

# Nonradiating normal modes in a classical many-body model of matter-radiation interaction

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## ABSTRACT

We consider a classical model of matter-radiation interaction, in which the matter is represented by a system of infinitely many dipoles on a one-dimensional lattice, and the system is dealt with in the so-called dipole (i.e. linearized) approximation. We prove that there exist normal-mode solutions of the complete system, so that in particular the dipoles, though performing accelerated motions, do not radiate energy. This comes about in virtue of an exact compensation which we prove to occur, for each dipole, between the “radiation reaction force” and a part of the retarded forces due to all the other dipoles. This fact corresponds to a certain identity which we name after Oseen, since it occurs that this researcher did actually propose it, already in the year 1916. We finally make a connection with a paper of Wheeler and Feynman on the foundations of electrodynamics. It turns out indeed that the Oseen identity, which we prove here in a particular model, is in fact a weak form of a general identity that such authors were assuming as an independent postulate.

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# 1 Introduction

In the framework of classical electrodynamics it is often given for granted that a charged particle, when accelerated under the influence of some external force, gives off radiation and thus loses energy. Actually, if a single particle is considered, this is only partially true. Indeed, if the contribution of the radiative interaction to the equation of motion is taken into account in the standard way through the familiar “radiation reaction force” proportional to the time derivative of the acceleration (or through its relativistic extension proposed by Dirac [1]), then a harmonic oscillator is easily proven to steadily lose energy and fall onto the center of attraction, while in the case of a Coulomb attraction the particle is found to start losing energy but finally to escape to infinity [2]. Anyway, in such two cases involving a single charged particle it has been proven that oscillatory motions do not exist.

In the present paper we investigate the existence of oscillatory motions in the case of  $N > 1$  charged particles. In fact, we even consider a system of infinitely many particles, for an extremely simplified model particularly suited for an analytical investigation. This is a system of equal particles (which we also call “dipoles” or “linear resonators”), that are all constrained to move on a line and that, in the absence of any electrodynamical interaction, would perform linear oscillations each about a proper site, the sites constituting a periodic lattice on the given line. The electrodynamic interaction is introduced in the standard way which was fixed by Dirac for a general system of particles and field, taking however the so-called “dipole approximation” (i.e. linearizing the equations of motion with respect to the displacements of the resonators, and to their time derivatives). In such a way, by the standard procedure of eliminating the field in the coupled equations for matter and field, one is reduced to a system of equations which are just the mechanical Newton equations for each dipole, in which the effect of the electromagnetic interaction appears through two contributions, namely: the familiar “radiation reaction force” on each dipole (in the dipole approximation, which is here equivalent to the nonrelativistic approximation), and a mutual retarded force between each pair of dipoles. An interaction with an external free field could also have been considered. But we drop it, because it mainly plays a role in connection with dispersion theory, with which we are not directly concerned here. The equations of motion of the model are written down below (see (1)). In fact, apart from the special choice of the disposition of the equilibrium positions of the resonators, the present model is nothing but the standard one that is usually employed for a microscopic molecular foundation of optics. Actually, this is only partially true, because the necessity of introducing the radiation reaction force in such models has been from time to time put in doubt. Here, not only do we introduce such a radiation reaction term into the model, relying on the authority of Dirac, but also claim that its role is clarified by the main result of the present paper.

Indeed we prove the rather surprising result that there exist solutions of the complete system of equations, in which the retarded electromagnetic forces produced on a given dipole by the fields “created” by all the other ones add up in such a way as to exactly compensate the reaction radiation force acting on it, so that one remains with no radiation at all. The relevant point is that such nonradiating solutions, which have the form of normal modes for the complete system, could not exist if the radiation reaction term had not been included in the model. Presumably, the present result might prove useful in establishing the existence of collective normal-mode motions also in a quantum mechanical version of the present model of matter–radiation interaction.

The way in which we came to conceive that nonradiating normal-mode solutions can exist at all might be of some interest. In the appendix we describe how this came about, starting from a critical reading of some old papers of Planck on the models of matter–radiation interaction describing a black-body. We also describe there the results of a bibliographical research that we made, after having found our result, in order to learn whether something analogous was already known. In fact, it turns out that the idea of an exact compensation of the radiation reaction term by the retarded forces due to all the other dipoles had been proposed by Oseen [3] already in the year 1916. Now, such a result was criticized and finally forgotten, whereas we show that its main content, in the form presented here, is correct. This is the reason why we name after Oseen the main identity which is at the basis of our result. A further point of possible interest of our result is its strong relation with the paper of Wheeler and Feynman [4] on the foundations of electrodynamics. Indeed it turns out that the Oseen identity, which is here proven in our model, is nothing but a weak form of an identity that plays a central role in the paper of Wheeler and Feynman, and is by them assumed as an independent postulate.

