① $\{a_n\} \to +\infty$ oppure $\to -\infty$

$$\lim_{\substack{n\to+\infty \\ \{a_n\}\to\pm\infty}} \left(1+\frac{1}{a_n}\right)^{a_n} = e$$

② $\{b_n\}\to 0$

$$\lim_{\substack{n\to+\infty \\ \{b_n\}\to 0}} (1+b_n)^{1/b_n} = e$$

③ $\{b_n\}\to 0$

$$\lim_{\substack{n\to+\infty \\ \{b_n\}\to 0}} \frac{\ln(1+b_n)}{b_n} = 1$$

④ $\{b_n\}\to 0$

$$\lim_{\substack{n\to+\infty \\ \{b_n\}\to 0}} \frac{e^{b_n}-1}{b_n} = 1$$

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nozione di successioni asintotiche:

$\{A_n\}$, $\{B_n\}$ successioni tali che

$\lim_{n\to+\infty} \frac{A_n}{B_n} = 1$: allora dico che $\{A_n\}$ e $\{B_n\}$ sono asintotiche.

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se $\{b_n\}\to 0$

$$\ln(1+b_n) \sim b_n$$
$$e^{b_n}-1 \sim b_n$$
$$\sin b_n \sim b_n$$

(1 (2

$$\lim_{n\to+\infty} \sqrt[7]{n^7-5n^6+3} - \sqrt[7]{n^7+n^4-2} = [\infty-\infty]$$

$$= \lim_{n\to+\infty} \sqrt[7]{n^7}\sqrt[7]{1-\frac{5}{n}+\frac{3}{n^7}} - \sqrt[7]{n^7}\sqrt[7]{1+\frac{1}{n^3}-\frac{2}{n^7}} =$$

$$= \lim_{n\to+\infty} n\left(\sqrt[7]{1-\frac{5}{n}+\frac{3}{n^7}} - \sqrt[7]{1+\frac{1}{n^3}-\frac{2}{n^7}}\right) = [\infty\cdot 0]$$

$$= \lim_{n\to+\infty} n\left[\left(1+\left(\frac{3}{n^7}-\frac{5}{n}\right)\right)^{1/7} - 1 + 1 - \left(1+\left(\frac{1}{n^3}-\frac{2}{n^7}\right)\right)^{1/7}\right] =$$

$$= \lim_{n\to+\infty} n\left[\left(1+\underbrace{\left(\frac{3}{n^7}-\frac{5}{n}\right)}_{b_n}\right)^{1/7}-1\right] + \lim_{n\to+\infty}\left[1-\left(1+\underbrace{\left(\frac{1}{n^3}-\frac{2}{n^7}\right)}_{b_n}\right)^{1/7}\right]$$

$$= \lim_{n\to+\infty} n\cdot\frac{1}{7}\left(\frac{3}{n^7}-\frac{5}{n}\right) - n\cdot\frac{1}{7}\left(\frac{1}{n^3}-\frac{2}{n^7}\right) = -\frac{5}{7}$$

⑤

$\{b_n\}\to 0$, $\forall t\in\mathbb{R}$

$$\lim_{\substack{n\to+\infty \\ \{b_n\}\to 0}} \frac{(1+b_n)^t-1}{b_n} = t$$

$$\lim_{\substack{n\to+\infty \\ \{b_n\}\to 0}} \frac{(1+b_n)^t-1}{t\, b_n} = 1$$

$$(1+b_n)^t - 1 \sim t\, b_n$$
$$(1+b_n)^t \sim 1 + t\, b_n$$

Infatti: se $\boxed{\{b_n\}\to 0}$ $\{1+b_n\}\to 1$ $\{1+b_n\}^t \to 1^t = 1$

$\Rightarrow$ NUMERATORE $\to 0$

$(1+b_n)^t = e^{a_n} \Rightarrow a_n = \ln(1+b_n)^t = \boxed{t\ln(1+b_n)}$

e quindi $\boxed{\{a_n\}\to 0}$.

