NONUNIQUENESS PROPERTIES OF THE PHYSICAL SOLUTIONS OF THE LORENTZ–DIRAC EQUATION

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ABSTRACT

The solutions of the Lorentz–Dirac equation are investigated, for the problem of a one–dimensional scattering of a charged particle by a potential barrier, and a phenomenon is found having some similarity to the quantum weak–reflection effect. Namely, there exists an energy strip, slightly above the maximum of the barrier, such that for any given initial energy in the strip there is a certain number of physical (or nonrunaway) solutions of two types, i.e. those of mechanical type, transmitted beyond the barrier, and those of nonmechanical type, reflected by the barrier. From the mathematical point of view, the existence of this phenomenon is related to the nonuniqueness of the physical solutions of the Lorentz–Dirac equation for given initial data of position and velocity. This in turn is strictly related to a property recently pointed out, namely the asymptotic character of the relevant series expansions occurring for that equation. Correspondingly, the width of the energy strip where the phenomenon occurs is found to decrease exponentially fast, as the small parameter entering the problem tends to zero.

PACS numbers: 03.50.D, 03.65.B, 02.30.M
Running title: Nonuniqueness properties in the Lorentz–Dirac equation

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1. Introduction. The main result described in the present paper is that classical physics allows for situations where a kind of indeterminacy is manifested having a certain similarity to that of quantum mechanics. The instance in which this is exhibited is the so called "weak reflection effect" [1], which occurs in the one–dimensional scattering of a point particle by a potential barrier, when the initial particle's energy is slightly larger than the barrier's height. In such a situation, according to quantum mechanics the final state of the particle is in a sense unpredictable, because the particle can suffer either transmission or reflection, and the theory affords only predictions of a statistical type, providing probabilities for each of the two possible cases. In purely classical mechanics the particle has instead constant energy and so is always transmitted. But we found that a situation somehow similar to that of quantum mechanics occurs if one abandons the domain of pure mechanics and takes into account the role of electromagnetism. The point is that electrodynamics has necessarily to be taken into account for any particle having a charged structure whatsoever, because of selfinteraction with its own field; for example, for a charged point particle it is usually assumed that in some approximation the selfinteraction be described in a universal way by the Lorentz–Dirac equation. Here we study such an equation, showing that its solutions exhibit an effect qualitatively similar to the quantum one of weak reflection. We recall that, in the nonrelativistic limit, to which we will restrict our attention, such an equation has the form

$$\epsilon \dddot{\mathbf{x}} = \ddot{\mathbf{x}} - \mathbf{F}(\mathbf{x})/m \ , \tag{1}$$

where $\mathbf{x}$ is the position vector of the particle, m its (renormalized) mass, $\mathbf{F}$ an external field of force, and the "small parameter' ϵ, with the dimension of a time, is given by

$$\epsilon = \frac{2}{3}\frac{e^2}{mc^3} \ ,$$

e and c being the charge of the particle and the speed of light respectively.

We have insisted above in stressing the possible impact of our result for the foundations of physics, because this is indeed our main domain of interest. But our result could have been stated also in a purely mathematical context, as answering a problem on the Lorentz–Dirac equation itself, considered as early as 30 years ago by two specialists in the theory of ordinary differential equations, Hale and Stokes, [2] namely that of uniqueness of the so called physical (or nonrunaway, see below) solutions of the Lorentz–Dirac equation. Indeed, such authors investigated the general problem whether initial conditions on position and velocity, together with the global condition of nonrunaway, uniquely define the solution of the equation. They could prove only existence, and we show here, in the example of scattering by a potential barrier, that there are domains of initial position and velocity with any number of physical solutions.

There is however another mathematical aspect of the problem, which is related to a remark on the Lorentz–Dirac equation which was made very recently, [3] and actually stimulated the research that led to our result. We refer to the circumstance that the series expansions in the parameter ϵ usually given for the solutions of that equation in general are divergent and asymptotic, as should be expected on general grounds due to the singular character of the equation. Now, as was pointed out in ref. [3], this has the consequence that the physical solutions of the Lorentz–Dirac equation should be classified as belonging to two qualitatively different classes: those well approximated by suitable finite truncations of the series expansion, which are thus perturbations of solutions of the purely mechanical Newton equation corresponding to $\epsilon = 0$, and were consequently called "of mechanical type"; and the remaining ones, qualified as " of nonmechanical type". On the other hand, it occurs that the solutions of mechanical type, being well approximated by finite truncations of the series, depend only on the initial data of position and velocity; so it is clear that there would be no indeterminacy if the series expansions were convergent. In other terms, it is just the asymptotic character of the series which allows for both the existence of the indeterminacy phenomenon alluded to above and the existence of solutions of nonmechanical type. However, the existence of solutions of nonmechanical type was only conjectured in ref. [3], no concrete example being available. Now, it was kindly suggested by an anonymous referee, as a free comment to that paper, that examples of solutions of nonmechanical type might possibly be found in the domain where the quantum tunnel effect occurs. The motivation was just a mathematical analogy, because in the framework of the semiclassical limit of the Schroedinger equation it is known that, for what concerns the tunnel effect, the existence of behaviours qualitatively distinct from purely mechanical ones (think of the orbits crossing the barrier) is due just to the asymptotic character of the relevant series expansions. The analogy turned out to be effective, because we could exhibit the existence of motions of nonmechanical type among the physical solutions of the Lorentz–Dirac equation in the problem of particle scattering by a barrier, for initial energies slightly larger than the height of the barrier. This is indeed the analog of the so called weak reflection effect; but in some cases, as will be shown below, we also found transmission for energies slightly lower than the height of the barrier, and this has a somehow greater similarity to the tunnel effect. An illustration of these facts is the scope of the present paper. We very gratefully acknowledge the influence of the anonymous referee in stimulating the research that led to the present work.

