# Exponentially long stability times for a nonlinear lattice in the thermodynamic limit 

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#### Abstract

In this paper, we construct an adiabatic invariant for a large 1-d lattice of particles, which is the so called Klein Gordon lattice. The time evolution of such a quantity is bounded by a stretched exponential as the perturbation parameters tend to zero. At variance with the results available in the literature, our result holds uniformly in the thermodynamic limit. The proof consists of two steps: first, one uses techniques of Hamiltonian perturbation theory to construct a formal adiabatic invariant; second, one uses probabilistic methods to show that, with large probability, the adiabatic invariant is approximately constant. As a corollary, we can give a bound from below to the relaxation time for the considered system, through estimates on the autocorrelation of the adiabatic invariant.


## 1 Introduction

One of the open problems of Hamiltonian perturbation theory is how to extend to infinite dimensional systems, at a finite specific energy (or temperature), the results known for systems with a finite number of degrees of freedom. Indeed there exist results both for infinite systems such as partial differential equation (see, for example, [1, 2]) or on infinite lattice systems (see [3, 4]), but only for a finite total energy of the sistem, i.e., at zero temperature.

In the present paper we provide perturbation estimates on the so called Klein Gordon lattice in the thermodynamic limit, at a finite temperature,

[^0]by controlling, in place of the usual $L^{\infty}$ norm, the $L^{2}$ norm relative to the Gibbs measure. If we denote by $H$ the Hamiltonian of the system and by $\mathcal{M}$ the corresponding phase space, the Gibbs measure is defined by
\[

$$
\begin{equation*}
\mu(\mathrm{d} x) \stackrel{\text { def }}{=} \frac{\exp (-\beta H(x))}{\mathcal{Z}(\beta)} \mathrm{d} x \tag{1}
\end{equation*}
$$

\]

where $\mathcal{Z}(\beta) \stackrel{\text { def }}{=} \int_{\mathcal{M}} \exp (-\beta H(x)) \mathrm{d} x$ is the partition function, and $\beta>0$ the inverse temperature.

We construct an adiabatic invariant whose time derivative has an $L^{2}$ norm exponentially small in the perturbation parameters (see Theorem 1 ). The construction of the adiabatic invariant is standard (see [5]), but the estimate of its time derivative in the $L^{2}$ norm involves some probabilistic techniques, which have been developed in the frame of statistical mechanics. In fact, since $L^{2}$ is not a Banach algebra, the usual scheme of perturbation estimates cannot be implemented. So the use of the algebra property is replaced here by a control of the decay of spatial correlations between the sites of the lattice, making use of techniques introduced by Dobrushin (see [6]). In particular, we are able to show that, for lattices in any dimension with finite range interaction (i.e., in which each particle interacts only with a finite numbers of neighbouring ones) the spatial correlations decay exponentially fast with the distance. This requires also an estimate on the marginal probability densities induced by the measure $\mu$ on subsystems of finite size: this is done by adapting to lattices the techniques introduced by Bogolyubov et al. (see [7]) in interacting gas theory.

The paper is organized as follows. The main result on the considered model, namely the construction of an adiabatic invariant in the thermodynamic limit, is stated in Section 2 (Theorem 1), togheter with two corollaries concerning a control on the time evolution of the adiabatic invariant and a lower bound to its time autocorrelation. Then, in Section 3, we present the scheme of the proof of Theorem 1, whereas the fundamental ingredients of the proof are separately given in the subsequent three sections. The first one (Section 4) concerns perturbation techniques and deals with the formal construction of the adiabatic invariant. The other two sections have a probabilistic nature: the estimate of the marginal probability is given in Section 5 together with the estimate of the norm of the time derivative of the adiabatic invariant. In Section 6, we state Theorem 2 in which the estimate of the spatial correlations is given, which enables us to give an estimate on the variance of the adiabatic invariant. The proof of Theorem 2 requires to apply a technique due to Dobrushin and Pechersky (see [8]), and is reported
in Appendix B.3. In Section 7 we discuss how a lower bound on the time autocorrelation provides information on the relaxation time to equilibrium. The conclusions follow in Section 8. Most of the proofs of a more technical character are given in two appendices.

## 2 Stabiliy estimate in the Klein Gordon lattice

In the literature, as a prototype of several models, the so called Klein Gordon lattice is studied (see [9]-[14]). From a physical point of view, it mimics a chain of particles, each free to move about a site of a lattice, subjected both to an on-site restoring nonlinear force and to a linear coupling with the nearest neighbours. It can also be seen as a discretization of the onedimensional $\Phi^{4}$ model, which plays a major role in field theory.

The Hamiltonian of such a system, in suitably rescaled variables, can be written as $H=H_{0}+H_{1}$, in which

$$
\begin{equation*}
H_{0} \stackrel{\text { def }}{=} \sum_{i=1}^{N} \omega\left(\frac{p_{i}^{2}}{2}+\frac{q_{i}^{2}}{2}\right) \quad \text { and } H_{1} \stackrel{\text { def }}{=} \varepsilon \sum_{i=1}^{N-1} \frac{q_{i} q_{i+1}}{\omega}+\sum_{i=1}^{N} \frac{q_{i}^{4}}{4 \omega^{2}}, \tag{2}
\end{equation*}
$$

where $p=\left(p_{1}, \ldots, p_{N}\right)$ and $q=\left(q_{1}, \ldots, q_{N}\right)$ are canonically conjugated variables in the phase space $\mathcal{M}$, and $\varepsilon$ is a positive parameter, while $\omega$ is defined by $\omega \stackrel{\text { def }}{=} \sqrt{1+2 \varepsilon}$. Since we don't want to face in this paper the problem of small divisors, which typically arises in perturbation theory, we confine ourselves to the case of small $\varepsilon$, i.e, of small coupling between the sites.

We aim at showing that, for small enough $\varepsilon$ and sufficiently large $\beta$, there exists an adiabatic invariant for $H$ (see Theorem 1 below). To come to a precise statement, we need some preliminaries.

As usual, $\langle X\rangle$ will denote the mean value of a dynamical variable $X$ with respect to the Gibbs measure $\mu$ relative to the given Hamiltonian $H$ at a given $\beta$, i.e.,

$$
\langle X\rangle \stackrel{\text { def }}{=} \int_{\mathcal{M}} X(x) \mu(\mathrm{d} x) .
$$

The $L^{2}(\mathcal{M}, \mu)$ norm of $X$ is then $\|X\| \stackrel{\text { def }}{=} \sqrt{\left\langle X^{2}\right\rangle}$ and its variance $\sigma_{X}^{2}$ is defined according to $\sigma_{X}^{2} \stackrel{\text { def }}{=}\left\langle X^{2}\right\rangle-\langle X\rangle^{2}$. Finally, we also recall that the correlation coefficient of two dynamical variables $X$ and $Y$ is

$$
\begin{equation*}
\rho_{X, Y} \stackrel{\text { def }}{=} \frac{\langle X Y\rangle-\langle X\rangle\langle Y\rangle}{\sigma_{X} \sigma_{Y}}, \tag{3}
\end{equation*}
$$

and that $X$ and $Y$ are said to be uncorrelated if $\rho_{X, Y}=0$.
We can now state our main theorem, in which $[\cdot, \cdot]$ denotes Poisson bracket,

Theorem 1 (Estimate on the adiabatic invariant) There exist positive constants $\varepsilon^{*}, \kappa$, independent of $N$, such that if $\varepsilon<\varepsilon^{*}$ and $\beta>\varepsilon^{-1}$, then there exists a polynomial function $\bar{X}$ uncorrelated with $H$ such that

$$
\begin{equation*}
\frac{\|[\bar{X}, H]\|}{\sigma_{\bar{X}}} \leq \exp \left[-\left(\frac{1}{\kappa\left(\varepsilon+\beta^{-1}\right)}\right)^{1 / 4}\right] \stackrel{\text { def }}{=} \frac{1}{\bar{t}} . \tag{4}
\end{equation*}
$$

Remark. We require $\bar{X}$ to be uncorrelated with $H$ in order that our adiabatic invariant be sufficiently different from the Hamiltonian, which is obviously a constant of motion.
Before the proof, we point out immediately that this theorem has two relevant (and strictly related) consequences on the time evolution of the dynamical variable $\bar{X}$. They will make clear in which sense $\bar{t}$ at the r.h.s. of (4) can be seen as a stability time. The first consequence (Corollary 1) concerns the probability $\mathbf{P}$ that the value of the variable $\bar{X}$ changes significantly from its original value. Indeed, it entails that the probability of such a change is practically negligible if $t<\bar{t}$. The second consequence (Corollary 2) is a lower bound on the time autocorrelation of $\bar{X}$. We take here as definition of time autocorrelation of a dynamical variable the following one:

$$
C_{X}(t) \stackrel{\text { def }}{=} \rho_{X_{t}, X}
$$

where $X_{t}(x) \stackrel{\text { def }}{=} X\left(\Phi^{t} x\right), \Phi^{t}$ is the flow generated by $H$ and $\rho$ the correlation coefficient defined by (3). We have chosen to rescale the usual definition, dividing it by $\sigma_{X}^{2}$, because the variance of $X$ is the natural scale of its autocorrelation, since $C_{X}(0)=1$ and the inequality $\left|C_{X}(t)\right| \leq 1$ holds for any $t$.

We report here both results, which follow from Theorem 1 and from the simple estimate $\left\|X_{t}-X\right\|^{2} \leq t^{2}\|[X, H]\|^{2}$. The latter can be found in the proof of Theorem 1 of paper [15] and is however reported here in Section 7 in order to make that section self contained.

Corollary 1 In the hypotheses of Theorem 1, for any $\lambda>0$ one has

$$
\mathbf{P}\left(\left|\bar{X}_{t}-\bar{X}\right| \geq \lambda \sigma_{\bar{X}}\right) \leq \frac{1}{\lambda^{2}}\binom{t}{\bar{t}}^{2}
$$

Corollary 2 In the hypotheses of Theorem 1, one has

$$
C_{\bar{X}}(t) \geq 1-\frac{1}{2}\left(\frac{t}{\bar{t}}\right)^{2}
$$

Remark. We observe that the notion of stability time for dynamical systems is not unambiguously defined. In Section 7 we will provide a definition of "relaxation time" in terms of time autocorrelation of dynamical variables, which seems to us significant from a physical point of view. With such a definition, Theorem 1 turns out to mean that the "relaxation time" is exponentially long in the perturbation parameters.

## 3 Scheme of the proof of Theorem 1

First we use a variant of the classical construction scheme of approximate integrals of motion (see [16]) in order to perform the construction of the adiabatic invariant as a formal power series. Precisely, we use the scheme developed by Giorgilli and Galgani for a direct construction of integrals of motion (see [5] and Section 4 for the actual implementation). It is well known that the series thus obtained are, in general, divergent, so that the standard procedure consists in using as approximate integral of motion a truncation of the series. Denoting by $Y_{n}$ the series truncated at order $2 n+2$, it turns out that it has the form

$$
\begin{equation*}
Y_{n} \stackrel{\text { def }}{=} H_{0}+\sum_{j=1}^{n} P_{j}(p, q) \tag{5}
\end{equation*}
$$

where $P_{j}$ are suitable polynomials. In order to make such a quantity uncorrelated with $H$, it is convenient to consider $X_{n} \stackrel{\text { def }}{=} Y_{n}-H$ instead of $Y_{n}$ itself.

In order to make the construction rigorous, one has to add rigorous estimates of the variance $\sigma_{X_{n}}^{2}$ of $X_{n}$, and of the $L^{2}$ norm of $\left[X_{n}, H\right]$. The first step to get such estimates consists in controlling the structure of the polynomials $P_{j}$ (which, in particular, contain only finite range couplings) and the size of their coefficients. This is done recursively, by a variant of the technique of the paper [17], which is implemented in Section 4 (see Lemma 2). We emphasize that, at variance with the original paper, we obtain here estimates independent of the number of degrees of freedom.

Then, due to the structure of the polynomials $P_{j}$, to get the needed $L^{2}$ estimates one has to compute the $L^{2}$ norm with respect to the Gibbs measure
of the monomials appearing in $P_{j}$. The key step for this computation consists in giving an upper bound independent of $N$ to the marginal probabilities of the Gibbs measure. Such an estimate is obtained by adapting techniques developed by Bogolyubov and Ruelle (see [7] and [18]) and is reported in Lemma 4 of Section 5. One thus obtains the following bound

$$
\begin{equation*}
\left\|\dot{X}_{n}\right\| \leq \sqrt{N}(\sqrt{2} \beta)^{-1}(n!)^{4}\left(\beta^{-1}+\varepsilon\right)^{n} \kappa_{1}^{n}, \tag{6}
\end{equation*}
$$

which is valid for a suitable constant $\kappa_{1}>0$, provided $\varepsilon$ is small enough and $\beta$ large enough (see Lemma 3 of Section 5).

We emphasize the presence of the factor $\sqrt{N}$ and that $\kappa_{1}$ is independent of $N$. It will be shown that actually the l.h.s. of (6) is of order $\sqrt{N}$ even if it is the square root of a sum of $O\left(N^{2}\right)$ terms. This is due to the fact that most of the terms have zero mean because the measure is even in $p$ and furthermore the $p$ 's are independent variables.

To get the Theorem, one also needs an estimate of $\sigma_{X_{n}}$ from below. This is obtained in two steps, which are based on the remark that $\sigma_{X_{n}} \geq \sigma_{X_{1}}-\sigma_{\mathcal{R}}$, where $\mathcal{R} \stackrel{\text { def }}{=} X_{n}-X_{1}$ is a remainder.

First we compute explicitly $X_{1}$ and estimate from below $\sigma_{X_{1}}$, obtaining a bound proportional to $\sqrt{N}$. Then, we estimate from above $\sigma_{\mathcal{R}}$. Precisely, we use techniques introduced by Dobrushin in papers $[6,8]$ to show that $\sigma_{\mathcal{R}}$ behaves as $\sqrt{N}$ (see Lemma 8 of Section 6). We remark that this is the analogue of the law of large numbers. We recall that Dobrushin's techniques enable us to show that spatial correlations between variables pertaining to different lattice sites decrease exponentially with the distance between the sites, so that the monomials appearing in $P_{j}$ are essentially independent, and the variance of $P_{n}$ is essentially the sum of the variances of each monomial. This leads to Lemma 7 of Section 6, which shows that, for small enough $\varepsilon$ and large enough $\beta$, for $n<\kappa_{2}^{-1 / 4}\left(\varepsilon+\beta^{-1}\right)^{-1 / 4}$ there holds

$$
\begin{equation*}
\sigma_{X_{n}} \geq \sqrt{N}\left(\varepsilon+\beta^{-1}\right) /(8 \beta) \tag{7}
\end{equation*}
$$

where again $\kappa_{2}$ is a positive constant.
Then one finds the optimal $n$, call it $\bar{n}$, such that the ratio $\left\|\left[X_{\bar{n}}, H\right]\right\| / \sigma_{X_{\bar{n}}}$ takes the minimal value. Notice that, as $n$ belongs to a bounded domain, the minimum can be attained at the boundary. The optimization is immediately done, once the estimates are given both for the $L^{2}$ norm $\left\|\left[X_{n}, H\right]\right\|$ of the time-derivative of the quasi integral of motion $X_{n}$, and for its variance $\sigma_{X_{n}}^{2}$. Then, the function $\bar{X}$ satisfying (4) of Theorem 1 is simply given by $\bar{X} \stackrel{\text { def }}{=} X_{\bar{n}}-H \rho_{X_{\bar{n}}, H} \sigma_{X_{\bar{n}}} / \sigma_{H}$. The identity $\sigma_{\bar{X}}^{2}=\left(1-\rho_{X_{\bar{n}}, H}^{2}\right) \sigma_{X_{\bar{n}}}^{2}$, together
with the upper bound to $\rho_{X_{\bar{n}}, H}$ given by Lemma 7 , enables us to extend all conclusions from $X_{\bar{n}}$ to $\bar{X}$.

## 4 Construction of the adiabatic invariant

Following [5], we look for the formal integral of motion by looking for a sequence of polynomials $\chi=\left\{\chi_{s}\right\}_{s \geq 1}$ such that

$$
\begin{equation*}
\left[H, T_{\chi} H_{0}\right]=0 \quad \text { at any order, } \tag{8}
\end{equation*}
$$

where $T_{\chi}$ is a linear operator, whose action on a polynomial function $f$ is formally defined by ${ }^{1}$

$$
\begin{equation*}
T_{\chi} f \stackrel{\text { def }}{=} \sum_{s \geq 0}\left(T_{\chi} f\right)_{s}, \quad \text { with }\left(T_{\chi} f\right)_{0} \stackrel{\text { def }}{=} f, \quad\left(T_{\chi} f\right)_{s} \stackrel{\text { def }}{=} \sum_{j=1}^{s} \frac{j}{s}\left[\chi_{j},\left(T_{\chi} f\right)_{s-j}\right] . \tag{9}
\end{equation*}
$$

Inserting the expansion of $T_{\chi} H_{0}$ and $H$ in (8) and equating terms of equal order one gets the system

$$
\begin{equation*}
\Theta_{0}=H_{0}, \quad \Theta_{s}-L_{0} \chi_{s}=\Psi_{s} \quad \text { for } s>0 \tag{10}
\end{equation*}
$$

where

$$
\begin{align*}
& \Psi_{1} \stackrel{\text { def }}{=} H_{1}, \\
& \Psi_{s} \stackrel{\text { def }}{=}-\sum_{l=1}^{s-1} \frac{l}{s}\left[\chi_{l},\left(T_{\chi} H_{0}\right)_{s-l}\right]-\sum_{l=1}^{s-1}\left(T_{\chi} \Theta_{l}\right)_{s-l} \quad \text { for } s \geq 2, \tag{11}
\end{align*}
$$

$L_{0} \stackrel{\text { def }}{=}\left[H_{0}, \cdot\right]$ is the homological operator and (10) has to be read as an equation for the unknowns $\chi_{s}, \Theta_{s}$, which have to belong, respectively, to the range and to the kernel of the operator $L_{0}$. By defining the projections $\Pi_{\mathcal{N}}, \Pi_{\mathcal{R}}$, respectively on the kernel $\mathcal{N}$ and on the range $\mathcal{R}$ of $L_{0}$, one thus determines recursively

$$
\begin{equation*}
\chi_{s}=-L_{0}^{-1} \Pi_{\mathcal{R}} \Psi_{s}, \quad \Theta_{s}=\Pi_{\mathcal{N}} \Psi_{s} \quad \text { for } s \geq 1 \tag{12}
\end{equation*}
$$

The approximate integral of motion is then obtained by truncating the sequence $T_{\chi} H_{0}$ at a suitable order.

