MR1961630 (2004e:70008) 70F05 (37J15 53D50 70G45 70G65 81S10)
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★**The Kepler problem.**
Group theoretical aspects, regularization and quantization, with application to the study of perturbations.
Progress in Mathematical Physics, 29.
Birkhäuser Verlag, Basel, 2003. *xviii*+439 *pp.* $89.95. *ISBN* 3-7643-6902-7

This is an interesting book, which well organizes the group-geometric aspects of the Kepler problem on which a great number of articles have been published along with the advance of symmetry theory. The key ideas the author takes to form a unified view of aspects of the Kepler problem are the following: The phase space of the regularized n-dimensional Kepler problem is symplectomorphic to T^+S^n, the cotangent bundle of the n-dimensional sphere with the zero section deleted, which is realized as a coadjoint orbit of the Lie group $\mathrm{SO}(2,n+1)$. Further, the geodesic flow on the cotangent bundle of the n-dimensional sphere can be transformed into the regularized Kepler flow for negative energy.

This book consists of four parts. Part I discusses elementary theory including basic facts about the Kepler motion, separation of variables and action-angle variables, quantization of the Kepler problem, and regularization and symmetry. Among these subjects, the quantization and the regularization of the Kepler problem are discussed sequentially. The reason for this is that the Fock method for studying the quantized Kepler problem and the Moser method for regularizing the Kepler problem are linked closely to each other, as the author stresses. In fact, in the Fock method one performs a Fourier transform of the Schrödinger equation for the quantized Kepler problem to express the equation in the momentum space, and then applies the stereographic projection to put the transformed equation in the form of an integral equation on S^n. In contrast with this, in the Moser method one transforms the geodesic motion on S^n into the regularized Kepler motion with negative energy by means of the stereographic projection. The exposition of elementary theory is given in a rather computational manner. Group-geometric aspects of these subjects are given in Part II in a unified manner from the viewpoint of a dynamical group.

Part II is devoted to group-geometric theory, in which the author deals with conformal regularization, spinorial regularization, return to separation of variables, geometric quantization, and the Kepler problem with a magnetic monopole. In particular, conformal regularization plays a central role in this book. The key idea is as follows: The group $\mathrm{SO}(2,n+1)$ acting on $\mathbf{R}^{2,n+1}$ induces an action on M, the space of null rays. Further, the group $\mathrm{SO}(2,n+1)$ may be identified with the conformal group of the Minkowski space $\mathbf{R}^{1,n}$. The cotangent bundle, T^+S^n, of the unit sphere S^n with the zero section deleted is realized as a coadjoint orbit of $\mathrm{SO}(2,n+1)$ through the momentum map of $T^+M\to \mathrm{so}^*(2,n+1)$, where T^+M denotes the cotangent bundle of M with the zero section deleted. In addition to the theoretical exposition, the author carries out all the procedures explicitly in terms of local coordinates. The symmetry of the Kepler problem is treated for respective energies, negative, zero, and positive, in a unified manner in the procedure of conformal regularization. The author also discusses variations of the Kepler problem such as the Kepler problem with a magnetic monopole in association with the group $\mathrm{SO}(2,4)$ or $\mathrm{SU}(2,2)$.

Geometric quantization is a procedure in which one starts by identifying the phase space of a Hamiltonian system with a coadjoint orbit of a Lie group, called a dynamical group, and relevant dynamical quantities with the components of a momentum map, and then forms a unitary irreducible representation of the dynamical group in the Hilbert space of square-integrable functions defined on the base space of a polarization of the coadjoint orbit. The author performs this procedure for the quantization of geodesic flows on the sphere S^n exploiting the fact that T^+S^n is identified with a coadjoint orbit of $\mathrm{SO}(2,n+1)$, and he further discusses the quantization of the Kepler problem by using some techniques such as intertwining operators.

In his discussion of the return to separation of variables, the author shows that the theorems of Stäckel, Eisenhart, and Robertson and the symmetry of the Kepler problem provide four coordinate systems in which the Hamilton-Jacobi equation and the Schrödinger equation for the quantized Kepler problem are separable. As a by-product of the proof, one can find finitely-perturbed Kepler Hamiltonians which are still separable and hence integrable. The Euler and the Stark problems are given as examples.

Part III includes a discussion of perturbation theory, which consists of general perturbation theory, perturbations of the Kepler problem, and perturbations with axial symmetry. In the section concerning general perturbation theory, the author first reviews the perturbation method through Lie series and the so-called homological equation, and then discusses the subtle problem of the convergence of Lie series by describing the celebrated Kolmogorov theorem and the Nekhoroshev theorem, which are stated in terms of action-angle variables. In contrast with the general perturbation theory, the perturbations of the Kepler problem are discussed in terms of coordinates that are suited for the topology of the phase space, T^+S^3, of the regularized Kepler problem. In the regularization procedure, the perturbed Kepler Hamiltonian is replaced by a more convenient perturbed Hamiltonian which has the same trajectories as the original one up to a change of parameters. A truncated averaged Hamiltonian then determines a Hamiltonian system on the orbit space $S^2 \times S^2$ of the regularized Kepler problem of negative energy. The author shows that almost all three-dimensional orbits corresponding to orbits on $S^2 \times S^2$ surrounding an elliptic critical point can survive for the fully perturbed Hamiltonian. In conclusion, the author reviews perturbations with axial symmetry. If the perturbed Kepler Hamiltonian admits axial symmetry, one can reduce $S^2 \times S^2$ further to a symplectic manifold which is homeomorphic with S^2. However, due to the presence of fixed points for the axial symmetry, the Poisson reduction method works better than the symplectic reduction one. The author discusses the lunar problem, Stark and quadratic Zeeman effects, and a satellite around an oblate primary.

Part IV contains appendices on differential geometry, Lie groups and Lie algebra, Lagrangian dynamics, and Hamiltonian dynamics. In particular, in his discussion of Hamiltonian dynamics, the author reviews the symplectic and Poisson reductions, the Liouville-Arnold theorem, and the monodromy associated with the Liouville-Arnold fibration. These subjects are compactly described.

This book serves as a nice reference not only for graduate students but also for scientists who are interested in dynamical systems with symmetry.

{Remark: In my copy of the book, Figure 8.14 on p. 191 contains no graphs of the Jacobi elliptic functions.}

Reviewed by *Toshihiro Iwai*