The paper is organized as follows. In section 2 we recall the main facts about the Lorentz–Dirac equation, with particular reference to the problem of a one–dimensional scattering by a potential barrier; we recall the Hale and Stokes uniqueness problem, and give a preliminary estimate of the range where solutions of nonmechanical type can be expected. In section 3 we give an account of some preliminary numerical studies for a gaussian potential barrier, just with the aim of illustrating in a vivid pictorial way the phenomenon discussed here. In section 4 we give a mathematical discussion of the problem, in the framework of the qualitative theory of dynamical systems, showing how the essence of the phenomenon is easily explained in terms of notions there familiar, such as the stable and unstable manifolds of the unstable equilibrium point corresponding to the maximum of the potential barrier. In section 5 we consider a case for which a complete analytical discussion can be performed, namely that of a rectangular barrier; in particular we can there control what happens by varying the "small parameter" ϵ, and exhibit some peculiar properties of the solutions of nonmechanical type. Some further comments are deferred to the conclusive section 6.

2. Main facts about the Lorentz–Dirac equation. We will restrict our considerations to the case of a particle on a line under the action of a potential barrier $V(x)$, of characteristic height V_0 and size L. It is then convenient to take m, L and V_0 as units of mass, lenght and energy, which leads to $\sqrt{mL^2/V_0}$ as a time unit. The Lorentz–Dirac equation then reads

$$\epsilon\dddot{x} = \ddot{x} - V'(x) \ , \tag{2}$$

where the prime denotes derivative with respect to x, the potential V has height 1 and size 1 (think of $V(x) = \exp(-x^2)$), while the characteristic time ϵ has to be thought of as expressed in the proper time unit.

The singular character of the equation is due to the fact that the order of the equation reduces from three to two as the parameter ϵ vanishes. Correspondingly, for $\epsilon = 0$ the Cauchy problem is well posed in the "mechanical phase space" of position x and velocity $v = \dot{x}$, while for $\epsilon \neq 0$ it is well posed in the "enlarged (or extended) phase space" of position velocity and acceleration $a = \dot{v}$.

It was apparently first remarked by Dirac [4] in the year 1938 that the equation presents the difficulty of having generic solutions with the so called *runaway* property, i.e. with acceleration growing exponentially for $t \to +\infty$; the simplest example is just that of the free particle, where the equation reduces to $\epsilon\dot{a} = a$, with solution $a(t) = a_0 \exp(t/\epsilon)$. This fact is probably better understood by the following argument, of a general type. Write equation (2) as usual in normal form $\dot{z} = f(z)$, with $z = (x, v, a) \in \mathbf{R}^3$ and $f(z)$ the corresponding vector field, namely

$$\dot{x} = v \ , \quad \dot{v} = a \ , \quad \dot{a} = \frac{1}{\epsilon}(a - F(x)) \ , \tag{3}$$

where $F = -V'$ is the force field. For small ϵ it is obvious that, apart from a small layer situated about the two–dimensional "slow manifold", defined by $a - F(x) = 0$, the vector field $f(z)$ is essentially parallel to the a axis and directed away from the slow manifold. Thus the field of directions parallel to the a–axis constitutes the so called "fast foliation", along which the point in the enlarged phase space escapes exponentially fast to infinity, thus producing the generic runaway solutions.

However there can exist exceptional initial data, possibly in a small layer about the slow manifold, giving rise to orbits not escaping to infinity; for example, in the case of the free particle this occurs for $a_0 = 0$, namely for data lying exactly on the slow manifold. Dirac himself proposed that the generic solutions should be discarded, and that the only relevant or "physical solutions", should be the ones, if they exist, characterized by the global nonrunaway property $a(t) \to 0$ for $t \to +\infty$; this seems indeed to be a sensible prescription in scattering situations, when the force field vanishes at infinity.

This leads to the following mathematical problem. Given an initial datum (x_0, v_0) in the mechanical phase space, one asks whether there exists a value a_0 of the acceleration such that the corresponding initial datum $z_0 = (x_0, v_0, a_0)$ in the enlarged phase space gives rise to a motion $z(t)$ satisfying the Dirac prescription $a(t) \to 0$ for $t \to +\infty$. Notice that neither existence nor uniqueness is obvious, because this somehow resembles a problem of Sturm–Liouville type. Under mild conditions on the force field F it was shown by Hale and Stokes that existence is guaranteed, but they were not able to prove uniqueness. From the technical point of view, this is due to the fact that one has here a problem of fixed point type in a nonstandard form, because one deals with a mapping which is not a contraction, so that existence is proven by making recourse to topological nonconstructive methods. However, one might have the impression that the lack of uniqueness be not a real fact, but just an oddness due to technical difficulties in the proof. We will show instead that, in the presence of a potential barrier, one has really nonuniqueness.