[^1]We have to estimate the action of the operator $T_{\chi}$ on the class of functions $f(p, q)$ we are interested in, in a norm which is well suited for our problem. Such a norm is defined as follows. Let $\mathcal{H}_{s}^{r, i}$ denote the class of monomials ${ }^{2} p^{k} q^{l}$ of degree $s$, i.e., with $|k|+|l|=s$, which furthermore depend on sites that are at most $r$ lattice steps away from $i$, namely such that $k_{j}=l_{j}=0$ if $|i-j| \geq r$. We denote by $\mathcal{P}_{s, r}$ the set of all homogeneous polynomials of degree $s$ that can be decomposed as

$$
\begin{equation*}
f=\sum_{i=1}^{N} \sum_{j=1}^{\left|\mathcal{H}_{s}^{r, i}\right|} c_{i j} f_{i j} \tag{13}
\end{equation*}
$$

with $f_{i j} \in \mathcal{H}_{s}^{r, i}$, where $\left|\mathcal{H}_{s}^{r, i}\right|$ is the cardinality of $\mathcal{H}_{s}^{r, i}$. To $f \in \mathcal{P}_{s, r}$ we associate a norm, ${ }^{3}$ defined by

$$
\begin{equation*}
\|f\|_{+} \stackrel{\text { def }}{=} \min \left\{\max _{i \in\{1, \ldots, N\}} \sum_{j=1}^{\left|\mathcal{H}_{s}^{r, i}\right|}\left|c_{i j}\right|\right\} \tag{14}
\end{equation*}
$$

where the minimum is taken over all possible decompositions of $f$.
Now, we can estimate the action of $T_{\chi}$ on any function $f \in \mathcal{P}_{s, r}$ according to the following Lemma, which is proved in Appendix A.

Lemma 1 Let $T_{\chi}$ be the operator defined by (9), relative to the sequence $\chi=$ $\left\{\chi_{s}\right\}_{s \geq 0}$ which solves the system of equations (12-11) for the Hamiltonian (2). Then, for any $f(p, q) \in \mathcal{P}_{2 s+2, r}$, one has $\left(T_{\chi} f\right)_{n}=\sum_{l=0}^{n} f_{n}^{(s+l)}$, where $f_{n}^{(s+l)} \in \mathcal{P}_{2 s+2 l+2, r+n-l}$ and

$$
\begin{equation*}
\left\|f_{n}^{(s+l)}\right\|_{+} \leq 2^{6 n} 2^{5(n-1)} 2^{2 s+l+2} n!\frac{(n+r)!}{r!} \frac{(n+s)!}{s!} \frac{n!}{l!(n-l)!} \varepsilon^{n-l}\|f\|_{+} \tag{15}
\end{equation*}
$$

Lemma 2 below will give bounds to the adiabatic invariant obtained by truncating at a finite order the formal power series which defines $T_{\chi} H_{0}$. In particular, the adiabatic invariant will simply be $Y_{n}=\sum_{s=0}^{n}\left(T_{\chi} H_{0}\right)_{s}$, so that the polynomials $P_{j}$ appearing at the r.h.s. of (5) of Theorem 1 are

$$
\begin{equation*}
P_{j} \stackrel{\text { def }}{=}\left(T_{\chi} H_{0}\right)_{j} \tag{16}
\end{equation*}
$$

[^2]while the quantity we will focus on will be
\[

$$
\begin{equation*}
X_{n} \stackrel{\text { def }}{=} Y_{n}-H=-\Theta_{1}+\sum_{j=2}^{n} P_{j} \tag{17}
\end{equation*}
$$

\]

The time derivative of $X_{n}$ is then given by

$$
\begin{equation*}
\dot{X}_{n} \stackrel{\text { def }}{=}\left[X_{n}, H\right]=\left[P_{n}, H_{1}\right] \tag{18}
\end{equation*}
$$

which is a polynomial of order $2 n+4$. In order to obtain the estimates of the $L^{2}$-norm, eventually, it is of interest to take into account the parity properties of the operator $T_{\chi}$, with respect to the canonical coordinate $p$. So we define as $\mathcal{P}^{+}$the space of polynomials of even order in $p$, and $\mathcal{P}^{-}$the space of those of odd order in $p$.

Finally, we can state
Lemma 2 For the adiabatic invariant constructed through $T_{\chi} H_{0}$ (see (16)) one can write

$$
\begin{equation*}
P_{n}=\sum_{l=0}^{n} \frac{n!}{l!(n-l)!} \varepsilon^{n-l} P_{n}^{(l)} \tag{19}
\end{equation*}
$$

where $P_{n}^{(l)} \in \mathcal{P}^{+} \cap \mathcal{P}_{2 l+2, n-l}$ and

$$
\begin{equation*}
\left\|P_{n}^{(l)}\right\|_{+} \leq \mathcal{D}_{n}, \quad \text { with } \mathcal{D}_{n} \stackrel{\text { def }}{=} 2^{12 n}(n!)^{3} \tag{20}
\end{equation*}
$$

Furthermore, one has

$$
\left[X_{n}, H\right]=\sum_{l=0}^{n+1} \frac{(n+1)!}{l!(n+1-l)!} \varepsilon^{n+1-l} \dot{X}_{n}^{(l)}
$$

with $\dot{X}_{n}^{(l)} \in \mathcal{P}^{-} \cap \mathcal{P}_{2 l+2, n+1-l}$ and

$$
\begin{equation*}
\left\|\dot{X}_{n}^{(l)}\right\|_{+} \leq \mathcal{C}_{n}, \quad \text { with } \mathcal{C}_{n} \stackrel{\text { def }}{=} 48 \cdot 2^{12 n} n!((n+1)!)^{2} \tag{21}
\end{equation*}
$$

Proof. The proof of the upper bounds is mainly based on the application of Lemma 1 to the function $H_{0} \in \mathcal{P}_{2,0}$, together with the simple bound $\left\|H_{0}\right\|_{+}=\omega \leq 2$, which holds for small enough $\varepsilon$. This proves equations (19), (20). Then, we use the fact that $\left[X_{n}, H\right]=\left[P_{n}, H_{1}\right]$ and the upper bound to the norm of the Poisson brackets of two variables provided by Lemma 10 of Appendix A. This gives equation (21).

The parity properties are obtained by observing that $\left[\mathcal{P}^{ \pm}, \mathcal{P}^{ \pm}\right] \subset \mathcal{P}^{+}$ and $\left[\mathcal{P}^{ \pm}, \mathcal{P}^{\mp}\right] \subset \mathcal{P}^{-}$, as well as $\Pi_{\mathcal{N}}\left(\mathcal{P}^{+}\right) \subset \mathcal{P}^{+}$and $\Pi_{\mathcal{N}}\left(\mathcal{P}^{-}\right) \subset \mathcal{P}^{-}$and that the similar inclusions regarding $\Pi_{\mathcal{R}}$ hold, and then working recursively.
Q.E.D.

## 5 Marginal probability estimates

The aim of this section is to prove the bound on the norm of $\dot{X}_{n}$ given by the following

Lemma 3 There exist constants $\bar{\beta}>0, \bar{\varepsilon}>0, \kappa_{1}>0$ such that, for any $\beta>\bar{\beta}$ and for any $\varepsilon<\bar{\varepsilon}$, for $\dot{X}_{n}$ defined by (18) of Section 4 one has

$$
\begin{equation*}
\left\|\dot{X}_{n}\right\| \leq \sqrt{N}(\sqrt{2} \beta)^{-1}(n!)^{4}\left(\beta^{-1}+\varepsilon\right)^{n} \kappa_{1}^{n} \tag{22}
\end{equation*}
$$

The key tool of the proof is an estimate of the probability that the coordinates of a finite number $s$ of sites are near some fixed values. Such an estimate is given in the following Subection 5.1, whereas the proof of Lemma 3 is given in Subection 5.2.

### 5.1 Estimates on the marginal probability

Everything is trivial for the $p$ coordinates, for which the measure can be decomposed as a product: from a probabilistic point of view, this means that every $p_{j}$ is independent of the $q$ and of any $p_{i}$, for $i \neq j$. We focus, instead, on the $q$ coordinates, which are independent of the $p$, but depend on each other. Then, we must study the relevant part of the density, which is given by

$$
\begin{equation*}
D_{N}\left(q_{1}, \ldots, q_{N}\right) \stackrel{\text { def }}{=} \frac{1}{Z_{N}} \exp \left[-\beta U_{N}\left(q_{1}, \ldots, q_{N}\right)\right] \tag{23}
\end{equation*}
$$

where $Z_{N}$ is the "spatial" partition function

$$
\begin{equation*}
Z_{N} \stackrel{\text { def }}{=} \int_{-\infty}^{+\infty} \mathrm{d} q_{1} \ldots \int_{-\infty}^{+\infty} \mathrm{d} q_{N} \exp \left[-\beta U_{N}\left(q_{1}, \ldots, q_{N}\right)\right] \tag{24}
\end{equation*}
$$

and $U_{N}$ the part of Hamiltonian (2) which depends on $q$, namely, the potential

$$
U_{N}\left(q_{1} \ldots, q_{N}\right) \stackrel{\text { def }}{=} \sum_{i=1}^{N}\left(\omega \frac{q_{i}^{2}}{2}+\frac{q_{i}^{4}}{4 \omega^{2}}\right)+\varepsilon \sum_{i=1}^{N-1} \frac{q_{i} q_{i+1}}{\omega}
$$

The main point is then to estimate the marginal probability $F_{s, \mathfrak{r}}^{(N)}\left(q_{i_{1}}, \ldots, q_{i_{s}}\right)$ that we are going to define. Given a set of indices $i_{1}<i_{2}<\ldots<i_{s}$ we
say that they form a connected block if $i_{j+1}=i_{j}+1$, i.e., if they label a "connected" chain. We say that a sequence of indices $i_{1}<i_{2}<\ldots<i_{s}$ form $\mathfrak{x}$ blocks if the set $\left\{i_{j}\right\}_{j=1}^{s}$ can be decomposed into $\mathfrak{x}$ connected blocks, which furthermore are not connected to each other. Given a set of indices $i_{1}<i_{2}<\ldots<i_{s}$ we define

$$
\begin{equation*}
F_{s, \mathfrak{v}}^{(N)}\left(q_{i_{1}}, \ldots, q_{i_{s}}\right) \stackrel{\text { def }}{=} \int_{-\infty}^{+\infty} \mathrm{d} q_{i_{s+1}} \ldots \int_{-\infty}^{+\infty} \mathrm{d} q_{i_{N}} D_{N}\left(q_{1}, \ldots, q_{N}\right), \tag{25}
\end{equation*}
$$

where $\mathfrak{x}$ is the number of blocks in the set $\left\{i_{j}\right\}_{j=1}^{s}$. We remark here that such a quantity depends on the number of particles, $N$, but we will find for it an upper bound independent of $N$. In fact, the estimate will depend only on $s$ and $\mathfrak{x}$, but not on the precise choice of the sites.

Define the two functions

$$
\begin{align*}
& n_{s, \mathfrak{r}}\left(q_{i_{1}}, \ldots, q_{i_{s}}\right) \stackrel{\text { def }}{=} \exp \left[-\beta\left(\sum_{k=1}^{s}\left(\frac{q_{i_{k}}^{2}}{2 \omega}+\frac{q_{i_{k}}^{4}}{4 \omega^{2}}\right)+\varepsilon \sum_{k, l=1}^{s} \delta_{i_{l}, i_{k}+1} \frac{\left(q_{i_{k}}-q_{i_{l}}\right)^{2}}{2 \omega}\right)\right] \\
& \quad \leq \exp \left(-\beta \sum_{k=1}^{s} \frac{q_{i_{k}}^{2}}{2 \omega}\right),  \tag{26}\\
& \tilde{n}_{s, \mathfrak{r}}\left(q_{i_{1}}, \ldots, q_{i_{s}} \stackrel{\text { def }}{=} \exp \left[-\beta\left(\sum_{k=1}^{s}\left(\frac{\omega q_{i_{k}}^{2}}{2}+\frac{q_{i_{k}}^{4}}{4 \omega^{2}}\right)+\varepsilon \sum_{k, l=1}^{s} \delta_{i_{l}, i_{k}+1} \frac{q_{i_{k}} q_{i_{l}}}{\omega}\right)\right],\right. \tag{27}
\end{align*}
$$

where $\delta_{i, j}$ is the Krönecker delta.
Remark. Notice that $n_{s, \mathfrak{r}}$ is the configurational part of the Gibbs measure of the system with variables $q_{i_{1}}, \ldots, q_{i_{s}}$ and free boundary conditions, apart from the absence of the normalization factor (i.e., the partition function), whereas $\tilde{n}_{s, \mathfrak{x}}$ is the analogous quantity for the same system, but with fixed boundary conditions. Thus, they differ only because of the different dependence on the coordinates at the sites lying on the boundary of the blocks, the number of which, $\gamma$, satisfies $\mathfrak{x} \leq \gamma \leq 2 \mathfrak{x}$. If we denote by $m_{1}, \ldots, m_{\gamma}$ the indices of these sites, we can write the identity

$$
\begin{equation*}
\frac{n_{s, \mathfrak{r}}\left(q_{i_{1}}, \ldots, q_{i_{s}}\right)}{\tilde{n}_{s, \mathfrak{r}}\left(q_{i_{1}}, \ldots, q_{i_{s}}\right)}=\prod_{j=1}^{\gamma} \exp \left(\frac{\beta \varepsilon}{\omega} \alpha_{m_{j}} q_{m_{j}}^{2}\right) \leq \prod_{j=1}^{\gamma} \exp \left(\frac{\beta \varepsilon}{\omega} q_{m_{j}}^{2}\right), \tag{28}
\end{equation*}
$$

where the factor $\alpha_{m_{j}}$ is equal to 1 or $1 / 2$ according to whether the site $m_{j}$ is isolated (i.e., the block is composed of only that site) or not.

Then the following lemma, which is the main result of the present subsection, holds

Lemma 4 There exist constants $\bar{\beta}>0, \bar{\varepsilon}>0, K>0$ and a sequence $\mathfrak{C}_{\mathfrak{x}}>0$ such that, for any $\beta>\bar{\beta}$ and for any $\varepsilon<\bar{\varepsilon}$, one has the inequalities

$$
\begin{equation*}
F_{s, \mathfrak{x}}^{(N)}\left(q_{i_{1}}, \ldots, q_{i_{s}}\right) \leq \mathfrak{C}_{\mathfrak{x}} K^{s}\left(\frac{\beta}{2 \pi \omega}\right)^{s / 2} n_{s, \mathfrak{x}}\left(q_{i_{1}}, \ldots, q_{i_{s}}\right) \tag{29}
\end{equation*}
$$

and

$$
\begin{equation*}
F_{s, \mathfrak{x}}^{(N)}\left(q_{i_{1}}, \ldots, q_{i_{s}}\right) \geq \frac{1}{\mathfrak{C}_{\mathfrak{x}}}\left(\frac{\beta}{2 \pi \omega}\right)^{s / 2} \tilde{n}_{s, \mathfrak{x}}\left(q_{i_{1}}, \ldots, q_{i_{s}}\right) \exp \left(-8 \varepsilon \mathfrak{x} \sqrt{\frac{\beta}{2 \omega}} \sum_{j=1}^{\gamma}\left|q_{m_{j}}\right|\right) \tag{30}
\end{equation*}
$$

The proof of such a lemma is based on the techniques of paper [7], which apply quite simply to the case of periodic boundary conditions (see Lemma 5 below), on account of the translational invariance. Thus, it is also useful to introduce the density $\tilde{D}_{N}$ relative to the periodic system, defined by

$$
\begin{equation*}
\tilde{D}_{N}\left(q_{1}, \ldots, q_{N}\right) \stackrel{\text { def }}{=} \frac{1}{Q_{N}} \exp \left[-\beta U_{N}\left(q_{1}, \ldots, q_{N}\right)+\beta \varepsilon q_{1} q_{N}\right] \tag{31}
\end{equation*}
$$

In this definition there appears the partition function for the periodic case

$$
\begin{equation*}
Q_{N} \stackrel{\text { def }}{=} \int_{-\infty}^{+\infty} \mathrm{d} q_{1} \ldots \int_{-\infty}^{+\infty} \mathrm{d} q_{N} \exp \left[-\beta U_{N}\left(q_{1}, \ldots, q_{N}\right)+\beta \varepsilon q_{1} q_{N}\right] \tag{32}
\end{equation*}
$$

For the periodic system it is simple to estimate two relevant quantities. The former is the ratio between the partition function for $N-1$ particles and that for $N$ particles, i.e., the ratio $Q_{N-1} / Q_{N}$. The relation between $Q_{N}$ and $Z_{N}$ is then obtained as a particular case of Lemma 6, which will be stated later on. The latter is the probability, evaluated with respect to the density for $N$ particles, that the coordinates of $r$ particles have an absolute value smaller than $\Theta \sqrt{2 \omega / \beta}$, for a given $\Theta$. In other terms, we need an estimate of the following quantity

$$
\begin{gather*}
\mathbf{P}_{N}\left(\left|q_{1}\right|<\Theta \sqrt{\frac{2 \omega}{\beta}} \wedge \ldots \wedge\left|q_{r}\right|<\Theta \sqrt{\frac{2 \omega}{\beta}}\right) \stackrel{\text { def }}{=} \int_{-\infty}^{+\infty} \mathrm{d} q_{1} \ldots \int_{-\infty}^{+\infty} \mathrm{d} q_{N} \mathbf{1}_{\left|q_{1}\right|<\Theta \sqrt{2 \omega / \beta}} \\
 \tag{33}\\
\times \ldots \times \mathbf{1}_{\left|q_{r}\right|<\Theta \sqrt{2 \omega / \beta}} \tilde{D}_{N}\left(q_{1}, \ldots, q_{N}\right)
\end{gather*}
$$

in which $\mathbf{1}_{A}$ is the indicator function of the set $A$. We can now give the mentioned estimates by the following lemma, whose proof is deferred to Appendix B.1.