In Section 2 the model is defined. In Section 3 the Oseen identity and the existence of normal modes are proven, while in Section 4 the form of the dispersion relation is discussed. In Section 5 the Oseen identity is expressed in terms of the forces acting on the dipoles, while the connection with the paper of Wheeler and Feynman is discussed in Section 6. Some conclusive comments follow in Section 7.

## 2 The model

We consider an infinite number of charged particles of equal mass  $m$  and charge  $e$  constrained on a line, and denote by  $x_j$ , with  $j \in \mathbf{Z}$ , their cartesian coordinates. We assume there exists an equilibrium configuration with positions  $x_j = r_j$ , where  $r_j = ja$  and  $a$  is a positive parameter (the lattice step), corresponding to a balance of the mutual Coulomb forces and of other

possible mechanical forces. In the absence of further electrodynamical interactions, we assume that each particle performs a linear oscillation with the same characteristic angular frequency  $\omega_0$  about its equilibrium position  $r_j$ . The interaction with the electromagnetic field is taken into account in the dipole approximations as described below, to the effect that the final mathematical model has as unknowns only the displacements  $q_j = x_j - r_j$  of the particles from their equilibrium positions, and that such displacements satisfy the infinite system of delayed differential equations (for  $j \in \mathbf{Z}$ )

$$m(\ddot{q}_j + \omega_0^2 q_j - \varepsilon \ddot{q}_j) = 2e^2 \sum_{k \neq j} \left[ \frac{q_k(t - r_{jk}/c)}{r_{jk}^3} + \frac{1}{c} \frac{\dot{q}_k(t - r_{jk}/c)}{r_{jk}^2} \right], \quad (1)$$

where the sum is extended over  $k \in \mathbf{Z}$ ,  $k \neq j$ . Here  $r_{jk} = |r_j - r_k|$  is the distance between particles  $k$  and  $j$ , evaluated at their equilibrium positions,  $c$  is the speed of light, and the familiar parameter

$$\varepsilon = \frac{2}{3} \frac{e^2}{mc^3}$$

has been introduced. As mentioned in the Introduction, these equations, apart from the special choice of the equilibrium positions of the particles, are the ones that were commonly used for a molecular foundation of optics in the years 1915–1925. Actually, this is completely true only for what concerns the right hand sides of the equations, because a general agreement had not yet been reached concerning the choice of the radiation reaction term  $-m\varepsilon \ddot{q}_j$  appearing at the l.h.s. of each equation.

Equations (1) are obtained in a well known way, by eliminating the field in the coupled equations for matter and field; we however recall it here briefly for the sake of completeness. Working in the dipole approximation means to linearize the system with respect to the displacements  $q_j$  and to their time derivatives. So the field “created” by particle  $k$  is obtained from the Maxwell equations by taking as sources the density and the current density given by

$$\begin{aligned} \rho_k(\mathbf{x}) &= e\delta(\mathbf{x} - r_k \mathbf{i}) - eq_k \mathbf{i} \cdot \nabla \delta(\mathbf{x} - r_k \mathbf{i}) \\ \mathbf{j}_k(\mathbf{x}) &= e\dot{q}_k \mathbf{i} \delta(\mathbf{x} - r_k \mathbf{i}). \end{aligned} \quad (2)$$

respectively,  $\delta$  being the usual “delta function”. The first source term  $e\delta(\mathbf{x} - r_k \mathbf{i})$  in the density gives rise to the static Coulomb field, which was already taken into consideration. One then solves the Maxwell equations with the other terms in the sources (2), and the retarded fields thus come out in a natural way if the solutions are evaluated, as usual, at a time  $t > t_0$  where  $t_0$  is an “initial time”; however, advanced fields too could have been used, by matching the initial data through a suitable free field. The analytical computations leading to the retarded fields are classical, and are easily reproduced. The results can however be found in reference [5] (see page 284). The magnetic field turns out to be already of first order. So the magnetic

force is of second order, and drops out, and one remains with the electric force. The electric field naturally appears as decomposed into the sum of several terms, but due to the specific model considered here (displacements along a line, on which the dipoles themselves are lying), some terms cancel and other ones suitably add up, so that one remains with two terms only, which are the ones appearing at the r.h.s. of equation (1). In conformity to the dipole approximation, they are evaluated at the equilibrium positions of the particles.

We finally add a few words concerning the old problem of the “self-field”, i.e. the fact that the electric field “created” by any particle  $j$  diverges at the position of that particle itself, so that a suitable prescription is needed. Here we follow the long tradition, going back to Lorentz, Planck and Abraham, and definitely fixed by Dirac (see also [6]), according to which the force due to the self-field leads both to mass renormalization (so that the empirical mass  $m$  is introduced as an external parameter) and to the familiar radiation reaction force, which yields, in the dipole approximation, the term  $-m\varepsilon\ddot{q}_j$  at the l.h.s. of equations (1).

