①

$$g(t)=\frac{\ln(1+t)}{(1+t)\sqrt{t}}$$

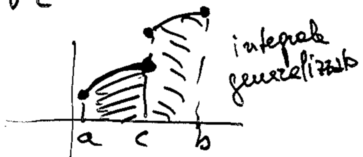
I.D. ...asintotici... $\int_{0^+}^{+\infty} g(t)\,dt$ converge?

I.D. $\begin{cases}1+t>0\\ t>0\end{cases} \Longrightarrow (0,+\infty)$

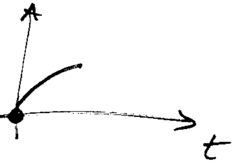
per $t\to 0^+$

$$g(t)=\frac{\ln(1+t)}{(1+t)\sqrt{t}} \sim \frac{t}{1\cdot\sqrt{t}}=\sqrt{t}$$

$$\lim_{t\to 0} g(t)=0$$



Completando per continuità il valore di $g(t)$ in $t=0$ trovo una funz integrabile tra 0 e $b>0$



$\overset{\shortparallel}{1}$ per comodità

Esamino l'integrale di 1ª specie $\int_1^{+\infty} g(t)\,dt$

per $t\to+\infty$

$$g(t)=\frac{\ln(1+t)}{(1+t)\sqrt{t}} \sim \frac{\ln t}{t^{3/2}}$$

Confronto $\frac{\ln t}{t^{3/2}}$ con $\frac{1}{t^k}$ ove $1<k<\frac{3}{2}$

$$\lim_{t\to+\infty}\frac{\ln t/t^{3/2}}{1/t^k}=\lim_{t\to+\infty}\frac{\ln t}{t^{3/2-k}}\underset{\substack{\uparrow\\ k<3/2}}{=}0$$

②

Quindi da un certo t_0 in poi

$$\frac{\ln t}{t^{3/2}}<\frac{1}{t^k}$$

e $\int_1^{+\infty}\frac{1}{t^k}\,dt$ è convergente poiché $k>1$

$\Longrightarrow$ per il criterio del confronto "forma disequazione"

$\int_1^{+\infty}\frac{\ln t}{t^{3/2}}\,dt$ converge

$\Longrightarrow$ per il criterio del confronto asintotico converge

$$\int_1^{+\infty} g(t)\,dt$$

$$\int_0^{+\infty} g(t)\,dt=\underbrace{\int_0^1 g(t)\,dt}_{\substack{\text{generalizzato}\\ \downarrow\\ \text{numero}}}+\underbrace{\int_1^{+\infty} g(t)\,dt}_{\substack{\text{l'improprio I specie}\\ \downarrow\\ \text{converge}}}$$

$=$ converge.

$$\frac{x}{y}+\frac{8}{x}-y = f(x,y)$$

I.D. $\mathbb{R}^2-\{x=0 \text{ e } y=0\}$ per f e f_x, f_y. I quattro quadranti sono anche le regioni di continuità di f, f_x, f_y

Studiare i punti critici.

$$\text{grad} f = (0,0)$$

$$f_x = \frac{1}{y}-\frac{8}{x^2}$$

$$f_y = \frac{-x}{y^2}-1$$

$$\begin{cases} \frac{1}{y}-\frac{8}{x^2}=0 \\ \frac{-x}{y^2}-1=0 \end{cases} \rightarrow \frac{1}{y}=\frac{8}{x^2}$$

$$\begin{cases} y=\frac{1}{8}x^2 \\ -x-y^2=0 \end{cases} \Leftrightarrow \begin{cases} y=\frac{1}{8}x^2 \\ -x-\frac{1}{64}x^4=0 \end{cases} \Leftrightarrow \begin{cases} y=\frac{1}{8}x^2 \\ \cancel{x}\left(1+\frac{1}{64}x^3\right)=0 \end{cases} \Leftrightarrow$$

$x \neq 0$ per I.D.