Lemma 5 There exist constants $\beta_{0}>0, \varepsilon_{0}>0, K_{0}>2$ such that, for any $\beta>\beta_{0}$ and $\varepsilon<\varepsilon_{0}$, one has

$$
\begin{equation*}
\frac{Q_{N-1}}{Q_{N}} \leq K_{0} \sqrt{\frac{\beta}{2 \pi \omega}} . \tag{34}
\end{equation*}
$$

Furthermore, if $\Theta \geq 2 \sqrt{r \log \left(4 r K_{0}\right)}$, one has

$$
\begin{equation*}
\mathbf{P}_{N}\left(\left|q_{1}\right|<\Theta \sqrt{\frac{2 \omega}{\beta}} \wedge \ldots \wedge\left|q_{r}\right|<\Theta \sqrt{\frac{2 \omega}{\beta}}\right) \geq \frac{1}{2} . \tag{35}
\end{equation*}
$$

This result enables us to give the proof of Lemma 4.
Proof of Lemma 4 First write

$$
\left.D_{N}\left(q_{1} \ldots, q_{N}\right)=n_{N-s, \mathfrak{z}^{\prime}}\left(q_{i_{s+1}}, \ldots, q_{i_{N}}\right)\right) n_{s, \mathfrak{k}}\left(q_{i_{1}}, \ldots, q_{i_{s}}\right) I\left(q_{1}, \ldots, q_{N}\right),
$$

for a suitable $\mathfrak{x}^{\prime}$, with $\mathfrak{x}-1 \leq \mathfrak{x}^{\prime} \leq \mathfrak{x}+1$ (the lower and the upper bound are attained, respectively, if both 1 and $N$ are contained in $i_{1}, \ldots, i_{s}$ or none of them), where $I$ contains the terms of interaction between the "internal" and the external part of the system. Remarking that $I \leq 1$, one gets

$$
\begin{align*}
F_{s, \mathfrak{r}}^{(N)}\left(q_{i_{1}}, \ldots, q_{i_{s}}\right) \leq & \frac{1}{Z_{N}}\left(\int_{-\infty}^{+\infty} \mathrm{d} q_{i_{s+1}} \ldots \int_{-\infty}^{+\infty} \mathrm{d} q_{i_{N}} \times\right. \\
& \left.\times n_{N-s, \mathfrak{r}^{\prime}}\left(q_{i_{s+1}}, \ldots, q_{i_{N}}\right)\right) n_{s, \mathfrak{r}}\left(q_{i_{1}}, \ldots, q_{i_{s}}\right) . \tag{36}
\end{align*}
$$

We now have to estimate the integral appearing in (36). More in general, in the course of the proof we need to estimate integrals of a similar type. This will be done in Lemma 6, which will be given in a while. Introduce the quantities:

$$
\begin{align*}
& \overline{\mathcal{Q}}_{M}^{\mathfrak{x}} \stackrel{\text { def }}{=} \inf _{B(M, \mathfrak{x})} \int_{-\infty}^{+\infty} \mathrm{d} q_{1} \ldots \int_{-\infty}^{+\infty} \mathrm{d} q_{M} n_{M, \mathfrak{x}}\left(q_{1}, \ldots, q_{M}\right),  \tag{37}\\
& \mathcal{Q}_{M}^{\mathfrak{x}} \stackrel{\text { def }}{=} \sup _{B(M, \mathfrak{x})} \int_{-\infty}^{+\infty} \mathrm{d} q_{1} \ldots \int_{-\infty}^{+\infty} \mathrm{d} q_{M} n_{M, \mathfrak{r}}\left(q_{1}, \ldots, q_{M}\right),
\end{align*}
$$

where $B(M, \mathfrak{x})$ denotes the collection of all possible partitions of $M$ indices in $\mathfrak{x}$ blocks. It is also convenient to consider the quantities defined in a similar way, by integrating $\tilde{n}_{M, \mathfrak{r}}$ in place of $n_{M, \mathfrak{r}}$, namely

$$
\begin{align*}
& \bar{Z}_{M}^{\mathfrak{r}} \stackrel{\text { def }}{=} \inf _{B(M, \mathfrak{r})} \int_{-\infty}^{+\infty} \mathrm{d} q_{1} \ldots \int_{-\infty}^{+\infty} \mathrm{d} q_{M} \tilde{n}_{M, \mathfrak{r}}\left(q_{1}, \ldots, q_{M}\right), \\
& Z_{M}^{\mathfrak{r}} \stackrel{\text { def }}{=} \sup _{B(M, \mathfrak{r})} \int_{-\infty}^{+\infty} \mathrm{d} q_{1} \ldots \int_{-\infty}^{+\infty} \mathrm{d} q_{M} \tilde{n}_{M, \mathfrak{r}}\left(q_{1}, \ldots, q_{M}\right) . \tag{38}
\end{align*}
$$

It is easily shown that for $\mathfrak{x}=1$ one has $\bar{Z}_{M}^{1}=Z_{M}^{1}=Z_{N}$. In order to link them to $Q_{N}$ and to each other, we use the following lemma, the proof of which is deferred to Appendix B.2.

Lemma 6 Let $\beta_{0}>0, \varepsilon_{0}>0$ and $K_{0}>2$ be constants such that Lemma 5 holds. Then, for any $\beta>\beta_{0}$ and any $\varepsilon<\varepsilon_{0}$, the inequalities

$$
\begin{equation*}
\frac{\overline{\mathcal{Q}}_{M}^{\mathfrak{x}}}{Q_{M}} \geq 1, \quad \frac{\bar{Z}_{M}^{\mathfrak{x}}}{Q_{M}} \geq \frac{1}{2}\left(8 \mathfrak{x} K_{0}\right)^{-32 \varepsilon_{0} \mathfrak{x}^{2}} \tag{39}
\end{equation*}
$$

hold. Furthermore, the chain of inequalities

$$
\begin{equation*}
\frac{Z_{M}^{\mathfrak{x}}}{Q_{M}} \leq \frac{\mathcal{Q}_{M}^{\mathfrak{x}}}{Q_{M}} \leq 2 K_{0}^{\mathfrak{x}} \exp \left(4 \mathfrak{x} \varepsilon_{0} \bar{\kappa}\left(\mathfrak{x}, K_{0}\right)\right) \tag{40}
\end{equation*}
$$

holds, where $\bar{\kappa}\left(\mathfrak{x}, K_{0}\right)$ is the solution of the equation

$$
\begin{equation*}
K_{0}^{2 \mathfrak{x}} \Gamma(\mathfrak{x}, \bar{\kappa})=\frac{1}{2} \tag{41}
\end{equation*}
$$

$\Gamma(s, x)$ being the upper regularized Gamma function

$$
\begin{equation*}
\Gamma(s, x) \stackrel{\text { def }}{=} \frac{1}{(s-1)!} \int_{x}^{+\infty} t^{s-1} e^{-t} \mathrm{~d} t \tag{42}
\end{equation*}
$$

The previous lemma enables one to see that $Z_{N}^{-1} \leq 2\left(8 K_{0}\right)^{32 \varepsilon_{0}} / Q_{N}$, while the integral appearing in (36) is estimated by

$$
\mathcal{Q}_{N-s}^{\mathfrak{x}^{\prime}} \leq 2 K_{0}^{\mathfrak{x}+1} \exp \left[4(\mathfrak{x}+1) \varepsilon_{0} \bar{\kappa}\left(\mathfrak{x}+1, K_{0}\right)\right] Q_{N-s}
$$

Thus, due to relation (34) of Lemma 5, one easily sees that

$$
\begin{equation*}
\frac{Q_{N-s}}{Q_{N}}=\prod_{i=1}^{s} \frac{Q_{N-i}}{Q_{N-i+1}} \leq K_{0}^{s}\left(\frac{\beta}{2 \pi \omega}\right)^{s / 2} \tag{43}
\end{equation*}
$$

so that (29) is proved, taking

$$
\begin{equation*}
\mathfrak{C}_{\mathfrak{x}} \geq 2^{96 \varepsilon_{0}+2} K_{0}^{\mathfrak{x}+1+32 \varepsilon_{0}} \exp \left[4(\mathfrak{x}+1) \varepsilon_{0} \bar{\kappa}\left(\mathfrak{x}+1, K_{0}\right)\right] \tag{44}
\end{equation*}
$$

We come now to the proof of $(30)$. To this end, we write

$$
\begin{aligned}
D_{N}\left(q_{1}, \ldots, q_{N}\right) & =\frac{Q_{N-s}}{Z_{N}} \tilde{D}_{N-s}\left(q_{i_{s+1}}, \ldots, q_{i_{N}}\right) \tilde{n}_{s, \mathfrak{x}}\left(q_{i_{1}}, \ldots, q_{i_{s}}\right) \\
& \times G\left(q_{m_{1}}, \ldots, q_{m_{\gamma}}, q_{l_{1}}, \ldots, q_{l_{\gamma^{\prime}}}\right)
\end{aligned}
$$

where the sites $l_{i}$ are the ones which are contiguous to the blocks, but not contained in them, taken by keeping the relative order. Furthermore, due to the periodicity there appear factors depending on $q_{1}$ and $q_{N}$, if the sites 1 and $N$ are not contained in $i_{1}, \ldots, i_{s}$. In this case, we put $q_{l_{1}}=q_{1}$ and $q_{l_{\gamma^{\prime}}}=q_{N}$. We denote by $\gamma^{\prime}$, with $\gamma^{\prime} \leq 2 \mathfrak{x}+2$, the number of such indices. The explicit expression of the function $G$ is complicated. Plainly, it represents the product of the factors $\exp \left(-\sum_{j} \beta \varepsilon q_{m_{j}} q_{l_{i}} / \omega\right)$ among all sites $l_{i}$ contiguous to $m_{j}$, and just the factor $\exp \left(\beta \varepsilon q_{l_{i}} q_{l_{i+1}} / \omega\right)$, when $l_{i}$ and $l_{i+1}$ belong to different blocks. ${ }^{4}$ In any case, a lower bound to $G$ in the region

$$
\mathcal{A} \stackrel{\text { def }}{=}\left\{\left|q_{l_{1}}\right|<2 \mathfrak{x} \sqrt{\log \left(4 K_{0}\right)} \sqrt{2 \omega / \beta} \wedge \ldots \wedge\left|q_{l_{\gamma^{\prime}}}\right|<2 \mathfrak{x} \sqrt{\log \left(4 K_{0}\right)} \sqrt{2 \omega / \beta}\right\}
$$

is given by

$$
G \geq \exp \left(-4 \varepsilon \mathfrak{x} \sqrt{\frac{\beta}{2 \omega}} \sqrt{\log \left(4 K_{0}\right)} \sum_{j=1}^{\gamma}\left|q_{m_{j}}\right|\right)\left(4 K_{0}\right)^{-8 \varepsilon\left(\mathfrak{x}^{2}+\mathfrak{x}\right)}
$$

So, we can write

$$
\begin{align*}
F_{s, \mathfrak{x}}^{(N)}\left(q_{i_{1}}, \ldots, q_{i_{s}}\right) & \geq \tilde{n}_{s}\left(q_{i_{1}}, \ldots, q_{i_{s}}\right)\left(4 K_{0}\right)^{-8 \varepsilon\left(\mathfrak{x}^{2}+\mathfrak{x}\right)} \frac{Q_{N-s}}{Z_{N}} \mathbf{P}_{N-s}(\mathcal{A}) \\
& \times \exp \left(-4 \varepsilon \mathfrak{x} \sqrt{\frac{\beta}{2 \omega}} \sqrt{\log \left(4 K_{0}\right)} \sum_{j=1}^{\gamma}\left|q_{m_{j}}\right|\right), \tag{45}
\end{align*}
$$

where the (positive) contribution of the integral over $\mathcal{A}^{c}$ was neglected.
The term with $\mathbf{P}_{N-s}(\mathcal{A})$ in (45) is bounded from below by relation (35) of Lemma 5. As for the fraction, by Lemma 6 we obtain

$$
Q_{N-s} \geq \frac{1}{2 K_{0} \exp \left(8 \varepsilon_{0} \bar{\kappa}\left(1, K_{0}\right)\right)} \mathcal{Q}_{N-s}^{1}
$$

Now, operating as in the deduction of formula (43), it is sufficient to observe that $\mathcal{Q}_{N-1}^{1} \geqq \sqrt{\beta /(2 \pi \omega)} \mathcal{Q}_{N}^{1}$ to obtain $\mathcal{Q}_{N-s}^{1} \geq(\beta / 2(\pi \omega))^{s / 2} \mathcal{Q}_{N}^{1}$. Then, choosing $\bar{\varepsilon}, \bar{\beta}$ such that $K_{0} \leq e^{4} / 4$ and observing that $\mathcal{Q}_{N}^{1} \geq Z_{N}$, one gets (30) with

$$
\begin{equation*}
\mathfrak{C}_{\mathfrak{x}} \geq\left(4 K_{0}\right)^{8 \varepsilon_{0}\left(\mathfrak{x}^{2}+\mathfrak{x}\right)+1} \exp \left(8 \varepsilon_{0} \bar{\kappa}\left(1, K_{0}\right) .\right. \tag{46}
\end{equation*}
$$

Finally, $\mathfrak{C}_{\mathfrak{x}}$ can be chosen as the maximum of the r.h.s. of (44) and of (46). This concludes the proof.

[^3]
### 5.2 Estimate of $\left\|\dot{X}_{n}\right\|$

We apply directly inequality (29) to get the proof of Lemma 3, using the fact that such a quantity is a sum of polynomials depending at most on $2 n+3$ sites, as can be seen by Lemma 2 of Section 4.
Proof of Lemma 3 The key ingredient of the proof is, as stated in Section 4, that the polynomials $P_{n}$ are even in the $p$ coordinates, so that the $\dot{X}_{n}$ 's are odd in the $p$. On account of that, $\dot{X}_{n}^{2}$ is a sum in which the terms coming from the product of two monomials depending on separated groups of sites contain at least one $p_{i}$ to an odd power. Since the measure is even with respect to any $p$, these terms have a vanishing integral.

We formalize this way of reasoning by decomposing $\dot{X}_{n}$ as $\dot{X}_{n}=\sum_{i=1}^{N} f_{i}$, where the $f_{i}$ 's are polynomials depending at most on the sites between $i-$ $n-1$ and $i+n+1$. Then, the $L^{2}-$ norm of $\dot{X}_{n}$ is expressed according to $\left\|\dot{X}_{n}\right\|^{2}=\sum_{i, j=1}^{N}\left\langle f_{i} f_{j}\right\rangle$. In this sum, all the terms with $|i-j|>2 n+2$ vanish, while the other ones are estimated in terms of $\left\|\dot{X}_{n}\right\|_{+}$in the following way.

On account of Lemma 2, we can write

$$
f_{i}=\sum_{l=0}^{n+1} \frac{(n+1)!}{l!(n+1-l)!} \varepsilon^{n+1-l} \sum_{s=1}^{\left|\mathcal{H}_{2 l+2}^{n+1-l, i}\right|} c_{i s, l} f_{i s}^{(l)}
$$

in which $f_{i s}^{(l)}$ is a monomial in $\mathcal{H}_{2 l+2}^{n+1-l, i}$ and the decomposition in these monomials is performed in such a way that $\sup _{i, l} \sum_{s}\left|c_{i s, l}\right| \leq \mathcal{C}_{n}$. Then, we sum on $j$ and obtain that

$$
\left\|\dot{X}_{n}\right\|^{2} \leq(4 n+5) \mathcal{C}_{n}^{2} \sum_{i=1}^{N} \sum_{l=0}^{n+1} \frac{(2 n+2)!}{l!(2 n+2-l)!} \varepsilon^{2 n+2-l} \sup _{g \in \mathcal{H}_{2 l+4}^{2 n+2-l, i}}\langle g\rangle,
$$

where we used the fact that the only nonvanishing contributions to the integral come from the product of $f_{j s}^{(l-r)} \in \mathcal{H}_{2 l-2 r+2}^{n+1-l+r, j}$ and $f_{k m}^{(r)} \in \mathcal{H}_{2 r+2}^{n+1-r, k}$, for $|j-k| \leq 2 n+2-l$, so that $g \stackrel{\text { def }}{=} f_{j s}^{(l-r)} f_{k m}^{(r)} \in \mathcal{H}_{2 l+4}^{2 n+2-l, i}$, for a suitable $i$ between $j$ and $k$.

