On the existence of scattering solutions for the Abraham–Lorentz–Dirac equation

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Abstract. It is well known that in the presence of an attractive force having a Coulomb singularity, scattering solutions of the nonrelativistic Abraham–Lorentz–Dirac equation having nonrunaway character do not exist for the case of motions on the line. By numerical computations on the full three dimensional case, we give indications that indeed there exists a full tube of initial data for which nonrunay solutions of scatterig type do not exist. We also give a heuristic argument which allows to estimate the size of such a tube of initial data. The numerical computations also show that in a thiny region beyond such a tube one has the nonuniqueness phenomenon, i.e. the "mechanical" data of position and velocity do not uniquely determine the nonrunaway trajectory.

1 Introduction.

It is usually assumed that the motion of a charged point particle in self-interaction with the electromagnetic field is described by the Abraham–Lorentz–Dirac equation. In its nonrelativistic version, to which we restrict our attention in the present paper, such an equation takes the form

$$m\ddot{\mathbf{x}} = \mathbf{F}^{ext}(\mathbf{x}) + \frac{e^2}{c^3}\dddot{\mathbf{x}} \ , \tag{1}$$

where $\mathbf{x}(t) \in \mathbf{R}^3$ denotes the position of the point charge at time t, $\mathbf{F}^{ext}$ is an external force field, while the term $e^2\dddot{\mathbf{x}}/c^3$ describes the reaction force due to the selfinteraction with the electromagnetic field; m, e are the mass and the charge of the particle respectively, while c is the speed of light. This is a third–order equation, so that the Cauchy problem requires to assign in addition to the usual "mechanical" initial data, namely position $\mathbf{x}_0$ and velocity $\mathbf{v}_0$, also the initial acceleration $\mathbf{a}_0$ (which should be understood as defined by the Cauchy data for the complete system of particle and field, as shown in [1]). On the other hand, it is well known that a generic initial acceleration $\mathbf{a}_0$ leads to absurd "runaway solutions", and in this connection we just follow the point of view of Dirac[2], which consists in considering as physically relevant only those solutions which have nonrunaway character, i.e. satisfy (in the case of scattering) the prescription $\ddot{\mathbf{x}} \to 0$ as $t \to +\infty$. Let us recall that, according to Dirac, such a prescription constitutes " *a striking departure from the usual idea of mechanics*", but on the other hand " *will lead to the most beautyful feature of the theory*".

The aim of the present paper is to investigate whether there exist nonrunaway solutions in the scattering of an electron from an attractive Coulomb center of force. Such a problem should be of interest in itself, but a special motivation comes from the desire of clarifying a rather paradoxical result, that was obtained long ago by Eliezer [3] (see [4] for the relativistic version), namely that, in the particular case of head–on collisions, the Abraham–Lorentz–Dirac equation *does not admit* any nonrunaway solution of scattering type. More precisely, it was actually proven that the only nonrunaway solutions are the motions that at a certain time spring out from the singularity. Instead, in the case of scattering, i.e. when it is assumed that there exists some asymptotic velocity for the particle at $t = -\infty$ directed towards the singularity, irrespective of the choice of the asymptotic acceleration the resulting motions were proven to be always of runaway type (i.e. non physical according to Dirac).

On the other hand, it is also true that the special initial data considered by Eliezer have zero mesure in the three–dimensional problem, so that they might be ignored from a physical point of view. So there naturally arises the question: if the initial "mechanical data" (i.e. position and velocity) are taken near those leading to a head–on collision, does there exist an initial acceleration leading to a nonrunaway solution, or rather will all motions be of runaway type, as in the case of head–on collisions? We will give numerical evidence that the set of mechanical states, for which nonrunaway solutions exist, is indeed open and, in a sense we will explain below, very large. We will also give an argument which allows to understand why this phenomenon occurs, and to give a rough estimate of the size of the region for which nonrunaway solutions do not exist.

The paper is organized as follows: in Section 2 a general mathematical discussion of the problem is given, together with an exposition of the argument mentioned above (the proofs being deferred to Appendix A); in Section 3 the numerical results are illustrated (error bounds being discussed in Appendix B), while a short discussion of the results is given in Section 4.