$$\begin{cases} y=\frac{1}{8}x^2 \\ x^3=-64 \end{cases} \Leftrightarrow \begin{cases} y=\frac{1}{8}\cdot 16=2 \\ x=-4 \end{cases} \qquad A=(-4,2)$$

$$f_{xx}=\frac{16}{x^3} \qquad f_{xy}=-\frac{1}{y^2} \qquad f_{yx}=-\frac{1}{y^2} \qquad f_{yy}=\frac{2x}{y^3}$$

uguali

$$H_f(x,y)=\begin{vmatrix} \frac{16}{x^3} & -\frac{1}{y^2} \\ -\frac{1}{y^2} & \frac{2x}{y^3} \end{vmatrix} \qquad H_f(-4,2)=\begin{vmatrix} -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -1 \end{vmatrix}=$$

$$=-\frac{1}{4}\begin{vmatrix} 1 & 1 \\ -\frac{1}{4} & -1 \end{vmatrix} = -\frac{1}{4}\left(-1+\frac{1}{4}\right) = \frac{3}{16}>0$$

estremante locale $\quad f_{xx}(A)<0 \Longrightarrow$ MAX locale forte

$$\begin{cases} y'+(t-2)y = e^{-t^2/2} \quad (*) \\ y(0)=1 \end{cases}$$

eq. diff lineare del 1° ordine completa

omog. associata: $z'+(t-2)z=0$

$$z'=(2-t)z \quad \text{var. sep.} \quad z=0$$

$$\int\frac{dz}{z}=\int(2-t)\,dt$$

$$\ln|z| = 2t-\frac{t^2}{2}+c$$

$$|z| = e^{2t-t^2/2}\cdot e^c$$

$$z=(\pm e^c)\,e^{2t-t^2/2}$$

0

soluzione:

$$z=k\,e^{2t-t^2/2}$$

variaz. delle costanti per trovare la sol. particolare

$$\bar{y}=k(t)\,e^{2t-t^2/2} \qquad \bar{y}'=k'e^{2t-t^2/2}+k(2-t)e^{2t-t^2/2}$$

Sostituisco in $(*)$

$$k'e^{2t-t^2/2}+\cancel{k(2-t)e^{2t-t^2/2}}+\cancel{(t-2)k\,e^{2t-t^2/2}}=e^{-t^2/2}$$

$$k'e^{2t-t^2/2}=e^{-t^2/2} \Rightarrow k'e^{2t}=1 \Rightarrow$$

$$k'=e^{-2t} \Rightarrow k=-\frac{1}{2}e^{-2t}$$

Sol part: $\bar{y}=-\frac{1}{2}\cancel{e^{-2t}}\,\cancel{e^{2t}}e^{-t^2/2}=-\frac{1}{2}e^{-t^2/2}$

Integr. generale $y=-\frac{1}{2}e^{-t^2/2}+k\,e^{2t-t^2/2}$

Condizione di Cauchy: $y(0)=1$

$$1=-\frac{1}{2}+k \qquad k=\frac{3}{2}$$

Sol probl. Cauchy $\varphi(t)=-\frac{1}{2}e^{-t^2/2}+\frac{3}{2}e^{2t-t^2/2}$.

$$y'' + 3y' = t^2 - 5t + 1$$

eq. diff lin. 2° ord. coef. cost. completa

omog. associata
$$z'' + 3z' = 0 \longrightarrow r^2 + 3r = 0 \longrightarrow \begin{matrix} r=0 \\ r=-3 \end{matrix}$$
equazione caratteristica

Sol particolari dell'omog. sono $e^{0\cdot t} = 1 \quad e^{-3t}$

$\downarrow$

int. generale dell'omogenea
$$C_1 + C_2 e^{-3t}$$

Ricerca della sol. particolare

$\bar{y}(t) = at^3 + bt^2 + ct + d \leftarrow$ (manca y nell'eq. $\Rightarrow$ polinomio di grado 2+1)