Then, we make use of (29) together with the estimate (26) for $n_{s, \mathfrak{x}}$ to
bound the mean value of any function in $\mathcal{H}_{2 l+4}^{2 n+2-l, i}$. In fact, one has

$$
\begin{aligned}
\sup _{g \in \mathcal{H}_{2 l+4}^{2 n+2-l, i}}\langle g\rangle & \leq \mathfrak{C}_{1} K^{4 n-2 l+5} \sqrt{\frac{\beta}{2 \pi \omega}} \int_{-\infty}^{\infty} x^{2 l+4} \exp \left(-\frac{\beta}{2 \omega} x^{2}\right) \mathrm{d} x \\
& =\mathfrak{C}_{1} K^{4 n-2 l+5}\left(\frac{2 \omega}{\beta}\right)^{l+2} \frac{(2 l+3)!!}{2^{l+2}} .
\end{aligned}
$$

So, the inequality

$$
\left\|\dot{X}_{n}\right\|^{2} \leq \mathfrak{C}_{1} K^{4 n+5}(4 n+5)(2 n+4)!(2 \omega)^{2 n+4} \beta^{-2}\left(\varepsilon+\frac{1}{\beta}\right)^{2 n+2} N \mathcal{C}_{n}^{2}
$$

holds. Thus, choosing a suitable $\kappa_{1}>0$ and using the value of $\mathcal{C}_{n}$ given by (21) of Lemma 2, inequality (22) is satisfied.
Q.E.D.

## 6 Estimate of the variance of the adiabatic invariant

In the present Section we prove the following Lemma 7, which was used in the proof of Theorem 1. The lemma concerns estimates on the variance $\sigma_{X_{n}}^{2}$ and on the correlation $\rho_{X_{n}, H}$ of the adiabatic invariant and reads

Lemma 7 There exist positive constants $\tilde{\varepsilon}>0, \kappa_{2}>0, \kappa_{3}>1$, such that, for any $\varepsilon<\tilde{\varepsilon}$, for any $\beta>\varepsilon^{-1}$ and for $n<\kappa_{2}^{-1 / 4}\left(\varepsilon+\beta^{-1}\right)^{-1 / 4}$, with $X_{n}$ defined by (17), the following inequalities hold:

$$
\begin{equation*}
\sigma_{X_{n}} \geq \sqrt{N} \frac{\varepsilon+\beta^{-1}}{8 \beta} \tag{47}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|\rho_{X_{n}, H}\right| \leq\left(1+\frac{1}{\kappa_{3}} \frac{\varepsilon^{2}}{\left(\varepsilon+\beta^{-1}\right)^{2}}\right)^{-1 / 2} \tag{48}
\end{equation*}
$$

The proof of this lemma requires the study the spatial correlations between quantities depending on two separate blocks. The study of these properties has to be performed within the general frame of Gibbsian fields and conditional probabilities. In the present Section we provide the necessary notions and give a proposition of a general character concerning the
decay of spatial correlations for lattices with finite range of interaction, i.e, Theorem 2 of Subsection 6.2 , from which it will be possible to finally come to the proof of Lemma 7, which will be given in Subsection 6.4.

Our treatment of conditional probabilities is inspired in particular by the work of Dobrushin (see [6]). More precisely, we will make reference to paper [6] for the main ideas, and to the subsequent beautiful but underestimated subsequent paper [8], by Dobrushin and Pechersky, for a more direct relation to our problem. As a matter of fact, most of the ideas of this section are already contained in works [6] and [8], but the explicit result on the spatial correlations given here required some additional work. We recall that, since Gibbsian fields and the related techniques were introduced in order to deal with infinite lattices, our result holds even if the number of sites tends to infinity.

The present section is structured as follows: in Subsection 6.1 the link between spatial correlations and conditional probabilities is shown, and in Subsection 6.2 we state Theorem 2, whose proof is deferred to Appendix B.3. Such a result is used in order to obtain an upper bound to $\sigma_{P_{n}}$, stated in Lemma 8 of Subection 6.3 , whence the proof of Lemma 7 easily follows, as shown in Subection 6.4.

### 6.1 Link between spatial correlations and conditional probability

In order to prove Lemma 7 we have to estimate quantities such as $\langle f g\rangle$ $\langle f\rangle\langle g\rangle$, relative to the Gibbs measure $\mu$, where $f$ is a function which depends on sites belonging to a set $\tilde{V}$, while $g$ depends only on sites in $V$, with $V \cap \tilde{V}=\emptyset$. Our aim is to show that such correlations decrease as the distance between $\tilde{V}$ and $V$ increases, where the distance $d(V, \tilde{V})$ is defined for example as $d(V, \tilde{V}) \stackrel{\text { def }}{=} \inf _{i \in V, j \in \tilde{V}}|i-j|$.

We start showing the relation between the spatial correlations and the conditional probability in a setting more general than ours. We consider as given a measure $\mu$ on $\mathbb{R}^{|T|}$, with $T \subset \mathbb{Z}^{\nu}$, which induces on the measurable set $A \subset \mathbb{R}^{|\tilde{V}|}$ the probability

$$
P_{\tilde{V}}(A) \stackrel{\text { def }}{=} \int_{\mathbb{R}|T|} \mathrm{d} \mu(x) \mathbf{1}_{A \times T \backslash \tilde{V}}(x),
$$

where $\mathbf{1}_{A}$ is the indicator function of the set $A$. One can express the quantity
we are interested in as

$$
\begin{equation*}
\langle f g\rangle-\langle f\rangle\langle g\rangle=\int_{\mathbb{R}^{\mid}|\bar{V}|} f(\mathbf{x}) P_{\tilde{V}}(\mathrm{~d} \mathbf{x})\left(\int_{\mathbb{R}^{|V|}} g(\mathbf{y}) P_{V}(\mathrm{~d} \mathbf{y} \mid \mathrm{d} \mathbf{x})-\int_{\mathbb{R}^{|V|} \mid} g(\mathbf{y}) P_{V}(\mathrm{~d} \mathbf{y})\right), \tag{49}
\end{equation*}
$$

where $P_{V}(B \mid A)$ represents the conditional probability of the measurable set $B \subset \mathbb{R}^{|V|}$, once $A$ is given. So, in order to estimate the correlation between two functions, it is sufficient to estimate the difference enclosed in brackets at the r.h.s. of (49). Now, we notice that for any pair of probabilities $P$ and $\tilde{P}$ on $\mathbb{R}^{|V|}$, one has

$$
\begin{align*}
\left|\int_{\mathbb{R}^{\mid} \mid} g(\mathbf{x}) P(\mathrm{~d} \mathbf{x})-\int_{\mathbb{R}^{|V|}} g(\mathbf{y}) \tilde{P}(\mathrm{~d} \mathbf{y})\right| \leq \int_{\mathbb{R}^{|V|} \mid \times \mathbb{R}^{|V|}} & (|g(\mathbf{x})|+|g(\mathbf{y})|)  \tag{50}\\
& \times \mathbf{1}_{\mathbf{x} \neq \mathbf{y}} Q(\mathrm{~d} \mathbf{x}, \mathrm{~d} \mathbf{y}),
\end{align*}
$$

in which $Q$ is any probability on $\mathbb{R}^{|V|} \times \mathbb{R}^{|V|}$ such that $P$ and $\tilde{P}$ are its marginal probabilities. In other terms $Q$ is a joint probability of $P$ and $\tilde{P}$, i.e., for any measurable $B \subset \mathbb{R}^{|V|}$ one has

$$
\begin{equation*}
P(B)=Q\left(B \times \mathbb{R}^{|V|}\right) \quad \text { and } \tilde{P}(B)=Q\left(\mathbb{R}^{|V|} \times B\right) \tag{51}
\end{equation*}
$$

Remark here that $Q$ is not unique: indeed, such a probability provides also a way to define a distance between two probabilities defined on the same set $V$ of indices, by

$$
\begin{equation*}
D(P, \tilde{P}) \stackrel{\text { def }}{=} \inf _{Q} \int_{\mathbb{R}^{|V|} \times \mathbb{R}|V|} \mathbf{1}_{\mathbf{x} \neq \mathbf{y}} Q(\mathrm{~d} \mathbf{x}, \mathrm{~d} \mathbf{y}) \tag{52}
\end{equation*}
$$

We stress that the infimum is attained, i.e., there exists a probability measure $\bar{Q}(\mathrm{~d} \mathbf{x}, \mathrm{~d} \mathbf{y})$ such that $D(P, \tilde{P})=\int \mathbf{1}_{\mathbf{x} \neq \mathbf{y}} \bar{Q}(\mathrm{~d} \mathbf{x}, \mathrm{~d} \mathbf{y})$ (see Lemma 1 of paper [8]). For the following, we suppose that it is possible to find a compact function $h,{ }^{5}$ with domain in $\mathbb{R}$, such that

$$
\begin{equation*}
|g(\mathbf{x})| \leq \sum_{i \in V} h\left(x_{i}\right) \tag{53}
\end{equation*}
$$

as is the case for the monomials we are dealing with. The bounds will then be given in terms of $h$. Now, observing that $\mathbf{1}_{\mathbf{x} \neq \mathbf{y}} \leq \sum_{i \in V} \mathbf{1}_{x_{i} \neq y_{i}}$, we can rewrite (50) as

$$
\begin{align*}
\left|\int_{\mathbb{R}^{|V|}} g(\mathbf{x}) P(\mathrm{~d} \mathbf{x})-\int_{\mathbb{R}^{|V|}} g(\mathbf{y}) \tilde{P}(\mathrm{~d} \mathbf{y})\right| \leq \sum_{i, j \in V} \int_{\mathbb{R}^{2}} & \left(h_{j}\left(x_{j}\right)+h\left(y_{j}\right)\right)  \tag{54}\\
& \times \mathbf{1}_{x_{i} \neq y_{i}} Q(\mathrm{~d} \mathbf{x}, \mathrm{~d} \mathbf{y}) .
\end{align*}
$$

[^4]This way, we can make a direct connectino with paper [8], in which the problem of estimating the r.h.s. of the above expression is dealt with. We summarize here the results and the methods we need.

### 6.2 Main argument and theorem on correlations

In the quoted work [8], the framework is more general than ours, because it deals with the problem of defining a "probability" for the configuration of an actually infinite $\nu$-dimensional lattice of particle, in terms of the set of conditional probabilities on each site, which is called the specification $\Gamma$. Now, our case is in principle different, because our lattice is finite, and the probability is defined through the Gibbs measure. In particular, the specification too is assigned by such a measure.

However, as proved in [8], under suitable assumptions assigning the specification uniquely determines the probability, i.e. the Gibbsian field, which, in our case, turns out to be precisely that of Gibbs. So, in our case, it is equivalent to speak in terms of specification or in terms of measure. Indeed, in this subsection we will speak in terms of specifications, and in the following one we will show that the specification determined by Gibbs measure (1) with the Hamiltonian (2) satisfies the assumptions of [8] (i.e. Conditions 1 and 2 below).

We notice that the r.h.s. of (54) can be bounded from above if one estimates the quantity

$$
\begin{equation*}
\lambda(\mathbf{j}, \mathbf{i}) \stackrel{\text { def }}{=} \max \left\{\mathbf{E}\left[\mathbf{1}_{\xi_{\mathbf{j}}^{1} \neq \xi_{\mathbf{j}}^{2}} h\left(\xi_{\mathbf{i}}^{1}\right)\right], \mathbf{E}\left[\mathbf{1}_{\xi_{\mathbf{j}}^{1} \neq \xi_{\mathbf{j}}^{2}} h\left(\xi_{\mathbf{i}}^{2}\right)\right]\right\} \tag{55}
\end{equation*}
$$

where $\xi^{1}$ and $\xi^{2}$ are two Gibbsian fields which assign, respectively, the probabilities $P$ and $\tilde{P}$ appearing in (54) and the expectations are obtained by integrating over a joint probability $Q$ of the two fields. Indeed, in [8] an upper bound just to $\lambda(\mathbf{j}, \mathbf{i})$ is given, by requiring that two suitable conditions are satisfied.

So, by adopting the same techniques of [8] we bound from above the r.h.s. (54) and, thus, the l.h.s. of (49). Such a bound is given in Theorem 2 below. In order to state it, we recall the main notations of [8].

First, we consider a lattice of sites contained in $T \subset \mathbb{Z}^{\nu}$, and a finiterange specification with a radius of interaction $r$ (this means that the conditional probability at site $\mathbf{i}$ does not depend on the conditioning at sites $\mathbf{j}$ for $|\mathbf{i}-\mathbf{j}|>r)$. Then, for a vector $\mathbf{x} \in \mathbb{R}^{|T|}$ we denote by $P_{\mathbf{i}, \mathbf{x}}(\mathrm{d} x)$ the probability distribution conditioned to $\mathbf{x}$ everywhere but at site $\mathbf{i}$. The specification $\Gamma$
is defined by

$$
\Gamma \stackrel{\text { def }}{=}\left\{P_{\mathbf{i}, \mathbf{x}}: \mathbf{i} \in T, \mathbf{x} \in \mathbb{R}^{|T|}\right\} .
$$

Furthermore, we will say that a continuous positive function $h$ on a metric space $\mathfrak{X}$ is compact if, for any $k \geq 0$, the set $\{x \in \mathfrak{X}: h(x) \leq k\}$ is compact.

For a fixed integer $r$, let $\partial_{r} V \stackrel{\text { def }}{=}\left\{\mathbf{j} \in T: \mathbf{j} \notin V, \min _{\mathbf{k} \in V}|\mathbf{j}-\mathbf{k}| \leq r\right\}$ be the boundary of thickness $r$ of a set $V \subset T$. We call $a$ the number of indices such that $|\mathbf{i}| \leq r, \mathbf{i} \neq 0$, where $r$ is the range of interaction.

If $Z_{0}$ is a maximal subgroup of $\mathbb{Z}^{\nu}$ satisfying the condition $|\mathbf{j}-\mathbf{k}|>r$, for $\mathbf{j}, \mathbf{k} \in Z_{0}$, we denote by $b$ the number of elements in the factor group $\mathbb{Z}^{\nu} \backslash Z_{0}$.

The conditions of paper [8] (which are hypotheses on the specification $\Gamma$, once the compact function $h$ is given) are the following

Condition 1 (Compactness) Let $h$ be a compact function on $\mathbb{R}$ and let $C \geq 0$ and $c_{\mathbf{j}} \geq 0$, for $|\mathbf{j}| \leq r, \mathbf{j} \neq 0$, be some constants. We suppose that

1. $\delta \stackrel{\text { def }}{=} \sum_{|\mathbf{j}| \leq r, \mathbf{j} \neq 0} c_{\mathbf{j}}<\frac{1}{a^{b}}$;
2. for any $\mathbf{i} \in T$ and any $\mathbf{x} \in \mathbb{R}^{|T|}$ one has

$$
\int_{\mathbb{R}} h(x) P_{\mathbf{i}, \mathbf{x}}(\mathrm{d} x) \leq C+\sum_{\mathbf{j} \in \partial_{r}\{\mathbf{i}\}} c_{\mathbf{j}-\mathbf{i}} h\left(x_{\mathbf{j}}\right) .
$$

Condition 2 (Contractivity) Let $\bar{K} \geq 0$ and $k_{\mathbf{j}}=k_{\mathbf{j}}(\bar{K}) \geq 0$, for $|\mathbf{j}| \leq r$, $\mathbf{j} \neq 0$, be constants and $h$ be a compact function. We suppose that

1. $\alpha \stackrel{\text { def }}{=} \sum_{|\mathbf{j}| \leq r, \mathbf{j} \neq 0} k_{\mathbf{j}}<1$;
2. for any $\mathbf{i} \in T$ and any pair of configurations $\mathbf{x}^{1}, \mathbf{x}^{2} \in \mathbb{R}^{|T|}$ such that

$$
\max _{\mathbf{j} \in \partial_{r}\{\mathbf{i}\}} \max \left\{h\left(x_{\mathbf{j}}^{1}\right), h\left(x_{\mathbf{j}}^{2}\right)\right\} \leq \bar{K},
$$

one has the inequality

$$
D\left(P_{\mathbf{i}, \mathbf{x}^{1}}, P_{\mathbf{i}, \mathbf{x}^{2}}\right) \leq \sum_{\mathbf{j} \in \partial_{r}\{\mathbf{i}\}} k_{\mathbf{j}-\mathbf{i}} \mathbf{1}_{x_{\mathbf{j}}^{1} \neq x_{\mathbf{j}}^{2}},
$$

where $D(\cdot, \cdot)$ is the distance defined by (52).

The set of specifications (i.e., the sets of conditional probabilities on every site) which satisfy Condition 1 for the constants $C, \delta$ and the compact function $h$ will be denoted by $\Theta(h, C, \delta)$. We will instead denote by $\Delta(h, \bar{K}, \alpha)$ the set of specifications which satisfy Condition 2 for the constants $\bar{K}, \alpha$ and the compact function $h$.

If the specification satisfies Conditions 1 and 2, Dobrushin and Pechersky show that the specification uniquely determines the probability (see Theorem 1 of [8]). In particular, also the marginal probability $P_{V}$ on $V$, and the probability $P_{V}(\cdot \mid \mathrm{d} \mathbf{x})$ conditioned to a vector $\mathbf{x}$ in $\tilde{V}$ are determined by $\Gamma$, and the maximum value taken by $\lambda(\mathbf{j}, \mathbf{i})$ of (55) is bounded from above.

As a matter of fact, in [8] the authors do not investigate the explicit dependence of $\lambda(\mathbf{j}, \mathbf{i})$ on $|\mathbf{i}-\mathbf{j}|$, which is what we need, but such a dependence can be obtained by a slight modification of their way of reasoning, which leads to the proof of Theorem 2 that appears in Appendix B.3. We need also to introduce an upper bound to the mean value of $h$ with respect to the probability $P_{V}$ on $V$ and to the probability $P_{V}(\cdot, \mathrm{~d} \mathbf{x})$, i.e.,

$$
\begin{equation*}
\langle h\rangle_{\mathbf{x}} \stackrel{\text { def }}{=} \sup _{\mathbf{i} \in V} \max \left\{\int_{\mathbb{R}} h(y) P_{\mathbf{i}}(\mathrm{d} y \mid \mathrm{d} \mathbf{x}), \int_{\mathbb{R}} h(y) P_{\mathbf{i}}(\mathrm{d} y), C\right\}, \tag{56}
\end{equation*}
$$

where $\mathbf{x} \in \tilde{V}$, while $C$ is the constant entering Condition 1 . So we can state
Theorem 2 (Decay of Correlations) Let $f$ be a measurable function from $\mathbb{R}^{|\tilde{V}|}$ to $\mathbb{R}$, depending on the sites lying in the set $\tilde{V}$. Let $g$ be a measurable function from $\mathbb{R}^{|V|}$ to $\mathbb{R}$, depending on the sites contained in the set $V$, with $V \cap \tilde{V}=\emptyset$, and $h$ be a compact function such that inequality (53) is satisfied.

