$$\int f g' \, dx = fg - \int f' g \, dx$$: altri esempi di utilizzo.

$$\int x e^x \, dx \qquad FF: x \qquad FD: e^x dx$$

$f = x \Rightarrow f' = 1$

$g' = e^x \Rightarrow g = e^x$

$$\Rightarrow \int x e^x \, dx = x e^x - \int 1 \cdot e^x \, dx = $$
$$= x e^x - e^x + c$$

$$\int x^n e^x \, dx = x^n e^x - \int n x^{n-1} e^x \, dx = \ldots$$

e proseguendo nell'integrazione per parti arriverò a $k \int x e^x \, dx$ e quindi so risolvere l'integrale.

$$\int x \cos x \, dx \qquad FF: x \qquad FD: \cos x \, dx$$

$f = x \Rightarrow f' = 1$

$g' = \cos x \Rightarrow g = \operatorname{sen} x$

$$\int x \cos x \, dx = x \operatorname{sen} x - \int 1 \cdot \operatorname{sen} x \, dx =$$
$$= x \operatorname{sen} x + \int (-\operatorname{sen} x) \, dx =$$
$$= x \operatorname{sen} x + \cos x + c$$

$$\int x^2 \cos x \, dx = \qquad FF: x^2 \qquad FD = \cos x \, dx$$
pp.

$f = x^2 \Rightarrow f' = 2x$

$g' = \cos x \Rightarrow g = \operatorname{sen} x$

$$\int x^2 \cos x \, dx = x^2 \operatorname{sen} x + \int 2x (\operatorname{sen} x) \, dx =$$
$$= x^2 \operatorname{sen} x + \left[2x \cos x - \int 2 \cos x \, dx \right] =$$
$$= x^2 \operatorname{sen} x + 2x \cos x - 2 \operatorname{sen} x + c$$

VERIFICA:

$$(x^2 \operatorname{sen} x + 2x \cos x - 2 \operatorname{sen} x)' =$$
$$\cancel{2x \operatorname{sen} x} + x^2 \cos x + \cancel{2 \cos x} - \cancel{2x \operatorname{sen} x} - \cancel{2 \cos x} = x^2 \cos x$$

$$\int e^x \cos x \, dx =$$ è indifferente chi scelgo come FF.

$\boxed{FF\ e^x}$ $$= e^x \operatorname{sen} x + \int e^x (-\operatorname{sen} x) \, dx =$$

$\boxed{FF\ e^x}$ $$= e^x \operatorname{sen} x + \left[e^x \cos x - \int e^x \cos x \, dx \right]$$

$$= e^x \operatorname{sen} x + e^x \cos x - \int e^x \cos x \, dx$$

$$\frac{2}{2} \int e^x \cos x \, dx = \frac{e^x \operatorname{sen} x + e^x \cos x}{2} + c$$

$\int e^x \operatorname{sen} x \, dx$ — è indifferente la scelta del FF.

$\boxed{FF: \operatorname{sen} x}$

$$\int \operatorname{sen} x \cdot e^x dx = \operatorname{sen} x \cdot e^x - \int \cos x \cdot e^x dx$$

= o dico che ho appena calcolato

$$\int \cos x \cdot e^x dx = \frac{e^x(\operatorname{sen} x + \cos x)}{2} + c$$

sostituisco:

$$\int \operatorname{sen} x \, e^x dx = e^x \left(\frac{2 \operatorname{sen} x - \operatorname{sen} x - \cos x}{2} \right) + c$$

$$= e^x \frac{\operatorname{sen} x - \cos x}{2} + c$$

oppure:

$\boxed{FF: \cos x}$

$$= \operatorname{sen} x \cdot e^x - \left[\cos x \cdot e^x - \int - \operatorname{sen} x \, e^x dx \right] =$$

$$= \operatorname{sen} x \cdot e^x - \cos x \cdot e^x - \int \operatorname{sen} x \cdot e^x dx =$$

$$\Rightarrow 2 \int \operatorname{sen} x \cdot e^x dx = (\operatorname{sen} x - \cos x) e^x + c$$

. . . .

