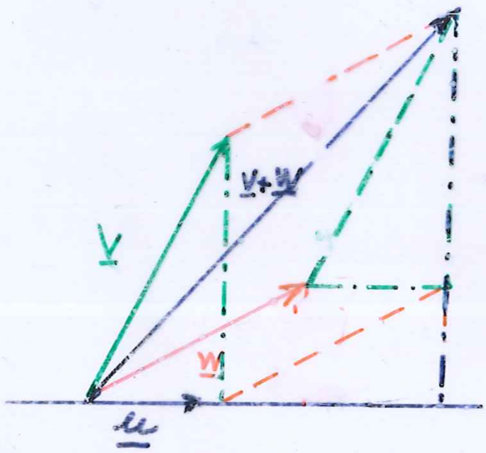
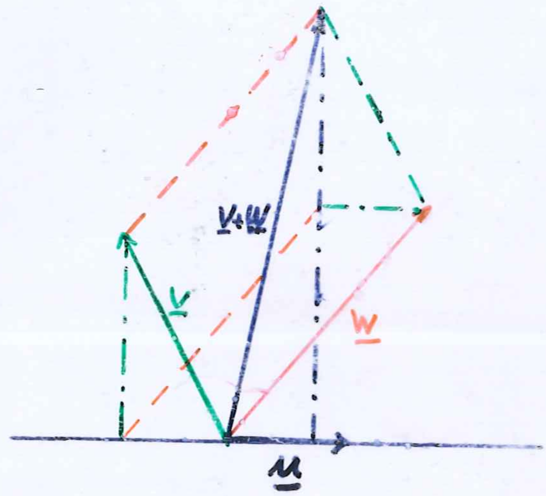
$\bullet : V \times V \longrightarrow \mathbb{R}$

Prodotto scalare di due vettori $\underline{v}, \underline{w}$

$\underline{v}\cdot\underline{w} = |\underline{v}|\cdot|\underline{w}|\cos\alpha$ ove $\alpha = \widehat{\underline{v}\,\underline{w}}$, $\alpha \in [0,\pi]$.

- commutativo
- distributivo: $\underline{u}\bullet(\underline{v}+\underline{w}) = \underline{u}\bullet\underline{v} + \underline{u}\bullet\underline{w}$ $\quad\forall \underline{u}, \underline{v}, \underline{w}$





$$\cancel{|\underline{u}|}\ |\underline{v}+\underline{w}| \cos\left(\widehat{\underline{u}\,(\underline{v}+\underline{w})}\right) \overset{\text{TESI}}{=} \cancel{|\underline{u}|}\cdot|\underline{v}|\cos(\widehat{\underline{u}\,\underline{v}}) + \cancel{|\underline{u}|}\cdot|\underline{w}|\cos(\widehat{\underline{u}\,\underline{w}})$$

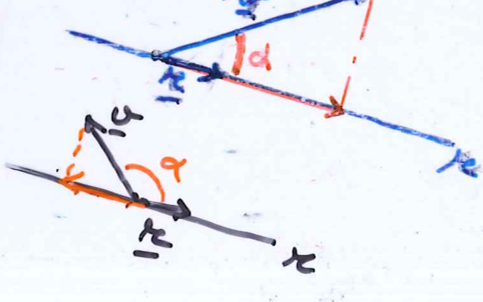
- $\forall t \in \mathbb{R}$, $\forall \underline{v}, \underline{w}$: $(t\underline{v})\bullet\underline{w} = t(\underline{v}\bullet\underline{w})$

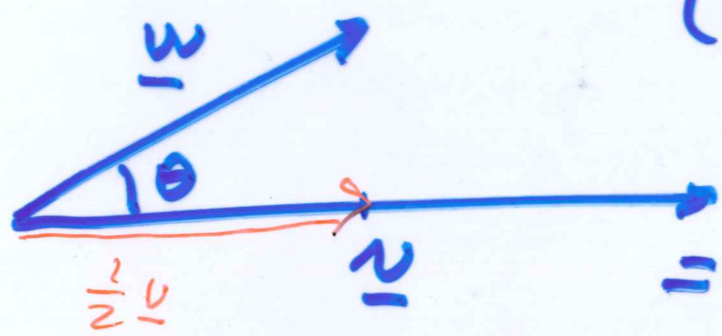
- $\underline{v}\bullet\underline{v} = |\underline{v}|\cdot|\underline{v}|\cos 0 = |\underline{v}|^2$

- $\underline{v}\bullet\underline{w} = 0$ e $\underline{v} \neq \underline{0}$, $\underline{w} \neq \underline{0}$ $\Rightarrow$ $\underline{v} \perp \underline{w}$