This property can be rephrased in the following way. Consider the subset of the enlarged phase space corresponding to motions satisfying the Dirac condition, and call it the "physical (or Dirac) manifold": uniqueness in the sense of Hale and Stokes would correspond to the physical manifold being a graph, say $a = g(x, v)$, while our result implies that it is folded. Moreover, it turns out that the initial data belonging to different branches of the folded physical manifold, and having the same position and velocity, give rise alternatively to motions of mechanical and of nonmechanical type, which are transmitted and reflected respectively.

We now add a few words about an estimate for the region where the solutions of nonmechanical type can be expected. As recalled above, the solutions of mechanical type are the ones that are well approximated by truncations of the series expansions in ϵ; thus a first order estimate for them is given by the requirement that the first order term of the expansion be smaller than the zero order term, namely that for all times t one has

$$\epsilon|\dot{x}(t)F'(x(t))| < |F(x(t))|$$

along the unperturbed orbit. For the problem at hand, with the potential barrier of height V_0, it is obvious that for $E > V_0$ there exists for the unperturbed orbit a time $\bar{t}$ such that $F(x(\bar{t})) = 0$, while it is always $|F'| > 0$. So the first order estimate leads to the possibile existence of nonmechanical orbits for $E > V_0$. On the other hand, it is clear that for $E \gg V_0$ one should have orbits of mechanical type, so these heuristic considerations lead to the expectation that orbits of nonmechanical type might exist for mechanical energies about V_0.

3. Preliminary numerical results. We report here the results of some numerical computations that were performed in a rather naive way, just as an exercise to introduce us to the subject, but in fact led us to meet with the phenomenon described in the present paper. The computations were performed for the exponential barrier $V(x) = \exp(-x^2)$.

In integrating the Lorentz–Dirac equation, one is first of all confronted with the technical problem that the instability of the flow due to the runaway property of the generic solutions makes any naive method unstable, leading rather soon to an overflow, even if one were able to select a priori the exceptional initial data lying on the physical (or Dirac) manifold; indeed, this practical difficulty is due to the fact that the physical manifold is a repellor for the flow of the considered dynamical system. An available way out, which was already considered in the literature [5], is based on the remark that the physical manifold is instead attractive, if one goes backward in time. So, taking a "final condition" in the enlarged phase space in a region of essentially vanishing potential and integrating backward, one obtains after a very short transient, for any choice of the "final" acceleration, the desiderd physical solution; in practice, we just took a vanishing "final" acceleration. Thus the whole discussion of the present section will always have a backward flavour, which is due to the trick used to overcome a numerical problem. As for problems of numerical precision, they are essentially irrelevant in the present context, just by virtue of the attractive character of the backward integration procedure; standard Runge–Kutta methods were used, but in fact the essential results were preliminarly obtained even using a first order Euler method.

The results are illustrated in Figs. 1–5, which all refer to $\epsilon = 1$. In Fig. 1 we report in abscissae the position x and in ordinates the mechanical energy $E = T + V$ of the particle, $T = \frac{1}{2}m\dot{x}^2$ being the kinetic energy; for reference, the potential energy $V(x)$ of the barrier is also drawn. For any final datum, a curve of E versus x is drawn, obtained by eliminating time from the functions $E(t), x(t)$, computed in a suitable range of time t; more precisely, the final position was always taken to be $x = +6$, and the computations were run up to reaching an initial position with $x = 10$ or $x = -10$. Indeed, it can occur that the particle be reflected by the barrier or transmitted, and this turns out to depend only on the final energy E_f; namely, there is a threshold in the final energy such that for lower energies the particle is always reflected, while for higher energies it is always transmitted. Clearly, the curves corresponding to reflection are characterized by showing a turning point, with vanishing kinetic energy (i.e. there exists a time $\bar{t}$ with $E(\bar{t}) = V(x(\bar{t}))$), while the kinetic energy is always positive for the curves corresponding to transmission. The most peculiar phenomenon exhibited by Fig. 1 is the existence on an "overlapping energy strip", characterized by the property that for the same given initial energy in that strip there are orbits of both types, with reflection and trasmission respectively; in the figure, the overlapping strip is contained in the interval $1.4 < E < 1.5$. In fact, in Fig. 1, for the sake of clarity of illustration we chose to exhibit only one reflected and one transmitted motion in the strip, but we will show below, both numerically and analytically, that for any positive integer n one can find a suitable strip of initial energies giving rise to exactly n different reflected motions and $n + 1$ transmitted motions.

Another relevant phenomenon is the existence of a kind of "inversion effect". Namely, in a certain range of energies the initial energy is observed to decrease if the final energy is increased. This is peculiar, because obviously at very high energies, when the motion is actually insensitive to the potential energy, the final and the initial energies have to be essentially equal, and the growing of the one corresponds to the growing of the other one.