FF: $\cos x$

$$\int (\cos x)^2 dx =$$

$$= \cos x \cdot \operatorname{sen} x - \int - \operatorname{sen} x \cdot \operatorname{sen} x \, dx =$$

$$= \cos x \operatorname{sen} x - \int \operatorname{sen} x \underbrace{(-\operatorname{sen} x)}dx =$$

$$= \cos x \operatorname{sen} x - \left[\operatorname{sen} x \cos x - \int \cos x \cdot \cos x \, dx \right]$$

$$= \cancel{\cos x \operatorname{sen} x} - \cancel{\operatorname{sen} x \cos x} + \int (\cos x)^2 dx$$

Quindi non posso procedere così!

$$\int (\cos x)^2 dx = \cos x \cdot \operatorname{sen} x + \int (\operatorname{sen} x)^2 dx =$$

$\operatorname{sen}^2 x = 1 - \cos^2 x$

$$= \cos x \operatorname{sen} x + \int (1 - \cos^2 x) dx =$$

$$= \cos x \operatorname{sen} x + x - \int \cos^2 x \, dx$$

$$\frac{2}{2} \int (\cos x)^2 dx = \frac{\cos x \operatorname{sen} x + x}{2} + c$$

$$\int (\operatorname{sen} x)^2 dx = \int (1 - (\cos x)^2) dx =$$

$$= x - \frac{\cos x \operatorname{sen} x + x}{2} + c =$$

$$= \frac{x - \cos x \cdot \operatorname{sen} x}{2} + c$$

METODO ALTERNATIVO

$$\int \cos^2 x \, dx$$

$$\cos 2x = \cos^2 x - \operatorname{sen}^2 x = 2\cos^2 x - 1 \implies \cos^2 x = \frac{\cos 2x + 1}{2}$$

$$\| $$

$$\int \left(\frac{1}{2}\cos 2x + \frac{1}{2}\right) dx = \frac{1}{2}\int \cos 2x \, dx + \frac{1}{2}x =$$

$$= \frac{1}{2}\left(\frac{1}{2}\operatorname{sen} 2x\right) + \frac{1}{2}x + c$$

$$= \frac{\operatorname{sen} x \cos x + x}{2} + c$$

Metodo di sostituzione VEDI!

Generalizzo. VEDI a lato. Nell'esempio prec.:

$$\int \cos 2x \, dx = \frac{1}{2}\int (\cos 2x) \cdot 2dx$$

$g(x) = 2x \quad g'(x) = 2$

$h'(t) = \cos t \quad h(t) = \operatorname{sen} t \quad t = 2x$

$$\int \frac{g'(x)}{g(x)} dx = \int \frac{1}{g(x)} \cdot \underline{g'(x)\, dx} =$$

$$= \ln |g(x)| + c$$

(1a)

$$\int \tan x \, dx = -\int \frac{-\operatorname{sen} x}{\cos x} dx = -\ln|\cos x| + c$$

$$\int \frac{1}{\tan x} dx = \int \frac{\cos x}{\operatorname{sen} x} dx = \ln|\operatorname{sen} x| + c$$

III. Integrazione per sostituzione

Ricordo: $D(h(g(t))) = h'(g(t)) \cdot g'(t)$

Sia ora $f : [a,b] \to \mathbb{R}$ la funzione continua di cui vogliamo trovare una primitiva H : $H'(x) = f(x)$ $\forall x \in [a,b]$

VERSIONE FACILE Se $f(x)$ ha la forma $h'(g(x)) \cdot g'(x)$ si ha

$$\int f(x)\,dx = \int h'(g(x)) \cdot g'(x)\,dx = h(g(x)) + c$$

e quindi $H(x) = h(g(x))$.

Esempi	Casi particolari:
1) $\int \frac{g'(x)}{g(x)} dx = \ln\|g(x)\| + c$	ⓐ $\int \operatorname{tg} x \, dx =$
	ⓑ $\int \frac{x\,dx}{x^2+b^2} =$
2) $\int \frac{g'(x)}{1+g^2(x)} dx = \operatorname{arctg}(g(x)) + c$	$\int \frac{dx}{x^2+2x+2} =$
VEDI PAG SEGUENTE	

In generale cerco di riprodurre questa situazione con una SOSTITUZIONE $x = g(t)$

VERSIONE GENERALE. Sia $g : [c,d] \to [a,b]$ una funz. derivabile con derivata 1^a continua e $\neq 0$ su $[c,d]$.

