

On the energy transfer to small scales in a discrete model of one-dimensional turbulence

A. Carati,¹ L. Galgani,¹ and F. Santolini²

¹*Dipartimento di Matematica, Università di Milano, Via Saldini 50, 20133 Milano, Italy*

²*Corso di Laurea in Fisica, Università di Milano, Via Celoria 12, 20133 Milano, Italy*

(Received 23 February 2009; accepted 1 June 2009; published online 24 June 2009)

We consider a discrete one-dimensional model that was investigated numerically by Daumont and Peyrard [Chaos **13**, 624 (2003)] as a model for turbulence in fluids, i.e., for the energy transfer from large to small scales. They found numerically that there exist two different regimes for the energy spectrum at high energies and low energies, respectively, and gave an analytical explanation for the high-energy spectrum. An analytical explanation is given here for the low-energy spectrum, which corresponds to the laminar regime. © 2009 American Institute of Physics.

[DOI: [10.1063/1.3156729](https://doi.org/10.1063/1.3156729)]

This paper aims at giving an analytical understanding of some numerical results found by Daumont and Peyrard in a series of papers (see Refs. 1 and 2), in which a particular one-dimensional lattice was studied as an extremely simplified tractable model for one-dimensional turbulence. This is based on the following idea. It is well known that in one-dimensional lattices, typically in the celebrated Fermi–Pasta–Ulam (FPU) system, the interaction between normal modes becomes negligible for low enough energies. So one can think that at low energies, such systems behave essentially in a linear way, especially for what concerns the high frequency modes. In fact, in this paper it will be shown that a purely linear analysis reproduces with a very good accuracy the numerical results found by Daumont and Peyrard¹ at low energies, i.e., in the “laminar regime.”

I. INTRODUCTION

In Ref. 2, Peyrard introduced a discrete mechanical model to study the phenomenon of energy transfer from large scales to small scales, which is of particular relevance for turbulence (see the references in the cited paper for a more comprehensive description of the model and the relations with the problem of turbulence). The model can be described as a linear chain of particles, each of which is acted upon by a nonlinear on-site force due to a Morse potential and also interacts linearly with its first neighbors. This actually amounts to a discrete version of an analog of the well known φ^4 model (see, for example, Refs. 3 and 4). To a such purely mechanical system, two further forces are added. The first one is a dissipation term acting on each particle, which is taken proportional to the particle velocity as to mimic a damping due to viscosity. The second one is a time-periodic forcing term which mimics an energy pumping at a large scale.

The problem is then to investigate in which way does energy, injected in the system by the time-periodic external force, flow to the small scales. A large part of the subsequent paper by Daumont and Peyrard¹ is devoted to investigate this

feature in terms of the spectrum: The authors compute it by solving numerically the equations of motion and then try to explain it by analytical means. Let us recall that the spectrum is defined as the curve which gives the time-averaged energies of the normal modes of the system versus wave number. Here, by normal modes we always mean those of the purely mechanical system (with no damping and forcing) linearized about the equilibrium position of the system.

First of all, the authors found that as time increases, the spectrum approaches a definite limit, which depends on the intensity of the forcing term. Moreover, they found that the form of the spectrum decays as an inverse power of the wave number if the forcing is below a certain threshold, while it decays exponentially fast above such a threshold. Then, the authors explain through some analytical arguments the exponential behavior at high forcing but are unable to find an analytical explanation for the power law behavior at low energies. In their own words, “This simple analysis ... does not explain why the energy exchange between the modes leads to the observed power law.”

The main result of the present paper is to show that a purely linear analysis, which might be expected to be the suited one at low enough energies, actually gives for the spectrum a power law decay and indeed with a power which fits very well the one found numerically by Daumont and Peyrard. Such an analysis is given in Sec. III. The model is recalled in Sec. II together with the results of some numerical computations which show that the linear approximation is actually justified at low energies. The conclusions are then given in Sec. IV.

