A LINEAR SYSTEM ON NARUKI'S MODULI SPACE OF MARKED CUBIC SURFACES

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Abstract. Allcock and Freitag recently showed that the moduli space of marked cubic surfaces is a subvariety of a nine dimensional projective space which is defined by cubic equations. They used the theory of automorphic forms on ball quotients to obtain these results. Here we describe the same embedding using Naruki's toric model of the moduli space. We also give an explicit parametrization of the tritangent divisors, we discuss another way to find equations for the image and we show that the moduli space maps, with degree at least ten, onto the unique quintic hypersurface in a five dimensional projective space which is invariant under the action of the Weyl group of the root system E_6.

Introduction

Recently Allcock, Carlson and Toledo [ACT] studied the moduli space of smooth cubic surfaces using the intermediate jacobian of the cubic threefold which is the triple cover of projective three space branched along a cubic surface. They show that this moduli space, as well as the moduli space of marked cubic surfaces $\mathcal{M}^0$ (that is, cubic surfaces with an ordered set of six skew lines) are open subsets of certain 4-ball quotients. The Weyl group $W(E_6)$ of the root system E_6 acts on $\mathcal{M}^0$ by permuting the markings on any given cubic surface, the quotient variety is the moduli space of cubic surfaces. The quasi projective variety $\mathcal{M}^0$ has a natural compactification $\mathcal{M}$ given by geometrical invariant theory. The projective variety $\mathcal{M}$ coincides with the Baily-Borel compactification of the ball quotient. The action of $W(E_6)$ extends to $\mathcal{M}$.

Using Borcherds' work on automorphic forms on ball quotients, Allcock and Freitag [AF] found a $W(E_6)$-equivariant embedding of $\mathcal{M}$ in a nine dimensional projective space. The action of $W(E_6)$ on the projective space is obtained from the unique ten dimensional irreducible linear representation of $W(E_6)$. This map actually already appears in a paper by A. B. Coble published in 1917 [C] (and see also [Y]) where $\mathcal{M}$ is identified with the moduli space of six points in the projective plane. The same embedding of $\mathcal{M}$ was also found by Matsumoto and Terasoma [MT] who used the theta constants associated to the intermediate jacobians.

An explicit smooth projective compactification $\mathcal{C}$ ('the cross ratio variety') of the moduli space $\mathcal{M}$ with a biregular action of the Weyl group was constructed by Naruki [N]. It is a modification of a toric variety associated to the root system D_4. Naruki constructs and studies his model as a subvariety of the product of 270 projective lines, each component of this map is given by a cross ratio (of certain tritangent planes containing a given line on the cubic surface). The Weyl group acts via permutations of these 270 projective lines.

In this paper we explicitly identify the nine dimensional linear system on Naruki's model $\mathcal{C}$ which defines the map F to $\mathbf{P}^9$ discovered by Coble, Allcock and Freitag (see Theorem 5.7)

$$F : \mathcal{C} \longrightarrow \mathcal{M} \quad (\subset \mathbf{P}^9).$$

We also give explicit formulas for the $W(E_6)$-action on this linear system in section 5.

A tritangent plane of a cubic surface is a plane which cuts out three lines on the surface. If these three lines meet in a point, that point is called an Eckart point. We obtain a nice parametrization, equivariant for the Weyl group of the root system F_4, of the 45 divisors in $\mathcal{M}$ which parametrize marked cubic surfaces with an Eckart point, see Theorem 6.5. A study of the linear relations between tritangent planes leads to the discovery that $\mathcal{M}$ is the singular locus of a variety X defined by six quintic polynomials, see 7.10. The group $W(E_6)$ acts on X and it would be very interesting to have a moduli interpretation for X.

The Weyl group of E_6 is defined as a reflection group on a real six dimensional vector space. Complexifying and projectivizing this vector space one obtains a biregular action of $W(E_6)$ on a $\mathbf{P}^5$. In his book [H], Bruce Hunt suggested an identification of the moduli space with the unique $W(E_6)$-invariant quintic hypersurface I_5 in $\mathbf{P}^5$. In section 8 we construct a dominant rational map $\Sigma : \mathcal{M} \longrightarrow \mathbf{P}^5$ which is equivariant for the action of $W(E_6)$ and we show that its image is I_5 (Thm. 8.6), but, unfortunately, this map has degree at least 10 (Thm 8.8).