### 3 The Oseen identity and the normal modes

So, our model is defined by the system of equations (1). For an analytical discussion we have to make use of the relation  $r_{jk} = |r_j - r_k| = |j - k|a$  holding in our specific one-dimensional model. With the relabeling  $k - j = n \in \mathbf{Z} \setminus \{0\}$ , the final form of the equations to be studied here is thus (for  $j \in \mathbf{Z}$ )

$$\ddot{q}_j + \omega_0^2 q_j - \varepsilon \ddot{q}_j = \frac{2e^2}{ma^3} \sum_{n \neq 0} \left[ \frac{q_{j+n}(t - |n|a/c)}{|n|^3} + \frac{a}{c} \frac{\dot{q}_{j+n}(t - |n|a/c)}{|n|^2} \right]. \quad (3)$$

The first point we make concerning system (3) is that it cannot present damped normal modes, contrary to what might be expected according to the generic presumption that charged particles, when accelerated, should radiate energy. In fact, if one looks for normal modes, i.e. introduces the *ansatz* (the real part should actually be taken later)

$$q_j(t) = u_j \exp(i\omega t), \quad (4)$$

where the parameter  $\omega$  is, a priori, a complex number, then (3) yields the system of equations

$$(-\omega^2 + \omega_0^2 + i\varepsilon\omega^3) u_j = \frac{2e^2}{ma^3} \sum_{n \neq 0} u_{j+n} \exp[-i|n|a\omega/c] \left( \frac{1}{|n|^3} + i\frac{a\omega}{c} \frac{1}{|n|^2} \right). \quad (5)$$

Now, if  $\omega$  had a *positive* imaginary part (which gives the familiar damped solution, when dealing with one single dipole subjected to an external forcing), then the terms of the series at the r.h.s. would grow exponentially fast, and the series would diverge. If instead  $\omega$  had a *negative* imaginary part, then one would be in presence of a so called runaway solution, i.e. a motion  $q_j = q_j(t)$  diverging for  $t \rightarrow +\infty$ . Following Dirac, Planck and the other classical authors, runaway solutions are discarded on physical grounds; and the same we also do. In more explicit terms, as an essential part of the definition of our model we restrict our consideration to motions  $q_j = q_j(t)$  that are solutions to the equations (3) and in addition satisfy the nonrunaway conditions of being bounded for all times.

So the problem of obtaining normal modes for our model is reduced to finding solutions of the form (4) to system (5), with  $\omega$  *real*. To this end we introduce the further usual *ansatz*

$$u_j = C \exp(i\kappa a j) , \quad (6)$$

with a given parameter (the “wave number”)  $\kappa \in [-\pi/a, \pi/a)$ . This corresponds to considering a “material wave” with *phase velocity*  $v = \omega/\kappa$ . So one is reduced to a single complex equation, namely

$$-\omega^2 + \omega_0^2 + i\varepsilon\omega^3 = \frac{2e^2}{ma^3} \left[ f(\kappa a, a\omega/c) + i g(\kappa a, a\omega/c) \right] , \quad (7)$$

where we have introduced the two functions

$$f(\alpha, \beta) = \sum_{n \neq 0} \left( \frac{\cos(n\alpha - |n|\beta)}{|n|^3} - \beta \frac{\sin(n\alpha - |n|\beta)}{|n|^2} \right) \quad (8)$$

$$g(\alpha, \beta) = \sum_{n \neq 0} \left( \frac{\sin(n\alpha - |n|\beta)}{|n|^3} + \beta \frac{\cos(n\alpha - |n|\beta)}{|n|^2} \right) . \quad (9)$$

This corresponds to two real equations, namely

$$-\omega^2 + \omega_0^2 = \frac{2e^2}{ma^3} f(\kappa a, a\omega/c) \quad (10)$$

$$\varepsilon\omega^3 = \frac{2e^2}{ma^3} g(\kappa a, a\omega/c) , \quad (11)$$

in the two real variables  $\omega$  and  $\kappa$ , with parameters  $a$  and  $\omega_0$  (while  $e$ ,  $c$ ,  $m$  and  $\varepsilon$  are thought of as fixed). So there should be no possibility for the solutions to define implicitly a curve in the  $(\kappa, \omega)$  plane, typically a function  $\omega = \omega(\kappa)$ , as expected for a dispersion relation in an infinite lattice. The situation turns out however to be quite fortunate, because it can be established that the second equation (11) actually is an identity (which we like to call the Oseen identity), so that one remains with only one equation, i.e. (10), in two variables.

