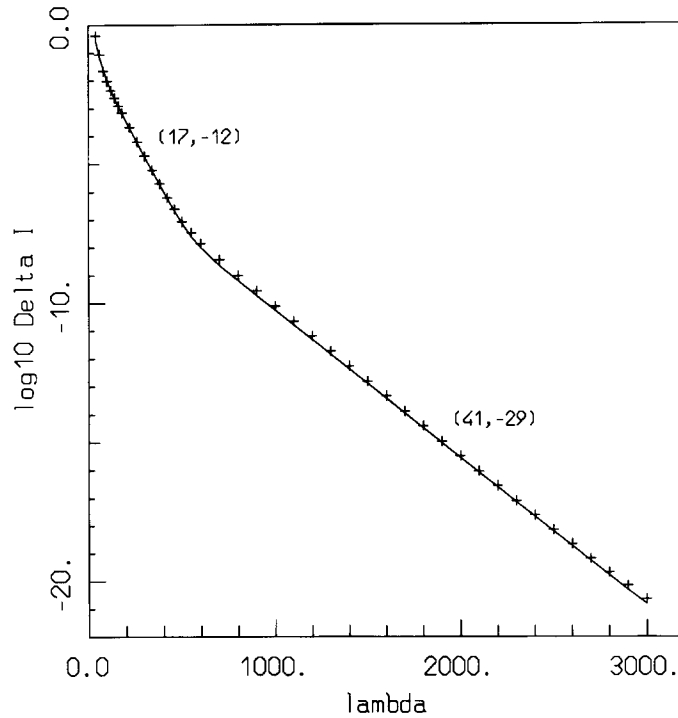
Fig. 6. The same as Fig. 3 for $\tau = 1$ and $\rho = 0.25$.

Table 1

τ	ρ	k	α	α'	β	β'	γ	γ'
1.0	1.00	(7, -5)	0.07110	0.07090	7.700	7.760	1.4000000	1.4000030
		(17, -12)	0.02942	0.02943	23.820	23.830	1.4166660	1.4166660
0.5	1.00	(7, -5)	0.03554	0.03551	7.760	7.780	1.4000000	1.4000100
		(17, -12)	0.01472	0.01473	23.870	23.880	1.4166666	1.4166666
1.0	0.50	(17, -12)	0.02944	0.02945	9.320	9.325	1.4166660	1.4166660
		(41, -29)	0.01220	0.01240	28.900	28.400	1.4193793	1.4193793
1.0	0.25	(17, -12)	0.02940	0.02960	2.070	2.100	1.4166000	1.4165000
		(41, -29)	0.01220	0.01220	11.400	11.100	1.4137931	1.4137931

Let us then consider the relevance of our choice of g . In principle, the JLT approach works for any g (even if not analytic), and (1.7) trivially generalizes into

$$\mathcal{I}_k \simeq \text{const. } k \mathcal{G}_k e^{-\lambda \tau |k \cdot \Omega|}.$$

But in the absence of nice assumptions on \mathcal{G}_k , this expression is essentially meaningless. If g is analytic in a strip of finite size ρ , even if the decay of the Fourier components with $|k|$ is not exactly exponential as in (1.4), we expect that our analysis is nevertheless correct, namely that $\overline{\Delta I}$ exhibits the broken line behavior, and that an asymptotic expression like (2.6) holds, provided, in some sense, the Fourier series is not too empty. Of course, the segments of the broken line will in general be shifted (but in a predictable way), and occasionally one of them, with \mathcal{G}_k vanishing

or particularly small, can be absent. A different interesting case is that of g integer, so that the Fourier components \mathcal{G}_k decay more rapidly than exponentially. In such a case the asymptotic behavior is expected to change: clearly, at least asymptotically the segments are shifted downward, and the asymptotic decay of $\overline{\Delta I}$ with λ gets faster.⁸

Another point, in fact a delicate one, is the role of the choice $\Omega = (1, \theta)$ with θ quadratic irrational, in the analysis of (1.7)–(2.2). Although we did not work out all details, we have no doubt that everything remains essentially unchanged if θ is irrational, and its continued fraction, although being not periodic, has nevertheless bounded entries. This covers a nondenumerable set of initial data, although a zero measure set. Another case which is easily handled is the case of θ rational: clearly, in such a case there will be a vector k , precisely the smallest integer vector for which $k \cdot \Omega$ exactly vanishes, which dominates for all sufficiently large λ ; correspondingly, for large λ , $\overline{\Delta I}$ gets constant (the last segment of the broken line is a horizontal line).⁹ The most interesting case would be that of θ diophantine, since this covers a set of initial data of full measure. But here the question is more delicate, and while it is easy to get upper bounds to $\overline{\Delta I}$, we do not see at the moment how to produce a good asymptotic expression. The case with three or more independent frequencies is, of course, even less clear.