2 The Abraham–Lorentz–Dirac equation in a central force field.

In order to study equation (1) in the case of the Coulomb force, i.e. with

$$\mathbf{F}^{ext} = -Ze^2 \frac{\mathbf{x}}{r^3} \qquad r = |\mathbf{x}| \ ,$$

where Z is the atomic number, one first of all reduces it, by a suitable change of units[1], to the dimensionless form

$$\ddot{\mathbf{x}} = -\frac{\mathbf{x}}{r^3} + \dddot{\mathbf{x}} \ . \tag{2}$$

It is easy to verify that the non–runaway solutions are indeed planar just as in the familiar Newton case, in which the selfinteraction force is neglected. This is seen as follow. Considering the triple product $\ddot{\mathbf{x}} \cdot (\dot{\mathbf{x}} \times \mathbf{x})$, it is easily checked that for any solution $\mathbf{x}$ of (2) one has

$$\frac{\mathrm{d}}{\mathrm{d}t}(\ddot{\mathbf{x}} \cdot (\dot{\mathbf{x}} \times \mathbf{x})) = \ddot{\mathbf{x}} \cdot (\dot{\mathbf{x}} \times \mathbf{x}) \ .$$

Thus either the triple product is zero for all times, so that the motion is planar, or it will grow exponentially, which is impossible for a nonrunaway solution (because the property $\ddot{\mathbf{x}} \to 0$ implies that the triple product can grow at most as a power).

The description of the scattering can be given, in the familiar way, in terms of the asymptotic velocity and of the impact parameter, or equivalently in terms of the asymptotic values of the energy E and of the angular momentum (actually, its component normal to the plane of motion, which we denote by L), defined as usual by

$$E = \dot{\mathbf{x}} \cdot \dot{\mathbf{x}}/2 - 1/r \quad , \quad L = \mathbf{x} \times \dot{\mathbf{x}} \cdot \mathbf{k} \ ,$$

$\mathbf{k}$ being the unit vector normal to the plane of motion. We concentrate our attention on the map $(E_i, L_i) \to (E_f, L_f)$ between the asymptotic values of E and L before and after the scattering respectively:

$$E_i = \lim_{t\to-\infty} E(t) \ , \quad E_f = \lim_{t\to+\infty} E(t) \, ,$$
$$L_i = \lim_{t\to-\infty} L(t) \ , \quad L_f = \lim_{t\to+\infty} L(t) \, .$$

Obviously such a map reduces to the identity if the selfinteraction is neglected, because the energy and the angular momentum are then constants of motion. This will no more be the case if the selfinteraction is taken into account, because the particle is then found to radiate both energy and angular momentum. Indeed we have the following

[1] One has to take m as unit of mass, $\tau_0 = 2e^2/3mc^3$ as unit of time and $r_0 = \sqrt[3]{4Z/9}\ e^2/mc^2$ as unit of lenght. Note that one has then a Z–dependent unit of action, namely $L_0 = \sqrt[3]{2Z^2/3}\ e^2/c$, which is not very dissimilar from Planck's constant $\hbar \simeq 137\, e^2/c$.

Theorem 1 *Assume there exists a nonrunaway solution* $\mathbf{x}(t)$ *of equation (2) with asymptotic data* E_i, L_i *for* $t \to -\infty$. *Then the asymptotic values* E_f, L_f *for* $t \to +\infty$ *exist, and one has*

$$\Delta E \stackrel{def}{=} E_f - E_i = -\int_{\mathbf{R}} \ddot{x}^2(t)\, dt \tag{3}$$

$$\Delta L \stackrel{def}{=} L_f - L_i = -\int_{\mathbf{R}} \frac{L(t)}{r^3(t)}\, dt \ . \tag{4}$$

In addition, $L(t)$ *is a monotonous function having a constant sign, and all non–runaway trajectories are convex.*

A sketch of the proof is given in Appendix A. We add here a comment on the fact that the existence of the two limits E_i and L_i for $t \to -\infty$ was assumed rather than proved. This is due to the fact that proving it turns out to be deeply linked to proving the very existence of scattering solutions, which in the present case is far from be trivial. Indeed, the standard existence theorem (see [5]) doesn't apply here because of the singular character of the force. One could prove the existence of such limits, but only for a certain set of initial asymptotic data. As one of the aims of the paper is to determine by numerical computations the set for which scattering solutions exist, we rely on the numerical evidence that indeed such solutions exist do (see the next section).