Proiezione ORTOGONALE di un vettore $\underline{v}$ lungo la direzione di una retta r (componente vettoriale di $\underline{v}$ lungo r):

$$\left[\underline{v}\bullet\left(\frac{\underline{r}}{|\underline{r}|}\right)\right]\frac{\underline{r}}{|\underline{r}|}$$





$$\left(\tfrac{1}{2}\underline{v}\right)\cdot\underline{w} = \left|\tfrac{1}{2}\underline{v}\right| |\underline{w}| \cos\theta = \left|\tfrac{1}{2}\right| |\underline{v}| |\underline{w}| \cos\theta = \left|\tfrac{1}{2}\right| (\underline{v}\cdot\underline{w})$$

$$(t\underline{v})\cdot\underline{w} = |t|(\underline{v}\cdot\underline{w})$$

$$\underline{v}\cdot(t\underline{w}) = |t|(\underline{v}\cdot\underline{w})$$

$$\underline{v}\cdot\underline{v} = |\underline{v}|\cdot|\underline{v}|\cdot 1 = |\underline{v}|^2$$

$$\underline{v}\cdot\underline{w} = 0 \implies ?$$

$$\parallel$$

$$|\underline{v}|\,|\underline{w}|\cos\theta = \begin{cases} |\underline{v}| = 0 \implies \underline{v} = \underline{0} \\ |\underline{w}| = 0 \implies \underline{w} = \underline{0} \\ \cos\theta = 0 \implies \theta = \frac{\pi}{2} \\ \quad \text{poiché } \theta \in [0, \pi] \end{cases}$$

Cioè:

$\underline{v}\cdot\underline{w} = 0$, se né $\underline{v}$ né $\underline{w}$ sono $= \underline{0}$, dice che $\underline{v}$ è ortogonale a $\underline{w}$

In generale:

$$\underline{v} \cdot \underline{w} = |\underline{v}|\,|\underline{w}| \cos\alpha$$

$$\Longrightarrow \cos\alpha = \frac{\underline{v} \cdot \underline{w}}{|\underline{v}|\,|\underline{w}|}$$

se scopro un diverso modo di calcolare $\underline{v} \cdot \underline{w}$ so come misurare l'angolo tra 2 vettori.

<u>Prod scalare</u> <u>in termini di componenti</u>. Vediamo il caso di $\mathbb{R}^2$.

$$\underline{v} = v_1 \underline{i} + v_2 \underline{j} = (v_1, v_2)$$

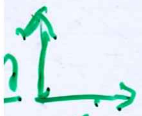
$$\underline{w} = (w_1, w_2) = w_1 \underline{i} + w_2 \underline{j}$$

$$\underline{v} \cdot \underline{w} = (v_1 \underline{i} + v_2 \underline{j}) \cdot (w_1 \underline{i} + w_2 \underline{j}) =$$

propr. distr.

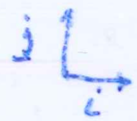
$$= (v_1 \underline{i}) \cdot (w_1 \underline{i}) + (v_1 \underline{i}) \cdot (w_2 \underline{j}) + (v_2 \underline{j}) \cdot (w_1 \underline{i}) + (v_2 \underline{j}) \cdot (w_2 \underline{j}) =$$

$$= (v_1 w_1)\underbrace{(\underline{i} \cdot \underline{i})}_{|\underline{i}|^2 = 1} + (v_1 w_2)\underbrace{(\underline{i} \cdot \underline{j})}_{0} + (v_2 w_1)\underbrace{(\underline{j} \cdot \underline{i})}_{0} + (v_2 w_2)\underbrace{(\underline{j} \cdot \underline{j})}_{|\underline{j}|^2 = 1}$$



$$= v_1 w_1 + v_2 w_2$$

In termini di componenti:



se $\underline{v}=(v_1,v_2)$ e $\underline{w}=(w_1,w_2)$

$$\underline{v}\cdot\underline{w}=(v_1\underline{i}+v_2\underline{j})\cdot(w_1\underline{i}+w_2\underline{j})=v_1\underline{i}\cdot(w_1\underline{i}+w_2\underline{j})+v_2\underline{j}\cdot(w_1\underline{i}+w_2\underline{j})=$$

$$=(v_1\underline{i})\cdot(w_1\underline{i})+(v_1\underline{i})\cdot(w_2\underline{j})+(v_2\underline{j})\cdot(w_1\underline{i})+(v_2\underline{j})\cdot(w_2\underline{j})=$$

$$=v_1w_1(\underline{i}\cdot\underline{i})+v_1w_2(\underline{i}\cdot\underline{j})+v_2w_1(\underline{j}\cdot\underline{i})+v_2w_2(\underline{j}\cdot\underline{j})=$$

$$=v_1w_1+v_2w_2$$

$\underline{i}\cdot\underline{j}=0=\underline{j}\cdot\underline{i}$

$\underline{i}\cdot\underline{i}=1=\underline{j}\cdot\underline{j}$

$$\boxed{\text{se } \underline{v}=(v_1,v_2,v_3)\ ;\ \underline{w}=(w_1,w_2,w_3)\ :\ \underline{v}\cdot\underline{w}=v_1w_1+v_2w_2+v_3w_3}$$

in $\mathbb{R}^3$

ES 1 Trovare l'angolo tra i due vettori $\underline{v}=(1,2,3)$ e $\underline{w}=(3,-1,2)$:

$$1\cdot3+2(-1)+3\cdot2=\underline{v}\cdot\underline{w}=|\underline{v}|\cdot|\underline{w}|\cos\alpha=\sqrt{1+4+9}\sqrt{9+1+4}\cos\alpha$$