The results of this paper are obtained from computations with rational functions on the toric variety, many of them computer assisted. It does lead to very explicit formulas and parametrizations, somewhat in contrast to the ball quotient approach where the modular forms in question are hard to describe explicitly.

I'm indebted to E. Freitag for suggesting to undertake this study and for many discussions. I would also like to thank him and E. Carlini for assistance with the computations.

1. Cubic surfaces their moduli space

1.1. We briefly recall the basics on cubic surfaces and E_6, see [H] and references given there for proofs. We relate this to the modular orthogonal geometry used by Allcock and Freitag.

1.2. **The 27 lines.** Any smooth cubic surface S has 27 lines and there are sets of six disjoint lines $\{a_1, \ldots, a_6\}$. Blowing down the lines a_i to points p_i defines a birational isomorphism $S \to \mathbf{P}^2$. The images of the other 21 lines on S are the 15 lines $< p_i, p_j >$ and the 6 conics which pass through all six points except one of the p_i. The corresponding lines are denoted by c_{ij} and b_j. The birational inverse $\mathbf{P}^2 \to S$ is given by the linear system of all cubics passing through the points $p_1, \ldots, p_6$.

1.3. **The root system E_6.** The Picard group of S is isomorphic to $\mathbf{Z}^7$ and a $\mathbf{Z}$-basis is given by the pull-back l of (the divisor class of) a line in $\mathbf{P}^2$ and the classes of the lines a_i. The intersection form is determined by

$$l^2 = 1, \quad a_i^2 = -1, \quad l \cdot a_i = 0, \quad a_i \cdot a_j = 0$$

for $i \neq j$. The classes of the lines are

$$c_{ij} = l - (a_i + a_j), \qquad b_i = 2l - (a_1 + \ldots + \hat{a}_i + \ldots + a_6).$$

The canonical class of S is $K_S := -3l + a_1 + \ldots + a_6$ and $K_S^2 = 3$. The class of a hyperplane section of S is $-K_S$. The primitive cohomology of S is thus the orthogonal complement of K_S. This $\mathbf{Z}$-module, with the bilinear form $(x, y) := -x \cdot y$, is isomorphic to the root lattice $Q(E_6)$ of the root system E_6:

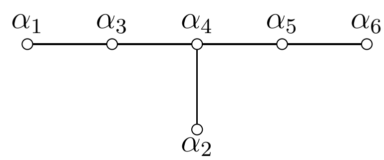
$$Q(E_6) \cong K_S^{\perp} := \{x \in Pic(S) : \ x \cdot K_S = 0 \ \}.$$

A $\mathbf{Z}$-basis for $Q(E_6)$ is given by:

$$\alpha_1 = a_2 - a_1, \qquad \alpha_3 = a_3 - a_2, \qquad \alpha_4 = a_4 - a_3, \qquad \alpha_5 = a_5 - a_4, \qquad \alpha_6 = a_6 - a_5,$$

$$\alpha_2 = l - a_4 - a_5 - a_6.$$

This is a basis of simple roots of E_6:



The set E_6^+ of positive roots of E_6 consists of the 36 elements in $Q(E_6)$ given by

$$h_{ij} := -a_i + a_j, \quad (i < j) \qquad h_{ijk} := l - a_p - a_q - a_r, \quad h := 2l - a_1 - \ldots - a_6,$$

where $\{i, j, k, l, p, q, r\} = \{1, 2, \ldots, 6\}$. In particular, $\alpha_2 = h_{123}$ and with this convention our notation is compatible with that of [H]. The root system $E_6 := E_6^+ \cup (-E_6^+) \subset Pic(S)$ contains 72 vectors, called roots.

1.4. **The Weyl group $W(E_6)$.** The Weyl group $W(E_6)$ is the subgroup of $GL(Q(E_6))$ generated by reflections in the roots. We denote by s_i the reflection in the hyperplane perpendicular to the root α_i. More generally we write s_α, with $\alpha \in E_6$, for the reflection in the hyperplane perpendicular to α.