(ciò garantisce che esiste $g^{-1} : [a,b] \to [c,d]$: PERCHÉ?)

Allora

$$\int f(x)\,dx = \left[\int f(g(t)) \cdot g'(t)\,dt\right]_{t = g^{-1}(x)}$$

Infatti, se $H'(x) = f(x)$ $\quad \int H'(g(t)) \cdot g'(t)\,dt = H(g(t)) + c$

e la sostituzione $t = g^{-1}(x)$ riporta proprio a $H(x) + c$.

Esempi

1) $\int \frac{dx}{x^2+a^2}$ VEDI pag 8 $\qquad$ 2) $\int \frac{dx}{x^2+x+1}$ vedi pag 8

7

$$\int \frac{x}{x^2+4}\,dx = \frac{1}{2}\int \frac{2x}{x^2+4}\,dx =$$

$$= \frac{1}{2}\ln|x^2+4| + c$$

1b
pag 6

Attenzione:

$$\int \frac{x+3}{x^2+x}\,dx = \frac{1}{2}\int \frac{2x+6}{x^2+x}\,dx = \quad (x^2+x)' = 2x+1$$

$$= \frac{1}{2}\int\left[\left(\frac{2x+1}{x^2+x}\right)+\left(\frac{5}{x^2+x}\right)\right]dx$$

$$= \frac{1}{2}\left(\ln|x^2+x| + 5\int\left(\frac{1}{x} - \frac{1}{x+1}\right)dx\right)$$

$$= \frac{1}{2}\left(\ln|x^2+x| + 5\ln\left|\frac{x}{x+1}\right|\right) + c$$

ecc.

$$\int \frac{g'(x)}{1+(g(x))^2}\,dx = \arctan(g(x)) + c$$

Esempio 2 pag 6

$$\int \frac{dx}{x^2+2x+2} = \int \frac{1\cdot dx}{1+(x+1)^2} = \arctan(x+1)+c$$

$$\int \frac{g'(x)}{g^2(x)} = -\frac{1}{g(x)} + c$$

Esempio 3

8

Funz raz fratta
Con denom. di 2° grado
$\Delta < 0$

$$\int \frac{1}{x^2+4}\,dx =$$

$$\frac{1}{4\left(\frac{x^2}{4}+1\right)} = \frac{1}{4}\cdot\frac{1}{t^2+1} \quad \text{ove } t = \frac{x}{2}$$

Allora sostituisco nell'integrale $x = 2t$
$dx = 2\cdot dt$

$$= \int \frac{1}{4t^2+4}\cdot 2\,dt = \frac{1}{2}\int\frac{dt}{t^2+1} =$$

$$= \frac{1}{2}\arctan t + c = \frac{1}{2}\arctan\left(\frac{x}{2}\right) + c$$

Esempi 1,2 in fondo a pag 6

FUNZ RAZ FRATTA
denom 2° grado
$\Delta < 0$

$$\int \frac{dx}{x^2+x+1} =$$

$$x^2+x+1 = x^2+2\cdot\frac{1}{2}x+\frac{1}{4}+\frac{3}{4} =$$

$$= (x+1/2)^2 + 3/4 = \frac{3}{4}\left(\frac{(x+1/2)^2}{\frac{3}{4}}+1\right)$$

arctan ??

sostituisco $\frac{x+1/2}{\sqrt{3}/2} = t$

$$\boxed{\begin{aligned} x+1/2 &= \frac{\sqrt{3}}{2}t \\ dx &= \frac{\sqrt{3}}{2}dt \end{aligned}}$$

$$= \int \frac{\frac{\sqrt{3}}{2}dt}{\frac{3}{4}t^2+\frac{3}{4}} = \frac{\sqrt{3}/2}{3/4}\int\frac{dt}{t^2+1} = \frac{2}{\sqrt{3}}\arctan t + c =$$

$$= \frac{2}{\sqrt{3}}\arctan\left(\frac{x+1/2}{\sqrt{3}/2}\right) + c$$

ESERCIZI

(Ricavare esercizi analoghi scambiando seno con coseno)