This is established as follows. While the series  $f$  entering the real part is not expressible in terms of elementary functions, it turns out that the series  $g$  entering the imaginary part can be summed without pain. This amounts to establishing the classical formulas

$$\sum_{n=1}^{+\infty} \frac{\sin(nx)}{n^3} = \frac{x^3}{12} - \frac{\pi x^2}{4} + \frac{\pi^2 x}{6} \quad (12)$$

and

$$\sum_{n=1}^{+\infty} \frac{\cos(nx)}{n^2} = \frac{x^2}{4} - \frac{\pi x}{2} + \frac{\pi^2}{6} , \quad (13)$$

which are known to hold in the fundamental domain  $x \in [0, 2\pi)$ ; see for example the handbook of Abramovitz and Stegun [7]. In such a way one obtains

$$g(\alpha, \beta) = \begin{cases} \beta^3/3 & \text{if } |\beta/\alpha| < 1 \\ \beta^3/3 + \pi/2(\alpha^2 - \beta^2) & \text{if } |\beta/\alpha| \geq 1 . \end{cases} \quad (14)$$

The first of these is proved directly using the formulas (12), (13), while the second one is established by translating the variable  $\alpha + \beta$  or the variable  $\alpha - \beta$ , when required, to the fundamental domain  $[0, 2\pi)$ .

Thus it turns out that, in the domain of the  $(\kappa, \omega)$  plane where  $|v|/c < 1$  (which corresponds to  $|\beta/\alpha| < 1$ ), everything combines in such a miraculous way that equation (11) rather turns out to be an identity. In such a domain, the dispersion relation is then defined implicitly by the real transcendental equation (10), which is discussed in the next Section. It is possible to check that in the complementary domain  $|v|/c \geq 1$  there are no further solutions to the complex equation (7). The waves having the property  $|v|/c < 1$  are known in optics as evanescent waves.

## 4 The dispersion relation

We come now to a discussion of the dispersion relation, namely the curve in the  $(\kappa, \omega)$  plane implicitly defined by equation (10), depending on the parameters  $a$  (the lattice step) and  $\omega_0$  (the proper frequency of the dipoles). The first thing to be established is whether there are values of the parameters for which a curve in fact exists at all; then one would like to determine some qualitative features, such as for example whether the curve is the graph of a function  $\omega = \omega(\kappa)$ , and how it differs from the constant function  $\omega(\kappa) = \omega_0$ .

An analytical study is actually not immediate, because the function  $f$  entering equation (10) turns out to be, at variance with  $g$ , non expressible in terms of elementary functions. So we turn to a numerical study. We have to look for possible intersections of the paraboloid  $z = -\omega^2 + \omega_0^2$  with the surface  $z = (2e^2/ma^3) f(\kappa a, \omega a/c)$ , by approximating the function  $f$  through a suitable truncation of the series (8).

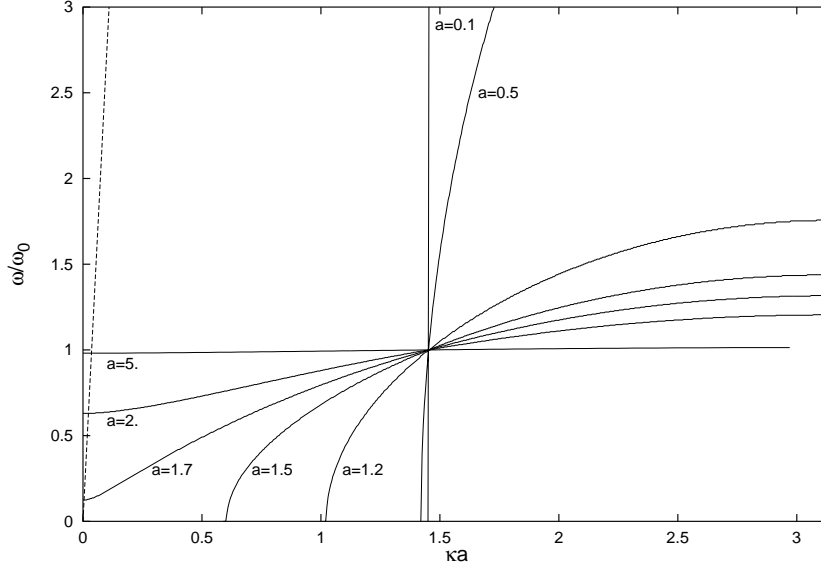


Figure 1: The dispersion curves in the plane  $(\kappa a, \omega/\omega_0)$ , for some values of  $a$ .