Let $\Gamma \in \Theta(h, C, \delta)$. Then, there exists a constant $\bar{K}_{0}$, depending on $h, C, a, b$ only, such that, if $\Gamma \in \Delta\left(h, \bar{K}_{0} C, \alpha\right) \cap \Theta(h, C, \delta)$ with any $\alpha$, one can find constants $D, c>0$, for which one has

$$
\begin{equation*}
|\langle f g\rangle-\langle f\rangle\langle g\rangle| \leq D|V|^{2}\left|\int_{\mathbb{R}^{|V|}} f(\mathbf{x})\langle h\rangle_{\mathbf{x}} P_{\tilde{V}}(\mathrm{~d} \mathbf{x})\right| \exp (-c d(V, \tilde{V})) \tag{57}
\end{equation*}
$$

The constant $D$ depends on $a, b, \alpha, \delta$ only, while one has

$$
\begin{equation*}
c \stackrel{\text { def }}{=}-\frac{1}{b r} \log \left[\frac{1}{2}\left(\max \left\{\alpha, \delta a^{b}\right\}+1\right)\right] \tag{58}
\end{equation*}
$$

### 6.3 Estimate of the variance of $P_{n}$

The previous way of proceeding can be fitted to our case by choosing as specification that given by the Gibbs measure relative to $H$. As $f$ and $g$,
we choose polynomials in $p$ and $q$, depending on two disjoint sets of sites. In fact, on account of Lemma 2, we know that the $P_{n}$ are constituted by a sum of such terms. This way we can study $\sigma_{P_{n}}$ and state that it is bounded from above as in the following

Lemma 8 There exist constants $\bar{\beta}>0, \bar{\varepsilon}>0, k^{\prime}>0$ such that, for any $\beta>\bar{\beta}$ and any $\varepsilon<\bar{\varepsilon}$, one has, for $n<1 / \varepsilon$,

$$
\begin{equation*}
\sigma_{P_{n}} \leq \sqrt{N} n!\left(k^{\prime}\right)^{n} \mathcal{D}_{n}(\sqrt{2} \beta)^{-1}\left(\varepsilon+\beta^{-1}\right)^{n} \tag{59}
\end{equation*}
$$

where the polynomials $P_{n}$ are defined by (16) and $\mathcal{D}_{n}$ are given in Lemma 2.
Proof. Lemma 2 provides the necessary estimates for the coefficients which appear in the sum defining $P_{n}$ (see (19-20)): we can write $P_{n}=\sum_{i=1}^{N} f_{i}$, where $f_{i}$ are polynomials depending at most on the sites between $i-n$ and $i+n$. The variance can be expressed as $\sigma_{P_{n}}^{2}=\sum_{i, j=1}^{N}\left(\left\langle f_{i} f_{j}\right\rangle-\left\langle f_{i}\right\rangle\left\langle f_{j}\right\rangle\right)$. We then consider the set $\mathcal{S}_{1} \stackrel{\text { def }}{=}\{(i, j):|j-i| \leq 2 n\}$ and $\mathcal{S}_{2}=\mathcal{S}_{1}^{c}$. The proof goes on by finding an upper bound separately for the contributions coming from these two sets: for the latter, we use the methods developed in the present section, while the terms of the former group are estimated in a way similar to that of Lemma 3.

We start from $\mathcal{S}_{1}$. Firstly, we observe that, in general, one has

$$
\left|\left\langle f_{i} f_{j}\right\rangle-\left\langle f_{i}\right\rangle\left\langle f_{j}\right\rangle\right| \leq \sigma_{f_{i}} \sigma_{f_{j}} \leq \max \left\{\left\langle f_{i}^{2}\right\rangle,\left\langle f_{j}^{2}\right\rangle\right\} ;
$$

so that it suffices to evaluate $\sup _{i}\left\langle f_{i}^{2}\right\rangle$ in a way similar to Lemma 3 and to sum over the $j$ with $|j-i| \leq 2 n$ to see that the contribution due to the terms in this set is smaller than

$$
\sum_{i, j \in \mathcal{S}_{1}}\left|\left\langle f_{i} f_{j}\right\rangle-\left\langle f_{i}\right\rangle\left\langle f_{j}\right\rangle\right| \leq \mathfrak{C}_{1} K^{4 n+1}(4 n+1)(2 n+1)!(2 \omega)^{2 n} \beta^{-2}\left(\varepsilon+\frac{1}{\beta}\right)^{2 n} N \mathcal{D}_{n}^{2}
$$

We come now to $\mathcal{S}_{2}$. We will show below that the specification coming from the Gibbs measure satisfies the hypotheses of Theorem 2, and we make use of it in estimating the terms in the set $\mathcal{S}_{2}$ in the following way. We separate the terms of a definite degree by writing

$$
f_{i}=\sum_{l=0}^{n} \frac{n!}{l!(n-l)!} \varepsilon^{n-l} \sum_{s=1}^{\left|\mathcal{H}_{2 l+2}^{n-l, i}\right|} c_{i s, l} f_{i s}^{(l)},
$$

in which $f_{i s}^{(l)}$ is a monomial in $\mathcal{H}_{2 l+2}^{n-l, i}$ and $\sup _{i, l} \sum_{s}\left|c_{i s, l}\right| \leq \mathcal{D}_{n}$. We fix an index $i$ and use Theorem 2 with $f=f_{i}$ and $g=f_{j s}^{(l)}$, for every $j \neq i$; then, we sum over $l, s, j$ and $i$, subsequently. For each $l$ we choose the compact function $h_{l}(x)$ as $|x|^{2 l+2}$, which satisfies (53) for any $f_{j s}^{(l)}$. We will show that, for $\beta$ large enough and $\varepsilon$ small enough, one has

$$
\begin{equation*}
\sum_{l=0}^{n}\left\langle h_{l}\right\rangle_{\mathbf{x}} \leq k^{n} n!\frac{1}{\beta}\left(\varepsilon+\frac{1}{\beta}\right)^{n} \exp \left(\sum_{j=1}^{|\tilde{V}|}\left(\frac{\beta \varepsilon}{\omega} x_{j}^{2}+8 \varepsilon \sqrt{\frac{\beta}{2 \omega}}\left|x_{j}\right|\right)\right) \tag{60}
\end{equation*}
$$

for a suitable constant $k$. Then the sum on $s$ brings in a factor $\mathcal{D}_{n}$. As regards the integration in $\tilde{V}$, we observe that we can write

$$
\beta \frac{1-2 \varepsilon}{2} x_{j}^{2}-8 \varepsilon \sqrt{\frac{\beta}{2 \omega}}\left|x_{j}\right| \geq \frac{\beta}{4} x_{j}^{2}-1,
$$

provided $\varepsilon$ is small enough. Thus, there exists $\bar{k}>0$ such that

$$
\left|\left\langle f_{i} f_{j}\right\rangle-\left\langle f_{i}\right\rangle\left\langle f_{j}\right\rangle\right| \leq \bar{k}^{n}(n!)^{2} \mathcal{D}_{n}^{2} \frac{1}{\beta^{2}}\left(\varepsilon+\frac{1}{\beta}\right)^{2 n} \exp [-c(|i-j|-2 n-1)]
$$

where the constant $c$ is defined by (58), and is, in the present case, equal to $\log (4 / 3) / 2$. Since the sum over $j$ of such terms converges as $N \rightarrow \infty$, the proof will be concluded if we show that (60) and the hypotheses of Theorem 2 are satisfied.

We proceed as in the proof of Theorem 2 of paper [8], starting from the explicit form of the conditional probability distribution given by the Gibbs measure: one has

$$
P_{i, \mathbf{x}}(x)=\frac{1}{Z_{\mathbf{x}}} \exp \left[-\beta\left(\frac{\omega x^{2}}{2}+\frac{x^{4}}{4 \omega^{2}}+\frac{\varepsilon}{\omega} x \cdot x_{i-1}+\frac{\varepsilon}{\omega} x \cdot x_{i+1}\right)\right],
$$

in which there appears the conditional partition function

$$
Z_{\mathbf{x}} \stackrel{\text { def }}{=} \int_{\mathbb{R}} \exp \left[-\beta\left(\frac{\omega x^{2}}{2}+\frac{x^{4}}{4 \omega^{2}}+\frac{\varepsilon}{\omega} x \cdot x_{i-1}+\frac{\varepsilon}{\omega} x \cdot x_{i+1}\right)\right] \mathrm{d} x .
$$

As regards Condition 1, we consider $\varepsilon<2^{-5}$ fixed and define

$$
\hat{y}(\mathbf{x}) \stackrel{\text { def }}{=} \max \left\{\left|x_{i-1}\right|,\left|x_{i+1}\right|\right\} \quad \text { and } y(\mathbf{x}) \stackrel{\text { def }}{=} \min \{1 / \hat{y}(\mathbf{x}), \sqrt{\beta}\} \text {. }
$$

So, it is easily proved that, for $\beta>1$, inequality $Z_{\mathbf{x}} \geq \bar{c} y(\mathbf{x}) / \beta$, holds for some $\bar{c}>0$ indipendent of $\mathbf{x}, \beta$ and $\varepsilon$. To prove it, it is sufficient to observe
that the integrand of $Z_{\mathbf{x}}$ is bounded away from zero if $|x| \leq y(\mathbf{x}) / \beta$, and then to integrate over such an interval. We now show item 2 of Condition 1 for $h(x)=|x|^{2 l+2}$. We note that

$$
\begin{aligned}
\int_{|x| \geq \hat{y}(\mathbf{x}) / 4} h_{l}(x) P_{i, \mathbf{x}}(x) \mathrm{d} x & \leq \frac{2}{Z_{\mathbf{x}}} \int_{1 /(4 y(\mathbf{x}))}^{+\infty} x^{2 l+2} \exp \left(-\beta \frac{x^{2}}{4}\right) \mathrm{d} x \\
& \leq \frac{2 \sqrt{\beta}}{\bar{c} y(\mathbf{x})} \frac{1}{\beta^{l+1}} \int_{\sqrt{\beta} /(4 y(\mathbf{x}))}^{+\infty} x^{2 l+2} e^{-x^{2} / 4} \mathrm{~d} x
\end{aligned}
$$

which, in turn, is smaller than $(l+1)!(B / \beta)^{l+1}$ for a suitable constant $B$, independent of $\varepsilon, \beta$ and $l$, since the integral decreases as an exponential function of $\sqrt{\beta} / y(\mathbf{x})$. Here, we have chosen $(l+1)$ ! so that the previous relation is satisfied independently of $l$. Furthermore, one has

$$
\int_{|x| \leq \hat{y}(\mathbf{x}) / 4} h_{l}(x) P_{i, \mathbf{x}}(x) \mathrm{d} x \leq 2^{-4 l-4}(\hat{y}(\mathbf{x}))^{2 l+2}
$$

and $\left(h_{l}\left(x_{i-1}\right)+h_{l}\left(x_{i+1}\right)\right) / h_{l}(\hat{y}(x)) \geq 1$, for any $\mathbf{x}$ : this implies that

$$
\int_{\mathbb{R}} h_{l}(x) P_{i, \mathbf{x}}(\mathrm{~d} x) \leq(l+1)!\left(\frac{B}{\beta}\right)^{l+1}+\frac{1}{16} h_{l}\left(x_{i-1}\right)+\frac{1}{16} h_{l}\left(x_{i+1}\right) .
$$

So, item 2 of Condition 1 holds with $C=(l+1)!(B / \beta)^{l+1}$ and $c_{1}=c_{-1}=$ $1 / 16$. Since we have $a=2, b=2, r=1$, item 1 of Condition 1 holds with $\delta a^{b}=1 / 2$.

Condition 2 is proved by computing two limits, first letting $\beta$ tend to infinity and then letting $\varepsilon \rightarrow 0$. In fact, let $\mathbf{x}^{(m)}$, for $m=1,2$, be two different configuration such that $\left|x_{i-1}^{(m)}\right| \leq \tilde{K} / e \sqrt{B l /(e \beta)}$ and $\left|x_{i+1}^{(m)}\right| \leq \tilde{K} / e \sqrt{B l /(e \beta)}$. Then it is easily checked that

$$
\lim _{\beta \rightarrow \infty} \int_{\mathbb{R}}\left|P_{i, \mathbf{x}^{(1)}}(x)-P_{i, \mathbf{x}^{(2)}}(x)\right| \mathrm{d} x=\int_{\mathbb{R}} \mathrm{d} z e^{-z^{2}}\left|f\left(\varepsilon, z, \mathbf{z}^{(1)}\right)-f\left(\varepsilon, z, \mathbf{z}^{(2)}\right)\right|,
$$

where

$$
f\left(\varepsilon, z, \mathbf{z}^{(m)}\right) \stackrel{\text { def }}{=} \exp \left(-\frac{\varepsilon}{\omega} z\left(z_{i-1}^{(m)}+z_{i+1}^{(m)}\right)+\frac{\varepsilon^{2}}{2 \omega^{2}}\left(z_{i-1}^{(m)}+z_{i+1}^{(m)}\right)^{2}\right)
$$

and $z=x \sqrt{\beta}, z_{j}^{(m)}=x_{j}^{(m)} \sqrt{\beta}$. Now, by the dominated convergence theorem, one has that the limit for $\varepsilon \rightarrow 0$ of $f\left(\varepsilon, z, \mathbf{z}^{(m)}\right)$ is equal to 1 . Here, use is made of the fact that $\varepsilon\left|z_{j}^{(m)}\right|$ can be bounded from above by
$\varepsilon \tilde{K} / e \sqrt{B l / e} \leq \tilde{K} / e \sqrt{B \varepsilon / e}$, because $l \leq n$, and $n$ is smaller than $1 / \varepsilon$, by hypothesis. So, for $\beta$ sufficiently large and $\varepsilon$ small enough one has

$$
\int_{\mathbb{R}}\left|P_{i, \mathbf{x}^{1}}(x)-P_{i, \mathbf{x}^{2}}(x)\right| \mathrm{d} x \leq \frac{1}{4} .
$$

We have chosen a bound to $\mathbf{x}^{m}$ of this particular form, because the constant $\bar{K}$ in Condition 2 turns out to be smaller than $\tilde{K}^{2 l+2}$, so that it is independent of $\beta$. So Condition 2 holds with $\bar{K}=\tilde{K}^{2 l+2}, k_{-1}=k_{1}=1 / 2$ and $\alpha=1 / 2$. Thus, Theorem 2 holds.

There still remains to estimate $\left\langle h_{l}\right\rangle_{\mathbf{x}}$ in our case. By looking at its definition (56), we notice that we have to estimate the integrals $\int h(y) P_{\mathbf{i}}(\mathrm{d} y \mid \mathrm{d} \mathbf{x})$ and $\int h(y) P_{\mathbf{i}}(\mathrm{d} y)$. Now, on account of Lemma 4 and relation (28), the distribution functions of $P_{\mathbf{i}}(\mathrm{d} y \mid \mathrm{dx})$ and $P_{\mathbf{i}}(\mathrm{d} y)$ can be bounded by

$$
K^{2 n+2} \mathfrak{C}_{2} \mathfrak{C}_{1} \sqrt{\frac{\beta}{2 \pi \omega}} \exp \left(\sum_{j=1}^{|\tilde{V}|}\left(\frac{\beta \varepsilon}{\omega} x_{j}^{2}+8 \varepsilon \sqrt{\frac{\beta}{2 \omega}}\left|x_{j}\right|\right)\right) e^{-\beta y^{2} /(2 \omega)}
$$

Then, we use the bound to $C$ previously found, together with the fact that

$$
\sqrt{\frac{\beta}{2 \pi \omega}} \int_{\mathbb{R}} y^{2 l+2} e^{-\beta y^{2} /(2 \omega)} \mathrm{d} y=\left(\frac{2 \omega}{\beta}\right)^{l+1} \frac{(2 l+1)!!}{2^{l+1}},
$$

and we get (60). This concludes the proof.
Q.E.D.

### 6.4 Estimate of the variance of $X_{n}$

Lemma 8 of Section 6.3 enables us to bound from below the variance of $X_{n}$ defined by (17) and to estimate the correlation coefficient $\rho_{X_{n}, H}$, according to Lemma 7, which we prove here.
Proof of Lemma $\mathbf{7}$ We start by recalling that, on account of (17), one has $X_{n}=-\Theta_{1}+\sum_{j=2}^{n} P_{j}$, with $\Theta_{1}$ defined by equations (12-11) of Section 4. It is easily seen that $\Theta_{1}=F+G+\mathcal{R}_{1}$, in which

$$
F \stackrel{\text { def }}{=}-\frac{\varepsilon}{2 \omega} \sum_{i=1}^{N-1} p_{i} p_{i+1} \quad \text { and } G \stackrel{\text { def }}{=} \frac{3}{32 \omega^{2}} \sum_{i=1}^{N} p_{i}^{4},
$$

and $\mathcal{R}_{1}$ is the remainder. Then, we study the properties of $F$ and $G$, for which the mean value, the variance and the correlation with $H$ can be computed almost exactly, and we extend such properties to $\Theta_{1}$, and to the whole $X_{n}$, by observing that, in some sense, $\Theta_{1}$ is the term of first order in $\varepsilon+\beta^{-1}$.