This second effect is not clearly visible by inspection of Fig. 1, and we could exhibit it by suitable dilatations of the figure. Instead, both phenomena are clearly exhibited if the initial energy E_i is drawn versus the final energy E_f; this is shown in Fig. 2. For very small and very large final energies the curve approaches that corresponding to the purely mechanical case $\epsilon = 0$, namely $E_i = E_f$, while the two new phenomena occur in a strip of initial energies centered near the value $E = 1$ of the maximum of the potential barrier. The inversion is particularly evident, as corresponding to a negative slope of the curve in a strip extending between a maximum and a minimum. Even more interesting is the fact that another inversion strip, with a corresponding pair of maxima and minima, seems to appear inside the main one, and this suggests that infinitely many higher order inversion strips might occur.

Fig. 1 is the result of a backward computation. But a better illustration is obtained by a suitable exhibition of just the same results in the spirit of a "forward attitude". Indeed, in performing the numerical integration we took all final data to the right of the barrier, and consequently the initial states occurred to be to the right for reflected motions and to the left for transmitted motions. On the other hand, as the equation is invariant with respect to the transformation $x \to -x$, it is clear that for each reflected motion there also exists a corresponding one with the same instantaneous energy and with initial and final states to the left of the barrier. So we might draw a new figure corresponding to Fig. 1 with all initial states to the left of the barrier. We prefer instead to draw the corresponding "mechanical phase portrait", namely the projection of the corresponding orbits in the mechanical phase space of position and velocity. This is shown in Fig. 3, where the two features pointed out above, namely existence of the overlapping strip and of inversion, are fairly evident.

The existence of inversion strips corresponds to the fact that for a pair of initial mechanical data x_0, v_0 of position and velocity there exist several initial data a_0 for the acceleration giving rise to nonrunaways solutions. In other terms, the physical manifold in the enlarged phase space corresponding to nonrunawy solutions is folded. In order to exhibit this fact visually, one can consider the section of the enlarged phase space with a plane $x_0 = \text{const}$ for a large negative value of the position x_0, and draw the points in the plane v_0, a_0 which give rise to nonrunaway solutions (computed as usual by the backward tool, by varying the final energy). The result is shown in Fig. 4 for several values of x_0, and the folding seems to be clearly exhibited. For the sake of illustration, a three-dimensional view of the complete physical manifold is reported in Fig. 5.

Obviously the width δE of the inversion strip depends on the parameters entering the problem; in particular, for a given potential, one can expect from heuristic considerations that it vanishes exponentially fast in the limit $\epsilon \to 0$. This was indeed checked numerically, and the best interpolating function was found to be

$$\delta E = \frac{a}{\epsilon^2} \mathrm{e}^{-b/\epsilon^2} \ ,$$

a and b being suitable constants, depending on the form of the potential. We point out that if ϵ is large enough, then the width of the strip is correspondingly large, so that it can occur that the strip of indeterminacy may extend to values of the energy below the maximum of the potential. So for large ϵ there exist orbits transmitted beyond the barrier although having initial energy smaller than the barrier height. This is a rather interesting fact which we actually observed, and is similar to the well known quantum tunnel effect.

We conclude this section with a remark concerning the behaviour of the mechanical energy $E = T + V$ during the motion of the particle. This concerns the widespread opinion that, because of radiation, energy should decrease steadily during forward motion; in Fig. 1 one sees instead that for a transmitted motion there is a time interval where energy increases, although one always has $E_f < E_i$. This can be understood in the following way. Recall that, through multiplication by $\dot{x}$, the Lorentz–Dirac equation gives the energy theorem in the form $\dot{E} = \epsilon \dot{x} \dddot{x}$, or equivalently

$$\dot{E} = -\epsilon (\ddot{x})^2 + \epsilon \frac{\mathrm{d}}{\mathrm{dt}} (\ddot{x} \dot{x}) \ . \tag{4}$$

Now, the first term at the right hand side corresponds to the usual energy loss according to Larmor formula, while the second one (known as Schott term) has no fixed sign. As in a scattering problem the acceleration vanishes for $|t| \to +\infty$, the contribution of Schott's term to the total energy loss vanishes, so that one always has $E_f < E_i$, and in this sense energy decreases. But this is not true in general for the instantaneous energy, and moreover it is clear a priori that in some circumstances the instantaneous energy should indeed increase with time. To see this, consider the first order approximation (in ϵ) of the original equation, where the term $\dddot{x} = \frac{\mathrm{d}\ddot{x}}{\mathrm{dt}}$ is approximated by $-\frac{\mathrm{d}}{\mathrm{dt}}V'(x(t)) = -V''(x(t))\,\dot{x}(t)$. This gives $\dot{E} = -\epsilon\dot{x}^2 V''(x)$, and so energy actually has to increase, in that approximation, in the regions where the potential energy is concave downward.