$$\frac{(1+b_n)^t-1}{b_n} = \frac{e^{a_n}-1}{b_n}\cdot\frac{a_n}{a_n} = \frac{e^{a_n}-1}{a_n}\cdot\frac{t\ln(1+b_n)}{b_n}$$

$$\left\{\frac{e^{a_n}-1}{a_n}\right\} \to 1 \quad \text{perché } \{a_n\}\to 0$$

$$\left\{\frac{\ln(1+b_n)}{b_n}\right\} \to 1 \quad \text{"} \quad \{b_n\}\to 0$$

$$\Rightarrow \left\{\frac{e^{a_n}-1}{a_n}\cdot\frac{t\ln(1+b_n)}{b_n}\right\} \to t .$$

$$\lim_{n\to+\infty}\left(\cancel{\frac{n}{7}\cdot\frac{3}{n^?}} - \frac{5}{7}\,\frac{n}{n} - \cancel{\frac{1}{7}\,\frac{n}{n^3}} + \cancel{\frac{2}{7}\,\frac{n}{n^?}}\right) = -5/7$$

$$\lim_{n\to+\infty} \underset{\substack{\downarrow\\ +\infty}}{n}\ \log_{10}\underbrace{\left(\frac{n+1}{n^2+2}\right)}_{-\infty} = -\infty$$

$$\lim_{n\to\infty}\left(\left(1+\frac{1}{n}\right)^{\sqrt{2}}-1\right)n = \lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^{\sqrt{2}}-1}{\frac{1}{n}} = \sqrt{2}$$

$$\lim_{n\to+\infty} n\left(5^{\frac{n+1}{n^2+1}}-1\right) = \lim_{n\to+\infty}\frac{e^{\frac{n+1}{n^2+1}\cdot\ln 5}-1}{\frac{1}{n}} \overset{1^\circ\text{ metodo}}{=}$$

$$\boxed{5^{a_n} = e^{a_n\ln 5}} = \lim_{n\to+\infty}\frac{e^{\frac{n+1}{n^2+1}\cdot\ln 5}-1}{\frac{n+1}{n^2+1}\cdot\ln 5}\cdot\frac{\frac{n+1}{n^2+1}\ln 5}{\frac{1}{n}} =$$

$$= \lim_{n\to\infty}\frac{n(n+1)}{n^2+1}\ln 5 = \ln 5 \qquad 1$$

2° metodo: uso degli asintotici

$$e^{a_n\ln 5}-1 \sim a_n\cdot\ln 5 \quad \text{se } \{a_n\}\to 0$$

$$\lim_{n\to+\infty} n\left(e^{\frac{n+1}{n^2+1}\ln 5}-1\right) = \lim_{n\to+\infty} n\cdot\frac{n+1}{n^2+1}\ln 5 =$$

$$= \ln 5$$

$$\lim_{n\to+\infty} n^2\left(\sqrt[3]{\frac{n^2-1}{n^2+1}}-1\right) =$$

$$= \lim_{n\to+\infty} n^2\left(\sqrt[3]{1-\frac{2}{n^2+1}}-1\right) =$$

$$= \lim_{n\to+\infty} n^2\left(\left(1\boxed{-\frac{2}{n^2+1}}\right)^{1/3}-1\right) = \text{applico l'asintotico}$$

$$= \lim_{n\to+\infty} n^2\left(-\frac{1}{3}\cdot\frac{2}{n^2+1}\right) = -\frac{2}{3}$$

$$\lim_{n\to+\infty}\left(1-\frac{1}{\sqrt{n}}\right)^n = [1^\infty]$$

$$= \lim_{n\to+\infty}\left(1-\frac{1}{\sqrt{n}}\right)^{(-\sqrt{n})(-\sqrt{n})} = \lim_{n\to+\infty}\left(\left(1-\frac{1}{\sqrt{n}}\right)^{-\sqrt{n}}\right)^{-\sqrt{n}} =$$

$$= e^{\lim -\sqrt{n}} = e^{-\infty} = 0^+$$

con i limiti derivati da quello di Nepero abbiamo operato un po' di confronti di infinitesimi.