The parameters  $a$  and  $\omega_0$  could a priori be taken each in the whole positive axis, but here we limit ourselves to the consideration of some values having an order of magnitude of interest for atomic physics. Actually, for  $\omega_0$  we just consider one value, i.e. the rotational frequency of the electron in the hydrogen atom in circular motion at the Bohr radius  $R_B$ ; for  $a$  we take several values ranging from  $0.1 R_B$  up to  $5 R_B$ . The results are illustrated in Fig. 1, where the dispersion curve is reported for several values of  $a$  (indicated in the Figure in units of  $R_B$ ).

The most significant result seems to be that the dispersion curve indeed exists; moreover its topology depends on the value of the lattice step  $a$ . All curves have a common behavior at the right extreme of the Figure, because they all intercept the vertical line  $\kappa a = \pi$  with a horizontal tangent. The situation is instead different in the region of small  $\kappa$ . Indeed, there exists a critical value  $a^* \simeq 1.7 R_B$  of  $a$ . For  $a > a^*$  the curve is the graph of a function  $\omega = \omega(\kappa)$  (in the whole admissible domain of  $\kappa$ ) which, for increasing  $a$ , tends to the horizontal curve  $\omega = \omega_0$ ; actually, the function essentially coincides with the constant function  $\omega(\kappa) = \omega_0$  already for  $a \simeq 5 R_B$ . Instead, for  $a < a^*$  the curve is the graph of a function  $\kappa = \kappa(\omega)$ , which has a central part tending to the vertical curve  $\kappa a = \pi/2$  as  $a$  decreases. Notice that in the  $(\kappa, \omega)$  plane the curves can exist only below the line  $\omega/\kappa = c$ . Such a line is reported in the Figure for the case  $a = 5 R_B$ . Notice that the slope increases as  $a$  diminishes, so that eventually the line becomes indistinguishable from the axis of the ordinates; in the Figure, this would already occur for  $a = R_B$ .



## 5 The Oseen identity in terms of the forces

It has been shown in Section 3 how the Oseen identity (namely the identically vanishing of the imaginary part of the complex equation (7) for the normal modes) allows for the existence of a dispersion relation. We now investigate how the identity reads in terms of the forces acting on each dipole. We show the quite significant result that such an identity provides a cancellation of the radiation reaction term  $-m\epsilon\ddot{q}_j$  pertaining to any dipole  $j$  by a resummation of a part of the retarded forces due to all the other dipoles  $k \neq j$ . In the next Section we will show that the identity can be expressed in another very enlightening way, which will allow us to make a strong connection with the work of Wheeler and Feynman.

Let us rewrite the equations of motion of our model in a perhaps more transparent way as follows:

$$m(\ddot{q}_j + \omega_0^2 q_j - \epsilon\ddot{q}_j) = e \sum_{k \neq j} E_{jk}^{\text{ret}}. \quad (15)$$

Here  $E_{jk}^{\text{ret}}$  is the (component along the  $\mathbf{i}$  vector of the) retarded electric field “created” by particle  $k$  and evaluated at the equilibrium position of particle  $j$ , in the dipole approximation, namely:

$$E_{jk}^{\text{ret}} = 2e \left[ \frac{q_k(t - r_{jk}/c)}{r_{jk}^3} + \frac{1}{c} \frac{\dot{q}_k(t - r_{jk}/c)}{r_{jk}^2} \right]. \quad (16)$$

Now, looking back at the way in which the existence of normal modes was proved, it is obvious that the result found in Section 3 can equivalently be expressed in the following way: There exist normal-mode solutions  $q_j(t) = A \cos(\kappa a j - \omega t)$  of system (15) (16) such that the sum of the retarded forces acting on any dipole  $j$  due to all other dipoles  $k \neq j$  decomposes as

$$e \sum_{k \neq j} E_{jk}^{\text{ret}} = \frac{2e^2}{a^3} f(\kappa a, a\omega/c) q_j(t) - m \epsilon \ddot{q}_j(t), \quad (17)$$

i.e. into a term that exactly compensates the “radiative term” at the l.h.s. of (15), and into another one that corrects the mechanical frequency  $\omega_0$  as to have  $\ddot{q}_j + \omega^2 q_j = 0$  (as obviously should be, by the definition itself of a normal-mode solution).

One could now ask whether it is possible to describe in some more transparent way such a splitting of the sum of the retarded forces acting on dipole  $j$  into a part compensating the radiative term, and another one correcting the mechanical frequency  $\omega_0$ . This is actually the door through which the advanced forces naturally enter the arena, and one is somehow compelled to take them into consideration, notwithstanding the fact that, following the traditional approach, only retarded forces had originally been introduced in the model.