II. THE MODEL

As mentioned above, the mechanical part of the model can be described as a chain of a finite number of equal particles, each of which is acted upon by an on-site Morse potential $V(y) = \Omega^2(e^{-y} - 1)^2$, where Ω is a positive constant, and furthermore interacts with its first neighbors through a linear force. For reasons that will appear clear soon, it is

convenient to label the particles with an index (n) ranging from 0 to $N+3$. The Hamiltonian of the model is thus

$$H = \sum_{n=0}^{N+3} \left[\frac{1}{2} \left(\frac{dy_n}{dt} \right)^2 + \frac{1}{4} K [(y_n - y_{n-1})^2 + (y_{n+1} - y_n)^2] + \Omega^2 (e^{-y_n} - 1)^2 \right], \quad (1)$$

where K is a positive constant. Here y_n with $n=0, \dots, N+3$ is the displacement of the n th particle from its equilibrium position, and one deals with fixed ends, i.e., one takes $y_0 = y_{N+3} = 0$. The linearized Hamiltonian can be analyzed in terms of the linear normal modes, which are defined as usual by

$$A_k = \sqrt{\frac{2}{N+3}} \sum_{n=1}^{N+2} y_n \sin\left(\frac{\pi k n}{N+3}\right), \quad k = 1, \dots, N+2, \quad (2)$$

the linear frequencies ω_k of the system being given by the dispersion relation

$$\omega_k^2 = 2\Omega^2 + 4K \sin^2\left(\frac{\pi k}{2(N+3)}\right). \quad (3)$$

In Ref. 1, a damping force (with a positive parameter ν)

$$F_n = \nu(\dot{y}_{n+1} + \dot{y}_{n-1} - 2\dot{y}_n)$$

acting on each particle was added in order to simulate the standard viscous damping, which is present in fluids. We shall dispense with this term in the rest of the paper for reasons which will appear clear in Sec. III. Here we just mention in passing that the authors of Ref. 1 themselves found out that such a term has no substantial influence on the spectrum, inasmuch as spectra computed for very different values of the “viscosity coefficient” ν , when stabilized after a sufficiently long time, turn out to essentially coincide.

In order to simulate a time-periodic force acting on the system at a large scale, Daumont and Peyrard made the choice of assigning the motion of particles 1 and $N+2$, i.e., of assigning the functions $y_1(t)$ and $y_{N+2}(t)$. Precisely, they made the choice

$$\begin{aligned} y_1(t) &= A \sin\left(\frac{2\pi}{N+3}\right) \sin(\omega_2 t), \\ y_{N+2}(t) &= -A \sin\left(\frac{2\pi}{N+3}\right) \sin(\omega_2 t), \end{aligned} \quad (4)$$

where ω_2 is the frequency of the second normal mode given by formula (3). These are the motions that the two considered particles would perform (in the linearized system with damping neglected) if only the second mode was excited. The aim is then to look at the way in which energy flows from the second normal mode to the higher-frequency ones. One might be tempted to understand this energy flowing as due to the nonlinear coupling among modes (and to the possible presence of damping), as is the case for fluids. We will show instead that this flowing is a purely linear phenomenon. The nonlinear coupling between the modes rather has, at least in certain cases, the inverse effect of preventing the

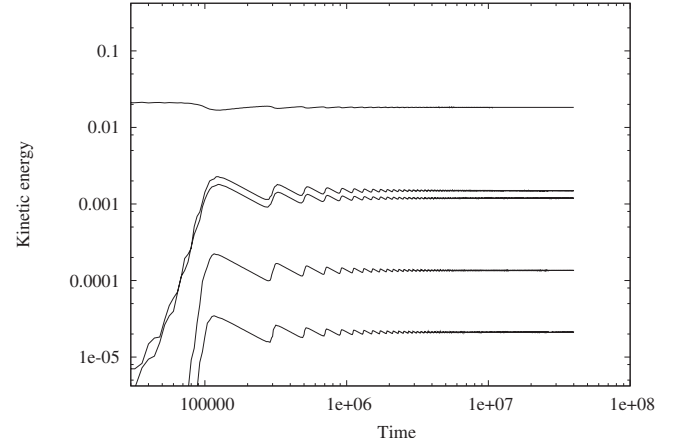


FIG. 1. Time averages (up to time t) of the kinetic energies of the first five normal modes vs time t for a small specific energy E/N , namely, $E/N = 1.6398 \times 10^{-4}$ and $N=256$, with the second mode only initially excited.

flow of energy. This is actually in agreement with the fact, observed and explained by Daumont and Peyrard, that the spectrum decays exponentially fast in the case of a high energy, i.e., when the nonlinear coupling prevails.