Confrontiamo i vari tipi di infiniti:

$\{\ln n\}$, $\{n^\alpha\}$, $\{e^n\}$, $\{n!\}$, n^n ....

confronto di potenze: $\alpha, \beta > 0$

$$\lim_{n\to+\infty} \frac{n^\alpha}{n^\beta} = \lim_{n\to+\infty} n^{\alpha-\beta} = \begin{cases} \alpha>\beta : n^\alpha \text{ è infinito di ord. sup. a } n^\beta \\ \alpha<\beta : n^\alpha \text{ è infinito di ord. inf. a } n^\beta \\ \alpha=\beta : \text{stesso ordine} \end{cases}$$

$a, b \in (1, +\infty)$

$$\lim_{n\to+\infty} \frac{a^n}{b^n} = \lim_{n\to+\infty} \left(\frac{a}{b}\right)^n = \begin{cases} a>b \Leftrightarrow \frac{a}{b}>1 \Rightarrow +\infty \\ a=b : 1 \\ a<b \Leftrightarrow \frac{a}{b}<1 \to 0^+ \end{cases}$$

tra due esponenziali di base >1 è infinito di ord. superiore quello con base maggiore

$\boxed{\text{a parità di esponente!}}$

confronto tra due logaritmi di base $a, b \in (1, +\infty)$

$$\lim_{n\to+\infty} \frac{\log_a n}{\log_b n} = \lim_{n\to+\infty} \frac{\log_a b \cdot \cancel{\log_b n}}{\cancel{\log_b n}} = \log_a b \neq 0$$

poiché $b \neq 1$

quindi due logaritmi hanno sempre ugual ordine di infinito.

Dico ma non dimostro che

$$\lim_{n\to+\infty} \frac{\ln(n)}{(n)} = 0 \Rightarrow \lim_{\substack{n\to+\infty \\ \{a_n\}\to+\infty}} \frac{\ln a_n}{a_n} = 0$$

$\{\ln n\}$ è infinito di ordine inf. a $\{n\}$

Esercizio

$$\lim_{n\to+\infty} \frac{(\ln n^{10})^{100}}{n^{\frac{1}{1000}}} = 0 \quad \text{infatti}$$

$$= \lim_{n\to+\infty} \left(\frac{\ln n^{10}}{n^{\frac{1}{10^5}}}\right)^{100} = \lim_{n\to+\infty} \left(\frac{\ln\left(n^{\frac{1}{10^5}}\right)^{10^6}}{n^{\frac{1}{10^5}}}\right)^{100} =$$

$$= \lim_{n\to+\infty} \left(\frac{10^6 \ln\left(n^{\frac{1}{10^5}}\right)}{n^{\frac{1}{10^5}}}\right)^{100} =$$

$$= \lim_{n\to+\infty} 10^{600} \cdot \left(\frac{\ln n^{\frac{1}{10^5}}}{n^{\frac{1}{10^5}}}\right)^{100} = 0$$

in generale ho verificato che $\forall \alpha, \beta, \gamma \in (0, +\infty)$

$$\lim_{n\to+\infty} \frac{(\ln n^\alpha)^\beta}{n^\gamma} = 0$$

Vale il criterio del rapporto:

se $a_n \geq 0$ e $\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = l < 1$ allora

$\lim_{n\to+\infty} a_n = 0$ (si dimostra usando il teor. della perm. del segno)

Cerco di provare che

$\lim_{n\to\infty} \frac{n^2}{e^n} = 0$ usando il criterio del rapporto.

$$a_n = \frac{n^2}{e^n} \qquad a_{n+1} = \frac{(n+1)^2}{e^{n+1}}$$

$$\left\{\frac{a_{n+1}}{a_n}\right\} = \left\{\frac{(n+1)^2}{e^{n+1}} \cdot \frac{e^n}{n^2}\right\} \longrightarrow \frac{1}{e} < 1$$

vale il criterio del rapporto cioè $\left\{\frac{n^2}{e^n}\right\} \to 0$

In generale

$$\lim_{n\to+\infty} \frac{n^\alpha}{c^n} = 0 \qquad \forall \alpha > 0,\ c > 1$$

Confronto fattoriale - esponenziale

$$a_n = \frac{10^n}{n!} \qquad a_{n+1} = \frac{10^{n+1}}{(n+1)!}$$

$$\left\{\frac{a_{n+1}}{a_n}\right\} = \left\{\frac{10^{n+1}}{\cancel{n!}(n+1)} \cdot \frac{\cancel{n!}}{10^n}\right\} = \left\{\frac{10}{n+1}\right\} \to 0 < 1$$