Indeed, the above decomposition of the total retarded force acting on dipole  $j$  into a term proportional to  $q_j$  and another one proportional to  $\ddot{q}_j$  turns out to actually constitute a decomposition into a symmetrical part and an antisymmetrical one with respect to time reversal. On the other hand a most natural decomposition of such a type for the single retarded forces themselves is

$$E_{jk}^{\text{ret}} = \frac{E_{jk}^{\text{ret}} + E_{jk}^{\text{adv}}}{2} + \frac{E_{jk}^{\text{ret}} - E_{jk}^{\text{adv}}}{2}, \quad (18)$$

where  $E_{jk}^{\text{adv}}$  is the advanced field, which, in our case, in the dipole approximation reads

$$E_{jk}^{\text{adv}} = 2e \left[ \frac{q_k(t + r_{jk}/c)}{r_{jk}^3} - \frac{1}{c} \frac{\dot{q}_k(t + r_{jk}/c)}{r_{jk}^2} \right]. \quad (19)$$

So the semidifference of the retarded and the advanced forces has the expression

$$\begin{aligned} e \sum_{k \neq j} \frac{E_{jk}^{\text{ret}} - E_{jk}^{\text{adv}}}{2} &= 2e^2 \sum_{k \neq j} \left[ \frac{q_k(t - r_{jk}/c) - q_k(t + r_{jk}/c)}{2r_{jk}^3} + \right. \\ &\quad \left. + \frac{1}{c} \frac{\dot{q}_k(t - r_{jk}/c) + \dot{q}_k(t + r_{jk}/c)}{2r_{jk}^2} \right], \end{aligned} \quad (20)$$

and it is immediately checked, using the Oseen identity, that along any normal-mode solution one has

$$e \sum_{k \neq j} \frac{E_{jk}^{\text{ret}} - E_{jk}^{\text{adv}}}{2} = -m\varepsilon \ddot{q}_j. \quad (21)$$

Due to the linearity of the equations of motion, this result can be extended to any combination of linear modes, and so one is led to the main result of the present Section, namely: In a generic solution of system (15) (16), the “reaction radiation force” acting on each dipole is exactly compensated by the sum of the semidifferences of the retarded and the advanced forces due to all the other dipoles, i.e. the relation (21) holds.

So we have the following situation. In the original definition of our model, the force acting on particle  $j$  had been defined, in the familiar way, as the Lorentz force (in the dipole approximation) due to the electromagnetic field. By the standard procedure of eliminating the field in the coupled equations of matter and field, such a force was then represented as the sum of the retarded forces due to all other particles  $k \neq j$ . On the other hand, such a resultant retarded force can be looked upon as being split into the combination (18) of the semisum and the semidifference of the retarded and the advanced forces due to all the other particles. But the Oseen identity in the form (21) then shows that the semidifferences just add up in such a way as to exactly cancel the reaction radiation term pertaining to particle  $j$ .

This has the important consequence that in the original system defining the model the radiation reaction term appearing in each equation can be dropped, provided that the r.h.s. be changed in a corresponding way, namely with each retarded force replaced by the corresponding semisum of retarded and advanced forces. So the original system of equation (15) can equivalently be rewritten in the form

$$m(\ddot{q}_j + \omega_0^2 q_j) = e \sum_{k \neq j} \frac{E_{jk}^{\text{ret}} + E_{jk}^{\text{adv}}}{2} . \quad (22)$$

## 6 The Oseen identity as a weak form of the Wheeler–Feynman identity

We now rewrite the Oseen identity (21) in a more perspicuous form. To this end we have to introduce the quantity  $[E_{jj}^{\text{ret}} - E_{jj}^{\text{adv}}]/2$ . Apparently, this is not defined, inasmuch as it involves two diverging terms. However, one immediately sees that such singularities are removable, that the quantity is correctly defined, and in fact one has

$$e \frac{E_{jj}^{\text{ret}} - E_{jj}^{\text{adv}}}{2} = m\epsilon \ddot{q}_j . \quad (23)$$

Indeed the actual original quantities of interest are the fields  $E_{(k)}^{\text{ret}}(\mathbf{x})$  and  $E_{(k)}^{\text{adv}}(\mathbf{x})$  “created” by particle  $k$  and evaluated at the current point  $\mathbf{x}$ , because the quantities entering the model are nothing but such fields evaluated at the equilibrium position  $\mathbf{r}_j$  of particle  $j$ , i.e.  $E_{jk}^{\text{ret}} = E_{(k)}^{\text{ret}}(\mathbf{r}_j)$ , and the corresponding advanced quantity. Now, evidently  $E_{(j)}^{\text{ret}}(\mathbf{x})$  diverges as  $\mathbf{x} \rightarrow \mathbf{r}_j$ , but from the explicit expression one immediately checks that the limit exists for the semidifference, and that its value is given according to (23). This in fact just is the nonrelativistic version of the result already found, in the relativistic case, by Dirac[1].