We finally close this section by showing that the Daumont and Peyrard model, in its purely mechanical version defined by Hamiltonian (1), i.e., with no damping and forcing, exhibits the so-called FPU phenomenon (see, for example, Refs. 5 and 6). This concerns the motions corresponding to initial data in which only a few low-frequency normal modes are initially excited and amounts to the following. There exists an energy threshold E_c , such that if the initial energy E is less than E_c , then the energy remains confined to a certain packet of low-frequency modes, while substantially no energy flows to the high-frequency ones, whose energies remain essentially constant. In this sense one can say that below such a threshold, the normal modes (or rather the high-frequency ones) are not actually interacting, and the behavior of the system is essentially linear. This fact is illustrated in Figs. 1 and 2, where the time averages (up to time t) of the normal mode kinetic energies are reported versus time

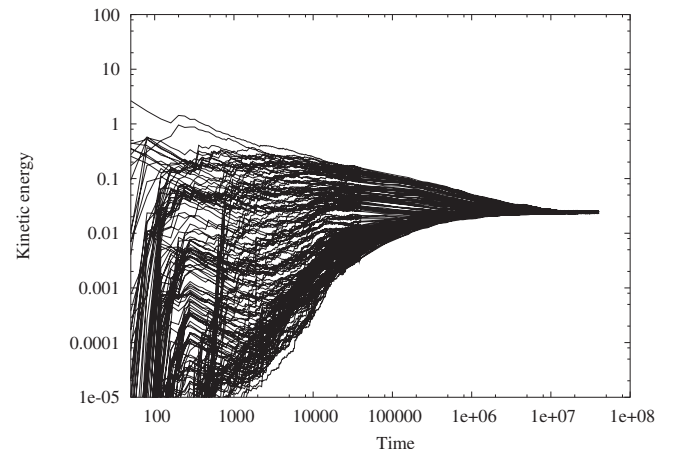


FIG. 2. Same as Fig. 1 for a large specific energy, namely, for $E/N = 4.198 \times 10^{-2}$. The approach to equilibrium (energy equipartition) is clearly exhibited.

t in log-log scale for a case of low energy ($E/N=1.6398 \times 10^{-4}$) and a case of high energy ($E/N=4.198 \times 10^{-2}$), respectively, with $N=256$ (the same number considered by Daumont and Peyrard). In the case of low energy one sees that not only energy equipartition is not attained within the observation time but also the system appears to have actually relaxed to some state in which the time averages of the single normal modes remain constant somehow, as would occur for the linearized system. Instead, in the case of high energy one clearly sees that equipartition is already attained (i.e., the time averages of the energies of all modes attain the same value) after $t=10^7$. This shows that the present model also presents what is usually called the FPU phenomenon. Further details can be found in Ref. 6.

The equations of motions corresponding to Hamiltonian (1) were integrated using the standard leap-frog method with a time step $h=0.08$, which insures an energy conservation of 1%.

III. LINEAR ANALYSIS

In the present section we consider the system linearized about its equilibrium position and solve the problem of finding the motions $y_n(t)$ of particles $n=2, \dots, N+1$ when the motions $y_1(t)$ and $y_{N+2}(t)$ of particles 1 and $N+2$ are assigned. In particular, we give an explicit expression for the functions $y_n(t)$ when the motions $y_1(t)$ and $y_{N+2}(t)$ have the form

$$\begin{aligned} y_1(t) &= A \sin\left(\frac{\pi}{N+3}\right) \sin(\omega_1 t), \\ y_{N+2}(t) &= A \sin\left(\frac{(N+2)\pi}{N+3}\right) \sin(\omega_1 t), \end{aligned} \quad (5)$$

which corresponds to exciting mode 1 in place of mode 2.

The general method consists of determining a change in coordinates (actually a simple translation will suffice) such that particles 1 and $N+2$ become fixed, and the system just appears as subject to a time-dependent forcing. Then one can pass to the normal modes of the corresponding chain with fixed ends, so that each normal mode satisfies the equation of a harmonic oscillator subject to a time-dependent forcing. Solving the corresponding trivial equation and going back to the original variables $y_n(t)$, the spectrum can be finally computed.

