

26/2/2015

Base di  $U \subseteq \mathbb{R}^4$  one

$$U = \langle u_1, u_2, u_3, u_4 \rangle$$

$$u_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, u_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, u_3 = \begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \end{pmatrix}, u_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

trovare un ssp  $V$  di  $\mathbb{R}$  t.c.  $\mathbb{R}^4 = U + V$ .

Algoritmo degli scarti successivi:

$$u_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \text{ è indip.}$$

 $\{u_2, u_4\}$  è indip.

$$\{u_1, u_2, u_3\} \quad \underbrace{\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = a \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + b \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{\text{sono indip.}} \Rightarrow \begin{array}{l} a=1 \\ a=2 \end{array} \text{IMP.}$$

$$\{u_1, u_2, u_3, u_4\} \quad \begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \end{pmatrix} = a \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + b \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + c \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \Leftrightarrow$$

$$u_3 = \begin{pmatrix} a+b+c \cdot 0 \\ 2a+b+c \cdot 0 \\ 3a+b+c \\ 4a+b+c \cdot 0 \end{pmatrix}$$

$$\begin{cases} a+b = -2 \\ 2a+b = -1 \\ 3a+b+c = 0 \\ 4a+b = 1 \end{cases}$$

$$\left( \begin{array}{ccc|c} 1 & 1 & 0 & -2 \\ 2 & 1 & 0 & -1 \\ 3 & 1 & 1 & 0 \\ 4 & 1 & 0 & 1 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c} 1 & 1 & 0 & -2 \\ 0 & -1 & 0 & 3 \\ 0 & -2 & 1 & 6 \\ 0 & -3 & 0 & 9 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c} 1 & 1 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$c=0 \quad b=-3 \quad a=1$$

$$\left( \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} | -3 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -8 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right) \Rightarrow u_3 \text{ dipende da } \{u_1, u_2, u_4\}$$

$\Rightarrow$  base di  $U = \{\underline{u}_1, \underline{u}_2, \underline{u}_4\}$

$\dim U = 3$

ALTERNATIVO: elenco i vettori per riga

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ -2 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

riduco a forma triangolare applicando una strategia alla G.A.B.S.

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 0 \end{pmatrix} \leftarrow R_2 - R_1 \quad : \underline{u}_1 - \underline{u}_2 \\ \leftarrow R_3 + 2R_1 \quad \underline{u}_3 + 2\underline{u}_2$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \leftarrow R_3 - R_2 \quad \underline{u}_3 + 2\underline{u}_2 - (\underline{u}_2 - \underline{u}_1) = 0 \\ \underline{u}_3 = -3\underline{u}_2 + \underline{u}_1 \\ \underline{u}_1 = \underline{u}_3 + 3\underline{u}_2$$

Altre basi possibili:  $\{\underline{u}_2, \underline{u}_3, \underline{u}_4\}$

Ottura, essendo  $\underline{u}_2 = \frac{1}{3}(\underline{u}_1 - \underline{u}_3)$ ,

$$\{\underline{u}_1, \underline{u}_3, \underline{u}_4\}$$

$\mathbb{R}^4 = U + V$ ? Per quale ssp.  $V$ ?

Risolvo il problema se cerco una base di  $\mathbb{R}^4$  che convegga tra i suoi vettori una base di  $U$ :

$\{\underline{u}_1, \underline{u}_2, \underline{u}_4, \underline{u}\}$  base di  $\mathbb{R}^4 \Rightarrow V = \langle \underline{u} \rangle$

è una soluzione del problema posto.

$$\underline{u}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \quad \underline{u}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad \underline{u}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{Completa la base di } U$$

generatori di  $\mathbb{R}^4$ :  $\{ \underline{u}_1, \underline{u}_2, \underline{u}_3, \underline{e}_1, \underline{e}_2, \underline{e}_3 \}$

$$\underline{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \underline{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \underline{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \underline{e}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\left( \begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 \\ 4 & 1 & 0 & 0 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -1 & 0 & -2 \\ 0 & -2 & 1 & -3 \\ 0 & -3 & 0 & -4 \end{array} \right) \rightarrow$$

$$\rightarrow \left( \begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{array} \right) \quad \Rightarrow \underline{e}_1 \text{ è indif.} \\ \text{de } \{\underline{u}_1, \underline{u}_2, \underline{u}_3\}$$

$\{\underline{u}_1, \underline{u}_2, \underline{u}_3, \underline{e}_1\}$  è un sist di 4 vett. indip. e quindi è una base.

(nota: verificare che poteva andar bene anche  $\{\underline{u}_1, \underline{u}_2, \underline{u}_4, \underline{e}_2\}$  oppure  $\{\underline{u}_1, \underline{u}_2, \underline{u}_4, \underline{e}_3\}$ )

$$V = \langle \underline{e}_1 \rangle : \quad \mathbb{R}^4 = \langle \underline{u}_1, \underline{u}_2, \underline{u}_3 \rangle + \langle \underline{e}_1 \rangle = U + V$$

Quanto detto sopra dice che non c'è un solo sottospazio di dimensione 1 che sommato a  $U$  dà  $\mathbb{R}^4$ .