The conclusion is that, in virtue of (23), the Oseen identity (21) now reads

$$\sum_{k \in \mathbf{Z}} \frac{E_{jk}^{\text{ret}} - E_{jk}^{\text{adv}}}{2} = 0 , \quad j \in \mathbf{Z} . \quad (24)$$

We now come to the connection with the work of Wheeler and Feynman [4]. The authors point out that there exist two a priori different formulations of electrodynamics of point particles, namely what they call “*the theory of Schwarzschild and Fokker*” on the one hand, and the “*theory of Dirac*” on the other. The latter, which is the traditional one, includes the radiation reaction term  $-m\epsilon \ddot{q}_j$  and introduces retarded forces; the first one drops the radiation reaction term and introduces the semisum of the retarded and the advanced forces. In our model, such theories amount to nothing but

equations (22) and (15) respectively. The declared aim of Wheeler and Feynman (see page 170 of their paper) was to prove “*a complete equivalence between the theory of Schwarzschild and Fokker on the one hand and the usual formalism of electrodynamics (i.e. that of Dirac) on the other*”.

They are able to prove the equivalence by making use of an hypothesis, which they describe in physical terms as corresponding to the existence of an “*absorbing universe*”. In mathematical terms such an hypothesis is formulated as requiring the identical vanishing of the semidifference of the fields created by all the particles, i.e. the identity

$$\sum_{k \in \mathbf{Z}} \frac{E_{(k)}^{\text{ret}}(\mathbf{x}) - E_{(k)}^{\text{adv}}(\mathbf{x})}{2} = 0, \quad \mathbf{x} \in \mathbf{R}^3. \quad (25)$$

In fact, for the equivalence it is sufficient to assume that the above relation just holds at the positions of all the particles and not in the full space  $\mathbf{R}^3$ . Now, in our model we have shown that the identity in the latter weaker form is not an additional hypothesis, but rather a theorem. Thus the equivalence of the two formulations of electrodynamics of point particles according to Schwarzschild–Fokker and to Dirac is proven in our model.

We finally add a comment, concerning the way in which Wheeler and Feynman discuss the equivalence of the two formulations. They give four arguments, with headings “*The radiative reaction: derivation I, II, III, IV*”. The fourth “derivation” is essentially the one given here in Section 5 (apart from the fact that they take as a postulate the identity which we prove). On the other hand, in the previous “derivations” they attempt essentially at proving (instead of postulating) what we called the Oseen identity in its first form (21). So, it will not appear strange that we happened to understand the whole paper of Wheeler and Feynman, and in particular their “derivations”, only after we happened to prove the Oseen identity in our model. The conclusion is thus that we prove in a special model what they argument on general grounds. Conversely, this seems to be a strong indication that the Oseen identity might be proved, as a real theorem, for a much larger class of models.

## 7 Conclusions

So we have proven, at least for our particular linearized model of many-body matter radiation interaction, that there exist nonradiating normal modes, i.e. solutions to the equations of motion of the complete system, particles plus field, in which the mechanical energy of the particles remains constant, notwithstanding the fact that all particles perform accelerated motions. A preliminary analytical investigation shows that the same phenomenon occurs in a different model, in which, at variance with the present one, also the far fields decaying as  $1/r$  play a role. A natural guess seems to be that the

same should occur with a three dimensional crystal. What should occur for a disordered system or a gas, is instead, apparently, completely open. Another open question concerns the Wheeler–Feynman identity (25). Indeed the Oseen identity was shown here to be equivalent to a weak form of it, and thus naturally the question arises whether the Wheeler–Feynman identity itself, in its general form (25), holds.

Some further comments concerning the connection of the present work with the problem of a microscopic foundation of optics, especially for the theories of dispersion and of extinction of light are given in the appendix.

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## Appendix

Here, we first describe the way in which we conceived that nonradiating normal-mode solutions should exist for models of matter–radiation interaction; then we give some information of a historical character on the Oseen identity; finally we add some comments concerning the theories of dispersion and of extinction of light.

1. We were interested in the models of matter–radiation interaction that Planck had actually been considering as mimicking a black body. We thus found out that his models, although in principle involving  $N$  resonators, were actually dealing with a single resonator acted upon by an external field: indeed Planck explicitly made the assumption that the  $N$  resonators act “independently of one another”. At first sight, such an assumption might appear just an innocuous one, perhaps constituting an excessive simplification of the physical problem, but for the rest acceptable. Things changed however when we suddenly realized [8] that such a simplified model is actually inconsistent. Indeed, if the resonators are supposed to act incoherently, then, at a given point, the far fields radiated by each of them are easily seen to add up to give a divergent contribution (or a contribution proportional to the volume in the case of finite  $N$ ), so that one meets here with a paradox analogous to that of Olbers. Thus it seemed to us that the inconsistency could be removed only if the system of resonators coupled by radiation was considered in its globality, and special solutions were looked for, which present a coherent character, i.e. are in the form of normal modes. Indeed, with suitable coherent motions one might obtain that the multipoles of higher and higher orders are made to vanish as  $N$  is increased, so that the total emitted field would vanish in the limit  $N \rightarrow \infty$  ([9], see also [10]). These two arguments agree in indicating that a large number of particles should be considered if normal modes have to be looked for. The choice of taking an infinite number of them was just made in order to obtain a simpli-