4. Analysis of the problem in light of the qualitative theory of dynamical systems. We show here how the numerical results illustrated above are rather easily explained in terms of standard concepts of the qualitative theory of dynamical systems. In order to go to the heart of the problem, we concentrate our attention on a nonanalytic particularly meaningful case, namely that of a force vanishing for $|x| > 1$ and linear for $|x| < 1$, i.e with potential energy

$$V(x) = 1 - x^2 \quad \text{for} \quad |x| < 1 \ , \quad V(x) = 0 \quad \text{for} \quad |x| > 1 \ ,$$

with continuity conditions for the acceleration at the points $x = \pm 1$. Nonanalytic cases of this type were often studied, but the description given here seems to be new, because in the literature the attention was concentrated on the analytical expression of the solutions rather than in a qualitative discussion, which became so familiar since the year sixties, with the discovery of strange attractors an so on. To determine the global physical solution, one takes initial data x_0, v_0 in the mechanical phase space, with $x_0 < -1$, and investigates the solutions for positive times, exploring all possible initial accelerations a_0. It is evident first of all that there exists an interval of negative values of a_0 which lead to motions never reaching the barrier for positive times, and all these are runaway solutions (apart from the trivial ones with $v_0 < 0$ and $a_0 = 0$). For all other values of a_0, there exists instead a positive time t_1 such that $x(t_1) = -1$, and it then always occurs, as will be shown shortly, that there also exists a time $t_2 > t_1$ with either $x(t_2) = -1$ and $v(t_2) < 0$ or $x(t_2) = 1$ and $v(t_2) > 0$; in such cases the nonrunaway solution has clearly to be selected by the prescription $a(t_2) = 0$. Our aim is to give a qualitative description for such solutions without making recourse to their explicit analytical expressions, which could be written down explicitly, but are not easily discussed. A complete analytical discussion will be given in next section, for the simpler case of a rectangular barrier.

After time t_1 (and before t_2) one has to solve the linear problem $\dot{z} = Az$ where $z = (x, v, a) \in \mathbf{R}^3$ and A is the 3 by 3 matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2/\epsilon & 0 & 1/\epsilon \end{pmatrix} \ .$$

The eigenvalues of A turn out to be real for small ϵ, while an interesting bifurcation occurs at $\epsilon = \sqrt{2/27}$, because for larger values of ϵ the matrix A has a real negative eigenvalue and two complex conjugate eigenvalues with positive real part. So, in the case of "large ϵ" to which we now concentrate our attention, for the corresponding linear system the phase space turns out to be the direct sum of a one dimensional "stable linear space" E^{s} and a two dimensional "unstable linear space" E^{u}, the restriction of the system to E^{u} being an unstable focus. The dispositions of such linear spaces are easily determined,(*) and it is of interest to locate some special subsets of them. The first one is the point, say $\bar{z}$, defined by the intersection of the stable linear space E^{s} with the plane $x=-1$. Indeed, recall that for a nonlinear system the stable manifold W^{s} of an equilibrium is defined as the set of points which give rise to orbits tending to the equilibrium for $t \to +\infty$. So, in our case a branch of the stable manifold W^{s} of the origin is just the union of the unique orbit arriving from the left to $\bar{z}$, and the segment of E^{s} joining $\bar{z}$ to the origin; analogously, one also has the symmetrical branch located at the right of the barrier. By the way, these are the unique nonrunaway solutions satisfying the Dirac prescription $a(t) \to 0$ for $t \to +\infty$, while being not solutions of scattering type; they are in fact the analogs of the well known separatrices of the purely mechanical problem, approaching the equilibrium from the left and from the right. A relevant role will also be played by the analogs of the two further separatrices of the mechanical problem, characterized by the property of tending to the equilibrium for $t \to -\infty$, i.e. constituting the unstable manifold W^{u} of the origin. In the purely mechanical case, the unstable manifold is constituted by just the two separatrices, while in the nonmechanical case it is constituted by the union of that part of the plane E^{u} having $|x|<1$, and its continuation for $|x|>1$ through solutions of the free particle problem. So, the generic solutions belonging to the unstable manifold W^{u} will be of runaway type, and the only nonrunaway solutions on W^{u} are the two ones passing through the points, say z^+ and z^-, which are the intersections of the plane E^{u} with either the straight line $a=0$, $x=+1$, or the straight line $a=0$, $x=-1$ respectively.

With this preliminary information in mind, it is now quite easy to describe the nonrunaway solutions which are either transmitted or reflected. Indeed, consider the orbits arriving from left and inciding onto the plane $x=-1$ near the point $\bar{z}$, namely the orbits arriving from left near W^{s}. As the orbits on W^{s} tend to the origin for $t \to +\infty$, by continuity the orbits passing near $\bar{z}$ also start going toward the origin, but in the meantime also go on spiraling in broader and broader arms around E^{s}, and at a certain moment t_2 reach either the plane $x=+1$ (having $v>0$), or the plane $x=-1$ (having $v<0$), and then are transmitted or reflected respectively. But the only nonrunaway solutions among them will be the ones that, in reaching such planes, will have exactly a vanishing acceleration. The nonrunaway solutions not lying on E^{s} will thus be characterized by suffering either transmission (i.e. by meeting with $x=+1$) or reflection (meeting with $x=-1$), and by having performed a certain positive integer number n of turns around E^{s} in the region $|x|<1$.