From the above Theorem 1 one can deduce rather easily an inequality which will play a central role in the present paper. For definiteness let us consider the case in which L_i is positive (which can always be obtained by a suitable choice of the orientation of the reference frame); then one has

$$L_i > \int_\gamma \frac{\mathrm{d}\varphi}{r} \ , \tag{5}$$

where γ is the trajectory parameterized by the polar angle φ; this is possible because, as stated in Theorem 1, L has constant sign so that the angular velocity never vanishes. The inequality (5) follows from relation (4), remembering that L_f is nonnegative and using φ as an independent variable.

Inequality (5) suggests that, having fixed the initial energy E_i (or equivalently the initial velocity v_i), the allowed values of the impact parameter b cannot be too small, because, if one lets b decrease to zero, the l.h.s also tends to zero, while the integral at the r.h.s is expected to diverge. In other terms the possible values of L_i leading to scattering nonrunaway motions are expected to have a positive lower bound (depending on the initial energy E_i). We thus expect that there exists a function $L^{cr}(E_i)$ such that nonrunaway scattering motions can exists only if the initial angular momentum L_i is larger than $L^{cr}(E_i)$. Equivalently, one can say that nonrunaway motions cannot exist within a tube of a certain critical radius $b^{cr}(v_i)$ around the straight line of head–on collisions. In particular the result of Eliezer is thus recovered. In other terms, the Dirac manifold (i.e. the manifold of the nonrunaway motions) is “pierced”.

The existence of the critical functions $L^{cr}(E_i)$, $b^{cr}(v_i)$ just introduced will be supported by the numerical computations that will be illustrated in the next section. In what follows we give instead a qualitative argument for the divergence of the integral at the r.h.s of relation (5) when the impact parameter tends to zero. From the convexity property of the trajectories one expects that they are “closer” to the center of force with respect to the asymptotes. In this way, substituting to the true trajectory the straight line corresponding to one of the asymptotes the value of the integral would decrease. As the asymptote has equation (in polar coordinates) $r = b/\sin\varphi$, one gets

$$\int_\gamma \frac{\mathrm{d}\varphi}{r} \geq \int_0^\pi \mathrm{d}\varphi \, \frac{\sin\varphi}{b} = \frac{2}{b} \ ,$$

which diverges as b tends to zero. Using this rough estimate for the integral, one obtains from (5) a lower bound for the function L^{cr}, namely $L^{cr}(E) > \sqrt[4]{E/2}$.

3 Numerical results.

As we saw in the previous section, the problem of the existence of scattering nonrunaway solutions can be addressed by studying the map from (E_i, L_i) to (E_f, L_f). Actually it happens that this map is non regular, presenting several branches. This can be explained as follows: in order to obtain a nonrunaway solution one has to give besides the mechanical initial data of position $\mathbf{x}_0$ and velocity $\mathbf{v}_0$, the particular initial acceleration $\mathbf{a}_0$ which should give rise to a solution with $\mathbf{a}(t) \to 0$. But it may happen that, for a given mechanical state, there exist several such initial accelerations. In such a case the physical manifold is folded, in the sense that to a given mechanical state there correspond several physical motions; the different motions will then have different final mechanical states so that the map from (E_i, L_i) to (E_f, L_f) will have several branches. We recall that for the scattering on a line by a potential barrier it was proved (see [6], [7]) that the folding is present for a certain set of initial data and of parameters of the potential. We will see that indeed numerical evidence suggests that also in the present case one meets with the phenomenon of nonuniqueness of nonrunaway solutions. For this reason, it is more convenient, from the mathematical point of view, to study the inverse map $(E_f, L_f) \stackrel{\Phi}{\to} (E_i, L_i)$, which turns out to be a regular function. Moreover, the function Φ is the object which is actually determined by the numerical computations. In fact, it is very difficult to integrate directly equation (2), because nonrunaway solutions are unstable, so that the integration errors grow rapidly and the motion quickly becomes