$\Rightarrow \cos\alpha=$ vedi pag V5.1

In generale:

ES 2 Trovare un vettore ortogonale a $\underline{v}=(-1,2,1)$ e a $\underline{w}=(3,1,4)$ di modulo 1.

$\underline{u}=(x,y,z)$

$\underline{u}\perp\underline{v} \iff$

$\underline{u}\perp\underline{w} \iff$ Vedi pag V5.2

$|\underline{u}|=\ ..$

ES 3 I vettori $\underline{u},\underline{v},\underline{w}$ dell'esercizio precedente sono indipendenti?

(VS.1

Trovare l'angolo tra $\underline{v} = (1, 2, 3)$ e $\underline{w} = (3, -1, 2)$.

Sol.

$\underline{v} \cdot \underline{w} = |\underline{v}|\,|\underline{w}| \cos\alpha$ ove $\alpha = \widehat{\underline{v}\,\underline{w}}$

$$\Rightarrow \cos\alpha = \frac{\underline{v} \cdot \underline{w}}{|\underline{v}|\,|\underline{w}|}$$

$$\underline{v} \cdot \underline{w} = (1,2,3)(3,-1,2) = $$
$$= 1 \cdot 3 + 2 \cdot (-1) + 3 \cdot 2 = 3 - 2 + 6 = 7$$

$$|\underline{v}| = \sqrt{1+2^2+3^2} = \sqrt{3^2+(-1)^2+(2)^2} = |w|$$

$$|\underline{v}|\,|\underline{w}| = 1+4+9 = 14$$

$$\cos\alpha = \frac{7}{14} = \frac{1}{2} \Rightarrow \alpha = \frac{\pi}{3}$$

Se invece prendessi

$\underline{u} = (-3, 1, -2) = -\underline{w}$ e $\beta = \widehat{\underline{u}\,\underline{v}}$

$$\cos\beta = \frac{\underline{v} \cdot \underline{u}}{|\underline{v}|\,|\underline{u}|} = \frac{-7}{14} = -\frac{1}{2}$$

$\beta = \frac{2\pi}{3}$: il coseno (cioè il prodotto scalare) permette di distinguere gli angoli acuti da quelli OTTUSI

V5.2

Trovare un vettore ortogonale a $\underline{v}=(-1,2,1)$ e $\underline{w}=(3,1,4)$ e avente modulo 1.

Sol.

$\underline{u}=(x,y,z)$ incognito

$$\begin{cases} \underline{u}\cdot\underline{v}=0 & (\text{perché } \underline{u}\perp\underline{v}) \\ \underline{u}\cdot\underline{w}=0 & (\text{"} \quad \underline{u}\perp\underline{w}) \\ |\underline{u}|=1 \end{cases}$$

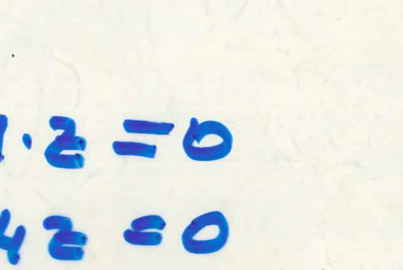
$$\begin{cases} -1\cdot x+2\cdot y+1\cdot z=0 \\ 3x+1\cdot y+4z=0 \\ x^2+y^2+z^2=1 \end{cases}$$

Sommo 3 volte la 1ª alla 2ª eq.

piano per l'origine ⇒ l'intersezione è una retta.

$$\begin{cases} -x+2y+z=0 \\ 7y+7z=0 \\ x^2+y^2+z^2=1 \end{cases} \qquad \begin{cases} x=2y-y=y \\ z=-y \\ x^2+y^2+z^2=1 \end{cases}$$

sfera con centro nell'origine



... Si prevedono, geometricamente, 2 soluzioni diametralmente opposte.

$$\begin{cases} x=y \\ z=-y \\ y^2+(-y)^2+y^2=1 \end{cases} \qquad y^2=\frac{1}{3} \begin{cases} y=\frac{1}{\sqrt{3}} \\ y=\frac{-1}{\sqrt{3}} \end{cases}$$

Sono soluzioni:

$\underline{u}=\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right)$ o $-\underline{u}$