A LAGRANGIAN FORMULATION FOR THE ABRAHAM–LORENTZ–DIRAC EQUATION

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ABSTRACT

It is shown that a Lagrangian exists for the nonrelativistic version of the Abraham–Lorentz–Dirac equation. The method used is an easy modification of the procedure used by Levi Civita a century ago to construct a Lagrangian for the damped harmonic oscillator. It is then shown how a trivial adaptation of the method allows also to give a Lagrangian for the corresponding relativistic equation in the case of one space dimension.

Running title: Lagrangian formulation for ALD equation

1. Introduction. As is well known, the Abraham–Lorentz–Dirac equation is usually assumed to correctly describe the motion of a point charge in interaction with the electromagnetic field, when the radiation reaction is taken into account. Many times the question was asked whether it be possible to deduce it from some variational principle (see for exemple the refs. [1], [2], [3], or the more recent ref. [4]); there are inded several motivation for this, the most important one possibly being the new insight it would give to the problem of the elimination of the run–away solutions, which are well known to plague microscopic electrodynamics. However, to my knowledge, no one was able to find a Lagrangian formulation up to now. This is witnessed, for example, by the following quotation from ref. [4] (page 91):"*Attempts to construct the Hamiltonian or the Lagrangian leading to Lorentz–Dirac equation have not as yet been succesful. It is possible that this cannot be done because the classical Lorentz–Dirac equation includes the frictional reaction of radiation, and therefore describes a non conservative system.*".

In the present paper the Lagrangian for the Abraham–Lorentz–Dirac equation in its non–relativistic version is explicitly exhibited; furthermore, it is shown how, starting from such a Lagrangian, the relativistic Lagrangian is easily inferred for the particular case of motion on a straight line.

Let us recall [5,6] that the Abraham–Lorentz–Dirac equation is

$$\frac{2e^2}{3mc^3}(\dddot{x}_\mu + \frac{\ddot{x}_\nu \ddot{x}^\nu}{c^2}\dot{x}_\mu) = \ddot{x}_\mu - \frac{e}{mc}F_{\mu\,\nu}\dot{x}^\nu \qquad (\mu = 0,1,2,3) \tag{1}$$

where the derivatives are intended with respect to proper time, e and m are the charge and the renormalized mass of the point charge respectively, c is the speed of light, and the antisymmetric tensor $F_{\mu\,\nu}$ represents an external (electromagnetic) field of force. In the non–relativistic case, equation (1) reduces to

$$\varepsilon\dddot{\mathbf{x}} = \ddot{\mathbf{x}} - \mathbf{F}(\mathbf{x})/m \ , \tag{2}$$

where now the derivatives are intended with respect to the observer time, $\mathbf{x} \in \mathbf{R}^3$, $\mathbf{F}$ is a mechanical force field, and the parameter $\varepsilon = \frac{2e^2}{3mc^3}$ was introduced.

It will be shown here that the Lagrangian corresponding to equation (2) is

$$\mathcal{L} = e^{-t/\varepsilon}\Big(\frac{\varepsilon\dot{\mathbf{a}}^2}{2} - \mathbf{a}\cdot(\dot{\mathbf{x}}\cdot\partial_{\mathbf{x}})\mathbf{F}(\mathbf{x}) + \frac{\mathbf{F}(\mathbf{x})\cdot\mathbf{F}(\mathbf{x})}{2\varepsilon}\Big) + \mathbf{p}\cdot(\mathbf{v}-\dot{\mathbf{x}}) + \mathbf{q}\cdot(\mathbf{a}-\dot{\mathbf{v}}) \ , \tag{3}$$

where $\mathbf{v}$ and $\mathbf{a}$ are the velocity and the acceleration of the particle, and $\mathbf{p}$ and $\mathbf{q}$ are vector parameters whose significance will be illustrated in the next section; furthermore, the dot $\cdot$ denotes the scalar product and $\partial_{\mathbf{x}}$ denotes the gradient operator. More precisely, the Euler–Lagrange equation for Lagrangian (3) turns out to coincide with the nonrelativistic Abraham–Lorentz–Dirac equation (2) if one fixes, in a way consistent with the Euler–Lagrange equation itself, the value of the variables $\mathbf{p}$ and $\mathbf{q}$ equal to zero for all times. In other words, the solutions of the Abraham–Lorentz–Dirac equation are a subset of the solutions of the Euler–Lagrange equation of Lagrangian (3) for a particular choice of the initial data. The reason for such a complication will be explained in the next section.

In Section 2 the Lagrangian (3) for equation (2) is deduced, while in Section 3 the Lagrangian corresponding to equation (1) in the case of motion on a straight line will be discussed.

