III. Integrazione per sostituzione

Ricordo: $D(h(g(t))) = h'(g(t))\cdot g'(t)$

Sia ora $f:[a,b]\to\mathbb{R}$ la funzione continua di cui vogliamo trovare una primitiva H: $H'(x)=f(x)$ $\forall x\in[a,b]$

VERSIONE FACILE Se $f(x)$ ha la forma $h'(g(x))\cdot g'(x)$ si ha

$$\int f(x)\,dx = \int h'(g(x))\cdot g'(x)\,dx = h(g(x))+c$$

e quindi $H(x)=h(g(x))$.

Esempi	Casi particolari:
1) $\int \frac{g'(x)}{g(x)}\,dx = \ln\lvert g(x)\rvert + c$	(a) $\int tg\,x\,dx =$ VEDI P4.1
	(b) $\int \frac{x\,dx}{x^2+b^2} =$ VEDI P4.1
2) $\int \frac{g'(x)}{1+g^2(x)}\,dx = arctg(g(x))+c$	$\int \frac{dx}{x^2+2x+2} =$ VEDI P4.2

In generale cerco di riprodurre questa situazione con una SOSTITUZIONE $x=g(t)$

VERSIONE GENERALE Sia $g:[c,d]\to[a,b]$ una funz. derivabile con derivata 1ª continua e $\neq 0$ su $[c,d]$.

(ciò garantisce che esiste $g^{-1}:[a,b]\to[c,d]$: PERCHÉ?)

Controinominale del TEOR. degli ZERI

Allora

$$\int f(x)\,dx = \left[\int f(g(t))\cdot g'(t)\,dt\right]_{t=g^{-1}(x)}$$

Infatti, se $H'(x)=f(x)$ $\quad\int H'(g(t))\cdot g'(t)\,dt = H(g(t))+c$

e la sostituzione $t=g^{-1}(x)$ riporta proprio a $H(x)+c$.

Esempi

1) $\int \frac{dx}{x^2+a^2}$ VEDI P4.6

2) $\int \frac{dx}{x^2+x+1}$ VEDI P4.4 - P4.5

$$\int \tan x \, dx = \int \frac{\operatorname{sen} x}{\cos x} dx \qquad (\cos x)' = -\operatorname{sen} x$$

$$= \int \frac{-(\cos x)'}{\cos x} dx = -\ln|\cos x| + c$$

ogni primitiva sarà definita su ciascuno degli intervalli in cui è continua la funz. integranda $\tan x$, cioè $\left(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi\right)$

$$\int \frac{1}{\tan x} dx = \int \frac{\cos x}{\operatorname{sen} x} dx = \int \frac{(\operatorname{sen} x)'}{\operatorname{sen} x} dx =$$

$$= \ln|\operatorname{sen} x| + c$$

$$\int \frac{x}{x^2+4} dx = \qquad (x^2+4)' = 2x$$

$$= \frac{1}{2} \int \frac{2x}{x^2+4} dx = \frac{1}{2} \ln(x^2+4) + c$$

$\boxed{x^2+4>0}$

$$\int \frac{dx}{x^2+2x+2} =$$

DEN ha $\Delta < 0$

$$= (x^2+2x+1)+1 = (x+1)^2+1$$

$$g(x) = x+1$$
$$g'(x) = 1$$

$$\int \frac{1 \cdot dx}{1+(x+1)^2} = \operatorname{arctg}(x+1)$$

$$\int \frac{2x}{1+x^4}\,dx$$

$(1+x^4)' \neq 2x$

no logaritmo!

ma $(x^2)' = 2x$

$$= \int \frac{(x^2)'}{1+(x^2)^2}\,dx = \arctan x^2 + c$$

$$\int x \cos x^2\,dx =$$

$(x^2)' = 2x$

$$\frac{1}{2}\int 2x \cdot \cos x^2\,dx = \frac{1}{2}\operatorname{sen} x^2 + c$$