As regards formula (47), we notice that, since $F$ is odd in the momenta, while $G, \mathcal{R}_{1}$ and the measure are even, then $F$ is uncorrelated both with $G$ and with $\mathcal{R}_{1}$. Furthermore, one can observe that

$$
\left\langle G R_{1}\right\rangle-\langle G\rangle\left\langle R_{1}\right\rangle=\frac{9}{2^{10} \omega^{4}} \sum_{i=1}^{N}\left\langle q_{i}^{2}\right\rangle\left(\left\langle p_{i}^{6}\right\rangle-\left\langle p_{i}^{4}\right\rangle\left\langle p_{i}^{2}\right\rangle\right),
$$

and use the estimates of Lemma 4 to bound from above $\left\langle q_{i}^{2}\right\rangle$, in order to prove that $\sigma_{\Theta_{1}}^{2} \geq \sigma_{F}^{2}+\sigma_{G}^{2}+2 C_{G, \mathcal{R}_{1}} \geq N\left(\varepsilon^{2}+\beta^{-2}\right) /\left(8 \beta^{2}\right)$, where the second inequality holds for $\varepsilon$ and $\beta^{-1}$ small enough. On the other hand, making use of (59) of Lemma 8 together with the estimate for $\mathcal{D}_{n}$ given by (20) of Lemma 2, one has

$$
\sigma_{X_{n}} \geq \sigma_{\Theta_{1}}-\sum_{j=2}^{n} \sigma_{P_{j}} \geq \sqrt{N} \frac{\varepsilon+\beta^{-1}}{4 \beta}\left(1-\sum_{j=2}^{n}(j!)^{4}\left(\varepsilon+\beta^{-1}\right)^{j-1} \kappa_{2}^{j}\right)
$$

for a suitable constant $\kappa_{2}$, if $\beta^{-1}$ and $\varepsilon$ are sufficiently small. Now, for $n<\kappa_{2}^{-1 / 4}\left(\varepsilon+\beta^{-1}\right)^{-1 / 4}$, the sum is smaller than a constant multiplied by $\varepsilon+\beta^{-1}$ and this proves (47).

As for (48), we observe that, since $H$ is even in the momenta, $F$ and $H$ are uncorrelated, so that, using $\rho_{X}, Y<1$, one gets

$$
\left|\rho_{X_{n}, H}\right| \leq \frac{1}{\sigma_{X_{n}} \sigma_{H}}\left(\left|C_{\Theta_{1}-F, H}\right|+\sum_{j=2}^{n}\left|C_{P_{j}, H}\right|\right) \leq \frac{\sigma_{\Theta_{1}-F}}{\sigma_{\Theta_{1}}} \frac{\sigma_{\Theta_{1}}}{\sigma_{X_{n}}}+\frac{\sum_{j=2}^{n} \sigma_{P_{j}}}{\sigma_{X_{n}}}
$$

As we have just shown, for $n<\kappa_{2}^{-1 / 4}\left(\varepsilon+\beta^{-1}\right)^{-1 / 4}$ the last term at the r.h.s. tends to zero as $\varepsilon+\beta^{-1}$, and in the same way behaves $\sigma_{\Theta_{1}} / \sigma_{X_{n}}-1$. So, we limit ourselves to study $\sigma_{\Theta_{1}-F} / \sigma_{\Theta_{1}}=1 / \sqrt{1+\sigma_{F}^{2} / \sigma_{\Theta_{1}-F}^{2}}$. By computing explicitly $\sigma_{F}^{2}$ and applying the upper bound (59) to $\sigma_{\Theta_{1}}^{2} \geq \sigma_{\Theta_{1}-F}^{2}$, we get that there exists a constant $\bar{\kappa} \geq 1$ such that

$$
\frac{\sigma_{\Theta_{1}-F}}{\sigma_{\Theta_{1}}} \leq\left(1+\frac{1}{\kappa} \frac{\varepsilon^{2}}{\left(\varepsilon+\beta^{-1}\right)^{2}}\right)^{-1 / 2}
$$

Since the r.h.s. differs from 1 by a quantity larger than $\varepsilon^{2} \beta^{2}$, the corrections given by the other terms can be neglected if $\beta \geq \varepsilon^{-1}$. This completes the proof.

## 7 Relation between stability estimates and relaxation times

In the present section we discuss which implications the existence of an adiabatic invariant has in the frame of ergodic theory. The main point is that it can provide a lower bound to the relaxation time to equilibrium. Since there is no agreement in the literature on the definition of relaxation time, we will give here a mathematically clear form to such a concept. To this end, we need a preliminary discussion of an a priori bound to the time autocorrelations (see Theorem 3). This is provided in Section 7.1. Then, in Section 7.2, we define the concept of relaxation time.

### 7.1 Relaxation times and time correlations

One of the open problems in statistical mechanics is that of thermalization, i.e., to establish whether a system, starting from a given microscopic state, does attain thermodynamic equilibrium, and, if this is the case, to estimate the time scale needed to reach it. Such a time scale is usually called the relaxation time. From a physical point of view the situation is complicated, because certain systems, for example gases, reach equilibrium on a very short time scale, while others, for example glasses, are believed to reach equilibrium on geological time scales.

Linear response theory (see $[19,20]$ ) shows that susceptibilities can be expressed in terms of the time autocorrelations of suitable dynamical variables (namely, those conjugated to the perturbing field). In particular, the susceptibilities assume the equilibrium values only for measurements which last a time large enough, i.e., larger than the time needed by the time autocorrelations to become negligible.

Now, the time correlations between pairs of dynamical variables are widely studied in the case of chaotic systems (see, for example, [21, 22] or the monograph [23]). For such systems, the correlations are known to tend to zero, as $t \rightarrow \infty$, and one of the problems is to estimate the decay rate, for long times, of the time autocorrelations of all dynamical variables. This however amounts to give an upper bound to the time autocorrelations. From the standpoint of linear response theory, it is also significant to bound from below the time autocorrelations of suitably chosen dynamical variables, because this leads to a lower bound to the relaxation time.

Corollary 2 of Theorem 1 gives an estimate of such a kind, showing that, for the system here considered, the relaxation time is larger than a constant $\bar{t}$, which is exponentially large in the perturbation parameters (and moreover does not depend on the number of degrees of freedom of the system). Results analogous to Corollary 2 are of a general type. In fact one has
Theorem 3 (Bound to the autocorrelation of a dynamical variable)
Suppose that, for a dynamilcal variable $X$, there exists a constant $\eta>0$ such that

$$
\begin{equation*}
\|[X, H]\| \leq \eta \sigma_{X} \tag{61}
\end{equation*}
$$

then one has

$$
\begin{equation*}
C_{X}(t) \geq 1-\frac{1}{2} \eta^{2} t^{2} \tag{62}
\end{equation*}
$$

Remark. This theorem is a slight modification of Theorem 1 of [15] and is proved in the same way. On the other hand, we think that the decision to focus on the time autocorrelation, which we make here at variance with paper [15], is crucial, if one aims at obtaining significant estimates in the thermodynamic limit.

Proof. Introduce the difference $\delta \stackrel{\text { def }}{=} X_{t}-X$. As $X_{t}$ satisfies the Liouville equation and $X$ is time-independent, one has $\partial_{t} \delta=\partial_{t} X_{t}=-\left[H, X_{t}\right]$, which in terms of $\delta$ takes the form

$$
\begin{equation*}
\partial_{t} \delta=-[H, \delta]+Y, \tag{63}
\end{equation*}
$$

with $Y \stackrel{\text { def }}{=}-[H, X]$. It is well known that, $\mu$ being invariant, the solutions of the Liouville equation are generated by a one-parameter group $\hat{U}(t)$ of unitary operators in the sense that $X_{t}=\hat{U}(t) X$. As $\delta(0)=0$, the solution of equation (63) is given by

$$
\delta=\int_{0}^{t} \hat{U}(t-s) Y \mathrm{~d} s
$$

so that, $\hat{U}$ being unitary, one gets the estimate

$$
\|\delta\| \leq \int_{0}^{t}\|\hat{U}(t-s) Y\| \mathrm{d} s=t\|Y\| \leq \eta t \sigma_{X}
$$

Then, one gets the thesis by using the simple identity

$$
C_{X}(t)=1-\frac{\left\|X_{t}-X\right\|^{2}}{2 \sigma_{X}^{2}}
$$

Q.E.D.

### 7.2 Definition and evaluation of relaxation times

Taking into account the relation between susceptibilities and time autocorrelations, it is meaningful to introduce a parameter $a=a(t)$, with values in $[0,1]$, which estimates how much the system is close to equilibrium, after a finite time $t$. So, for the set $\mathcal{B} \subset L^{2} \cap \mathcal{C}^{\infty}$ of the dynamical variables uncorrelated with $H$, we define

Definition 1 (Correlation level) The correlation level $a(t)$ at time $t$ is defined as $a(t) \stackrel{\text { def }}{=} \sup _{X \in \mathcal{B}}\left|C_{X}(t)\right|$.

Remark. One can limit oneself to the smooth observables, because these are the physically relevant ones. ${ }^{6}$

In the chaotic case $a$ tends to 0 as $t \rightarrow \infty$ : thus, looking for the decay to zero of the correlations is equivalent to looking at the asymptotic behaviour about zero of $t(a)$, the inverse function of $a(t) .{ }^{7}$ On the other hand, according to linear response theory, it is more meaningful to look at the time after which the correlations are below a certain threshold. So, we introduce the following notion

Definition 2 (Relaxation time relative to level a) The relaxation time relative to level $a$ is defined as $t(a) \stackrel{\text { def }}{=} \inf t^{*}(a)$, where $t^{*}(a)$ is such that

$$
\sup _{X \in \mathcal{B}}\left|C_{X}(t)\right| \leq a \quad \text { for all } t \geq t^{*}(a)
$$

Remark. In order to provide a significant lower bound to the relaxation time $t(a)$ at level $a$, as previously defined, it is clearly sufficient to find the time at which the autocorrelation of at least one dynamical variable $X$ uncorrelated with the Hamiltonian is certainly larger than $a$. Now, the following corollary on the relaxation time descends immediately from Theorem 3:

Corollary 3 (Bound to the relaxation time) Suppose there exists a dynamical variable $X \in \mathcal{B}$ and a constant $\eta>0$ such that $\|[X, H]\| \leq \eta \sigma_{X}$; then one has

$$
t(a) \geq \sqrt{2(1-a)} \frac{1}{\eta}
$$

[^5]The point is that many Hamiltonian systems of interest for Solid State Physics reduce to integrable ones in some limit, while, on the other hand, for integrable systems one has $t(a)=+\infty$ for any $a<1$, since their integrals of motion remain correlated for all times. The question is then, what is the behaviour of such systems when the perturbation is small, i.e., to study the ergodic properties of slightly perturbed (or nearly integrable) Hamiltonian systems. It is natural to think that there exists a sort of continuity as the perturbation diminishes. Continuity can in fact be recovered in terms of the time needed for the system to reach thermalization (i.e., a sufficiently low correlation level). This is indeed the case in the system we have considered, because we can say that, as a consequence of Theorem 1, one has the lower bound

$$
t(a) \geq \sqrt{2(1-a)} \frac{\varepsilon}{\kappa} \exp \left(\frac{1}{\kappa\left(\varepsilon+\beta^{-1}\right)}\right)^{1 / 4}
$$

which goes to infinity as both $\varepsilon$ and $\beta^{-1} \rightarrow 0$.

## 8 Conclusions

In this paper, we have constructed, for the Klein Gordon lattice, an adiabatic invariant, i.e., a dynamical variable whose time derivative is small as a stretched exponential with the perturbation parameters. Thus our result is similar to those which are known in Hamiltonian perturbation theory in the case of a finite number of degree of freedom or in the case of an infinite number of them, but at a fixed total energy (see [25, 4]). The new feature of the present work is that our theorem remains valid in the thermodynamic limit, because the given bound turns out to be independent of the number of particles, and depends only on intensive quantities. As a corollary, we bound from below the stability time of such a model.

We now add some comments. The first one concerns the fact that in our model we have two perturbation parameters, $\varepsilon$ and $1 / \beta$. We believe however that the only really relevant parameter is $1 / \beta$. Indeed, at least formally the parameter $\varepsilon$ can be arbitrarily decreased by performing a suitable normal form change of coordinates ([26]), and it seems to us that such a normal form does not alter in any fundamental feature the perturbation $H_{1}$ (i.e., its local character). At the moment, however, we are unable to say anything definite on this point.

In any case, while the estimate of Section 6 could presumably be applied to models more general than that studied in this paper, the estimates on the marginal probabilities of Section 5 are especially adapted to our model.

It would be important to improve our method, making it more flexible in order to cover more general situations.

In our opinion, the real big problem that remains is the construction of adiabatic invariants for problems in which small denominators appear. This problem could be overcome in particular cases (see, for example, paper [15]), but no precise strategy exists yet for the general case. We plan to tackle this problem soon.

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## A Estimates for the construction of the adiabatic invariant

Here we intend to prove Lemma 1 of Section 4. In order to do that, we need to recollect the usual algebraic properties used in perturbation theory (see [17]), adapted to our norm $\|\cdot\|_{+}$, defined by (14). Such properties are stated in Lemmas 9-11 later on, then the proof of Lemma 1 is briefly sketched.

In order to develop the perturbation theory, a primary role is played by the action of the operator $L_{0}$ and by the projections on its kernel and its range (see Section 4), and these are more easily discussed in terms of the complex variables which diagonalize $L_{0}$. These are implicitly defined by

$$
\begin{equation*}
q_{l}=\frac{1}{\sqrt{2}}\left(\xi_{l}+i \eta_{l}\right), \quad p_{l}=\frac{1}{\sqrt{2}}\left(\xi_{l}-i \eta_{l}\right), \quad 1 \leq l \leq N \tag{64}
\end{equation*}
$$

and in such variables one has

$$
\begin{equation*}
L_{0} \xi^{j} \eta^{k}=i \omega(|k|-|j|) \xi^{j} \eta^{k} . \tag{65}
\end{equation*}
$$

We must, however, take into account the fact that the norm $\|\cdot\|_{+}$is not invariant under such a change of coordinates. In fact, such a norm is formally well defined also for polynomials depending on the variables $(\xi, \eta)$ if, in the definition of $\mathcal{H}_{s}^{r, i}$ and $\mathcal{P}_{s, r}$, we simply substitute for $(p, q)$ the pair $(\xi, \eta)$. In that case, denoting by $f^{\prime}$ the transform of $f$ via (64), one will have, in general, $\|f\|_{+} \neq\left\|f^{\prime}\right\|_{+}$. On the other hand, the following lemma,
whose proof is identical to that of Lemma A. 1 of paper [17], enables one to estimate the difference between the norms of the two functions.

Lemma 9 Let $f(q, p)$ be in $\mathcal{P}_{s, r}$ and let $f^{\prime}(\xi, \eta)$ be the transform of $f$ via (64). Then, one has $f^{\prime} \in \mathcal{P}_{s, r}$ and $\left\|f^{\prime}\right\|_{+} \leq 2^{\frac{s}{2}}\|f\|_{+}$. Moreover, let $g^{\prime}(\xi, \eta)$ be in $\mathcal{P}_{s, r}$ and let $g(q, p)$ be the transform of $g$ via the inverse of (64). Then, one has $g \in \mathcal{P}_{s, r}$ and $\|g\|_{+} \leq 2^{\frac{s}{2}}\left\|g^{\prime}\right\|_{+}$.

We need also the following lemmas
Lemma 10 Let $f$ be in $\mathcal{P}_{s, r}$ and $g$ in $\mathcal{P}_{s^{\prime}, r^{\prime}}$. Then, $[f, g] \in \mathcal{P}_{s+s^{\prime}-2, r+r^{\prime}}$ and one has, both in real and in complex variables, the inequality

$$
\|[f, g]\|_{+} \leq\left(2 r+2 r^{\prime}+1\right) s s^{\prime}\|f\|_{+}\|g\|_{+} .
$$

Proof. See Lemma A. 2 of [17], noticing that, for any fixed $i$, each term of $f$ contained in $\mathcal{H}_{s}^{r, i}$ has Poisson bracket different from 0 only with the monomials of $\mathcal{H}_{s^{\prime}}^{r^{\prime}, k}$ such that $|i-k| \leq r+r^{\prime}$. The number of such monomials appearing in the decomposition of $g$ is smaller than $2 r+2 r^{\prime}+1$.
Q.E.D.

Lemma 11 Let $f \in \mathcal{P}_{s, r}$ be a polynomial in complex variables. Then $\Pi_{\mathcal{N}} f$, $\Pi_{\mathcal{R}} f$ and $L_{0}^{-1} \Pi_{\mathcal{R}} f$ belong to $\mathcal{P}_{s, r}$ and the following inequalities hold:

$$
\left\|\Pi_{\mathcal{N}} f\right\|_{+} \leq\|f\|_{+}, \quad\left\|\Pi_{\mathcal{R}} f\right\|_{+} \leq\|f\|_{+}, \quad\left\|L_{0}^{-1} \Pi_{\mathcal{R}} f\right\|_{+} \leq\|f\|_{+}
$$

Proof. The fact that $L_{0}^{-1} f$ belongs to $\mathcal{P}_{s, r}$ comes directly from Lemma 10, as $H_{0}$ is in $\mathcal{P}_{2,0}$. The remaining statements are a consequence of the fact that $L_{0}$ is diagonal in complex coordinates and that the smallest eigenvalue of $L_{0}$ on $\mathcal{R}$ has modulus $\omega \geq 1$, in virtue of (65).
Q.E.D.

Proof of Lemma 1 We pass to complex variables via Lemma 9 and proceed by induction on $n$, checking at each step even two supplementary inductive hypotheses:
i) $\Psi_{n}$ can be decomposed as $\Psi_{n}=\sum_{l=0}^{n} \Psi_{n}^{(l)}$, where $\Psi_{n}^{(l)} \in \mathcal{P}_{2 l+2, n-l}$;
ii) the following bound holds

$$
\left\|\Psi_{n}^{(l)}\right\|_{+} \leq 2^{n} 2^{10(n-1)}(n!)^{2}(n-1)!\frac{n!}{l!(n-l)!} \varepsilon^{n-l}
$$

As a matter of facts, on account of Lemma 11, such an estimate enables one to control the contributions due to $\chi_{n}$ and $\Theta_{n}$, which appear in the recurrent procedure that determines $\chi_{s}$, for $s \geq n$. Then, we come back to real variables via lemma 9 again.
Q.E.D.