1.5. **The orthogonal geometry.** Allcock and Freitag use a non-degenerate quadratic form Q on the vector space $\mathbf{F}_3^5$ and its orthogonal group $O(5, 3)$ to describe the combinatorics of the lines on a cubic surfaces and of divisors on the moduli space $\mathcal{M}$. The basic facts are ([AF], section 2):

$$O(5, 3) \cong W(E_6) \times \{\pm 1\},$$

there are 72 vectors with $Q(x) = -1$, these are called the short roots (note $Q(x) = Q(-x)$). There are 90 vectors with $Q(x) = -2$, the long roots, and there are 80 nonzero vectors with $Q(x) = 0$, called isotropic vectors. (See also [MT], §3.)

1.6. **Boundary divisors of $\mathcal{M}$.** If a root is the class of an effective divisor on the blow up of $\mathbf{P}^2$, then this effective divisor is a $\mathbf{P}^1$ which is contracted to a node on the cubic surface. This sets up a correspondence between the set of irreducible divisors in $\mathcal{M}$ parametrizing nodal cubic surfaces and E_6^+. These divisors are labelled by pairs $\pm x$ of 'short roots' in [AF].

The divisor in $\mathcal{M}$ corresponding to $\alpha \in E_6^+$ is denoted by D_α (or by D_{ij} if $\alpha = h_{ij}$ etc.). These divisors are the fixed point sets of the corresponding reflections $s_\alpha \in W(E_6)$ in $\mathcal{M}$. The reflection $s_\alpha \in Aut(Pic(S))$ may be identified with the Picard-Lefschetz transformation associated to the general nodal cubic surface S_0 in D_α.

1.7. **Lines and weights.** Let $P(E_6) \subset Q(E_6) \otimes \mathbf{Q}$ be the weight lattice of E_6:

$$P(E_6) := \{x \in Q(E_6) \otimes_{\mathbf{Z}} \mathbf{Q} : \ (x, y) \in \mathbf{Z}, \ \forall y \in Q(E_6)\}.$$

The intersection number of the class c of a line on S with a root is an integer, hence c defines an element $x_c \in P(E_6)$. In this way one obtains a $W(E_6)$-orbit of 27 weights (which are also denoted by a_i, b_i, c_{ij} with $1 \leq i \leq 6$, $1 \leq i < j \leq 6$, cf. [H], § 6.1.3). Note that

a_1 is perpendicular to all simple roots except α_1 and that $(a_1, \alpha_1) = -1$, thus a_1 is minus a fundamental root of E_6.

1.8. **The tritangent planes and tritangent divisors.** Since hyperplane sections of S correspond to cubics on the p_i, it is easy to see that there are 45 planes, the tritangent planes, which intersect S in three lines, in Schläfli's notation these are denoted by:

$$(ij) = \{a_i, b_j, c_{ij}\}, \qquad (ij.kl.mn) = \{c_{ij}, c_{kl}, c_{mn}\},$$

where$\{i, \ldots, n\} = \{1, \ldots, 6\}$. Another labelling for the tritangents was given by Cayley and is used by Naruki. The dictionary between the labels is given in [Se], p.371. The 45 tritangent divisors in $\mathcal{M}$ are written as D_t where t is one of Schläfli's labels. The tritangent divisors correspond to pairs $\pm x$ of long roots of [AF].

Three lines lie in a tritangent plane iff the sum of their classes in $Pic(S)$ is $-K_S$ iff the corresponding weights are linearly dependent. The orthogonal complement in E_6 of the span of three such weights is a root system of type D_4. If the tritangent is labelled by t, we will denote this D_4 by $t^\perp$.

1.9. **The subsystem D_4.** An important example is the case that $t = (16) = \mathrm{w}$. In that case $t^\perp$ is the $D_4 \subset E_6$ spanned by the simple roots $\alpha_2,\ \alpha_3,\ \alpha_4$ and α_5. This root system is discussed in section 2.