4.2

A second question we would like to discuss is the possibility of understanding the behavior of our system, in particular the dependence of $\overline{\Delta I}$ on λ , by the usual methods of Hamiltonian perturbation theory. In fact, Hamiltonian (1.1) is a typical Hamiltonian which can be successfully studied within such a theory, λ^{-1} being the small parameter. Moreover, thanks to the fact that $f(t)$ tends to zero in an integrable way for $|t| \rightarrow \infty$, a trivial a priori estimate shows that the action $I(t)$ cannot move too far from the initial value I^0 , so a “local” study, namely a study for I in some neighborhood of the asymptotic value $I^0 = I(-\infty)$, turns out to be sufficient, and most complications of geometric nature, which are typical of Nekhoroshev theorem and come from the necessity of taking care simultaneously of the whole action space, here are avoided.

By proceeding in a very standard way, it is easy to obtain an elementary result like the following:

Consider the Hamiltonian (1.1) with f and g as described in Section 1; let $\omega(I^0) = \lambda\Omega$ with Ω diophantine and λ large, and denote

$$\tilde{f}(t) = \frac{e^{-(|t|-\tau)^2}}{\tau^2 + t^2}.$$

Then there exists a (time-dependent) canonical transformation

$$(I, \varphi) = (I', \varphi') + \lambda^{-1} \tilde{f}(t) \mathcal{W}(I', \varphi', t; \lambda),$$

\mathcal{W} being real analytic and bounded for $t \in \mathbb{R}$ and for $(I', \varphi') \in D(I^0) \times \mathbb{T}^n$, where $D(I^0)$ is a neighborhood of I^0 independent of λ , such that the new Hamiltonian H' takes the form

⁸ A special case is that of g with finite Fourier expansion. The first-order analysis is trivial: indeed, from a certain moment on, the dominating k no more changes, and the asymptotic behavior is $e^{-a\lambda}$. But (as is obvious, and also confirmed by numerical experiments) first order is, in these conditions, no more significant: combinations of the integer vectors entering the Fourier expansion are expected to appear, and give rise to a broken line behavior in this case, too.

⁹ However, it may be worthwhile observing that in this case there is a special combination of actions, precisely the “fast action” $I_\Omega = \Omega \cdot I$, such that, as one immediately deduces by the presence of the factor k in (1.7), the resonant harmonics do not contribute at all to its change. Moreover, if θ is rational and k is nonresonant, then the “small denominator” $k \cdot \Omega$ is bounded away from zero (one has indeed $|k \cdot \Omega| \geq 1$), and one immediately deduces that asymptotically $\Delta I_\Omega \sim e^{-a\lambda}$. In this case too several numerical computations, not reported here, confirm such a first-order analysis.

$$H'(I', \varphi', t) = \sum_{j=1}^n \frac{I'^2}{2A} + \tilde{f}(t)\mathcal{N}(I', t; \lambda) + e^{-(a\lambda)^\nu} \tilde{f}(t)\mathcal{R}(I', \varphi', t; \lambda), \quad (4.1)$$

where a and ν are positive constants, while \mathcal{N} and \mathcal{R} are bounded and real analytic for $(I', \varphi') \in D(I^0) \times \mathbb{T}^n$ and $t \in \mathbb{R}$.

The exponent ν depends only on the arithmetic properties of Ω , and for the special case $n = 2$ and Ω such that $|k \cdot \Omega| > \text{const. } |k|^{-1}$ for all $k \in \mathbb{Z}^2$ (in particular $\Omega = (1, \theta)$ with θ quadratic irrational), one has $\nu = \frac{1}{2}$.

The proof is completely standard and not reported.

From the normal form (4.1), using the fact that the φ -dependent term goes to zero (in an integrable way) for $|t| \rightarrow \infty$, while in the same limit the “deformation” \mathcal{W} also vanishes, one immediately deduces an exponential estimate for the variation of the actions. In particular, for $n = 2$ and θ quadratic irrational one gets

$$\overline{\Delta I} < \text{const.} e^{-\sqrt{a\lambda}}. \quad (4.2)$$

Such an expression clearly retains something of the asymptotic behavior (2.6), but the result is poor: indeed, (4.2) provides only an upper bound to ΔI , and moreover the constant a , as constructed in the proof, is much smaller than the (presumably) optimal value $a = c\tau\rho$, taken from (2.6). In addition, the characteristic “broken line behavior”, obtained heuristically and numerically, gets completely lost.