## B Technical proofs

## B. 1 Proof of Lemma 5

We start by proving formula (34). On account of the symmetry of the periodic system, one can pass from a system with $N-1$ particles to one with $N$ by inserting one more particle after the $i$-th site, for $i=1, \ldots, N-1$. The potential energy of the corresponding system is given by
$U_{N}\left(q_{1}, \ldots, q_{N}, q\right)=U_{N-1}-\frac{\varepsilon}{2 \omega}\left(q_{i+1}-q_{i}\right)^{2}+\frac{\varepsilon}{2 \omega}\left(q-q_{i}\right)^{2}+\frac{\varepsilon}{2 \omega}\left(q-q_{i+1}\right)^{2}+\frac{q^{2}}{2 \omega}+\frac{q^{4}}{4 \omega^{2}}$.
Neglecting the second term at the r.h.s. (which gives a contribution to the partition function which can be bounded from below by 1, and averaging over $i$ in order to get a traslational invariant system, one gets

$$
\begin{align*}
\frac{Q_{N}}{Q_{N-1}} \geq & \frac{1}{N-1} \sum_{i=1}^{N-1} \int_{-\infty}^{+\infty} \mathrm{d} q_{1} \ldots \int_{-\infty}^{+\infty} \mathrm{d} q_{N-1} \tilde{D}_{N-1}\left(q_{1}, \ldots, q_{N-1}\right) \times \\
& \times \int_{-\infty}^{+\infty} \mathrm{d} q \exp \left[-\frac{\beta}{2 \omega}\left(q^{2}+\frac{q^{4}}{2 \omega}+\varepsilon\left(q-q_{i}\right)^{2}+\varepsilon\left(q-q_{i+1}\right)^{2}\right)\right] \tag{66}
\end{align*}
$$

Here we have put $q_{N}=q_{1}$. Then, we introduce the function $\varphi_{q_{i}}(q) \stackrel{\text { def }}{=}$ $1-\exp \left[-\beta \varepsilon\left(q-q_{i}\right)^{2} /(2 \omega)\right]$, for which the inequality

$$
\exp \left[-\frac{\beta \varepsilon}{2 \omega}\left(q-q_{i}\right)^{2}+\frac{\beta \varepsilon}{2 \omega}\left(q-q_{i+1}\right)^{2}\right] \geq 1-\varphi_{q_{i}}(q)-\varphi_{q_{i+1}}(q)
$$

holds. We will show now that $\varphi_{q_{i}}(q)$ is small except for a set of small measure. Making use of the previuos inequality, relation (66) becomes

$$
\begin{array}{r}
\frac{Q_{N}}{Q_{N-1}} \geq a(\beta, \varepsilon)-\int_{-\infty}^{+\infty} \mathrm{d} q_{1} \ldots \int_{-\infty}^{+\infty} \mathrm{d} q_{N-1} \tilde{D}_{N-1}\left(q_{1}, \ldots, q_{N-1}\right) \times \\
\quad \times \int_{-\infty}^{+\infty} \mathrm{d} q \frac{2}{N-1} \sum_{i=1}^{N-1} \varphi_{q_{i}}(q) \exp \left[-\frac{\beta}{2 \omega}\left(q^{2}+\frac{q^{4}}{2 \omega}\right)\right] \tag{67}
\end{array}
$$

in which the function $a(\beta, \varepsilon)$ is defined by ${ }^{8}$

$$
a(\beta, \varepsilon) \stackrel{\text { def }}{=} \int_{-\infty}^{+\infty} \mathrm{d} q \exp \left[-\frac{\beta}{2 \omega}\left(q^{2}+\frac{q^{4}}{2 \omega}\right)\right]=\frac{\sqrt{2 \omega} e^{\frac{\beta}{8}}}{2} K_{\frac{1}{4}}\left(\frac{\beta}{8}\right)
$$

where $K_{\alpha}(x)$ is the Bessel modified function of second kind. The well known properties of $K_{\alpha}(x)$ imply that $a(\beta, \varepsilon)$ can be written as $a(\beta, \varepsilon)=$ $G(\beta, \varepsilon) \sqrt{2 \pi \omega / \beta}$, where $G$ is a function always smaller than 1 , approaching 1 , at fixed $\varepsilon$, as $\beta \rightarrow+\infty$. We go on by dealing with the integral in (67), first giving an upper bound for the innermost integral over $q$. We estimate it by splitting the phase space of the $N-1$ particles periodic system in two sets: we will fix $\kappa>0$ and consider $\Omega(N-1, \kappa)$, which is defined by

$$
\begin{equation*}
\Omega(N-1, \kappa) \stackrel{\text { def }}{=}\left\{\left(q_{1}, \ldots, q_{N-1}\right) \text { such that } \sum_{i=1}^{N-1} q_{i}^{2}<\frac{2 \omega}{\beta} \kappa(N-1)\right\} \tag{68}
\end{equation*}
$$

and its complement. In the latter set, the integral is simply bounded from above by $2 a(\beta, \varepsilon)$. On the other hand, in order to estimate the integral in the set $\Omega(N-1, \kappa)$, we observe that, for any $\kappa_{1}$, the number of particles for which $\left|q_{i}\right| \geq \sqrt{\kappa_{1} \kappa 2 \omega / \beta}$ holds cannot exceed $(N-1) / \kappa_{1}$. For these particles the integral is estimated again by $2 a(\beta, \varepsilon)$. For the purpose of estimating the contribution of the remaining particles, we introduce the function

$$
I\left(\beta, \varepsilon, \kappa, \kappa_{1}\right) \stackrel{\text { def }}{=} \frac{1}{a(\beta, \varepsilon)} \sup _{|y|<\sqrt{\kappa_{1} \kappa 2 \omega / \beta}} \int_{-\infty}^{+\infty} \varphi_{y}(q) \exp \left(-\frac{\beta}{2 \omega} q^{2}\right) \mathrm{d} q
$$

We point out that $I\left(\beta, \varepsilon, \kappa, \kappa_{1}\right)$ tends to 0 as $\varepsilon$ tends to 0 , for $\beta, \kappa, \kappa_{1}$ fixed. Then, in the region $\Omega(N-1, \kappa)$, for any $\kappa_{1}>1$, one has the bound

$$
\int_{-\infty}^{+\infty} \mathrm{d} q \frac{2}{N-1} \sum_{i=1}^{N-1} \varphi_{q_{i}}(q) \exp \left[-\frac{\beta}{2 \omega}\left(q^{2}+\frac{q^{4}}{2 \omega}\right)\right] \leq\left[\frac{2}{\kappa_{1}}+2 I\left(\beta, \varepsilon, \kappa, \kappa_{1}\right)\right] a(\beta, \varepsilon)
$$

[^6]We notice that we have provided estimates independent of $q_{i}$, so the integrals over $q_{1}, \ldots, q_{N-1}$ appearing in (67) can simply be estimated as the product of these upper bounds times the measures of the sets in which the bounds hold. Now, we observe that the measure of $\Omega^{c}(N-1, \kappa)$ is estimated by

$$
\int_{\Omega^{c}(N-1, \kappa)} \mathrm{d} q_{1} \ldots \mathrm{~d} q_{N-1} \tilde{D}_{N-1}\left(q_{1}, \ldots, q_{N-1}\right) \leq \frac{R_{N-1}(\beta, \kappa)}{Q_{N-1}}
$$

where the function $R_{N-1}(\beta, \kappa)$ is defined by

$$
\begin{align*}
R_{N-1}(\beta, \kappa) & \stackrel{\text { def }}{=} \int_{\Omega^{c}(N-1, \kappa)} \mathrm{d} q_{1} \ldots \mathrm{~d} q_{N-1} \exp \left(-\frac{\beta}{2 \omega} \sum_{i=1}^{N-1} q_{i}^{2}\right)  \tag{69}\\
& =\left(\frac{2 \pi \omega}{\beta}\right)^{\frac{N-1}{2}} \Gamma\left(\frac{N-1}{2}, \kappa(N-1)\right)
\end{align*}
$$

and $\Gamma(s, x)$ is defined by (42). This way one obtains, finally,

$$
\begin{equation*}
\frac{Q_{N}}{Q_{N-1}} \geq\left(1-\frac{2}{\kappa_{1}}-2 I\left(\beta, \varepsilon, \kappa, \kappa_{1}\right)-2 \frac{R_{N-1}(\beta, \kappa)}{Q_{N-1}}\right) a(\beta, \varepsilon) \tag{70}
\end{equation*}
$$

From this expression one can prove (34) by induction on $N$.
We now come to the proof of (35). We make use of the trivial inequality $\mathbf{P}\left(A_{1} \cap \ldots \cap A_{r}\right) \geq 1-\sum_{i=1}^{r} \mathbf{P}\left(A_{i}^{c}\right)$, which holds for any probability and any collection of sets $A_{1}, \ldots, A_{r}$. Consequently, we obtain
$\mathbf{P}_{N}\left(\left|q_{1}\right|<\Theta \sqrt{\frac{2 \omega}{\beta}} \wedge \ldots \wedge\left|q_{r}\right|<\Theta \sqrt{\frac{2 \omega}{\beta}}\right) \geq 1-r \cdot \mathbf{P}_{N}\left(\left|q_{1}\right| \geq \Theta \sqrt{\frac{2 \omega}{\beta}}\right)$.
because, due to the translation invariance of the periodic system, every set has the same measure. Recall that $\mathbf{P}_{N}\left(\left|q_{1}\right| \geq \Theta \sqrt{2 \omega / \beta}\right)$ is just the integral of $\tilde{D}_{N}$ times $\mathbf{1}_{\left|q_{i}\right| \geq \Theta \sqrt{2 \omega / \beta}}$. A bound to this integral can be found proceeding as above, i.e., by symmetrizing on $q_{i}$, fixing $\kappa>0$ and integrating separately over $\Omega(N, \kappa)$ and its complement (recall that $\Omega(N, \kappa)$ is defined by (68)). This way we get

$$
\begin{aligned}
\mathbf{P}_{N}\left(\left|q_{1}\right| \geq \Theta \sqrt{\frac{2 \omega}{\beta}}\right) \leq & \frac{1}{N Q_{N}} \sum_{i=1}^{N} \int_{\Omega(N, \kappa)} \mathbf{1}_{\left|q_{i}\right| \geq \Theta \sqrt{2 \omega / \beta}} \tilde{D}_{N}\left(q_{1}, \ldots, q_{N}\right) \\
& +\frac{1}{Q_{N}} R_{N}(\beta, \kappa)
\end{aligned}
$$

where $R_{N}$ is defined by (69), and we bound $\mathbf{1}_{\left|q_{i}\right| \geq \Theta \sqrt{2 \omega / \beta}}$ by 1 in $\Omega^{c}(N, \kappa)$. It is straightforward to notice that the number of sites for which $\left|q_{i}\right| \geq$ $\Theta \sqrt{2 \omega / \beta}$, in the interior of $\Omega(N, \kappa)$, cannot exceed $N \kappa / \Theta^{2}$. Therefore, the former term at the r.h.s. of the previous formula is smaller than $1 / 4 r$ if $\Theta \geq 2 \sqrt{\kappa r}$. As far as the latter is concerned, we can choose $\kappa$ such that $R_{N}(\beta, \kappa) / Q_{N} \leq 1 / 4 r$, as we have shown above. For example, we can fix $\kappa=\log \left(4 r K_{0}\right)$. This suffices to infer that, for $\Theta \geq 2 \sqrt{r \log \left(4 r K_{0}\right)}$, (35) is valid.

## B. 2 Proof of Lemma 6

The first inequality in (39) comes directly from the fact that the integrand appearing in the definition of $Q_{M}$ is smaller than the function $n_{M, \mathfrak{x}}$, i.e., the integrand in the definition of $\overline{\mathcal{Q}}_{M}^{\mathfrak{x}}$.

As regards the second inequality in (39), we note that the integrand of $Q_{M}$ is equal to the one of $\bar{Z}_{M}^{\mathfrak{x}}$ multiplied by $\mathfrak{x}$ terms of the form $\exp \left(-\beta \varepsilon q_{m_{i}} q_{m_{i+1}} / \omega\right)$ at the sites, in number $2 \mathfrak{x}$, on the boundary of the blocks, which we denote by $m_{1}, \ldots, m_{2 \mathfrak{x}}$, with the convention that $m_{2 \mathfrak{x}+1}=m_{1}$. Then, we integrate only in the region in which the $q$ coordinate of each of these sites is smaller than $\Theta \sqrt{2 \omega / \beta}$, with $\Theta=2 \sqrt{2 \mathfrak{x} \log \left(8 \mathfrak{x} K_{0}\right)}$, and we observe that

$$
\begin{aligned}
\bar{Z}_{M}^{\mathfrak{x}} & \geq \exp \left(-4 \mathfrak{x} \varepsilon \Theta^{2}\right) Q_{M} \mathbf{P}_{M}\left(\left|q_{m_{1}}\right|<\Theta \sqrt{2 \omega / \beta} \wedge \ldots \wedge\left|q_{m_{2 \mathfrak{x}}}\right|<\Theta \sqrt{2 \omega / \beta}\right) \\
& \geq \frac{Q_{M}}{2}\left(8 \mathfrak{x} K_{0}\right)^{-32 \varepsilon_{0} \mathfrak{x}^{2}}
\end{aligned}
$$

Here, $Q_{M}$ comes from the normalization of the probability, and in the second line use is made of Lemma 5 .

We come now to inequalities (40) and observe at once that the first one is trivial, because, on account of the identity in (28), one has $\tilde{n}_{M, \mathfrak{x}} \leq n_{M, \mathfrak{x}}$. The second one is more complicated: we begin by proving it in the case in which each block is constituted by an even number of elements.

In order to estimate $\mathcal{Q}_{M}^{\mathfrak{x}}$, we divide again the phase space of the system in the region $\tilde{\Omega}$ in which $\left|q_{m_{1}}\right|<\sqrt{2 \omega \kappa / \beta}, \ldots,\left|q_{m_{2 \mathfrak{r}}}\right|<\sqrt{2 \omega \kappa / \beta}$, where $\kappa>0$ is a constant to be determined, and in its complement $\tilde{\Omega}^{c}$. The integral over $\tilde{\Omega}$ is smaller than $Q_{M} \cdot \exp (4 \mathfrak{x} \varepsilon \kappa)$, while, as regards the complement, we notice that it is contained in the set in which $\sum_{i=1}^{2 \mathfrak{x}} q_{i}^{2} \geq 2 \kappa \omega / \beta$. Thus, the integral over such a region is bounded from above by $\mathcal{Q}_{M-2 \mathfrak{x}}^{\mathfrak{x}_{1}}(2 \pi \omega / \beta)^{\mathfrak{x}} \Gamma(\mathfrak{x}, \kappa)$, with $\mathfrak{x}_{1} \leq \mathfrak{x}$, where we have dropped some positive term in the potentials, then we have integrated first over $q_{m_{1}}, \ldots, q_{m_{2 \mathfrak{x}}}$ (which gives the term $(2 \pi \omega / \beta)^{\mathfrak{x}} \Gamma(\mathfrak{x}, \kappa)$, with $\Gamma(\mathfrak{x}, \kappa)$ defined by $\left.(42)\right)$; then the blocks made of just

2 particles disappear, so that the integration over the remaining positions gives the term $\mathcal{Q}_{M-2 \mathfrak{x}}^{\mathfrak{r}_{1}}$. This way we get

$$
\mathcal{Q}_{M}^{\mathfrak{x}} \leq Q_{M} \cdot \exp (4 \mathfrak{x} \varepsilon \kappa)+\left(\frac{2 \pi \omega}{\beta}\right)^{\mathfrak{x}} \Gamma(\mathfrak{x}, \kappa) \cdot \mathcal{Q}_{M-2 \mathfrak{x}}^{\mathfrak{x}_{1}}, \quad \text { with } \quad \mathfrak{x}_{1} \leq \mathfrak{x}
$$

Now, we apply the previous inequality to the function $\mathcal{Q}_{M-2 \mathfrak{x}}^{\mathfrak{r}_{1}}$ at the r.h.s, and we end up with a relation similar to the previous one, in which however there appears the function $\mathcal{Q}_{M-2 \mathfrak{x}-2 \mathfrak{x}_{1}}^{\mathfrak{x}_{2}}$, with $\mathfrak{x}_{2} \leq \mathfrak{x}_{1}$. So, we can iterate this procedure, observing that $\Gamma(\mathfrak{x}, \kappa)$ is an increasing function of $\mathfrak{x}$, and we get

$$
\mathcal{Q}_{M}^{\mathfrak{x}} \leq \exp (4 \mathfrak{r} \varepsilon \kappa) \sum_{j=0}^{J}\left(\frac{2 \pi \omega}{\beta}\right)^{\sigma_{j}} Q_{M-2 \sigma_{j}}(\Gamma(\mathfrak{x}, \kappa))^{j}, \quad \text { with } \quad Q_{0} \stackrel{\text { def }}{=} 1
$$

where we define $\sigma_{j}=\sum_{k=0}^{j} \mathfrak{x}_{k}$, with $\mathfrak{x}_{0}=\mathfrak{x}$, and $J$ represents the integer such that $\sigma_{J}=M / 2$. We make use of inequality (34) of Lemma 5 and finally get, if the series converges, $\mathcal{Q}_{M}^{\mathfrak{x}} \leq \exp (4 \mathfrak{x} \varepsilon \kappa) Q_{M} \sum_{j=0}^{\infty}\left(K_{0}^{2 \mathfrak{x}} \Gamma(\mathfrak{x}, \kappa)\right)^{j}$. We point out that the common ratio of this geometric series is a decreasing function of $\kappa$, which tends to 0 as $\kappa \rightarrow+\infty$ : thus, we choose $\bar{\kappa}=\bar{\kappa}\left(\mathfrak{x}, K_{0}\right)$ so as to satisfy (41), and obtain the relation

$$
\begin{equation*}
\mathcal{Q}_{M}^{\mathfrak{x}} \leq 2 \exp \left(4 \mathfrak{y} \varepsilon \bar{\kappa}\left(\mathfrak{x}, K_{0}\right)\right) Q_{M} . \tag{71}
\end{equation*}
$$

If a number $\lambda \leq \mathfrak{x}$ of blocks is constituted by an odd number of elements, we integrate on one of the sites on the boundary of each of these blocks, in order that each of the blocks, in number $\mathfrak{x}^{\prime}$, of the resulting lattice contains an even number of elements. By dropping some suitably chosen interaction terms in the potential, one gets $\mathcal{Q}_{M}^{\mathfrak{x}} \leq(2 \pi \omega / \beta)^{\lambda / 2} \mathcal{Q}_{M-\lambda}^{\mathfrak{r}^{\prime}}$, with $\mathfrak{x}^{\prime} \leq \mathfrak{x}$, where the blocks made of just one particles disappear. Now we can use (71) with $Q_{M-\lambda}$ instead of $Q_{M}$. Then, making use of Lemma 5 to express $Q_{M-\lambda}$ in terms of $Q_{M}$, we get (40).