1) $\int x^2 \operatorname{sen} x \, dx$

1bis) $\int x \operatorname{sen} x^2 \, dx$

2) $\int x^n \cos x \, dx$

3) $\int x \cos 2x \, dx$

4) $\int x \, (\cos x)^2 \, dx$

5) $\int e^x (\operatorname{sen} x)^2 \, dx$

6) $\int x e^{2x} \, dx$

7) $\int e^{-3x} \operatorname{sen} x \, dx$

8) $\int \frac{x-1}{2x^2-4x+5} \, dx$

9) $\int \frac{dx}{2x^2-4x+5}$

10) $\int \frac{dx}{x^2-4x-5}$

11) $\int \operatorname{arctg} x \, dx$

12) $\int x \operatorname{arctg} x \, dx$

13) $\int \operatorname{arcsen} x \, dx$

14) $\int \frac{dx}{\operatorname{tg} x}$

15) $\int \frac{dx}{\operatorname{sen} x \cos x}$

16) $\int \frac{dx}{\operatorname{sen} x}$ (SOST.)

17) $\int \frac{dx}{\cos x}$ (SOST.)

18) $\int \frac{\sqrt{x} - \sqrt[4]{x}}{1+\sqrt[4]{x}} \, dx$ (SOST.)

19) $\int \operatorname{sen}^3 x \sqrt{\cos x} \, dx$

20) $\int \left(\sqrt{\operatorname{tg}^3 x} + \frac{1}{\sqrt{\operatorname{tg} x}} \right) dx$

21) $\int \frac{e^x \ln(2+e^x)}{(e^x+1)^2} \, dx$ (...FRATTE)

22) $\int \sqrt{a^2-x^2} \, dx$

23) $\int 2x \ln(1-2x) \, dx$

24) $\int \left(x + (\operatorname{sen} 2x)^2 \right) \cos 2x \, dx$

25) $\int \left(e^{2x} + \sqrt{x^2+x} \right)(2x+1) \, dx$

26) $\int \frac{\operatorname{sen} x}{(\operatorname{sen} x)^2 + 2(\cos x)^2} \, dx$

27) $\int \ln\left(1 + \frac{x}{2}\right) dx$

28) $\int x \cos(2x-1) \, dx$

29) $\int x (\operatorname{sen} 2x - 1) \, dx$

30) $\int \frac{\sqrt{1+\ln x}}{x} \, dx$

31) $\int \left(e^{2x} - \sqrt{x^2-5} \right) dx$

32) $\int \frac{\cos x - x \operatorname{sen} x}{x^2 - (x \operatorname{sen} x)^2} \, dx$

$$\int \frac{x-1}{2x^2-4x+5}\,dx = \qquad (2x^2-4x+5)' = 4x-4$$

$$= \frac{1}{4}\int \frac{4x-4}{2x^2-4x+5}\,dx = \frac{1}{4}\ln|2x^2-4x+5| + c.$$

$$\int \frac{dx}{2x^2-4x+5} = \qquad \frac{\Delta}{4} = 4-10<0 \Rightarrow \text{arctan ?}$$

$$2(x^2-2x+1)+3 =$$

$$= 2\left[(x-1)^2+\tfrac{3}{2}\right] = 2\cdot\frac{3}{2}\left[\frac{(x-1)^2}{3/2}+1\right]$$

$$\frac{x-1}{\sqrt{3/2}} = t \quad \Rightarrow \quad \boxed{\begin{array}{l} x-1 = \sqrt{\tfrac{3}{2}}\,t \\ dx = \sqrt{\tfrac{3}{2}}\,dt \end{array}}$$

$$= \int \frac{\sqrt{3/2}\,dt}{3(t^2+1)} = \frac{1}{\sqrt{6}}\,\text{arctg}\,t + c =$$