We are convinced that a more careful inspection of the problem, based in an essential way on a strict control of the propagation of small divisors along perturbative series (see for example [16–18]), could significantly improve the results, and possibly recover¹⁰ the “broken line behavior” of $\log \overline{\Delta I}$. But the main limitation, namely the fact that such methods lead only to upper estimates (and to bad constants), in our opinion cannot be overcome.

4.3

As a final comment, let us mention the possibility of making rigorous the JLT approximation, by a perturbative approach different from Nekhoroshev theory. The basic idea is clearly to look for a series expansion of ΔI such that (2.2) is the first order. The task is not trivial since, for the first order to dominate, one needs the presence of the same exponential factor $e^{-\lambda|k \cdot \Omega|\tau - \rho|k|}$ (or a faster decreasing one) at all orders, and moreover the series should converge. A result in this direction can be found in [7], where one studies an isochronous problem with Hamiltonian of the form

$$\lambda\Omega \cdot I + \frac{1}{2}p^2 + U(x) + \varepsilon V(p, x, I, \varphi), \quad (p, x) \in \mathbb{R}^2, \quad I \in \mathbb{R}^n, \quad \varphi \in \mathbb{T}^n.$$

In fact, an expression similar to (1.7) is rigorously found in [7], but unfortunately, a strong disappointing assumption of the form $\varepsilon \leq \lambda^{-m}$, with sufficiently large m , is apparently necessary. In spite of this difficulty, we believe that the perturbative scheme exploited in [7] could lead to interesting results in the present case too.¹¹

¹⁰ In this task there is a relatively easy part, namely finding a sequence $\lambda_1, \lambda_2, \dots$, such that for $\lambda_k < \lambda < \lambda_{k+1}$ one has $\log \overline{\Delta I} < A_k \lambda + B_k$, the ratio of the A_k being the correct one, i.e., $A_k = \text{const.} \alpha_k$. The difficult part, where we have not been successful, is controlling the B_k in such a way so as to recover (as an upper estimate) a continuous piecewise linear function, that is the broken line. The sequence A_k corresponds to the sequence of “approximate convergence radii” of [16,17].

¹¹ Ref. [7] is in fact one of the numerous papers which, after the work by Eliasson [19,20], exploit the Lindstedt method (expanding solutions in series of a small parameter) in place of the Von Zeipel method (working on the Hamiltonian by canonical transformations), and make essential use of cancellations among the huge amount of terms which are produced at each perturbative order. See for example [6].

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Appendix A

We deduce here the asymptotic behavior (2.6); our analysis here is essentially equivalent to [10]. To this purpose, we start from (2.4), and compute the asymptotic behavior of

$$-\log \overline{\Delta I} = \alpha_k \lambda + \beta_k, \quad (\text{A.1})$$

$k = k(\lambda)$ being the dominant harmonic, precisely the (nearly resonant) integer vector which for a given λ minimizes the RHS of (A.1). Let $k^{(0)}, k^{(1)}, k^{(2)}, \dots$ denote the sequence of nearly resonant vectors; from the continued fraction expansion $\sqrt{2} = [1; 2, 2, \dots]$, one immediately finds

$$\alpha_{k^{(s)}} = \tau \eta^s, \quad \eta = \sqrt{2} - 1.$$

On the other hand, as remarked in Section 2, one has

$$\begin{pmatrix} k_1^{(s)} \\ k_2^{(s)} \end{pmatrix} = A \begin{pmatrix} k_1^{(s-1)} \\ k_2^{(s-1)} \end{pmatrix} \quad \text{with } A = \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix} \quad \text{and } k^{(0)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

So, denoting by μ the eigenvalue larger than one of A , and writing $k^{(0)} = k^+ + k^-$, the vectors k^+ and k^- belonging, respectively, to the expanding and contracting eigenspaces of A , asymptotically for large s one has

$$|k^{(s)}| = \mu^s |k^+| + O(\mu^{-s}), \quad \|k^{(s)}\| = \mu^s \|k^+\| + O(\mu^{-s}).$$