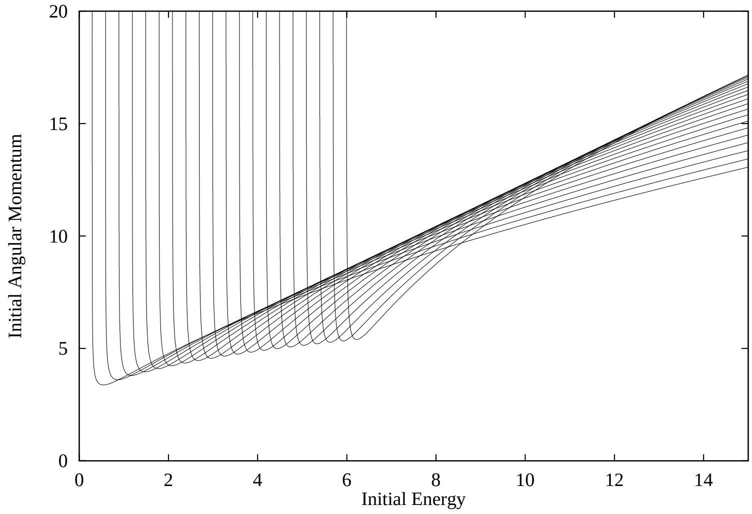

Figure 1: Plot of the map

runaway. Instead, the runaway solutions are stable for the backward flow, so that, if one integrates backward in time, after a small transient the motions practically coincide with non-runaway solutions. So, in order to integrate the equation, one chooses some final data, i.e. the final value E_f, L_f of energy and angular momentum, a final distance from the center of force larger than a certain threshold value R_0 and a vanishing final acceleration. Then one integrates backwards in time the equation, stopping the integration when the distance becomes again larger than the threshold R_0. One finally proceeds to compute the quantity E_i, L_i. Obviuosly the choice of the threshold value R_0 affects the results, but it can be estimated (see Appendix B) that the errors on the energy are of order $1/\sqrt{E_f}R_0^3$, while the errors on the angular momentum are of order $L_i/\sqrt{E_f}R_0^2$. In our computations we take $R_0 = 100$, which assures that the relative errors are less than 10^{-4}. The numerical integrations were performed using a standard Runge–Kutta fourth order method. The integration step h was chosen variable at every step using the formula $h = 10^{-3}|\mathbf{x}|$; this assures that the step is small only when the force tends to diverge, and relatively large when the particle is essentially free. In this way the integration times can be kept small. In order to check the consistency of our computations, we changed both the cut–off distance and the integration step, and the changes in the results were found to be within the mentioned bound. In addition we checked that, when the particle is far from the force center, it moves essentially following the Coulomb motions; moreover, by increasing the cut–off distance one obtains solutions which have nonrunaway character over increasing long distances, while the value of energy and angular momentum seem to settle to some definite asymptotic values. This gives a numerical support to the conjecture that scattering solutions exist and can be computed through backward integration.

In this way the function Φ can be built up point by point. From Theorem 1, one knows that the function Φ maps the first quadrant in a subset of the same quadrant. We chose to represent graphically this function in the following way (see Figure 1). Taking a segment in the plane (E_f, L_f), its image by Φ will be a curve in the plane (E_i, L_i). Correspondingly, taking a pencil of parallel segment we obtain a pencil of curves. In Figure 1 we report the curves obtained by applying Φ to a pencil of segments parallel to the L_f axis. The shape of the curves can be understood in the following way. For fixed energy, to high angular moments there correspond high impact parameters, so that the motions remain a large distance from the center of force. In this case the particle radiates a small amount of energy and angular momentum, so that the map Φ is very near to the identity, and the curves essentially coincide with the original pencils of segments. Now, if we decrease the final angular momentum, the distance from the singularity decreases and correspondingly the energy and momentum radiated increase and the curves become more distorted. The appearance of a minimum is explained as follows. As we have seen in the previous Section (inequality (5)), at a fixed energy the initial angular momentum L_i has a minimum value $L^{cr}(E_i)$; so, initially, to a decreasing of the final angular momentum there corresponds a decreasing also of the initial one, but after L_i reached its minimal value it starts increasing even if L_f is decreased.

Looking at Figure 1 one has a first information: the envelope of all curves gives the boundary of the region (i.e. the one below the envelope) in which nonrunaway scattering solutions do not exist. From the points of the graphic one can estimate that this boundary is well above the estimate for $L^{cr}(E_i)$ given at the end of Section 2; in particular it is not clear whether $L_i \to 0$ when $E_i \to 0$ as the estimate suggests. It may be possible that $L_i > L_0$ where L_0 is a constant independent of the initial energy.