Invece:

$$\int x^2 \cos x \, dx \underset{\text{P.P. F.F. } x^2}{=}$$

$$= x^2 \cdot \operatorname{sen} x - \int 2x \cdot \operatorname{sen} x \, dx =$$

$$= x^2 \operatorname{sen} x + 2\left(\int x \left(-\operatorname{sen} x\right) dx \right) =$$

$$= x^2 \operatorname{sen} x + 2\left(x \cos x - \int 1 \cdot \cos x \, dx \right) =$$

$$= x^2 \operatorname{sen} x + 2x \cos x - 2 \operatorname{sen} x + c$$

Prova:

$$\left(x^2 \operatorname{sen} x + 2x \cos x - 2 \operatorname{sen} x \right)' =$$

$$= x^2 \cos x + 2x \cdot \cancel{\operatorname{sen} x} - 2x \cancel{\operatorname{sen} x} + 2\cancel{\cos x} + - 2\cancel{\cos x} =$$

$$= x^2 \cos x$$

GIUSTO

$$\int \frac{dx}{x^2+a^2}$$

ui part: P4.4

$a=2$

$$\int \frac{dx}{x^2+4} = \int \frac{dx}{4\left(1+\frac{x^2}{4}\right)} = \boxed{\begin{matrix}\frac{x}{2}=t \text{ cioè} \\ x=2t \\ dx=2dt\end{matrix}}$$

$$= \frac{1}{4}\int \frac{2dt}{1+t^2} = \left(\frac{1}{2}\arctan t\right)_{t=\frac{x}{2}} + c =$$

$$= \frac{1}{2}\arctan\frac{x}{2} + c$$

$$\int \frac{dx}{x^2+x+1} =$$

$x^2+x+1=$

$=x^2+2\frac{x}{2}+\frac{1}{4}+\frac{3}{4}=$

$=\left(x+\frac{1}{2}\right)^2+\frac{3}{4}$

$$= \int \frac{dx}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}} = \frac{4}{3}\int \frac{dx}{\underbrace{\frac{4}{3}\left(x+\frac{1}{2}\right)^2}+1} =$$

$$t=\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right) \Rightarrow dt=\frac{2}{\sqrt{3}}dx \Rightarrow dx=\frac{\sqrt{3}}{2}dt$$

$$= \frac{4}{3}\int \frac{\frac{\sqrt{3}}{2}\,dt}{1+t^2} = \left(\frac{2}{\sqrt{3}}\arctan t\right)_{t=\dots} + c$$

$$= \frac{2}{\sqrt{3}}\arctan \frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right) + c$$

Uso dei vari metodi di integrazione.

$$\int x\cos 2x\,dx \underset{\text{P.P. F.F. } x}{=}$$

$$\boxed{\begin{array}{ll} f(x)=x & f'(x)=1 \\ g'(x)=\cos 2x & g(x)=\frac{1}{2}\,\mathrm{sen}\,2x \end{array}}$$

$$= \frac{1}{2}x\,\mathrm{sen}\,2x - \frac{1}{2}\int 1\cdot \mathrm{sen}\,2x\,dx =$$

$$= \frac{1}{2}x\,\mathrm{sen}\,2x + \frac{1}{4}\cos 2x + c$$

PROVA ... FARLA

ESERCIZI

(Ricavare esercizi analoghi scambiando seno con coseno)

1) $\int x^2 \operatorname{sen} x \, dx$

1bis) $\int x \operatorname{sen} x^2 dx$

→ vedi (con cos) P4.2 - P4.3

2) $\int x^n \cos x \, dx$

3) $\int x \cos 2x \, dx$ vedi P4.5

4) $\int x (\cos x)^2 dx$ vedi pag. 1 e 2 a seguire

5) $\int e^x (\operatorname{sen} x)^2 dx$

6) $\int x e^{2x} dx$ vedi pag 2

7) $\int e^{-3x} \operatorname{sen} x \, dx$ vedi pag 3

8) $\int \frac{x-1}{2x^2-4x+5} dx$ vedi pag 5 (e 4 per idee sulle fratte)

9) $\int \frac{dx}{2x^2-4x+5}$ vedi pag 6

10) $\int \frac{dx}{x^2-4x-5}$ vedi pag 7

1

$$\int (\cos x)^2 dx =$$

$$\boxed{\begin{array}{l}\cos 2x = 2(\cos x)^2 - 1 \\ \Rightarrow (\cos x)^2 = \dfrac{1+\cos 2x}{2}\end{array}}$$

$$\frac{1}{2}\int (1+\cos 2x)\,dx = \frac{1}{2}\left(x + \frac{1}{2}\operatorname{sen} 2x\right) + c$$