This picture can also be described in the following way. Consider the half straight line γ^{tr}, defined by the intersection of the plane $x=+1$ with the plane $a=0$ and lying with respect to the plane E^{u} on the same side as the branch of E^{s} coming from left; then let γ^{tr} evolve backward according to the flow of the linear system, i.e. consider the orbits $\exp(At)z$, $t<0$, $z \in \gamma^{\mathrm{tr}}$. Due to the inclinations of the plane E^{u} and the straight line E^{s} with respect to γ^{tr}, it turns out that the backward orbits $\exp(At)z$, $t<0$, $z \in \gamma^{\mathrm{tr}}$ intersect

(*) To fix ideas, for $\epsilon=1$ one has the eigenvalue -1 with eigenvector $(1,-1,1)$ and the eigenvalues $1 \pm i$ with eigenspace spanned by $(1,1,0)$ and $(0,1,2)$.

the plane $x = -1$ along a certain curve, say $\tilde{\gamma}^{\mathrm{tr}}$; such a curve is by definition the intersection (or, as we say, the trace) of the physical manifold, corresponding to transmission, with the plane $x = -1$. The trace corresponding to reflection is obtained analogously, considering the finite segment γ^{re} on the straight line $x = -1$, $a = 0$, lying with respect to the plane E^{u} on the same side as the branch of E^{s} coming from left, and furthermore such that $v < 0$; then let γ^{re} evolve backward along the linear flow, up to reaching again the plane $x = -1$ on a certain curve $\tilde{\gamma}^{\mathrm{re}}$. As the points of the half straight line γ^{tr} and of the segment γ^{re} which are situated near the plane E^{u} have by continuity to perform many turns around E^{s} before reaching backward the plane $x = -1$, it is clear that the two traces of the physical manifold have to wind indefinitely around $\bar{z}$.

In conclusion, the intersection (or trace) of the physical manifold with the plane $x = -1$ is constituted by two curves, $\tilde{\gamma}^{\mathrm{tr}}$ and $\tilde{\gamma}^{\mathrm{re}}$, which both spiral about the same point $\bar{z}$. So there is an interval of values of velocity where such an intersection is folded, i.e. the acceleration is not uniquely determined by the velocity. This is indeed the nonuniqueness property of the physical solutions for given x_0 and v_0, which explains the phenomenon of the existence of an overlapping strip. Indeed it is quite clear how to prove that for any positive integer n there exist velocity intervals I_n^+ and I_n^- such that there are n possible accelerations for which the physical solution is transmitted or reflected respectively.

Moreover, it is just this spiral structure that explains the inversion phenomenon described above, concerning the initial and final energies E_i and E_f. Indeed, moving continuously along the trace of the physical manifold, with its winding around $\bar{z}$, corresponds to an alternate increasing and decreasing of the "initial" energy, while the "final" energy tends monotonically towards a well defined value, namely that corresponding to the exceptional nonrunaway solution constituted by W^{s}. This just explains the main features of Fig. 1.

So much for what concerns the trace or intersection of the physical manifold with the plane $x = -1$. The complete part of the physical manifold itself in the region $x < 1$ is then trivially obtained by the analytical solution of the problem of the free particle. The main feature seems to be that for $x \to -\infty$ the whole manifold turns out to be squeezed on the plane $a = 0$, which constitutes the physical manifold for the free particle. This has the consequence that in a scattering experiment it would be practically impossible to control the additional nonmechanical parameter, namely the initial acceleration a_0, in order to predict whether one will have transmission or reflection.

For what concerns the analytical case of any potential barrier, by virtue of the stable manifold theorem it is quite clear that the qualitative behaviour should not be dissimilar from that described above, but we renounce to give here any precise mathematical statement of this.

5. Analytical discussion for the case of a rectangular barrier. After the indications given by the numerical computations, and the general discussion in terms of the qualitative theory of ordinary differential equations, it seems useful to have available a case for which an almost complete analytical discussion with explicit formulae can be provided. The simplest case seems to be that of a rectangular barrier

$$V(x) = 1 \quad \text{for} \quad -1 \leq x \leq 0, \quad V(x) = 0 \quad \text{elsewhere} \;, \tag{5}$$

because one has then to deal just with the problem of the free particle for all times for which it is $x(t) \neq -1$ or $x(t) \neq 0$; a global solution is then obtained by suitable matching conditions. These are continuity of position x and velocity v; instead, as is easily seen from the energy equation (4), for the acceleration a it turns out that one has to require a jump of amplitude $\pm 1/\epsilon|v|$, with the plus sign at $x = -1$ and the minus sign at $x = 0$.

The analytical manipulations are easy and some details are deferred to the appendix. The results can be described by making reference to Fig. 6, where the initial velocity v_i is plotted versus the final velocity v_f, for several values of $\epsilon < 1$ (the solid line corresponds to $\epsilon = 0.2$, the dotted line to $\epsilon = 0.4$, and the dashed one to $\epsilon = 0.6$). One sees that there appear two branches, one with $v_f > 1$ and another one with $v_f < 1$; from the analytical discussion reported in the appendix one can see that the former corresponds to transmission, while the latter one corresponds to reflection. Notice in particular that the curves occur only above a minimum for v_i, and this means that nonrunaway solutions exist only for values of v_i larger than that minimum. Another remarkable fact is that the second branch turns out to be independent of ϵ, while the other one squeezes down towards the line $v_i = v_f$ as ϵ decreases. So the branch with $v_f > 1$ corresponds to the solutions of mechanical type, while the other one corresponds to the solutions of nonmechanical type, which persist for any small value of $\epsilon \neq 0$. In other terms, as should be expected, the nonmechanical solutions are the reflected ones, and moreover they turn out to be independent of ϵ.