## B. 3 Proof of Theorem 2

As already said, the proof is performed by bounding from above every term at the r.h.s. of (54), i.e., $\lambda(\mathbf{j}, \mathbf{i})$, for $\mathbf{j}, \mathbf{i} \in V$.

We point out that the expectations in the definition of $\lambda(\mathbf{j}, \mathbf{i})$ depend on the choice of $Q$, which is not completely fixed by its marginal probabilities (see comments on relations (51)). In fact, the main part of paper [8] consists in introducing a suitable reconstruction operator (on the space of the joint
probabilities) which enables one to find a joint probability distribution that minimizes $\lambda$, starting from an initially chosen one. We adopt the same technique, with the only difference that we apply it not at all sites of the lattice, but only at those lying on the complement of a fixed set $\bar{V}$ (we will call it $\bar{T} \stackrel{\text { def }}{=} T \backslash \bar{V})$.

We also need to control, together with $\lambda(\mathbf{j}, \mathbf{i})$, the auxiliary quantity

$$
\gamma(\mathbf{i}) \stackrel{\text { def }}{=} \mathbf{E}\left[\mathbf{1}_{\xi_{\mathbf{i}}^{1} \neq \xi_{\mathbf{i}}^{2}}\right]
$$

where $\xi^{1}$ and $\xi^{2}$ are the same Gibbsian fields entering in (55).
We introduce, then, the main tool of the proof, i.e., the reconstruction operator $U_{\mathbf{i}}$, with $\mathbf{i} \in T$, which will enable us to construct the joint probabilities $Q(\mathrm{~d} \mathbf{x}, \mathrm{~d} \mathbf{y})$ on $V$ (see formula (54)). This operator is defined on a couple of fields $\left(\xi^{1}, \xi^{2}\right)$ having the same conditional probability at $\mathbf{i}$, as follows. For each pair of configurations $\mathbf{x}^{1}, \mathbf{x}^{2} \in \mathbb{R}^{|T|}$, we denote by $P_{\mathbf{x}^{1}, \mathbf{x}^{2}}^{\mathbf{i}}$ the measure on $\mathbb{R}^{2}$ for which the minimum of the distance between $P_{\mathbf{i}, \mathbf{x}^{1}}$ and $P_{\mathrm{i}, \mathrm{x}^{2}}$ is attained, i.e., such that, for any measurable $B \subset \mathbb{R}$, one has

$$
\begin{aligned}
& P_{\mathbf{x}^{1}, \mathbf{x}^{2}}^{\mathbf{i}}(\mathbb{R} \times B)=P_{\mathbf{i}, \mathbf{x}^{1}}(B), \quad P_{\mathbf{x}^{1}, \mathbf{x}^{2}}^{\mathbf{i}}(B \times \mathbb{R})=P_{\mathbf{i}, \mathbf{x}^{2}}(B), \\
& \text { and } \int_{\mathbb{R}^{2}} \mathbf{1}_{x \neq y} P_{\mathbf{x}^{1}, \mathbf{x}^{2}}^{\mathbf{i}}(\mathrm{d} x, \mathrm{~d} y)=D\left(P_{\mathbf{i}, \mathbf{x}^{1}}, P_{\mathbf{i}, \mathbf{x}^{2}}\right)
\end{aligned}
$$

Such a definition enables us to describe the action of $U_{\mathbf{i}}$, because this operator maps the couple $\left(\xi^{1}, \xi^{2}\right)$ into $\left(\hat{\xi}^{1}, \hat{\xi}^{2}\right)$ such that, for any measurable $C \subset \mathbb{R}^{2}$,

$$
P\left(\left(\hat{\xi}_{\mathbf{i}}^{1}, \hat{\xi}_{\mathbf{i}}^{2}\right) \in C \mid \hat{\xi}_{T \backslash\{\mathbf{i}\}}^{1}=\mathbf{x}_{T \backslash\{\mathbf{i}\}}^{1}, \hat{\xi}_{T \backslash\{\mathbf{i}\}}^{2}=\mathbf{x}_{T \backslash\{\mathbf{i}\}}^{2}\right)=P_{\mathbf{x}^{1}, \mathbf{x}^{2}}^{\mathbf{i}}(C),
$$

and, for any finite $V \subset T$ not containing $\mathbf{i}$, the joint probability distribution of $\left(\hat{\xi}_{V}^{1}, \hat{\xi}_{V}^{2}\right)$ coincides with that of $\left(\xi_{V}^{1}, \xi_{V}^{2}\right)$.

The effect of $U_{\mathbf{i}}$ on $\gamma(\mathbf{i})$ and $\lambda(u, \mathbf{i})$ is described in detail in Lemmas 2, 3 and 4 of the work [8]. Following such a paper, we adopt the convention that the quantities relative to the reconstructed couple ( $\hat{\xi}^{1}, \hat{\xi}^{2}$ ) are distinguished from the corresponding ones relative to $\left(\xi^{1}, \xi^{2}\right)$ by adding the symbol ${ }^{\wedge}$. For every set $S \subset T$ we define the operator

$$
\begin{equation*}
U_{S} \stackrel{\text { def }}{=} U_{\mathbf{i}_{1}} \circ U_{\mathbf{i}_{2}} \circ \ldots \circ U_{\mathbf{i}_{m}}, \tag{72}
\end{equation*}
$$

where the order of the points $\mathbf{i}_{1}, \ldots, \mathbf{i}_{m}$, contained in $S \cup \partial_{b r} S$ is chosen in a suitable way. This is described in full detail in the proof of the following

Lemma 12. If we define $\gamma_{S} \stackrel{\text { def }}{=} \sup _{\mathbf{i} \in S} \gamma(\mathbf{i})$ and $\lambda_{S} \stackrel{\text { def }}{=} \sup _{\mathbf{j}, \mathbf{i} \in S} \lambda(\mathbf{j}, \mathbf{i})$, we can describe the action of $U_{S}$ on a couple of fields having the same conditional probability on $S \cup \partial_{b r} S$ accordingly to the following lemma, which is proved in Appendix B.4.

Lemma 12 Let $\left(\xi^{1}, \xi^{2}\right)$ be a couple of fields having the same conditional probability on $S \cup \partial_{b r} S$, given by a specification $\Gamma \in \Theta(h, C, \delta) \cap \Delta(h, \bar{K} C, \alpha)$, and $\left(\hat{\xi}^{1}, \hat{\xi}^{2}\right)=U_{S}\left(\xi^{1}, \xi^{2}\right)$. Then, one has

$$
\binom{\hat{\gamma}_{S}}{\hat{\lambda}_{S}} \leq A\binom{\gamma_{S \cup \partial_{b r} S}}{\lambda_{S \cup \partial_{b r} S}}
$$

in which the matrix $A$ is defined by

$$
A \stackrel{\text { def }}{=}\left(\begin{array}{cc}
\alpha+N \bar{K}^{-1} & C^{-1} M \bar{K}^{-1} \\
C\left(R+N \bar{K}^{-1}\right) & \delta a^{b}+M \bar{K}^{-1}
\end{array}\right),
$$

where $N, M$ and $R$ are constants depending on $a$ and $b$ only.
We remark that, if the eigenvalues of $A$ are smaller than 1 , the reconstructed quantities are smaller than the initial ones. So, we want to iterate the reconstruction procedure as much as possible. It turns out that we can iterate the procedure at most a number of times proportional to the distance between $V$ and $\tilde{V}$. The reason is the following.

In our case, $\xi^{1}$ is the field relative to the equilibrium Gibbs measure and $\xi^{2}$ that relative to the probability conditioned to the configuration $\mathbf{x}$ on $\tilde{V}$, which we consider as fixed. It is apparent that such fields have the same conditional probability on every set which does not intersect $\tilde{V}$, but not on the whole $T$; by hypothesis, this conditional probability is that given by $\Gamma \in \Theta(h, C, \delta) \cap \Delta(h, \bar{K} C, \alpha)$. Since the reconstuction procedure shrinks the set $S$ on which we can control $\gamma$ and $\lambda$, we can iterate it until $V \subset S$. So, the maximum number of iterations is attained if we start by reconstructing on $V \cup \partial_{n b r} V$, where $n$ is the largest number such that $\partial_{(n+1) b r} V \cap \tilde{V}=\emptyset$. We use Lemma 12 as the first step of a recurrent scheme, by applying each time $U_{V_{m}}$, where $V_{m+1}=V_{m} \cup \partial_{b r} \tilde{V}_{m}, V_{0}=V$. In virtue of Lemma 12, after the application of $U_{V_{m}}$, one has

$$
\binom{\hat{\gamma}_{V_{m}}}{\hat{\lambda}_{V_{m}}} \leq A\binom{\gamma_{V_{m+1}}}{\lambda_{V_{m+1}}} .
$$

Thus we get that the final values of $\gamma_{V}$ and $\lambda_{V}$ are smaller than the result of the application of the matrix $A^{n}$ to the vector with components $\gamma, \lambda$.

Moreover, we observe that we can write $A=J^{-1} \tilde{A} J$, where $\tilde{A}$ and $J$ are defined by

$$
\tilde{A} \stackrel{\text { def }}{=}\left(\begin{array}{cc}
\alpha+N \bar{K}^{-1} & M \bar{K}^{-1} \\
R+N \bar{K}^{-1} & \delta a^{b}+M \bar{K}^{-1}
\end{array}\right) \quad \text { and } J \stackrel{\text { def }}{=}\left(\begin{array}{cc}
C & 0 \\
0 & 1
\end{array}\right) .
$$

This way we get $A^{n}=J^{-1} \tilde{A}^{n} J$. As the component $\lambda_{V}$, which is the one we are interested in, is not affected by the action of $J^{-1}$, we can write that it is the second component of the matrix product

$$
\tilde{A}^{n}\binom{C \gamma}{\lambda} .
$$

Since the eigenvalues of $\tilde{A}$ are smaller than $G \stackrel{\text { def }}{=} \max \left\{(\alpha+1) / 2,\left(\delta a^{b}+\right.\right.$ 1) $/ 2\}<1$, if $\bar{K}$ is large enough, there exists $\bar{K}_{0}$ such that

$$
\lambda_{V} \leq(D / 2) \max \{\lambda, C \gamma\} G^{n},
$$

where $D$ is a constant depending on $a, b, \alpha, \delta$ only. On the other hand, $n=d(V, \tilde{V}) /(b r)$ and $\gamma \leq 1$, from which there follows

$$
\lambda_{V} \leq \frac{D}{2}\langle h\rangle_{\mathbf{x}} \exp (-c d(V, \tilde{V})),
$$

where $c$ is defined in (58).
In order to show (57), we need only the use of (54) in estimating the term in brackets of (49). We then observe that the r.h.s. of (54) is smaller than $2|V|^{2} \lambda_{V}$, for the joint probability we have just found, and this concludes the proof.

## B. 4 Proof of Lemma 12

Lemma 5 of work [8] shows the result of the application of $U_{\mathbf{i}}$, in a suitably chosen order, to every site of $T$ in sequence: one obtains that, for the couple of fields $\left(\xi^{1}, \xi^{2}\right)$, with the same specification $\Gamma \in \Theta(h, C, \delta) \cap \Delta(h, \bar{K} C, \alpha)$ and $\bar{K} \geq 1$, and for the reconstructed couple $\left(\hat{\xi}^{1}, \hat{\xi}^{2}\right)$ the following matrix relation holds ${ }^{9}$

$$
\begin{equation*}
\binom{\hat{\gamma}_{T}}{\hat{\lambda}_{T}} \leq A\binom{\gamma_{T}}{\lambda_{T}} . \tag{73}
\end{equation*}
$$

[^7]In the proof of Lemma 5 of [8], the order of the $U_{\mathrm{i}}$ 's is chosen in the following way: the lattice is partitioned in $b$ disjoint sublattice, $Z_{0}, \ldots, Z_{b-1}$, which are the cosets in $T$ of $\mathbb{Z}^{\nu}$ with respect to $Z_{0}$. Then the reconstruction is applied in sequence to each sublattice, and use is made of the fact that, if $\mathbf{i} \in Z_{l}$, there exists a bound to $\gamma(\mathbf{j}), \lambda(\mathbf{j}, \mathbf{k})$ and $\lambda(\mathbf{k}, \mathbf{j})$ for $\mathbf{j} \in Z_{l} \backslash\{\mathbf{i}\}$ and $\mathbf{k} \in T \backslash\{\mathbf{i}\}$ which does not change after the application of $U_{\mathbf{i}}$. In particular, this implies that the bounds do not change for the already reconstructed sites in $Z_{l}$. Neither does the reconstruction at site $\mathbf{i}$ change, on account of Lemma 4 of paper [8], the value of $\lambda(\mathbf{i}, \mathbf{j})$ and $\lambda(\mathbf{j}, \mathbf{i})$, for $|\mathbf{j}-\mathbf{i}|>r$. In this sense the reconstruction is local.

So, the values of $\hat{\gamma}_{V}$ and $\hat{\lambda}_{V}$, after one application of $U_{\mathbf{i}}$, depend at most on the values of $\gamma(\mathbf{i})$ and $\lambda(\mathbf{j}, \mathbf{i})$ in $V \cup \partial_{r} V$. It is thus appearent that we can control the values of the reconstructed quantities only in a set $V$ smaller than the set $V^{\prime}$ on which we control $\gamma$ and $\lambda$ initially. In particular, $V$ can be chosen so that $V^{\prime}=V \cup \partial_{r} V$. Therefore, for any $V \subset S$, we define for $l=0, \ldots, b-1$ a nested sequence of sets $V_{l+1} \stackrel{\text { def }}{=} V_{l} \cup \partial_{r} V_{l}$, with $V_{0} \stackrel{\text { def }}{=} V$ and the operator $U_{V}$, as $U_{V}=U_{V_{0} \cap Z_{0}} \circ \ldots \circ U_{V_{b-1} \cap Z_{b-1}}$, and notice that (see the above remarks) the order in which the sites in $V_{l} \cap Z_{l}$ are chosen does not matter. Then, after the application of $U_{V}$, one has that

$$
\binom{\hat{\gamma}_{V}}{\hat{\lambda}_{V}} \leq A\binom{\gamma_{V_{b}}}{\lambda_{V_{b}}}
$$

for the same matrix $A$ appearing in (73). This concludes the proof.

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[^1]:    ${ }^{1}$ Notice that in paper [5] the $\chi_{s}$ were required to be homogeneous polynomials of degree $s+2$.However, there is no problem in considering the present more general case.

[^2]:    ${ }^{2}$ We adopt here the multi-index notation: $k=k_{1}, \ldots, k_{N}$ and $l=l_{1}, \ldots, l_{N}$ are vectors of integers, with $|k|=\left|k_{1}\right|+\ldots+\left|k_{2}\right|$. So, $p^{k} q^{l}=p_{1}^{k_{1}} \cdot \ldots \cdot p_{N}^{k_{N}} q_{1}^{l_{1}} \cdot \ldots \cdot q_{N}^{l_{N}}$.
    ${ }^{3}$ One can check that this is indeed a norm.

[^3]:    ${ }^{4}$ Notice that the index $i$ lies in $\left\{1, \ldots, \gamma^{\prime}\right\}$. But if in the expression $\exp \left(\beta \varepsilon q_{l_{i}} q_{l_{i+1}} / \omega\right)$ there appears $q_{\gamma^{\prime}+1}$ then one has to intend simply $l_{\gamma^{\prime}+1}$ as $l_{1}$.

[^4]:    ${ }^{5}$ See later for the definition of a compact function, according to the convention of paper [8].

[^5]:    ${ }^{6}$ Notice that one can find "pathological" functions for which the decay is arbitrarily slow (see [24]) even for strongly chaotic systems. For this reason, the control is usually restricted to a fixed continuity class.
    ${ }^{7}$ As a matter of fact, we cannot guarantee that $a(t)$ is invertible, but we will give below a meaningful univocal definition of $t(a)$ (see definition 2).

[^6]:    ${ }^{8}$ Remark that the function $a(\beta, \varepsilon)$ depends on $\varepsilon$ only via the term $\omega=\sqrt{1+2 \varepsilon}$.

[^7]:    ${ }^{9}$ As a matter of fact, $A$ is not the same matrix which appears in [8], since we needed to make the dependence on $C$ explicit. Our statement can be proved by checking, in proving the induction (11)-(14) of [8], that the constants $N(\cdot, \cdot)$ are proportional to $C^{2}$, the constants $N(\cdot)$ and $M(\cdot, \cdot)$ are proportional to $C$ and the constants $M(\cdot)$ are independent of $C$.