Alternativa

$$\int (\cos x)(\cos x)\,dx \underset{p.p.}{=}$$

$$= \operatorname{sen} x \cdot \cos x - \int \operatorname{sen} x\,(-\operatorname{sen} x)\,dx$$

Non posso più integrare pp poichè torno a $\int \cos^2$

$$\boxed{\cos^2 x - 1 = -\operatorname{sen}^2 x}$$

$$= \operatorname{sen} x \cdot \cos x - \int \left[(\cos x)^2 - 1\right] dx =$$

$$= \operatorname{sen} x \cos x + x - \int (\cos x)^2 dx$$

$$\Rightarrow 2\int (\cos x)^2 dx = \operatorname{sen} x \cos x + x + c_1$$

$$\Rightarrow \int (\cos x)^2 dx = \frac{\operatorname{sen} x \cos x + x}{2} + c$$

$$\int x(\cos x)^2 dx =$$

MEGLIO CHE P.P.

$$\boxed{(\cos x)^2 = \frac{\cos 2x + 1}{2}}$$

$$= \frac{1}{2}\int (x\cos 2x + x)\,dx =$$

$$= \frac{1}{2}\left(\int x\cos 2x\,dx + \frac{x^2}{2}\right) + c =$$

$$= \frac{1}{2}\left(\frac{1}{2}x\,\text{sen}\,2x + \frac{1}{4}\cos 2x + \frac{x^2}{2}\right) + c$$

$$= \frac{1}{4}x\,\text{sen}\,2x + \frac{1}{8}\cos 2x + \frac{x^2}{4} + c$$

$$\int x e^{2x}\,dx \underset{\text{P.P. F.F. } x}{=}$$

$$= \frac{1}{2}xe^{2x} - \frac{1}{2}\int 1\cdot e^{2x}\,dx =$$

$$= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c$$

$$\int e^{-3x} \operatorname{sen} x \, dx =$$

P.P. FF e^{-3x}

$$\boxed{\begin{array}{ll} f(x) = e^{-3x} & f'(x) = -3e^{-3x} \\ g'(x) = \operatorname{sen} x & g(x) = -\cos x \end{array}}$$

$$= -e^{-3x} \cos x + \int (-3e^{-3x}) \cdot \cos x \, dx \underset{PP}{=}$$

$$\boxed{\begin{array}{ll} f(x) = e^{-3x} & f'(x) = -3e^{-3x} \\ g'(x) = \cos x & g(x) = \operatorname{sen} x \end{array}}$$

$$= -e^{-3x} \cos x - 3\left[e^{-3x} \operatorname{sen} x + 3\int e^{-3x} \operatorname{sen} x \, dx\right]$$

Cioè

$$\int e^{-3x} \operatorname{sen} x \, dx = e^{-3x}(-\cos x - 3 \operatorname{sen} x) + \\ - 9 \int e^{-3x} \operatorname{sen} x \, dx$$

$\Rightarrow$

$$10 \int e^{-3x} \operatorname{sen} x \, dx = -e^{-3x}(\cos x + 3 \operatorname{sen} x) + c$$

$$\Rightarrow \int e^{-3x} \operatorname{sen} x \, dx = -\frac{e^{-3x}}{10}(\cos x + 3 \operatorname{sen} x) + c$$

INTEGRALI di FUNZ. RAZ. FRATTE

- $\int \frac{1}{ax+b}\,dx$, $a \neq 0$

$$= \frac{1}{a} \int \frac{a}{ax+b}\,dx = \frac{1}{a} \ln|ax+b| + c$$

- $\Delta > 0$: $x^2 - a^2$

$\Delta = 0$: x^2 oppure $(x+a)^2$

$\Delta < 0$: $x^2 + a^2$

sono i prototipi

$\Delta = 0$

$$\int \frac{1}{(x+a)^2}\,dx = -\frac{1}{x+a} + c$$

oppure $$\int \frac{dx}{(ax+b)^2} = \frac{1}{a} \int \frac{a\,dx}{(ax+b)^2} =$$

$$= -\frac{1}{a} \cdot \frac{1}{ax+b} + c$$

$$\int \frac{1}{x^2 - a^2}\,dx = \int \left(\frac{A}{x-a} + \frac{B}{x+a} \right) dx$$

$$= \frac{1}{2a} \int \left(\frac{1}{x-a} + \frac{-1}{x+a} \right) dx = \frac{1}{2a} \log\left| \frac{x-a}{x+a} \right| + c$$