1.10. **The $W(F_4)$ and tritangents.** To a tritangent t one associates an element $\gamma(\mathrm{t}) \in W(E_6)$ which is the product of the relections in 4 orthogonal roots in $t^\perp \cong D_4$. Thus $\gamma(t)$ is $-I$ on the span of $t^\perp$ and is $+I$ on the orthogonal complement which is the span on the subspace spanned by the weights corresponding to the lines in t. For $t = (16) = \mathrm{w}$ one may take $\gamma(\mathrm{w}) = s_2 s_5 s_3 (s_4 s_5 s_3 s_4) s_2 (s_4 s_3 s_5 s_4)$. The $\gamma(t)$'s are a conjugacy class of 45 elements in $W(E_6)$ which correspond (via their $+1$-eigenspace) with the tritangents. The centralizer of a $\gamma(t)$ in $W(E_6)$ is isomorphic to the Weyl group $W(F_4)$. The fixed point set of a $\gamma(t)$ on $\mathcal{C}$ is the tritangent divisor D_t which parametrises cubic surfaces for which the three lines in t meet in one point, called an Eckart point ([N] §8).

2. The toric variety

2.1. For general facts on toroidal compactifications we refer to [Fu], for root systems see [Hu].

2.2. **The torus.** The D_4-adjoint torus

$$T \xrightarrow{\cong} (\mathbf{C}^*)^4, \qquad t \longmapsto (\lambda(t), \mu(t), \nu(t), \rho(t))$$

comes with a natural identification of its character group $\mathrm{Hom}(T, \mathbf{C}^*) \cong \mathbf{Z}^4$ with the sublattice

$$M := \langle\, e_1 - e_2,\ e_2 - e_3,\ e_3 - e_4,\ e_3 + e_4 \rangle \subset \oplus_{i=1}^4 \mathbf{Z} e_i.$$

The lattice M, with the scalar product induced by the standard inner product on $\oplus \mathbf{Z} e_i$, is the root lattice $Q(D_4)$ of D_4. We often use:

$$\mathrm{Hom}(T, \mathbf{C}^*) \xrightarrow{\cong} M, \qquad \lambda \mapsto e_1 - e_2,\ \mu \mapsto e_3 + e_4,\ \nu \mapsto e_3 - e_4,\ \rho \mapsto e_2 - e_3.$$

For $\alpha \in M$ we define a regular function on T by:

$$f_\alpha := \lambda^a \mu^b \nu^c \rho^d \qquad \text{with} \quad \alpha = a(e_1 - e_2) + b(e_2 - e_3) + c(e_3 - e_4) + d(e_3 + e_4) \in M.$$

2.3. **The root system.** The root system D_4 consists of the following 24 vectors in M:

$$D_4 = \{ \pm e_i \pm e_j \in M : \quad 1 \leq i < j \leq 4 \}.$$

The set

$$\Delta_0 := \{e_1 - e_2,\, e_2 - e_3,\, e_3 - e_4,\, e_3 + e_4\} \quad (\subset D_4)$$

is a fundamental system (or base of the root system), that is any root is a linear combination of these 4 vector with all coefficients either positive (such a root is called positive) or negative. Let $N = M^*$ be the dual lattice of M,

$$N := \mathrm{Hom}_{\mathbf{Z}}(M, \mathbf{Z}) = \{x \in (\oplus \mathbf{Z} e_i)^* \otimes_{\mathbf{Z}} \mathbf{R} : \; \langle x, \alpha \rangle \in \mathbf{Z} \quad \forall \alpha \in M \},$$

here $\langle .,. \rangle$ is the pairing between $(\oplus \mathbf{Z} e_i)^* \otimes_{\mathbf{Z}} \mathbf{R}$ and its dual. Let $\{\epsilon_1, \ldots, \epsilon_4\} \subset (\oplus \mathbf{Z} e_i)^* \otimes_{\mathbf{Z}} \mathbf{R}$ be the dual basis of $\{e_1, \ldots, e_4\}$. Then the basis of N which is dual to Δ_0 is

$$\epsilon_1, \quad (\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4)/2, \quad (\epsilon_1 + \epsilon_2 + \epsilon_3 - \epsilon_4)/2, \quad \epsilon_1 + \epsilon_2 \qquad (\in N).$$

2.4. **The Weyl group.** The Weyl group $W(D_4)$ of the root system is the subgroup of $GL(M \otimes \mathbf{R})$ generated by the reflections in the roots (so $s_\alpha(\beta) = \beta - (\beta, \alpha)\alpha$ and $(.,.)$ is the standard inner product on $\oplus \mathbf{Z} e_i$). This group has 192 elements and is a semidirect product of S_4 (permuting the e_i) and $(\mathbf{Z}/2\mathbf{Z})^3$ (changes the sign of an even number of the e_i). The Weyl group acts simply transitively on the fundamental systems.