The figure also suggests that in some region of the phase space the nonuniqueness phenomenon is present. In fact the figure shows that there are different curves that do intersect, so that an initial datum corresponding to a point of intersection gives rise to different solutions with different final data (because the final points lie on different segments). In other terms, one has that the nonrunaway manifold is folded. In particular it seems that at least three branches are present. At the moment we have no theoretical explanation for this folding, and we are not even sure of the number of the branches which are present (three or more).

4 Further comments.

We have shown that there exists a set of mechanical initial data (position and velocity) for which no nonrunaway scattering solutions exists. One can wonder whether, for such initial data, there exist other type of nonrunaway solutions. For example, in the relativistic Coulomb case, if the initial angular momentum is below a certain threshold, the particle falls on the center of force. But, in our case, in a recent paper (see [8]) a theorem was proven according to which it is impossible for a solution of the Abraham–Lorentz–Dirac to fall on the center of force in a finite or infinite time. There remains open the possibility of having motions which are bounded in a open region and never fall on the center of force. But it is in general believed by physicists that motions of this type are forbidden for energy conservation reasons (the particle would radiate an infinite amount of energy). So it may be the case that really there aren't any nonrunaway solutions for such initial data, all solutions being of runaway type. The presence of a set of initial data for which no nonrunaway solutions exist is very pleasent from the mathematical point of view; because it would show that the Eliezer result is not an oddness, but is a particular case of a more general situation.

From the physical point of view such a situation would be much less pleasent, because it is usually assumed that it is possible to assign initial data at will, in particular the values of the possible initial data cannot be affected by the presence or by the absence of a force in the future (but in this connection see also [9]). On the other hand, we want to stress that, when discussing scattering experiments, a cut–off on the impact parameter b (or equivalently a cut–off in the angular momentum) is often imposed in order to correctly compute a quantity of interest, for example in computing the emission spectrum of bremsstrahlung. In general such cut–offs are imposed by considerations external to the theory itself. For example, it is assumed that the theory in no more valid when the initial angular momentum is below $\hbar$, i.e. Planck's constant, so that initial impact parameters below a certain threshold have to be ignored. Instead, the Abraham–Lorentz–Dirac equation seems to offer an internal criterion in imposing the cut–off, whose order of magnitude by the way turns out to be not very dissimilar from $\hbar$ (see also [10]).

Obviously, one might ask to what extent can the results thus found be physically relevant, because in the region of interest the particle has a velocity larger than the speed of light, so that one should use the full relativistic Abraham–Lorentz–Dirac equation. It is known (see [4]) that even in the relativistic case there exist no nonrunaway scattering solutions for head–on collisions; the mathematical study of the full threedimensional case is more difficult than in the nonrelativistic case, and at present we are unable to say anything definite. We have performed some preliminary numerical computations, which seem to indicate that even in the relativistic case there exists a region in which there are no nonrunaway solutions of scattering type. We plan in the future to study the relativistic case, both analytically and numerically, in order to give some definite answer to the question of the existence of nonrunaway scattering solutions.

Appendix A. Proof of Theorem 1.

Multiplying (2) by $\dot{\mathbf{x}}$ one has

$$\frac{\mathrm{d}}{\mathrm{d}t}(\dot{\mathbf{x}}^2/2 - 1/r - \ddot{\mathbf{x}}\cdot\dot{\mathbf{x}}) = -\ddot{\mathbf{x}}^2 \ ,$$

so that by integration one obtains

$$\dot{\mathbf{x}}^2/2 - 1/r - \ddot{\mathbf{x}}\cdot\dot{\mathbf{x}} = E_i - \int \ddot{\mathbf{x}}(t)\,\mathrm{d}t \ , \tag{6}$$

which can be written has

$$\frac{1}{2}(\dot{\mathbf{x}} - \ddot{\mathbf{x}})^2 = E_i - \int \ddot{\mathbf{x}}^2(t)\mathrm{d}t + \frac{1}{r} + \frac{1}{2}\ddot{\mathbf{x}}^2 \ .$$

The r.h.s converges when $t \to +\infty$, because by assumption $r \to +\infty$ and $\ddot{\mathbf{x}} \to 0$, while the integral of a positive function has always a limit (finite or infinite). In particular the limit of the integral has to be finite, otherwise the the r.h.s. would be negative and this in contraddiction with the fact that the l.h.s. is always positive. Then, as the r.h.s. has a limit, also the l.h.s. admits a limit, and it is easy to check that the limit of l.h.s. is just E_f, and thus the relation (3) is proven. It also follows that E_f is finite, and this obviuosly implies that also $|\dot{\mathbf{x}}|$ has a finite limit.