$\Rightarrow \lim_{n\to+\infty} 10^n/n! = 0$: $n!$ è infinito di ordine sup. a 10^n

$$\left\{\frac{n^n}{n!}\right\} = \left\{\frac{e^{n \ln n}}{n!}\right\} \longrightarrow +\infty$$

$$a_n = \frac{n!}{n^n} \qquad a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} =$$

$$= \frac{n+\cancel{1}}{(n+1)^{n+\cancel{1}}} \cdot n^n = \left(\frac{n}{n+1}\right)^n =$$

$$= \left(1 + \frac{-1}{n+1}\right)^n$$

$$\lim_{n\to+\infty} \left(1 + \frac{-1}{n+1}\right)^n = e^{-1} < 1 \Rightarrow \left\{\frac{n!}{n^n}\right\} \to 0$$

alternativa

$$\lim_{n\to+\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to+\infty} e^{n \ln\left(\frac{n}{n+1}\right)} =$$

$$= \lim_{n\to+\infty} e^{n \ln\left(1 - \frac{1}{n+1}\right)} = \lim_{n\to+\infty} e^{n\cdot\left(-\frac{1}{n+1}\right)} = e^{-1}$$

Significa che in ordine crescente ho i seguenti infiniti

$$\ln n,\ n^\alpha,\ c^n,\ n!,\ n^n$$

$\boxed{\alpha > 0}$ $\boxed{c > 1}$

Fare:

$$\lim_{n\to+\infty} \frac{\sqrt[3]{n}+\sqrt[6]{n}}{\sqrt{n}+1} = \dots$$

$$\lim_{n\to+\infty} \frac{(\log_{10} n)^5 + 5(\ln n)^4}{\sqrt[10]{n-1} - 20} =$$

$$= \lim_{n\to+\infty} \frac{(\log_{10} e)^5(\ln n)^5 + 5(\ln n)^4}{(n-1)^{1/10} - 20} =$$

$$= \lim_{n\to+\infty} \frac{(\log_{10} e)^5(\ln n)^5\Big(1 + \underbrace{\frac{5}{(\log_{10} e)^5 \cdot \ln n}}_{\to 0}\Big)}{(n-1)^{1/10}\Big(1 \underbrace{- \frac{20}{(n-1)^{1/10}}}_{\to 0}\Big)}$$

$$= \lim_{n\to+\infty} \frac{(\log_{10} e)^5(\ln n)^5}{(n)^{1/10}\underbrace{\left(1-\frac{1}{n}\right)^{1/10}}_{\to 1}} = 0$$

per confronto tra la potenza di un log. e la potenza del suo argomento

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$$\lim_{n\to+\infty} \frac{\ln(1+e^n)}{2n - e^{-n}+1} = \lim_{n\to+\infty} \frac{\ln(e^n(1+e^{-n}))}{2n}$$

$$= \lim_{n\to+\infty} \frac{\ln e^n + \ln(1+e^{-n})}{2n} = \lim_{n\to+\infty} \frac{n}{2n} = \frac{1}{2}$$

$$\lim_{n\to+\infty} \frac{\operatorname{sen}(n^2-3n^3)}{n^3} = 0$$

$|\operatorname{sen}(n^2-3n^3)| \leq 1$; denom. $\to +\infty$

Non si può usare l'asintotico $\operatorname{sen} a_n \sim a_n$ poiché $\{a_n\} \not\to 0$

$$\lim_{n\to+\infty} \frac{\sqrt{n} - n\ln(1+n)}{n} \leftarrow \boxed{\infty-\infty \text{ : quale di ordine superiore?}}$$

$$= \lim_{n\to+\infty} \frac{\cancel{\sqrt{n}}(1-\sqrt{n}\ln(1+n))}{\cancel{\sqrt{n}}\sqrt{n}} =$$

$$= \lim_{n\to+\infty} \frac{1-\sqrt{n}\ln(1+n)}{\sqrt{n}} = \lim_{n\to+\infty} -\ln(1+n) = -\infty$$

In questo caso conveniva scrivere la succ. come

$$\left\{\frac{\sqrt{n}}{n} - \ln(1+n)\right\} \to -\infty$$

$$\lim_{n\to+\infty} \left(\ln\sqrt{n+9n^2} - \ln n\right) = [\infty-\infty]$$

$$= \lim_{n\to+\infty} \ln\left(\frac{\sqrt{n+9n^2}}{n}\right) \underset{\substack{\boxed{n>0} \\ \sqrt{n^2}=n}}{=} \lim_{n\to+\infty} \ln\sqrt{\frac{n+9n^2}{n^2}}$$

$$= \ln 3$$