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2. Deduction of the Lagrangian in the nonrelativistic case. Some preliminary remarks are in order:

i) The Lagrangian (3) depends on time t as should be expected, because the Abraham–Lorentz–Dirac equation is non conservative, so that energy cannot be a costant of motion. The most familiar example of such a situation is the damped harmonic oscillator, i.e. the linear equation

$$\ddot{x} + \gamma\dot{x} + \omega^2 x = 0 \ ,$$

which, as is well known (see refs. [7],[8] [2]), can be obtained from the Lagrangian

$$\mathcal{L} = e^{\gamma t}\Big(\frac{\dot{x}^2}{2} - \frac{\omega^2 x^2}{2}\Big) \ .$$

This example will guide us in determining the expression (3) for the Lagrangian of the Abraham–Lorentz–Dirac equation.

ii) It is obvious that it is impossible to find a Lagrangian leading directly to an equation of motion of the third order, because the highest order will always be even, at least if one considers Lagrangians quadratic in the highest order term.† This problem is overcome if one considers instead the equation of fourth order which is obtained by differentiating the Abraham–Lorentz–Dirac equation with respect to time. Obviously one has then to add some supplementary conditions in order to obtain a solution of the original equation (2), and this is indeed the reason for the restriction of the allowed initial data mentioned above.

iii) It is easy to obtain a Lagrangian for the differentiated equation, if one thinks of the acceleration $\mathbf{a}$ as an indipendent variable; obviously, this will then require to add the constraints $\dot{\mathbf{x}} = \mathbf{v}$ and $\dot{\mathbf{v}} = \mathbf{a}$. However, these constraints are non holonomic, and so, to save the Lagrangian formalism, one is forced to introduce the so called "Lagrangian multipliers" (see for example the classical treatise of Whittaker [9]), which were denoted by $\mathbf{p}$ and $\mathbf{q}$ in expression (3).

Following the above remarks, one differentiates equation (2) with respect to time obtaining

$$\varepsilon\ddot{\mathbf{a}} = \dot{\mathbf{a}} - (\dot{\mathbf{x}}\cdot\partial_{\mathbf{x}})\mathbf{F}(\mathbf{x}) \ ,$$

† Clearly one could also look for Lagrangians linear in the highest order term, but I prefer to restrict the attention to quadratic Lagrangians.

where $\mathbf{a} = \ddot{\mathbf{x}}$. This equation, using the paradigma of the damped harmonic oscillator, naturally leads to the Lagrangian

$$\mathcal{L} = \exp(-t/\varepsilon)\Big(\frac{\varepsilon\dot{\mathbf{a}}^2}{2} - \mathbf{a}\cdot(\dot{\mathbf{x}}\cdot\partial_{\mathbf{x}})\mathbf{F}(\mathbf{x})\Big) + \mathcal{L}_1 \ ,$$

where $\mathcal{L}_1$ is a function to be determined. In order to respect the constraints, one is naturally led to write $\mathcal{L}_1$ in the form

$$\mathcal{L}_1 = \mathbf{p}(\mathbf{v} - \dot{\mathbf{x}}) + \mathbf{q}(\mathbf{a} - \dot{\mathbf{v}}) + \mathcal{L}_2 \ ,$$

with a still indeterminated function $\mathcal{L}_2$. In turn, $\mathcal{L}_2$ is easily determined from a quick inspection of the resulting equation of motion for the variable $\mathbf{x}$, which leads to

$$\mathcal{L}_2 = e^{-t/\varepsilon}\frac{\mathbf{F}(\mathbf{x})\cdot\mathbf{F}(\mathbf{x})}{2\varepsilon} \ ,$$

namely to the Lagrangian (3).

The corresponding Euler–Lagrange equations are

$$\begin{aligned}
&e^{-t/\varepsilon}\Big(\varepsilon\ddot{\mathbf{a}} - \dot{\mathbf{a}} + \dot{\mathbf{x}}\cdot\partial_{\mathbf{x}}\mathbf{F}(\mathbf{x})\Big) - \mathbf{q} = 0\\
&\frac{1}{\varepsilon}e^{-t/\varepsilon}(\partial_{\mathbf{x}}\mathbf{F}(\mathbf{x}))\Big(\varepsilon\dot{\mathbf{a}} - \mathbf{a} + \mathbf{F}(\mathbf{x})\Big) - \dot{\mathbf{p}} = 0\\
&\dot{\mathbf{x}} = \mathbf{v}\\
&\dot{\mathbf{v}} = \mathbf{a}\\
&\dot{\mathbf{q}} = \mathbf{p} \ .
\end{aligned} \tag{4}$$

Now, with very few calculations, one verifies that if $\mathbf{x}(t)$ is a solution of (2) then $\mathbf{p}(t) = 0$, $\mathbf{q}(t) = 0$, $\mathbf{x}(t)$, $\mathbf{v}(t) = \dot{\mathbf{x}}(t)$, $\mathbf{a}(t) = \ddot{\mathbf{x}}(t)$ is a solution of system (4). Conversely, one immediately verifies that, given a solution of (4) with initial data $\mathbf{p}_0 = 0$, $\mathbf{q}_0 = 0$, $\mathbf{v}_0 = \dot{\mathbf{x}}_0$, $\mathbf{a}_0 = \ddot{\mathbf{x}}_0$ and $\dot{\mathbf{a}}_0 = \mathbf{a}_0 - \mathbf{F}(\mathbf{x}_0)$, then the motion $\mathbf{x}(t)$ is also a solution of equation (2) as stated in Section 1.