The proof of the second relation (4) is a little more complicated. Multiplying equation (2) vectorially by $\mathbf{x}$, we obtain the following equation for the angular momentum L

$$\dot{L} = \ddot{L} - \ddot{\mathbf{x}}\times\dot{\mathbf{x}} \ . \tag{7}$$

At variance with the Newton case, the r.h.s. does not vanish. The term $\ddot{\mathbf{x}}\times\dot{\mathbf{x}}$ can be expressed as a function of the angular momentum in the following way: multiplying now equation (2) vectorially by $\dot{\mathbf{x}}$ one obtains

$$\ddot{\mathbf{x}}\times\dot{\mathbf{x}} = -\frac{\mathbf{L}}{r^3} + \frac{\mathrm{d}}{\mathrm{d}t}(\ddot{\mathbf{x}}\times\dot{\mathbf{x}}) \ ,$$

which can be rewritten, for a nonrunaway solution, as

$$\ddot{\mathbf{x}}\times\dot{\mathbf{x}} = \int_t^{+\infty} \mathrm{d}s\,\frac{e^{t-s}}{r^3}\mathbf{L} \ . \tag{8}$$

Note that the quantity $\ddot{\mathbf{x}}\times\dot{\mathbf{x}}$ has a geometric meaning, being linked to the curvature radius R by $|\ddot{\mathbf{x}}\times\dot{\mathbf{x}}| = \dot{\mathbf{x}}^2/R$. In particular, as L can be shown to have a constant sign, the curvature does not vanish, so the trajectory is convex. Now, using expression (8) in (7), one finds an integro–differential equation, which can be put in the form

$$\dot{L} = -\int_t^{+\infty} \mathrm{d}s\,(t-s)\frac{e^{t-s}}{r^3}L\ , \tag{9}$$

or, integrating once more, in the form:

$$\begin{aligned} L(t_1) - L(t_0) = & - \int_{t_0}^{t_1} \mathrm{d}s\,\frac{L}{r^3} + \int_{t_1}^{+\infty} \mathrm{d}s\,(t_1 - s - 1)e^{t_1 - s}\frac{L}{r^3} + \\ & - \int_{t_0}^{+\infty} \mathrm{d}s\,(t_0 - s - 1)e^{t_0 - s}\frac{L}{r^3}\ . \end{aligned} \tag{10}$$

Now, we have just shonw that $|\dot{\mathbf{x}}|$ has a finite limit, so that $L/r^3 \le |\dot{\mathbf{x}}|/r^2$ tends to zero as $t \to \pm\infty$. Then, letting $t_0 \to -\infty$ and $t_1 \to +\infty$, by applying the Lebesgue theorem to the second and third integral at the r.h.s, one sees that the two integrals vanish so that one has (4).

We show now that $L_f = \lim_{t\to+\infty} L(t)$ indeed exists (i.e. the integral in (4) is finite). First of all note that from equation (9) one has the following alternative: either i) $L(t)$ has constant sign, or ii) the set $\{t_k\}$ of the zeroes of $L(t)$ is not limited from above. In fact, if $\sup\{t_k\} = \bar{t} < +\infty$, one has $L(\bar{t}) = 0$, because L is a continuous function, and, for $t > \bar{t}$, $L(t)$ has constant sign, for example $L > 0$. But this is impossible because from (9) one has $\dot{L}(\bar{t}) < 0$, so one would have $L(t) < 0$ for $t > \bar{t}$. The same occurs if one supposes that $L(t) < 0$ for $t > \bar{t}$. Now, in case i) one has that $L(t)$ is a monotone function, $|L(t)|$ is decreasing so L_f is finite.