$\bar{y}'(t) = 3at^2 + 2bt + c$

$\bar{y}''(t) = 6at + 2b$ sostituisco

$$6at + 2b + 9at^2 + 6bt + 3c \equiv t^2 - 5t + 1 \quad \forall t$$

$$t^2(9a-1) + t(6a+6b+5) + 2b + 3c - 1 \equiv 0$$

$$\begin{cases} 9a - 1 = 0 \\ 6a+6b+5 = 0 \\ 2b+3c-1 = 0 \end{cases} \qquad \begin{matrix} a = 1/9 \\ b = -5/6 - 1/9 \\ c = \frac{1}{3} - \frac{2}{3}\left(-5/6 - 1/9\right) \end{matrix}$$

d è qualunque! $\Rightarrow$ lo pongo $=0$

Integrale generale

$$y(t) = c_1 + c_2 e^{-3t} + \frac{1}{9}t^3 - \frac{17}{18}t^2 + \frac{\cancel{96}}{\cancel{27}}{}^{??}\, t$$

$$y'' - 2y' + y = te^t$$

$\downarrow$ omog. assoc.
$$z'' - 2z' + z = 0 \Longrightarrow r^2 - 2r + 1 = 0 \quad r = 1$$
eq caratteristica

$z(t) = c_1 e^t + c_2 te^t$ soluz. dell'omog. assoc.

Sol. particolare: la cerco tra quelle del tipo $\big(\underbrace{a+bt}_{\text{sol dell'omogenea}} + ct^2\big)e^t$

$\bar{y}(t) = ct^2 e^t$

$\bar{y}' = c\left(2t\, e^t + t^2 e^t\right) = c\,(2t+t^2)e^t$

$\bar{y}'' = c\left(2+2t+2t+t^2\right)e^t = c\,(2+4t+t^2)e^t$

$$c\left[2 + \cancel{4t} + \cancel{t^2} - \cancel{4t} - \cancel{2t^2} + \cancel{t^2}\right]e^t = te^t$$

$\boxed{2c = t}$ non è ancora sufficiente!

$\bar{y}(t) = f(t)e^t$

$\bar{y}'(t) = \left(f'(t) + f(t)\right)e^t$

$\bar{y}''(t) = \left(f'' + 2f' + f\right)e^t$

$$\left(f'' + \cancel{2f'} + \cancel{f} - \cancel{2f'} - \cancel{2f} + \cancel{f}\right)\cancel{e^t} = t\cancel{e^t}$$

$$f'' = t \qquad f' = \frac{t^2}{2} \qquad f = \frac{t^3}{6}$$

$$\bar{y}(t) = \frac{t^3}{6}e^t \quad \Longrightarrow y(t) = \frac{t^3}{6}e^t + c_1 e^t + c_2 te^t$$

$$A_k = \begin{pmatrix} 3 & 3k & -2 & k+1 \\ -k & -1 & \boxed{3} & \boxed{0} \\ 3k & 3 & \boxed{-2k} & \boxed{2} \end{pmatrix} \qquad rg\, A_k \,?$$ (7)

e risolvere $A_k \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{vmatrix} 3 & 0 \\ -2k & 2 \end{vmatrix} = 6 \neq 0 \Rightarrow 2 \le rg\, A_k \le 3 = \min(n,m)$$

Orlo la matrice appena evidenziata con le restanti righe e colonne

$$\begin{vmatrix} 3k & -2 & k+1 \\ -1 & 3 & 0 \\ 3 & -2k & 2 \end{vmatrix} = 1 \cdot \begin{vmatrix} -2 & k+1 \\ -2k & 2 \end{vmatrix} + 3 \begin{vmatrix} 3k & k+1 \\ 3 & 2 \end{vmatrix} =$$

$$= -4 + 2k^2 + 2k + 18k - 9k - 9 =$$

$$= 2k^2 + 11k - 13 = 0$$

$$k = \frac{-11 \pm \sqrt{121 + 104}}{4} = \frac{-11 \pm 15}{4} = \begin{cases} 1 \\ -\frac{13}{2} \end{cases}$$

per $k \neq 1$ e $\neq -\frac{13}{2}$ $\quad rg\, A_k = 3$

$$\begin{vmatrix} 3 & -2 & k+1 \\ -k & 3 & 0 \\ 3k & -2k & 2 \end{vmatrix} = k \begin{vmatrix} -2 & k+1 \\ -2k & 2 \end{vmatrix} + 3 \begin{vmatrix} 3 & k+1 \\ 3k & 2 \end{vmatrix} =$$