One immediately computes

$$\mu = \eta^{-1}, \quad k^+ = \frac{1}{2\sqrt{2}} \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix},$$

and thus

$$|k^{(s)}| = c' \eta^{-s} + O(\eta^s), \quad \|k^{(s)}\| = c'' \eta^{-s} + O(\eta^s)$$

with

$$c' = \frac{2 + \sqrt{2}}{4}, \quad c'' = \frac{\sqrt{3}}{2\sqrt{2}}.$$

It follows that

$$\beta_{k^{(s)}} = c' \rho \eta^{-s} - \log \eta^{-s} - \log C c'' + O(\eta^s),$$

so, denoting $x = \eta^s$, the RHS of (A.1) can be written

$$\lambda \hat{\tau} x + c' \rho x^{-1} + \log x - \log C c'', \quad \hat{\tau} = \tau(1 + O(\lambda^{-1})). \quad (\text{A.2})$$

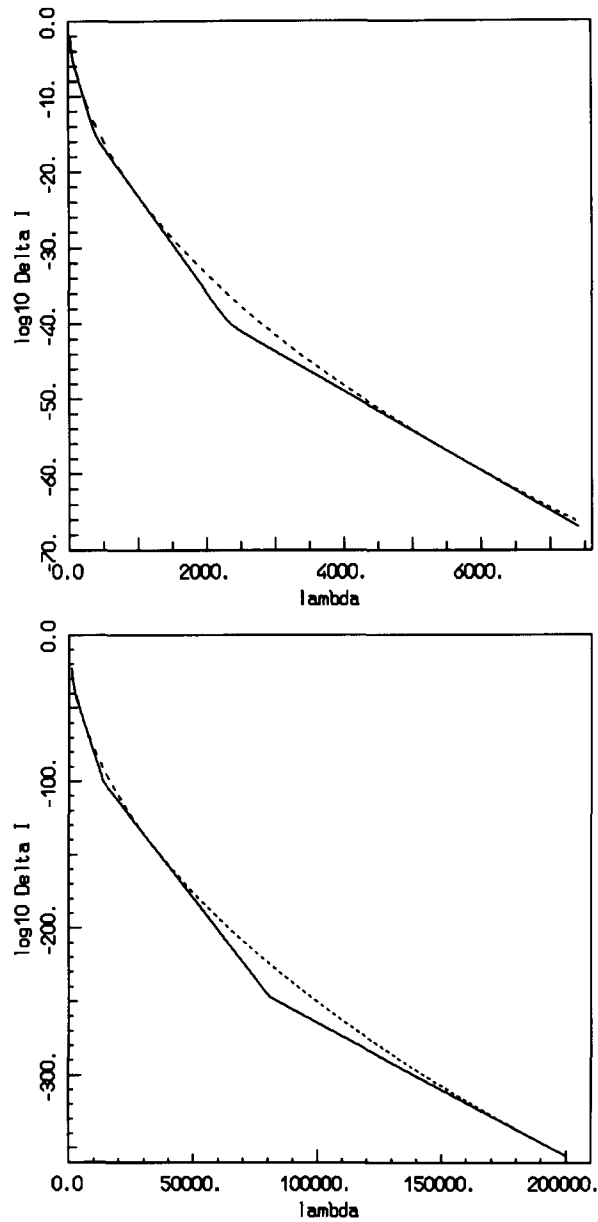


Fig. 7. A comparison between the curve (A.3), dashed line, and the broken line (2.5), solid line, for $\tau = \rho = 1$.

Now, in order to find the dominant harmonic, we treat x as a continuous variable, and minimize (A.2). The minimum \bar{x} solves the equation $\lambda \hat{\tau} x^2 + x - c' \rho = 0$, and is given by

$$\bar{x} = \frac{\sqrt{4c'\rho\lambda\hat{\tau}} - 1}{2\lambda\hat{\tau}} = \sqrt{\frac{c'\rho}{\lambda\tau}} - \frac{1}{2\lambda\tau} + O(\lambda^{-3/2});$$

correspondingly one gets

$$\lambda \tau \bar{x} + c' \rho \bar{x}^{-1} = 2\sqrt{c' \rho \lambda \tau} + O(\lambda^{-1/2})$$

(the next dominant terms cancel), and finally

$$\overline{\Delta I} = A \frac{e^{\tau^2}}{\rho \tau} (1 + O(\lambda^{-1/2})) \sqrt{c' \rho \lambda \tau} e^{-\sqrt{c' \rho \lambda \tau}} \quad (\text{A.3})$$

with

$$A = \frac{1}{2} \sqrt{3} \pi (\sqrt{2} - 1), \quad c = 4c' = 2 + \sqrt{2}.$$

This deduction of the asymptotic behavior (A.3) from (2.4) is apparently questionable when we treat x as a continuous variable; but a little reflection shows that:

- (i) In any case, the minimization procedure assures that (A.3) is an upper bound to (2.4).
- (ii) Once in each interval, \bar{x} must coincide with η^s , and at that moment (A.3) is exact [asymptotically, that is disregarding the term $O(\lambda^{-1/2})$].