An interesting remark is that the presence of the gap described above is due to the nonanalyticity of the potential. Indeed, if the rectangular barrier is smoothed, it occurs that the gap disappears, and there exist nonrunaway solutions for any value of the initial velocity. This is shown in Fig. 7, where we report the results of numerical computations for a potential barrier interpolating the rectangular one, obtained by smoothing the discontinuity with two semi–gaussians, namely $V(x) = 1$ if $x \in [-1, 0]$, $V(x) = \exp\left[-(x+1)^2/\sigma\right]$ if $x < -1$ and $V(x) = \exp\left(-x^2/\sigma\right)$ if $x > 0$. The results refer to $\sigma = 10^{-5}$ and $\epsilon = 0.5$.

6. Further comments. So we have shown how, when the selfinteraction of a charged particle with the electromagnetich field is taken into account through the Lorentz–Dirac equation, classical physics leads to situations in which a kind of indeterminacy is manifested, having a certain similarity with that of quantum mechanics; namely, in an experiment of scattering by a potential barrier, the control of the initial particle energy is not sufficient to predict whether the particle will be transmitted or reflected. This is due to the fact that, for given initial position and velocity, the requirement for the solution to be physical, i.e. nonrunaway, does not determine uniquely the other initial datum ,i.e. the acceleration, and so the solution of the equation. In this connection we would like to express here a personal opinion of ours, without pretending to substantiate it at present. Namely, we conjecture that a similar property should occur also for the complete system describing the electromagnetic field and the particle, and not only for the reduced equation for the particle discussed in the present paper; in such a more general context, the indeterminacy on the particle motion should be due to a lack of control of the initial data of the field, which should act somehow as hidden variables. By the way, as indicated by the concrete example of the Lorentz–Dirac equation discussed here, it is clear that hidden parameters of such a type should have quite peculiar properties, the most interesting one being that the values they can take are strongly correlated to the mechanical state of the particle.

However, it turns out that such a similarity between classical physics and quantum mechanics for what concerns the indeterminacy illustrated here is only a qualitative one, because one can see that a quantitative comparison fails. To make such a comparison, one should define somehow, in the classical case, the analogs of the transmission and of the reflection coefficients; this is a very interesting subject which we hope to be able to discuss in the future. On the other hand, for a given initial particle position one has available a clear definition for the range of energy where the phenomenon of indeterminacy occurs, namely the width of the overlapping strip; and this has a quantum analog, namely the energy range where the transmission and the reflection coefficients are both significantly different from zero. As such coefficients depend exponentially on energy, the latter energy range is rather well defined, and can be compared with its classical analog. Now, if one considers a given particle such as the electron, with the corresponding physical values for its parameters, acted upon by a concrete potential barrier occurring in physical instances, one easily checks that the classical energy range of indeterminacy and its quantum analog differ by several orders of magnitude.

Nevertheless, we believe that the phenomenon illustrated in the present paper might be of a certain interest, inasmuch as it shows, in the domain of electrodynamics, that classical physics presents a richness of behaviours, which apparently were not fully appreciated up to now. This state of affairs is very similar to that occurring in classical statistical mechanics, where it was presumed that equipartition should obtain, while it has now become common knowledge that the situation is much more complicated, because one meets with several phenomena having a qualitative similarity to those described by quantum mechanics; we refer to the freezing of the degrees of freedom with high frequencies, which is now understood in terms modern perturbation theory (see for example [6], [7]).

NOTE ADDED IN PROOFS.

The discovery of multiple physical solutions to the Lorentz–Dirac equation was made alraedy in the year 1943 by F. Bopp who, for the nonrelativistic equation, found two solutions in the example of the one–dimensional potential step (see [8], page 596). A little variant of that example (a potential step increasing linearly between the two constant values) was studied by R. Haag [9] in the year 1955. Analogous results were then found for the potential step in the relativistic case by Baylis and Huschilt, [10] apparently unaware of the previous works of Bopp and Haag. All such examples were studied by the quoted authors in a very elementary way, along lines similar to those followed in the present paper for the example of the rectangular barrier dealt with in the appendix. We are very grateful to D. Noja for indicating to us the beautiful review article of the year 1961 by T. Erber, where the results of Bopp and Haag are briefly discussed (see [11], page 355). The thesis maintained by Erber is that, among the two "physical solutions" of Bopp and Haag, by some reason one should be retained and the other discarded. The main contribution of the present paper seems then to be that in general there are not just two, but rather any number of physical solutions; so there is no hope to determine a privileged one among them, and some new interpretation is required. Another point is that the existence of multiple solutions is not confined to any particular model, but is a general characteristic occurring when the external potential presents a maximum.

Strangely enough, the results of Bopp and Haag are mentioned neither by Hale and Stokes, in their mathematical paper devoted to prove existence of physical solutions, nor by Rohrlich in his book [12] which is often considered a the standard reference in this field (although the papers of Bopp and Haag are quoted). Even stranger is the fact that in the review paper by Plass exactly the model of Haag is considered and the analytical computations are actually performed (page 54), without finding the phenomenon of multiple solutions. Apparently this is due to the lack of remarking that a bifurcation occurs when the parameters are changed. In such a way, the phenomenon of the existence of multiple solutions turned out to be essentially forgotten.