The Weyl group acts on N and the 4 elements of the dual basis above are in distinct orbits of lengths 8, 8, 8 and 24 respectively. We define

$$S := \{\pm \epsilon_i\} \cup \{(\pm \epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4)/2\}, \qquad R := \{\pm \epsilon_i \pm \epsilon_j\},$$

S and R each have 24 elements.

2.5. **The Weyl chambers.** The (closed) Weyl chamber $C(\Delta)$ of a fundamental system Δ ($\subset D_4$) is the (maximal) cone in $N \otimes_{\mathbf{Z}} \mathbf{R} = (\oplus \mathbf{Z} e_i)^* \otimes_{\mathbf{Z}} \mathbf{R}$ defined by:

$$C(\Delta) := \{x \in N \otimes_{\mathbf{Z}} \mathbf{R} \;:\; \quad \langle x, \alpha \rangle \geq 0 \quad \forall \alpha \in \Delta\}.$$

If $\Delta = \{\alpha_1, \ldots, \alpha_4\}$ then the edges (i.e. the one dimensional faces) of $C(\Delta)$ are the 4 half-lines $\mathbf{R}_{\geq 0}\tau_i$ with $\{\tau_1, \ldots, \tau_4\}$ the dual basis of Δ. The decomposition

$$N \otimes_{\mathbf{Z}} \mathbf{R} = \cup_\Delta C(\Delta)$$

is a regular cone decomposition of the vector space $N \otimes_{\mathbf{Z}} \mathbf{R}$, it defines in a fan in N whose faces are the faces of the 192 Weyl chambers. This fan has 48 edges which correspond to the elements of $S \cup R$.

2.6. **The toroidal compactification.** Associated to this fan is a toric variety $\tilde{T}$,

$$\tilde{T} = \cup_\Delta A(\Delta), \qquad A(\Delta) \cong \mathbf{C}^4$$

and the inclusion $T \subset A(\Delta)$ is defined by the inclusion of the rings of regular functions

$$\mathbf{C}[A(\Delta)] := \langle\, f_\alpha \;: \alpha \in M, \quad \langle x, \alpha \rangle \geq 0 \quad \forall x \in C(\Delta)\,\rangle \;\hookrightarrow\; \mathbf{C}[T] := \mathbf{C}[\lambda^{\pm 1},\, \mu^{\pm 1},\, \nu^{\pm 1},\, \rho^{\pm 1}].$$

For example $\mathbf{C}[A(\Delta_0)] = \mathbf{C}[\lambda,\, \mu,\, \nu,\, \rho]$. Each edge $\mathbf{R}_{\geq 0}\tau$, with $\tau \in S \cup R$, defines a divisor $V(\tau)$ in $\tilde{T}$ ([Fu], §3.3) and these 48 divisors are the complement of T in $\tilde{T}$:

$$\tilde{T} - T = \cup_{\tau \in S \cup R} V(\tau).$$

The regular functions f_α, $\alpha \in M$, on T extend to rational functions on $\tilde{T}$. The divisor of f_α is given by:

$$(f_\alpha) = \sum_\tau n_\tau V(\tau) \qquad \text{with} \quad n_\tau := \langle \tau, \alpha \rangle.$$

2.7. **Example.** The divisor of $\lambda = f_{e_1 - e_2}$ is given by:

$$(\lambda) = D_\lambda^+ - D_\lambda^- \quad \text{with} \left\{ \begin{array}{l} D_\lambda^+ = V(\epsilon_1) + V(-\epsilon_2) + \sum_{\pm,\pm} V((\epsilon_1 - \epsilon_2 \pm \epsilon_3 \pm \epsilon_4)/2) + D' \\ D_\lambda^- = V(-\epsilon_1) + V(\epsilon_2) + \sum_{\pm,\pm} V((-\epsilon_1 + \epsilon_2 \pm \epsilon_3 \pm \epsilon_4)/2) + D'' \end{array} \right.$$

where D' and D'' are combinations of the divisors $V(\tau)$ with $\tau \in R$ with coefficients in $\{-2, -1, 0, 1, 2\}$.