3. The relativistic case for motions on a straight line. The problem of finding a Lagrangian in the relativistic case with only one space dimension can be reduced, at a formal level, to the non relativistic case by using the constancy of the modulus of the velocity $\dot{x}^\mu$, namely

$$c^2\dot{t}^2 - \dot{x}^2 = c^2 \ , \tag{5}$$

to reduce the order of the equation of motion. Indeed, first of all condition (5) naturally leads to introduce a new variable z, usually called "rapidity" of the motion, such that $\dot{x} = c\mathrm{Sh}\,(z)$ and $\dot{t} = \mathrm{Ch}\,(z)$. Then one uses the fact that the tensor field $F_{\mu\nu}$ in (1) is antisymmetric, so that, in the one–dimensional case, the only independent nonvanishing component is $F_{01} = -F_{10} \stackrel{\text{def}}{=} F(x)$. Thus, in the one–dimensional case, in terms of x and z equation (1) becomes

$$
\begin{aligned}
\varepsilon \ddot{z} &= \dot{z} - \frac{eF(x)}{mc} \\
\dot{x} &= c \,\mathrm{Sh}\, z \ ,
\end{aligned}
$$

or, equivalently

$$
\begin{aligned}
\varepsilon \dot{a} &= a - \frac{eF(x)}{mc} \\
\dot{z} &= a \\
\dot{x} &= c \,\mathrm{Sh}\, z \ .
\end{aligned}
$$

The latter system of equations shows that the relativistic equation on the line can be obtained from the non–relativistic one (3), if one imposes the non–linear constraint $\dot{x} = c\mathrm{Sh}\, z$ instead of the linear one $\dot{x} = v$. If one prefers to avoid a transcendental constraint, it is possible to use a rational parametrization of the curve $\dot{x}^\mu \dot{x}_\mu = c^2$, by introducing the variable v defined by $\dot{x} = \frac{c}{2}(v - 1/v)$, which leads also to $\dot{t} = \frac{1}{2}(v + 1/v)$. The equations of motion then read

$$
\begin{aligned}
\varepsilon \dot{a} &= a - \frac{eF(x)}{mc} \\
\dot{v} &= av \\
\dot{x} &= \frac{c}{2}\Big(v - \frac{1}{v}\Big) \ .
\end{aligned}
\tag{6}
$$

It is thus clear that, in the same sense discussed in the previous section, the Lagrangian for this system is then

$$
\mathcal{L} = e^{-t/\varepsilon}\Big(\frac{\varepsilon \dot{a}^2}{2} - a\dot{x}\partial_x F(x) + \frac{F(x)^2}{2\varepsilon}\Big) + p\Big(\frac{c}{2}(v - 1/v) - \dot{x}\Big) + q\Big(av - \dot{v}\Big) \ , \tag{7}
$$

which is obtained from (3) by changing the constraints in the way just explained. This formula solves the problem of finding the Lagrangian for the relativistic motion on a line.

There remains still open the problem of finding the relativistic Lagrangian for the full three–dimensional problem, but the result of this section strongly seems to suggest that it might be possible to solve it.

References

[1] P. Caldirola, *Nuovo Cim.* **18**, 393 (1941).
[2] P. Caldirola, *Rend. Ist. Lomb. Sc* **A 93**, 439 (1959).
[3] F. Röhrlich, *Classical charged particles*, Addison–Wesley (Reading, 1965).
[4] A.A. Sokolov, I.M. Ternov, *Radiation from relativistic electrons*, American Institute of Physics (New York, 1986).
[5] P.A.M. Dirac, *Proc. Royal Soc. (London)* A**167**, 148–168 (1938).
[6] H.A. Lorentz, *The theory of electrons*, Dover (New York, 1952); first edition 1909.

[7] R.M. Santilli, *Foundations of theoretical mechanics*, Springer–Verlag (New York 1978).

[8] T. Levi Civita, *Atti R. Ist. Veneto Sc.* **53**, 1004 (1896), in

[9] E.T. Whittaker, *A A treatise on the analytic dynamics of particles and rigid body*, Dover (New York, 1944).