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APPENDIX

Analytical computations for a rectangular barrier. Our aim is to find the scattering solutions of the Lorentz–Dirac equation (2) when $V(x)$ is the rectangular potential barrier, $V(x)=1$ if $x\in[-1,0]$ and $V(x)=0$ elsewhere. For $x\neq -1,0$ one deals with the free particle problem, with general solution

$$x(t)=A\mathrm{e}^{t/\epsilon}+Bt+C\ ; \tag{6}$$

thus the nonrunaway scattering solutions are characterized by the existence of a time t_F (with $x(t_F)=-1$ or $x(t_F)=0$) such that for $t>t_F$ they reduce to uniform motion $x(t)=Bt+C$. We can take $t_F=0$. Assuming that the particle incides on the barrier from the left, the nonrunaway solutions can be classified as being of two types, according to whether $x(0)=0$ (transmission) or $x(0)=-1$ (reflection).

Consider first the case of the transmitted solutions, which are characterized by $x(t)=v_f t$ for $t\geq 0$, where $v_f>0$ is the final velocity. For $t<0$ and $x>-1$ (i.e. "inside the barrier") the solution has the form (6) where A,B,C have to be determined in terms of the final velocity v_f by the matching conditions at $x=0$. The solution is immediately checked to be

$$x(t)=-\frac{\epsilon}{v_f}(1-\mathrm{e}^{t/\epsilon})+\left(v_f-\frac{1}{v_f}\right)t\ .$$

As the solution has to correspond to transmission, the range of possible values of v_f has to be constrained by the condition that the particle comes from $x=-1$, namely that the equation

$$-\frac{\epsilon}{v_f}(1-\mathrm{e}^{t^*/\epsilon})+\left(v_f-\frac{1}{v_f}\right)t^*=-1$$

has a solution $t^*<0$. It is easy to check that for $\epsilon<1$ this condition is verified if and only if $v_f>1$, so that there are no transmitted particles with final velocity less than 1. We have now to impose the matching conditions at $x=-1$, knowing that at the left of the barrier, i.e. for $t<t^*$, the solution has the general form (6) with $B=v_i$, where v_i is the "initial" velocity of the particle. Denoting $z=t^*/\epsilon$, this leads to the system

$$\begin{aligned} v_i &= \left(v_f-\frac{1}{v_f}\right)+\left[v_f+\left(v_f-\frac{1}{v_f}\right)z+\frac{1}{\epsilon}\right]^{-1}\\ \frac{1}{\epsilon} &= -\frac{1}{v_i}(1-\mathrm{e}^{-z})-\left(v_f-\frac{1}{v_f}\right)z\ , \end{aligned}$$

with $v_f>1$. Here v_f has to be considered as a parameter, the unknowns being v_i and z, so that the solution of the system provides in particular an expression for v_i as a function of v_f, which is the relation of interest for us. The system is easily solved numerically, and the curves v_i versus v_f are reported in Fig. 6.

The reflected solutions are calculated in an analogous way. For $t>0$ they have the form $x(t)=v_f t-1$ with $v_f<0$. One then looks for the solution "inside the barrier", by imposing the matching conditions at $x=-1$, and looks for the constraints on the final velocity such that the equation $x(t^*) = -1$ has a solution $t^* < 0$; one finds that this condition is verified if and only if $v_f \in (-1,0)$. Then one has to impose again the matching conditions, at $x = -1$, with the solution corresponding to $t < t^*$, and one obtains the dependence of the initial velocity on the final one, by solving the system

$$
\begin{aligned}
& v_i = \left(v_f - \frac{1}{v_f}\right) + \left[v_f + \left(v_f - \frac{1}{v_f}\right) z\right]^{-1} \\
& \frac{1}{v_i}(1 - \mathrm{e}^{-z}) + \left(v_f - \frac{1}{v_f}\right) z = 0
\end{aligned}
$$

with $v_f \in (-1,0)$. Notice that now (at variance with the case of transmitted solutions) the system, and so the relation between v_i and v_f, does not depend on ϵ.

FIGURE CAPTIONS

Fig. 1. Particle energy versus position for several numerical solutions of the Lorentz–Dirac equation with $\epsilon = 1$, for a gaussian potential barrier (also drawn). Notice the existence of an “overlapping strip”. Results of numerical computations for the same case are illustrated in the figures 2–5.

Fig. 2. Initial energy E_i versus final energy E_f. Notice the “inversion effect”.

Fig. 3 Projections of the orbits of the system on the mechanical phase space of position and velocity.

Fig. 4. Sections of the physical manifold with several planes $x = \mathrm{const.}$

Fig. 5. A three–dimensional representation of the physical manifold in the enlarged phase space.

Fig. 6. Initial velocity v_i versus final velocity v_f for the rectangular barrier, for several values of ϵ.

Fig 7. Initial velocity v_i versus final velocity v_f computed numerically for a smooth interpolation of the rectangular barrier.