2.8. **The cross ratio variety.** Naruki's (smooth, projective) cross ratio variety $\mathcal{C}$ is obtained from the toric variety $\tilde{T}$ as follows ([N], §10-12):

$$\mathcal{M} \longleftarrow \mathcal{C} \xleftarrow{r} \hat{T} \xrightarrow{\pi''} \tilde{T}'' \xrightarrow{\pi'} \tilde{T}' \xrightarrow{\pi_e} \tilde{T}.$$

The map π_e is the blow up of $\tilde{T}$ in the identity element $e \in T$. The exceptional divisor $\pi_e^{-1}(e) \cong \mathbf{P}^3$ is denoted by $\mathbf{P}^3_{\mathrm{w}}$. The image in $\mathcal{M}$ of its strict transform in $\tilde{T}''$ is the tritangent divisor $D_{\mathrm{w}} = D_{(16)}$.

The map π' is the blow up of $\tilde{T}'$ in the strict transforms in $\tilde{T}'$ of the 12 curves in the $W(D_4)$-orbit of the curve in $\tilde{T}$ defined by $\lambda = \nu = \rho = 1$. The morphism r contracts the strict transforms in $\hat{T}$ of the 12 exceptional divisors in $\tilde{T}''$ to surfaces in $\mathcal{C}$ and is an isomorphism on the complement ([N], Prop. 11.3).

The map π'' is the blow up in the strict transform in $\tilde{T}''$ of the 16 surfaces in the $W(D_4)$-orbit of $\mu = \rho = 1$. The 16 exceptional divisors in $\hat{T}$ map under r to divisors in $\mathcal{C}$, their $W(E_6)$-orbit consists of 40 divisors, the other 24 are the images under r of the strict transforms of the $V(\tau)$'s with $\tau \in R$ ([N], Prop. 11.2). We call these 40 divisors the cusp divisors of $\hat{T}$.

There is a morphism $\mathcal{C} \to \mathcal{M}$, where $\mathcal{M}$ is the moduli space of semistable marked cubic surfaces, which contracts the 40 cusp divisors to points (cf. [N], Introduction and §12), the cusps of $\mathcal{M}$. The Weyl group $W(E_6)$ acts biregularly on $\mathcal{C}$ and $\mathcal{M}$ and the morphism $\mathcal{C} \longrightarrow \mathcal{M}$ is $W(E_6)$-equivariant.

3. The $W(E_6)$-action on boundary divisors.

3.1. According to Naruki [N], Prop. 11.3', the boundary $\mathcal{C} - \mathcal{M}^0$ consists of two $W(E_6)$-orbits of divisors, one orbit is formed by the 36 boundary divisors D_α with $\alpha \in E_6^+$. The other orbit consists of the 40 cusp divisors and will not be of interest for us. In Naruki's toroidal construction, the 36 D_α's are parametrized by the 12 positive roots D_4^+ of D_4 and by the 24 elements of a set of S (see 2.4) of weights of D_4. In this section we determine the corresponding $W(D_4)$-equivariant bijection between $E_6/\{\pm 1\}$ and $(D_4/\{\pm 1\}) \cup S$, see table 9.2 for the final result.

3.2. To do the required computations, it is sufficient to work on the blow up of $\tilde{T}$ in the origin, rather then on $\mathcal{C}$ or $\mathcal{M}$, cf. 2.8. For each positive root $\alpha \in D_4$ the closure in $\tilde{T}$ of the subtorus defined by $f_\alpha = 1$ in T is an irreducible divisor. Since it contains e, its pull-back to $\tilde{T}'$ has two irreducible components, one is $\mathbf{P}^3_{\mathrm{w}}$ and the other is its strict transform which we will denote by D^1_α. The image in $\mathcal{M}$ of the strict transform of D^1_α in $\hat{T}$ is D_α, so these twelve divisors are labelled via $D_4 = \mathrm{w}^\perp \subset E_6$.