We now implement such a method for a forcing given by Eq. (5). Performing the translation

$$x_n(t) = y_n(t) - A \sin\left(\frac{\pi}{N+3}\right) \sin(\omega_1 t) \quad (6)$$

with $n=1, \dots, N+2$, one obtains $x_1=x_{N+2}=0$, while for the variables x_n ($n=2, \dots, N+1$) one gets the equations

$$\begin{aligned} \ddot{x}_n &= K(x_{n+1} + x_{n-1} - 2x_n) - 2\Omega^2 x_n \\ &+ (\omega_1^2 - 2\Omega^2) A \sin\left(\frac{\pi}{N+3}\right) \sin(\omega_1 t). \end{aligned} \quad (7)$$

Introducing the normal modes a_k , which now refer to a chain of $N+2$ particles with fixed ends by

$$a_k = \sqrt{\frac{2}{N+1}} \sum_{n=1}^N x_n \sin\left(\frac{\pi k n}{N+1}\right), \quad k=1, \dots, N, \quad (8)$$

one finds for each a_k the equation of a harmonic oscillator with an assigned forcing, i.e.,

$$\ddot{a}_k + \tilde{\omega}_k^2 a_k = \frac{(\omega_1^2 - 2\Omega^2) A \sin\left(\frac{\pi}{N+3}\right) \cot\left(\frac{\pi k}{2(N+1)}\right)}{\sqrt{\frac{2}{N+1}}} \sin(\omega_1 t), \quad (9)$$

where $\tilde{\omega}_k$ are now the frequencies for a system of $N+2$ particles with fixed ends, which are given by

$$\tilde{\omega}_k^2 = 2\Omega^2 + 4K \sin^2\left(\frac{\pi k}{2(N+1)}\right).$$

Notice that the angular frequencies $\tilde{\omega}_k$ and ω_k are slightly different because they refer to chains of slightly different lengths. Nevertheless, it will be seen that such a difference has a relevant impact on the form of the spectrum, inasmuch as it has the effect of producing in its power decay an exponent which actually fits the numerical results.

The solution of Eq. (9) is given by

$$a_k(t) = \frac{(\omega_1^2 - 2\Omega^2) A \sin\left(\frac{\pi}{N+3}\right) \cot\left(\frac{\pi k}{2(N+1)}\right)}{\sqrt{\frac{2}{N+1}} (\tilde{\omega}_k^2 - \omega_1^2)} \sin(\omega_1 t). \quad (10)$$

Notice that a solution of the corresponding homogeneous equation should have been added, but we put it equal to zero just to mimic the effect of a damping (however small), which was present in the original model and was neglected here. We note in passing that the damping would change the factor $\tilde{\omega}_k^2 - \omega_1^2$ in the denominator of Eq. (10) into the complex expression $\tilde{\omega}_k^2 - \omega_1^2 + i\nu\omega_1$ and, analogously, the factor $\omega_1^2 - 2\Omega^2$ of the numerator into $\omega_1^2 - 2\Omega^2 + i\nu\omega_1$. So the solution (10) is the asymptotic one for large times. This produces a corresponding “final” spectrum in which the damping coefficient did disappear, in agreement with the fact that also the numerical spectrum discussed in Ref. 1 is an asymptotic one, which was found not to depend on the damping coefficient ν . Inverting the transformation (8) for the functions $y_n(t)$ ($n=2, \dots, N+1$) one finds the expression

$$y_n(t) = A \sin\left(\frac{\pi}{N+3}\right) \sin(\omega_1 t) + \sum_{k=1}^N a_k \sin\left(\frac{\pi k n}{N+1}\right).$$

Having determined the motions $y_n(t)$ of all particles (remember that $y_0=y_{N+3}=0$) one can now compute the normal modes of the original chain of $N+4$ particles. In particular, one can compute the kinetic energies E_l^K of the modes and their time-averages \overline{E}_l^K (i.e., the spectrum), which are given by

$$\begin{aligned} \overline{E}_l^K = & \frac{1}{2(N+3)} \left\{ \omega_1^2 D^2 \cot^2 \left(\frac{\pi l}{2(N+3)} \right) + 2D\omega_1^2 \cot^2 \left(\frac{\pi l}{2(N+3)} \right) \sum_{k=1, k \text{ odd}}^N \left[B_k \sum_{n=1}^N \left(\sin \left(\frac{\pi kn}{N+1} \right) \sin \left(\frac{\pi l(n+1)}{N+3} \right) \right) \right] \right. \\ & \left. + \omega_1^2 \sum_{k=1, k \text{ odd}}^N \left[B_k \sum_{n=1}^N \left(\sin \left(\frac{\pi kn}{N+1} \right) \sin \left(\frac{\pi l(n+1)}{N+3} \right) \right) \right]^2 \right\}, \end{aligned} \quad (11)$$

where

$$D = A \sin \left(\frac{\pi}{N+3} \right)$$

and

$$B_k = \frac{(\omega_1^2 - 2\Omega^2)D \cot \left(\frac{\pi k}{2(N+1)} \right)}{\sqrt{\frac{2}{N+1}}(\tilde{\omega}_k^2 - \omega_1^2)}.$$

This expression is rather cumbersome, but it can easily be estimated numerically. The spectrum, i.e., \overline{E}_l^K versus $q_l = l/(N+3)$, is reported (in log-log scale) in Fig. 3, where the power decay is clearly exhibited at least for not too large q . In the figure there is also the curve $E_l^K = Aq_l^{-2.193}$, which is the fit obtained in Ref. 1. The agreement seems to be very good. In the conclusion some comments will be added to the behavior of the spectrum for the high values of q .