$$= -4k + 2k^3 + 2k^2 + 18 - 9k^2 - 9k =$$

$= 2k^3 - 7k^2 - 13k + 18$. quanto vale per $k=1$ e per $k = -13/2$

per $k=1$: $2 - 7 - 13 + 18 = 0$

per $k = -\frac{13}{2}$: $-\frac{13^3}{4} - \frac{7 \cdot 13^2}{4} + \frac{13^2}{2} + 18 =$

$$-\frac{13^2}{\cancel{4}} \cdot \cancel{20}^{\,5} + \frac{13^2}{2} + 18 \neq 0$$

$\Rightarrow$ per $k = -\frac{13}{2}$ $\quad rg\, A_k = 3$

per $k = 1$ $\quad rg\, A_1 = 2$ poiché si annullano entrambi i det. che si ottengono orlando la matrice 2x2 a det$\neq 0$ (KRONECKER)

Soluzioni del sist omogeneo

se $k \neq 1$ dipendono da $4 - 3 = 1$ parametri

$k = 1$ " $4 - 2 = 2$ "

Se $k = 1$ $\quad \begin{pmatrix} 3 & 3 & -2 & 2 \\ -1 & -1 & \boxed{3} & \boxed{0} \\ 3 & 3 & \boxed{-2} & \boxed{2} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

le 2e due righe sono indip. posso dimenticare la 1a che dipende da esse.

$$\begin{cases} -x - y + \boxed{3z} = 0 \\ 3x + 3y \boxed{-2z + 2w} = 0 \end{cases}$$

x, y pensati come parametri

$$\begin{cases} x = s \\ y = t \\ z = \frac{s+t}{3} \\ w = \frac{s+t}{3} - \frac{3}{2}(s+t) \end{cases} \qquad \begin{cases} x = s \\ y = t \\ z = \frac{s+t}{3} \\ w = -\frac{7}{6}s - \frac{7}{6}t \end{cases}$$

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se $\begin{cases} k \neq 1 \\ k \neq -\frac{13}{2} \end{cases}$ fisso x come parametro e risolvo

$$\begin{cases} x = t \\ 3ky - 2z + (k+1)w = -3t \\ -y + 3z = kt \\ 3y - 2kz + 2w = -3kt \end{cases}$$

con Cramer.

se $k = -\frac{13}{2}$ fisso y come parametro e risolvo

$$\begin{cases} y = t \\ 3x - 2z + (k+1)w = -3kt \\ -kx + 3z = t \\ 3kx - 2kz + 2w = -3t \end{cases}$$

con Cramer.

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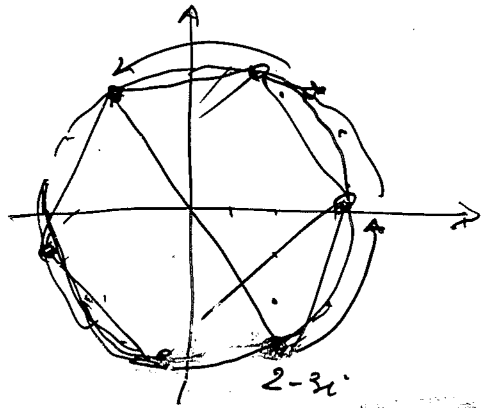
$z = 2-3i$ sia una radice 6ª di w.

Quali sono le altre radici 6ª di w?



$$Z_k = z\left(\cos\frac{k\pi}{3} + i\sin\frac{k\pi}{3}\right)$$

$k=1$

$k=2$

$k=3 \quad z_3 = -z$

$k=4 \quad z_4 = -z_1$

$k=5 \quad z_5 = -z_2$

$$z_1 = (2-3i)\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$

$$z_2 = (2-3i)\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$

(11)