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\frac{x}{a} + c$$

in generale se $\Delta < 0$ avrò un arctan.

$$\int \frac{x-1}{2x^2-4x+5}\,dx =$$

il numeratore è derivata del den?

$$(2x^2-4x+5)' = 4x-4 = 4(x-1)$$

Si posa

$$\frac{1}{4}\int \frac{(2x^2-4x+5)'}{2x^2-4x+5} = \frac{1}{4}\ln|2x^2-4x+5| + c$$

$$\int \frac{x-2}{2x^2-4x+5}\,dx$$ come sopra.

$$\|$$

$$\int \frac{x-1}{2x^2-4x+5}\,dx - \int \frac{dx}{2x^2-4x+5}\ ?$$

VEDI SOPRA

$$\frac{\Delta}{4} = 4 - 2\cdot 5 < 0$$

il denom è

$$2x^2-4x+5 = 2(x^2-2x+1)+3 =$$

$$= 2(x-1)^2+3 = 3\left[\underbrace{\frac{2}{3}(x-1)^2}_{t^2}+1\right]$$

$$t = \sqrt{\frac{2}{3}}\,(x-1)$$

$$dt = \sqrt{\frac{2}{3}}\,dx \quad \Rightarrow \quad dx = \sqrt{\frac{3}{2}}\,dt$$

$$\int \frac{dx}{2x^2-4x+5} = \frac{1}{3}\int \frac{\sqrt{3/2}\,dt}{t^2+1} =$$

$$= \frac{1}{3}\sqrt{\frac{3}{2}}\arctan t + c =$$

$$= \frac{1}{\sqrt{3\cdot 2}}\arctan\left(\sqrt{\frac{2}{3}}\,(x-1)\right) + c$$

sostituire e sommare:

$$\frac{1}{4}\ln(2x^2-4x+5) - \frac{1}{\sqrt{6}}\arctan\left(\sqrt{\frac{2}{3}}\,(x-1)\right) + c$$

$$\int \frac{x-1}{x^2-4x-5}\,dx =$$

$$\boxed{(x^2-4x-5)' = 2x-4 = 2(x-2)}$$

$$= \frac{1}{2}\int \frac{2(x-2)+2}{x^2-4x-5}\,dx =$$

$$= \frac{1}{2}\int \frac{(x^2-4x-5)'}{x^2-4x-5}\,dx + \int \frac{1}{x^2-4x-5}\,dx =$$

$$\boxed{\begin{array}{l} x^2-4x-5=0 \implies x_1=-1 \quad x_2=5 \\ x^2-4x-5 = (x+1)(x-5) \end{array}}$$

$$\frac{1}{(x+1)(x-5)} = \frac{A}{x+1} + \frac{B}{x-5} = \frac{x(A+B)+B-5A}{(x+1)(x-5)} =$$

$$\begin{cases} A+B=0 \\ B-5A=1 \end{cases} \quad \begin{cases} A=-B \\ 6B=1 \end{cases} \quad \begin{cases} B=1/6 \\ A=-1/6 \end{cases}$$

$$= \frac{1}{2}\ln|x^2-4x-5| + \frac{1}{6}\int \frac{1}{x-5} - \frac{1}{x+1}\,dx =$$

$$= \frac{1}{2}\ln|x^2-4x-5| + \frac{1}{6}\ln\left|\frac{x-5}{x+1}\right| + c$$

11) $\int \operatorname{arctg} x \, dx$ vedi pag 9

12) $\int x \operatorname{arctg} x \, dx$ vedi pag 9

13) $\int \operatorname{arcsen} x \, dx$

14) $\int \frac{dx}{\operatorname{tg} x}$

15) $\int \frac{dx}{\operatorname{sen} x \cos x}$

16) $\int \frac{dx}{\operatorname{sen} x}$ (SOST.)