fication in the mathematical treatment of the problem, as it occurs in the so called “thermodynamic limit” of statistical mechanics. This is the way we happened to find out the existence of normal modes. The core of the result is the identity which gives an exact compensation (or cancellation) of the reaction radiation term of each dipole, by a part of the sum of the retarded fields produced by all the other dipoles.

2. Having found such a result, we started a bibliographical research with the aim of learning whether something analogous was already known. Through the book of Born and Wolf [11] we went back to the celebrated papers of Ewald [12] and Oseen [13]. The best source of information proved however to be the long review written by Jaff  [14] for the *Handbuch der Experimentalphysik*. So we learned first of all that the kind of model considered by us (dipoles with mutual retarded electromagnetic interactions) was completely standard (apart from discussions about the choice of the reaction radiation term) in the many studies on the microscopic molecular foundation of optics, that were filling up a consistent part of the issues of *Annalen der Physik* in the years 1915–1925. It should be mentioned however that essentially all such papers (with the exception of that of Ewald, which on the other hand apparently might present serious drawbacks) actually had a somehow mixed character. Indeed, in estimating the total force acting on a single resonator due to all the other ones, use was made of an approximation in which the relevant sums were replaced by appropriate integrals, somehow in the spirit of continuum mechanics. Thus the problem could not be dealt with in a mathematically clearcut way, as needed when dealing with a theorem for a system of differential equations. From the review of Jaff  we also learned that our main identity producing the mentioned cancellation had already been introduced by Oseen [3] in a paper subsequent to the one mentioned by Born and Wolf. The proof was given however in the spirit quoted above of continuum mechanics, so that in particular it was not clear whether the identity applies only to crystals or also to gases, or even whether it holds at all. Furthermore it occurred that Oseen was making use of his result in a critique [15] to a previous theory of Planck on the dispersion of light. A long debate followed, in which the two questions, truth of the identity and soundness of Planck’s dispersion theory, were accomanated. The final conclusion, explicitly stated in the review of Jaff  (see page 266), was that the theory of Oseen was actually wrong (*irrig*). As a consequence, the identity itself was apparently discarded by the scientific community, and eventually forgotten. Because of all these considerations concerning the identity in question which we like to call the Oseen identity (i.e. lack of an explicit proof in a concrete model of differential equations, and its having been discarded by the scientific community), we came to the conclusion that our proof, produced as a theorem for a concrete model, might be worth of publication.

3. We finally add a further comment, concerning the connection of the present work with the problem of a microscopic foundation of optics, espe-

cially for the theories of dispersion and of extinction of light. At first sight one might be tempted to believe that the handbook of Born and Wolf did already say the last word, at least for what concerns the general aspects of the problem. But an accurate analysis shows that actually this is not the case. Indeed they do not deal with a clearly defined mathematical model, and somehow oscillate between a continuum phenomenological description of matter on the one hand, and the consideration of single dipoles on the other; moreover, they do not even introduce an actual dynamical equation for the dipoles. According to Born and Wolf, the dynamical foundation was given by Ewald [12] and by Oseen (in his first paper [13]). Now, in our opinion no one of these two works is consistent. Indeed they both neglect the radiation reaction term, and nevertheless pretend that normal modes do exist. But we have shown that, at least for a one-dimensional crystal, a part of the retarded forces acting on any given dipole due to all the other ones, just add up in such a way as to produce the “dissipative” term  $-m\epsilon\ddot{q}_j$ , which would exactly compensate the reaction radiation term, if this had originally been included in the model. This is indeed the reason for the very existence of normal modes in our model. But conversely, just for the same reason, normal modes cannot exist if the reaction radiation term had not been included in the model. In our opinion, there should be some mistake hidden in the two quoted works. It seems to us that the new relevant step after such works is just the one performed by Oseen in his subsequent paper [3], where the idea of the cancellation was introduced. Now, it happened that this second work of Oseen was finally discarded as wrong, and his proposal forgotten. On the other hand, the cancellation is proven here as a theorem in a concrete model. In our opinion, the status of the microscopic foundation of optics, which should lead to an explanation of the dispersion and the extinction of light in molecular terms, should perhaps be reconsidered.

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