$U \cap V = \{0\}$  perché  $\underline{e}_1$  è indif. dei vettori di  $V$ . Ma posso anche dire che

$$\mathbb{R}^4 = U + \langle \underline{u}_3, \underline{e}_4 \rangle \Rightarrow U \cap V = \langle \underline{u}_3 \rangle$$

o anche, banalizzando al massimo:

$$\mathbb{R}^4 = U + \mathbb{R}^4$$

$$\underline{\alpha}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \underline{\alpha}_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \underline{\alpha}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

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T4

$$\underline{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \underline{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \underline{v}_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

Già  $f: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  lineare definita da  
 $f(\underline{\alpha}_i) = \underline{v}_i \quad \forall i=1,2,3$

Det. la matrice che rappresenta  $f$  rispetto alla base canonica in  $\mathbb{R}^3$  (tanto come sp. di partenza che di arrivo)

$f(\underline{\alpha}_1)$  ? per prima cosa devo esprimere  $\underline{\alpha}_1$  come comb. lin. di  $\underline{\alpha}_1, \underline{\alpha}_2, \underline{\alpha}_3$ : quali sono gli  $x, y, z \in \mathbb{R}$  tali che:

$$x \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + y \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + z \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} ?$$

$$\begin{cases} x+z=1 \\ -y+z=0 \\ y+z=0 \end{cases} \quad \begin{cases} x=1-z \\ y=z \\ 1-z+z=0 \end{cases} \quad \begin{cases} x=2 \\ y=-1 \\ z=-1 \end{cases}$$

$$\underline{\alpha}_1 = 2\underline{\alpha}_1 + (-1)\underline{\alpha}_2 + (-1)\underline{\alpha}_3 = (\underline{\alpha}_1, \underline{\alpha}_2, \underline{\alpha}_3) \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$$

$$f(\underline{\alpha}_1) ? \quad \begin{cases} x+z=0 \Rightarrow x=-1 \\ -y+z=1 \Rightarrow z=1 \\ y+z=0 \Rightarrow y=0 \end{cases}$$

Come sopra

$$\underline{\alpha}_2 = -\underline{\alpha}_1 + \underline{\alpha}_3$$

$$f(\underline{\alpha}_2) ? \quad \begin{cases} x+z=0 \quad x=-1 \\ -y+z=0 \quad z=1 \\ x+y+z=1 \quad \Rightarrow y=1 \end{cases}$$

come sopra

$$\underline{\alpha}_3 = -\underline{\alpha}_1 + \underline{\alpha}_2 + \underline{\alpha}_3$$

Il testo del problema dice che:

$$(f(\underline{\alpha}_1), f(\underline{\alpha}_2), f(\underline{\alpha}_3)) = \left( \underline{e}_1, \underline{e}_2, \underline{e}_3 \right) \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix} \quad \text{A} \quad (\text{T4})$$

adesso:

$$f(\underline{\alpha}_1) = \underline{e}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 0 \cdot \underline{e}_1 + 0 \cdot \underline{e}_2 + 1 \cdot \underline{e}_3$$

Ma dovo trovare B:

$$(f(\underline{\alpha}_1), f(\underline{\alpha}_2), f(\underline{\alpha}_3)) = (\underline{e}_1, \underline{e}_2, \underline{e}_3) B ?$$

$$f(\underline{\alpha}_1) = f(2\underline{\alpha}_1 - \underline{\alpha}_2 - \underline{\alpha}_3) = 2f(\underline{\alpha}_1) - f(\underline{\alpha}_2) - f(\underline{\alpha}_3) =$$

$$= (f(\underline{\alpha}_1), f(\underline{\alpha}_2), f(\underline{\alpha}_3)) \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \quad \text{ecc. per } \begin{matrix} f(\underline{\alpha}_2) \\ f(\underline{\alpha}_3) \end{matrix}$$

(vedi  $\underline{e}_2, \underline{e}_3$  a pag. T3)

$$(f(\underline{\alpha}_1), f(\underline{\alpha}_2), f(\underline{\alpha}_3)) = \underbrace{(f(\underline{\alpha}_1), f(\underline{\alpha}_2), f(\underline{\alpha}_3))}_{\text{B}} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

$$= (\underline{e}_1, \underline{e}_2, \underline{e}_3) \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix} \quad \text{P}^{-1}$$

è la matrice richiesta,  
ove  $P^{-1}$  è esattamente l'in-  
versa della matrice dei coeff.

$$(\underline{\alpha}_1, \underline{\alpha}_2, \underline{\alpha}_3) = (\underline{e}_1, \underline{e}_2, \underline{e}_3) P$$

di  $\underline{\alpha}_1, \underline{\alpha}_2, \underline{\alpha}_3$   
rispetto alla  
base canonica

Versione semplificata.