As a matter of fact (and this is rigorous) in the log scale (A.3) appears as a convex curve, which is asymptotically tangent to the broken line defined by (2.4) in each of its segments. The same of course is true if one takes for $\overline{\Delta I}$ the definition (2.5), as far as, for large λ , the two definitions coincide. Fig. 7 shows (A.3) compared to the broken line (2.5).

References

- [1] J.H. Jeans, On the vibrations set up in molecules by collisions, *Phil. Mag.* 6 (1903) 279
- [2] L. Landau and E. Teller, On the theory of sound dispersion, *Physik. Z. Sowjetunion* 10 (1936) (34); in: *Collected Papers of L.D. Landau*, ed. D. Ter Haar (Pergamon Press, Oxford, 1965) p. 147.
- [3] L. Boltzmann, On certain questions of the theory of gases, *Nature* 51 (1895) 413.
- [4] G. Benettin, Adiabatic invariants and time scales for energy sharing in models of classical gases, in: *Integrability and Chaotic Behavior in Hamiltonian Systems*, ed. J. Seimenis, NATO ASI Series B: Physics (Plenum Press, New York, 1994).
- [5] G. Benettin, Ergodicity and time-scales for statistical equilibrium in classical dynamical systems, *Progr. Theoret. Phys. Suppl.* 116 (1994) 207–234.
- [6] G. Gallavotti, Twistless KAM tori, quasiflat homoclinic intersections, and other cancellations in the perturbation series of certain completely integrable hamiltonian systems. A review, *Rev. Math. Phys.* 6 (1994) 343–411.
- [7] G. Benettin, A. Carati and G. Gallavotti, A rigorous implementation of the Jeans–Landau–Teller approximation for adiabatic invariants, *Nonlinearity*, in press.
- [8] M. Rudnev, Number-theoretical consideration of the asymptotic behavior of the series typically encountered in problems of exponentially small splitting separatrices in the weakly hyperbolic Hamiltonian systems with three degrees of freedom, preprint (1996).
- [9] A. Delshals, V. Gelfreich, A. Jorba and T.M. Seara, Exponentially small splitting of separatrices under fast quasiperiodic forcing, *Universitat de Barcelona, Math. Preprint Series No. 199* (1996).
- [10] C. Simò, Averaging under fast quasiperiodic forcing, in: *Hamiltonian Mechanics, Integrability and Chaotic Behavior*, ed. J. Seimenis, Nato ASI Series B 331 (Plenum Press, New York, 1994).
- [11] G. Benettin, A. Carati and P. Sempio, On the Landau–Teller approximation for the energy exchanges with fast degrees of freedom, *J. Statist. Phys.* 73 (1993) 175–192.
- [12] G. Benettin and A. Giorgilli, On the Hamiltonian interpolation of near to the identity symplectic mappings, with application to symplectic integration algorithms, *J. Statist. Phys.* 73 (1994) 1117–1144.
- [13] O. Baldan and G. Benettin, Classical “freezing” of fast rotations: Numerical test of the Boltzmann–Jeans conjecture, *J. Statist. Phys.* 62 (1991) 201.
- [14] D. Rapp, Complete classical theory of vibrational energy exchange, *J. Chem. Phys.* 32 (1960) 735–737.
- [15] A. Carati, G. Benettin and L. Galgani, Towards a rigorous treatment of the Jeans–Landau–Teller method for the energy exchanges of harmonic oscillators, *Comm. Math. Phys.* 150 (1992) 321–336.
- [16] G. Servizi, G. Turchetti, G. Benettin and A. Giorgilli, Resonances and asymptotic behavior of Birkhof series, *Phys. Lett. A* 95 (1983) 11.

- [17] A. Bazzani and G. Turchetti, Perturbation theory and analyticity of normalizing transformations for area preserving maps, in: *Chaotic Dynamics: Theory and Practice*, ed. T. Bountis (Plenum Press, New York, 1992).
- [18] A. Giorgilli, Rigorous results on the power expansions for the integrals of a Hamiltonian system near an elliptic equilibrium point, *Ann. Inst. H. Poincaré Phys. Théor.* 48 (1988) 423–439.
- [19] L.H. Eliasson, Absolutely convergent series expansions for quasi-periodic motions, Report 2–88, Department of Mathematics, University of Stockholm (1988).
- [20] L.H. Eliasson, *Hamiltonian Systems with Linear Normal Form Near An Invariant Torus*, ed. G. Turchetti (Bologna Conference, 1988) (World Scientific, Singapore, 1989).