$$= \frac{1}{\sqrt{6}}\arctan\left(\frac{x-1}{\sqrt{3/2}}\right) + c$$

$$\int \frac{dx}{x^2-4x-5} = \qquad \text{Radici del denom: } x=-1,5$$

$$\frac{1}{x^2-4x-5} = \frac{A}{x+1} + \frac{B}{x-5} = \frac{(A+B)x-5A+B}{(x+1)(x-5)}$$

$$\Leftrightarrow (A+B)x + B-5A = 1 \Leftrightarrow$$

$$\begin{cases} A+B=0 \\ B-5A=1 \end{cases} \Longrightarrow \begin{array}{l} A=-B \\ 6B=1 \end{array} \qquad \begin{array}{l} A=-1/6 \\ B=1/6 \end{array}$$

$$\int \frac{dx}{x^2-4x-5} = \int \frac{-1/6}{x+1} + \frac{1/6}{x-5}\,dx$$

$$= \frac{1}{6}\left(\ln|x-5| - \ln|x+1|\right) + c$$

$$= \frac{1}{6}\ln\left|\frac{x-5}{x+1}\right| + c$$

$$\int \frac{1}{\text{sen}^2x\cos^2x}\,dx = \int \frac{\cos^2x+\text{sen}^2x}{\text{sen}^2x\cos^2x}\,dx =$$

$$= \int \frac{1}{\text{sen}^2x}\,dx + \int \frac{1}{\cos^2x}\,dt = -\frac{1}{\tan x} + \tan x + c$$

$$\left(\frac{1}{\tan x}\right)' = \left(\frac{\cos x}{\text{sen}\,x}\right)' = \frac{-\text{sen}\,x\cdot\text{sen}\,x - \cos x\cdot\cos x}{\text{sen}^2x} =$$

$$= \frac{-1}{\text{sen}^2x}$$

$$\int \frac{dx}{tg\,x} = \ln|\operatorname{sen} x| + c$$

$$\int \frac{dx}{\operatorname{sen} x \cos x} = \int \frac{\operatorname{sen}^2 x + \cos^2 x}{\operatorname{sen} x \cos x}\, dx =$$

$$= \int \frac{\operatorname{sen} x}{\cos x}\, dx + \int \frac{\cos x}{\operatorname{sen} x}\, dx$$

$$= \ln|\operatorname{sen} x| - \ln|\cos x| + c =$$

$$= \ln|tg\,x| + c$$

$$\int \frac{1}{\operatorname{sen} x}\, dx \underset{\substack{x = 2t \\ dx = 2\,dt}}{=} \int \frac{2\,dt}{\operatorname{sen} 2t} =$$

$$= \int \frac{\cancel{2}\,dt}{\cancel{2}\operatorname{sen} t \cos t} = \ln|\tan t| + c =$$

$$= \ln\left|\tan \frac{x}{2}\right| + c$$

$$\cos x = \operatorname{sen}\left(\frac{\pi}{2} - x\right)$$

$$\int \frac{1}{\cos x}\, dx = -\int \frac{-1}{\operatorname{sen}\left(\frac{\pi}{2} - x\right)}\, dx = -\ln\left|\tan \frac{\pi/2 - x}{2}\right| + c$$

$$\int \arctan x\, dx = \text{pp.} \qquad \text{FF } \arctan x$$

$$= x \arctan x - \int x \cdot \frac{1}{1+x^2}\, dx =$$

$$= x \arctan x - \frac{1}{2} \int \frac{2x}{1+x^2}\, dx =$$

$$= x \arctan x - \frac{1}{2} \ln(1+x^2) + c$$

ES11

$$\int x \arctan x\, dx = \text{pp.} \qquad \text{FD}: x\,dx$$

$$= \frac{x^2}{2} \arctan x - \frac{1}{2} \int \frac{x^2}{1+x^2}\, dx$$

$$= \frac{x^2}{2} \arctan x - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2}\right) dx =$$

$$= \frac{x^2}{2} \arctan x - \frac{1}{2} x + \frac{1}{2} \arctan x + c$$

ES12