(TS)

$$\underline{\alpha}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \underline{\alpha}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad f(\underline{\alpha}_1) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, f(\underline{\alpha}_2) = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$$
$$2\underline{e}_1 + 3\underline{e}_2 \quad 0\underline{e}_1 + 4\underline{e}_2$$

$$\forall \underline{v} \in \mathbb{R}^2 \quad \underline{v} = x\underline{\alpha}_1 + y\underline{\alpha}_2$$

$$f(\underline{v}) = f(x\underline{\alpha}_1 + y\underline{\alpha}_2) = x f(\underline{\alpha}_1) + y f(\underline{\alpha}_2)$$

$$(f(\underline{\alpha}_1), f(\underline{\alpha}_2)) = (\underline{e}_1, \underline{e}_2) \begin{pmatrix} 2 & 0 \\ 3 & 4 \end{pmatrix}$$

$A = \begin{pmatrix} 2 & 0 \\ 3 & 4 \end{pmatrix}$  è la matr. raffresentativa dif  
risp. a  $(\underline{\alpha}_1, \underline{\alpha}_2)$  in  $\mathbb{R}^2$  domini  
e  $(\underline{e}_1, \underline{e}_2)$  in  $\mathbb{R}^2$  codom.

Se voglio la matrice  $B$  che rappresenta  $f$   
risp. alle basi  $(\underline{e}_1, \underline{e}_2)$ ,  $(\underline{\alpha}_1, \underline{\alpha}_2)$

ho bisogno di conoscere  $f(\underline{e}_1)$ ,  $f(\underline{e}_2)$

I°) devo conoscere che è  $\underline{e}_1$  (ed  $\underline{e}_2$ ) rispetto a

$$(\underline{\alpha}_1, \underline{\alpha}_2) = (\underline{e}_1, \underline{e}_2) \underbrace{\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}}_P \Rightarrow (\underline{\alpha}_1, \underline{\alpha}_2) P^{-1} = (\underline{e}_1, \underline{e}_2) \underbrace{P^{-1}}_I$$

più in dettaglio cerco una comb. lin. di  $\underline{\alpha}_1, \underline{\alpha}_2$   
che mi dia  $\underline{e}_1$  e un'altra che mi dia  $\underline{e}_2$

$$\left( \begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array} \right) \rightarrow \left( \begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{array} \right) \rightarrow$$

$$\left( \begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 1 & -1/2 & 1/2 \end{array} \right) \rightarrow \left( \begin{array}{cc|cc} 1 & 0 & 1/2 & 1/2 \\ 0 & 1 & -1/2 & 1/2 \end{array} \right)$$

$$\underline{e}_1 = (\underline{\alpha}_1, \underline{\alpha}_2) \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix} \quad \underline{e}_2 = (\underline{\alpha}_1, \underline{\alpha}_2) \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$$

II°) devo trovare  $f(\underline{e}_1)$ ,  $f(\underline{e}_2)$

$$f(\underline{e}_1) = f\left(\frac{1}{2} \underline{\alpha}_1 + \frac{1}{2} \underline{\alpha}_2\right) = \frac{1}{2} f(\underline{\alpha}_1) - \frac{1}{2} f(\underline{\alpha}_2)$$

$$f(\underline{e}_2) = f\left(\frac{1}{2} \underline{\alpha}_1 + \frac{1}{2} \underline{\alpha}_2\right) = \frac{1}{2} f(\underline{\alpha}_1) + \frac{1}{2} f(\underline{\alpha}_2)$$

$$(f(\underline{e}_1), f(\underline{e}_2)) = \boxed{(f(\underline{\alpha}_1), f(\underline{\alpha}_2))} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

III) esprimere  $f(\underline{\alpha}_1, f(\underline{\alpha}_2))$  con le matrice

$$\begin{aligned} (f(\underline{e}_1), f(\underline{e}_2)) &= \boxed{(\underline{e}_1, \underline{e}_2) \begin{pmatrix} 2 & 0 \\ 3 & 4 \end{pmatrix} \circ} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \\ &= (\underline{e}_1, \underline{e}_2) \begin{pmatrix} 1 & 1 \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \end{aligned}$$

E' giusto?

$$f\begin{pmatrix} x \\ y \end{pmatrix} = (f(\underline{e}_1), f(\underline{e}_2)) \begin{pmatrix} x \\ y \end{pmatrix} = (\underline{e}_1, \underline{e}_2) \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

In particolare se:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \underline{\alpha}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} x \\ y \end{pmatrix} = \underline{\alpha}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$f\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad f\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$$

che è ciò da cui siamo partiti.