Let us show that ii) leads to a contradiction. First we prove that not only the limit L_f exists, but also that one has $L_f = 0$. In fact, using (10), with $t_0 = t_k$ and $t_1 = t_{k+1}$ one has

$$0 = L(t_{k+1}) - L(t_k) = \int_{t_k}^{t_{k+1}} \frac{L}{r^3}\,\mathrm{d}t + O(1)\ ,$$

because, as we have already remarked, when $k \to +\infty$ the second and third integrals at the r.h.s of (10) vanish. We can then conclude that

$$\lim_{k\to+\infty} \int_{t_k}^{t_{k+1}} \frac{L}{r^3}\,\mathrm{d}t = 0\ .$$

Now if $\bar{t}_k \in [t_k, t_{k+1}]$ is the point of maximum for $L(t)$ in that interval, one has

$$|L(\bar{t}_k)| \le |\int_{t_k}^{\bar{t}_k} \frac{L}{r^3}\,\mathrm{d}t| + O(1) < |\int_{t_k}^{t_{k+1}} \frac{L}{r^3}\,\mathrm{d}t| + O(1)\ ,$$

where the second inequality is due to the fact $L(t)$ has constant sign on the interval $[t_k, t_{k+1}]$; letting $k \to +\infty$ one has then $|L(\bar{t}_k)| \to 0$, so that $L_f = 0$.

Let us show now that L_f cannot be zero. Suppose that on the contrary $L_f = 0$; then, integrating (9) from t to $+\infty$, one gets

$$L(t) = \int_t^{+\infty} \mathrm{d}s\, \frac{1-(t-s-1)e^{t-s}}{r^3} L \ .$$

As $r \to +\infty$, there will be a $\tilde{t}$ such that for all $t \geq \tilde{t}$ one has

$$\int_t^{+\infty} \mathrm{d}s\, \frac{(1-(t-s-1)e^{t-s})}{r^3} < 1/2 \ .$$

Notice that the integrand is positive, so that one gets, for $t \geq \tilde{t}$,

$$|L(t)| \leq \sup_{t>\tilde{t}} |L| \int_t^{+\infty} \mathrm{d}s\, \frac{(1-(t-s-1)e^{t-s})}{r^3} \leq \frac{\sup_{t>\tilde{t}} |L|}{2} \ .$$

Now, taking the sup to l.h.s one has the contradiction $\sup |L| \leq 1/2 \sup |L|$. So the only possible alternative is i), which implies that L is monotone and has a definite sign. This concludes the proof of the theorem.

Appendix B. Estimate of the numerical errors.

As we have seen in Appendix A, defining $\mathcal{E} = \dot{\mathbf{x}}^2/2 - 1/|\mathbf{x}| - \dot{\mathbf{x}} \cdot \ddot{\mathbf{x}}$, one has (see equation (6))

$$|\mathcal{E}(\pm\infty) - \mathcal{E}(t_0)| = |\int_{t_0}^{\pm\infty} \ddot{\mathbf{x}}^2\, \mathrm{d}t| \ .$$

Now for $|\mathbf{x}| \gg 1$ one has $\ddot{\mathbf{x}} \simeq -\mathbf{x}/r^3$, so that one can make the estimate

$$|\mathcal{E}(\pm\infty) - \mathcal{E}(t_0)| \simeq |\int_{t_0}^{\pm\infty} \frac{1}{r^4}\, \mathrm{d}t| \ .$$

On the other hand, because of $r(t) > v_f(t-t_0) + R_0$ (where v_f denotes the final asymptotic velocity) one gets

$$|\mathcal{E}(\pm\infty) - \mathcal{E}(t_0)| \simeq \frac{1}{3 v_f R_0^3} \ .$$

Remembering that $\mathcal{E}(\pm\infty) = E(\pm\infty)$ and that $v_f < \sqrt{2E_f}$, one obtains the estimate for the energy error.

The estimate for the angular momentum error can be obtained from equation (9) using the fact that L is almost constant for $|\mathbf{x}| \gg 1$. So, integrating by parts, one gets

$$\dot{L} = -\frac{L}{r} + O(1)$$

where the term $O(1)$ decreases to zero more quickly than the first term. Now, using again the fact that $r(t) > v_f(t-t_0)+R_0$ and neglecting the $O(1)$ terms, by integration one finally finds

$$|L(\pm\infty) - L(t_0)| \leq \frac{L_i}{2v_f R_0^2} \leq \frac{L_i}{\sqrt{E_f} R_0^2} \ .$$

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