17) $\int \frac{dx}{\cos x}$ (SOST.)

18) $\int \frac{\sqrt{x} - \sqrt[4]{x}}{1 + \sqrt[4]{x}} \, dx$ (SOST.)

19) $\int \operatorname{sen}^3 x \sqrt{\cos x} \, dx$

20) $\int \left(\sqrt{\operatorname{tg}^3 x} + \frac{1}{\sqrt{\operatorname{tg} x}} \right) dx$

21) $\int \frac{e^x \ln(2+e^x)}{(e^x+1)^2} \, dx$ (... FRATTE)

22) $\int \sqrt{a^2 - x^2} \, dx$ vedi pag 10

$$\int \arctan x \, dx =$$

P.P. con FF $\arctan x$
FD $1 \cdot dx$

$$= x \cdot \arctan x - \int x \cdot \frac{1}{1+x^2} \, dx =$$

$$= x \cdot \arctan x - \frac{1}{2} \int \frac{2x}{1+x^2} \, dx =$$

$$= x \cdot \arctan x - \frac{1}{2} \ln(1+x^2) + c$$

$$\int x \arctan x \, dx =$$

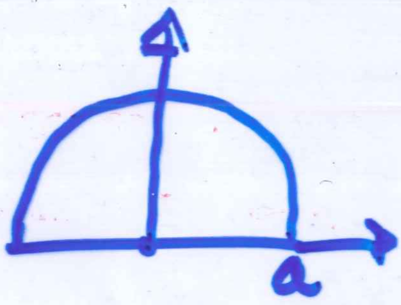
P.P. con F.D. $x\,dx$

$$= \frac{x^2}{2} \arctan x - \frac{1}{2} \int \frac{x^2}{1+x^2} \, dx =$$

$$= \frac{x^2}{2} \arctan x - \frac{1}{2} \int \left(\underbrace{\boxed{\frac{1+x^2}{1+x^2}}}_{1} - \frac{1}{1+x^2} \right) dx =$$

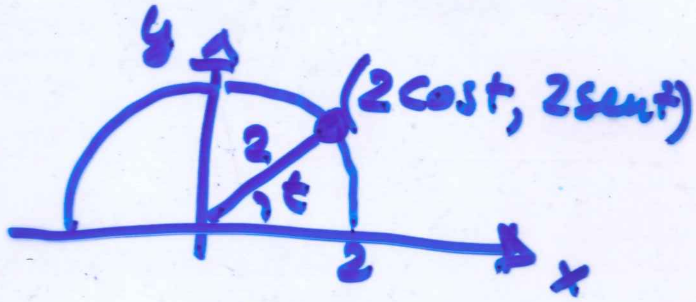
$$= \frac{x^2}{2} \arctan x - \frac{1}{2} x + \frac{1}{2} \arctan x + c$$

$\int \sqrt{a^2-x^2}\,dx$



$a = 2$

$\int \sqrt{4-x^2}\,dx =$



$x = 2\cos t$

$dx = -2\,\mathrm{sen}\,t$

$\sqrt{4-x^2} = 2\,\mathrm{sen}\,t$

se $t \in [0,\pi]$
cos t è decresc.
⇒ invertibile
con l'arccos y;
$\boxed{t = \arccos\frac{x}{2}}$

$\int 2\,\mathrm{sen}\,t \cdot (-2\,\mathrm{sen}\,t)\,dt = 4\int(\mathrm{sen}\,t)^2\,dt =$

$-(\mathrm{sen}\,t)^2 = (\cos t)^2 - 1$

$-\int(\mathrm{sen}\,t)^2\,dt = \int(\cos t)^2\,dt - t =$ VEDI PAG 1

$= \frac{\mathrm{sen}\,t\cos t + t}{2} - t + c = \frac{\mathrm{sen}\,t\cos t - t}{2} + c$

$= 2(\mathrm{sen}\,t\cos t - t) + c =$

$= \frac{1}{2}x\sqrt{4-x^2} - 2\arccos\frac{x}{2} + c.$