$i z^3 = \bar{z}$

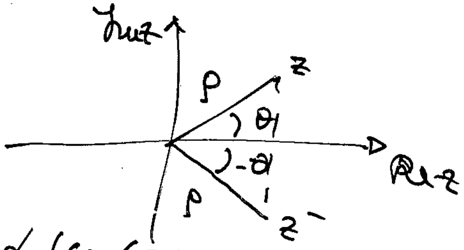
$z=0$ è una soluzione

$\mathrm{Im}\,\bar z = -\mathrm{Im}\,z$

$\mathrm{Re}\,\bar z = \mathrm{Re}\,z$

$z = \rho(\cos\theta + i\,\mathrm{sen}\,\theta)$

$\bar z = \rho(\cos(-\theta) + i\,\mathrm{sen}(-\theta))$



$i\rho^{\cancel{3}\,2}(\cos 3\theta + i\,\mathrm{sen}\,3\theta) = \cancel{\rho}(\cos(-\theta) + i\,\mathrm{sen}(-\theta))$

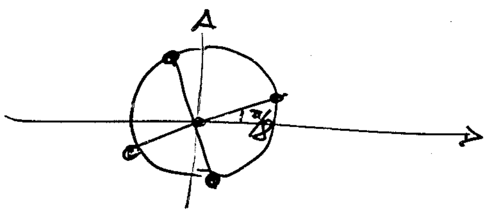
$\left(\cos\frac{\pi}{2} + i\,\mathrm{sen}\frac{\pi}{2}\right)$

$\rho^2\left(\cos\left(3\theta+\frac{\pi}{2}\right) + i\,\mathrm{sen}\left(3\theta+\frac{\pi}{2}\right)\right) = \cos(-\theta) + i\,\mathrm{sen}(-\theta)$

$\begin{cases} \rho^2 = 1 \\ 3\theta + \frac{\pi}{2} = -\theta + 2k\pi \quad k\in\mathbb{Z} \end{cases}$

$\begin{cases} \rho = 1 \\ 4\theta = -\frac{\pi}{2} + 2k\pi \quad k\in\mathbb{Z} \end{cases} \Rightarrow \theta = -\frac{\pi}{8} + k\frac{\pi}{2}$

$z_k = 1\left(\cos\left(-\frac{\pi}{8}+k\frac{\pi}{2}\right) + i\,\mathrm{sen}\left(-\frac{\pi}{8}+k\frac{\pi}{2}\right)\right) \quad k=0,1,2,3$

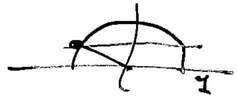


(12)

$z = \left(\frac{i-\sqrt{3}}{1-i}\right)^{20} = \frac{(i-\sqrt{3})^{20}}{(1-i)^{20}}$

$w_1 = i - \sqrt{3}$ $\qquad |w_1| = \sqrt{3+1} = 2$

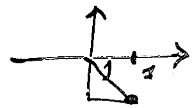
$\arg w_1 = \frac{5}{6}\pi$



$(w_1)^{20} = 2^{20}\cdot\left(\cos\left(20\cdot\frac{5}{6}\pi\right) + i\,\mathrm{sen}\left(20\cdot\frac{5}{6}\pi\right)\right)$
$= 2^{20}\left(\cos\left(\frac{2\pi}{3}+12\pi\right) + i\,\mathrm{sen}\left(\frac{2\pi}{3}\right)\right)$

$w_2 = 1 - i$ $\qquad |w_2| = \sqrt{2}$

$\arg w_2 = -\frac{\pi}{4}$



$(w_2)^{20} = (\sqrt{2})^{20}\left(\cos 20\cdot\left(\frac{-\pi}{4}\right) + i\sin 20\left(\frac{-\pi}{4}\right)\right)$
$= (\sqrt{2})^{2}\left(\cos\pi + i\,\mathrm{sen}\,\pi\right)$

$z = \frac{2^{20}}{(\sqrt{2})^{20}} \frac{\cos\frac{2\pi}{3} + i\,\mathrm{sen}\frac{2\pi}{3}}{\cos\pi + i\,\mathrm{sen}\,\pi} =$

$= (\sqrt{2})^{20}\left(\cos\left(-\frac{\pi}{3}\right) + i\,\mathrm{sen}\left(-\frac{\pi}{3}\right)\right) =$

$= 2^{10}\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 512 - 512\sqrt{3}\,i.$