There remains the problem of explaining analytically how it occurs that expression (11) actually leads to an exponent very near to the value of 2. This is achieved by approximating the sums entering Eq. (11) through corresponding integrals, a procedure which is asymptotically exact in the

limit of large N . In principle the computations are simple because the integrals can be computed through the standard residue technique. One thus obtains rather cumbersome expressions, which can actually be estimated, giving the final result

$$|E_l^K| \approx C/l^2 \quad (12)$$

with a positive constant C . This is a clear indication that a power decay with exactly the exponent 2 should be expected to occur in the limit of an infinitely long chain. This might be confirmed by numerical computations involving increasingly large numbers N . The analytical details leading to estimate (12) are quite long and not particularly significant, so they are omitted here.

IV. CONCLUSIONS

So we have shown how the power decaying of the spectrum in the Daumont and Peyrard model at low energies is very well described by the corresponding linearized system, in which the assigned motion of the extreme particles has just the effect of forcing independently each mode. In particular the agreement is very good for not large values of q .

For what concerns the high q , it turns out that the spectrum falls off faster than a power, as one can check both from our Fig. 3 and from Fig. 2 of Ref. 1. Here, checking the agreement is a much more difficult problem because Daumont and Peyrard neither do study the spectrum in this region nor do they give the raw data. It seems to us that in this region the agreement is not as good as in the power decay region. Perhaps, the reason is that in this extreme region, the small contribution due to the nonlinearities is non-negligible with respect to the linear one because of the smallness of the linear contribution itself. We leave for a future work a further study of this interesting problem.

As the spectrum at high energies is well described in terms of the breather mechanism, as explained by Daumont and Peyrard, it might be interesting to explore an intermediate energy region in which the modes do exchange energy and the breather mechanism is not yet effective.

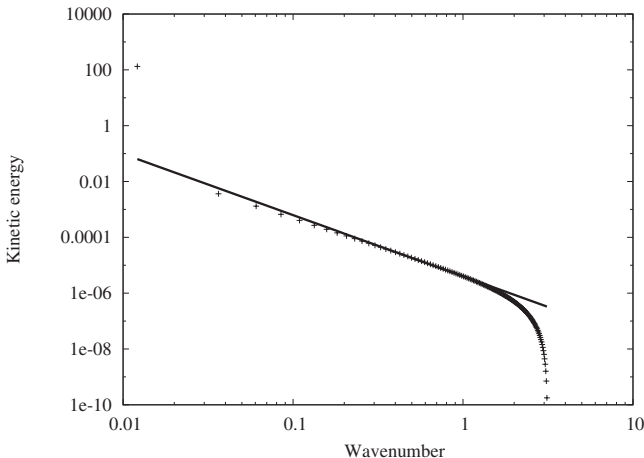


FIG. 3. \overline{E}_l^K vs $q_l = l/(N+3)$ in log-log scale for $A=0.001$ and $N=256$. Continuous line is the curve $E_l^K = Aq_l^{-2.193}$.

¹I. Daumont and M. Peyrard, *Chaos* **13**, 624 (2003).

²M. Peyrard, *Physica D* **193**, 265 (2004).

³E. Fucito, F. Marchesoni, E. Marinari, G. Parisi, L. Peliti, S. Ruffo, and A. Vulpiani, *J. Phys. (France)* **43**, 707 (1982).

⁴G. Parisi, *Europhys. Lett.* **40**, 357 (1997).

⁵A. Carati, L. Galgani, and A. Giorgilli, *Chaos* **15**, 015105 (2005).

⁶G. Benettin, A. Carati, L. Galgani, and A. Giorgilli, *The Fermi-Pasta-Ulam Problem: A Status Report*, Lecture Notes in Physics Vol. 728, edited by G. Gallavotti (Springer-Verlag, Berlin, 2007).