The other 24 boundary divisors in $\mathcal{M}$ are the images in $\mathcal{M}$ of the strict transforms of the $V(\beta)$ with $\beta \in S$ ([N], Prop. 11.1). The Weyl group $W(D_4)$ has three orbits on S and it suffices to identify one divisor from each orbit. That is done in the following lemma. The resulting labelling of all 36 divisors is given in table 9.2.

3.3. **Lemma.** Let s_1, s_6 be the reflections in $W(E_6)$ defined by the roots $\alpha_1 = h_{12}$, $\alpha_6 = h_{56}$ respectively. Then we have:

$$s_1^* D^1_\lambda = V(-\epsilon_2)$$

hence $V(-\epsilon_2) = D_{13}$. Similarly we have:

$$\begin{array}{rcl} s_6^* D^1_{\lambda\nu\rho} & = & V((\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4)/2), \\ s_1^* V((\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4)/2) & = & V((\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4)/2) \end{array}$$

and thus $V((\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4)/2) = D_{26}$, $V((\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4)/2) = D_{16}$.

Proof. The divisor of the rational function $\lambda - 1$ on $\tilde{T}'$ is

$$(\lambda - 1) = D^1_\lambda + \mathbf{P}^3_{\mathrm{w}} - D^-_\lambda,$$

where D^-_λ is as in 2.7. Therefore $s_1^*(\lambda - 1)$ will have exactly two effective components, one being $s_1^* D^1_\lambda$ which must be in the orbit of length 36 and the other will be a tritangent divisor. From [N], p. 13 we have:

$$s_1 : \ \lambda \longmapsto \frac{\lambda\mu\nu\rho^2(1-\lambda)}{\lambda\mu\nu\rho^2 - 1}$$

and hence that

$$s_1 : \ \lambda - 1 \longmapsto f_1 := \frac{1 - \lambda^2\mu\nu\rho^2}{\lambda\mu\nu\rho^2 - 1}.$$

Since $\lambda^2\mu\nu\rho^2 = f_{2e_1}$ (note that $\lambda^2\mu\nu\rho^2$ is not a root) and $\lambda\mu\nu\rho^2 = f_{e_1+e_2}$ we see that the denominator has a pole of order one on $V(-\epsilon_2)$ but the numerator has vanishing order zero on that divisor, hence $V(-\epsilon_2)$ must be one of the two effective components of (f_1). The other effective component is defined by $1 - \lambda^2\mu\nu\rho^2 = 0$, which is the local equation of the tritangent divisor $D_{\bar{\mathrm{x}}}$ ($\bar{\mathrm{x}} = (26)$, cf. Table 3 of [N]). Note that $\lambda = f_{e_1-e_2}$ and $e_1 - e_2 = h_{23}$, so $D^1_\lambda = D_{23}$ and that s_1 permutes the indices 1 and 2 of an h_{ij}, hence $s_1^* D_{23} = D_{13}$ and $s_1^* D_{(16)} = D_{(26)}$.

Using the formulas from [N], p. 13 again we get:

$$s_6 : \ 1 - \lambda\nu\rho \longmapsto f_2 := \frac{1 - \lambda\mu\nu^2\rho^2}{1 - \mu\nu\rho}.$$

Since $\lambda\mu\nu^2\rho^2 = f_{e_1+e_2+e_3-e_4}$ and $\mu\nu\rho = f_{e_2+e_3}$, we see that $V((\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4)/2)$ is one of the two effective components of (f_2). The other component corresponds to the tritangent divisor $D_{\bar{\mathrm{z}}} = D_{(15)}$ defined by $1 - \lambda\mu\nu^2\rho^2 = 0$. Note that $\lambda\nu\rho = f_{e_1-e_4}$ and $e_1 - e_4 = h_{25}$, so $D^1_{\lambda\nu\rho} = D_{25}$ and that s_6 permutes the indices 5 and 6 of an h_{ij}, hence $s_6^* D_{25} = D_{26}$ and $s_6^* D_{(16)} = D_{(15)}$.

Next we apply s_1 to f_2 and obtain:

$$s_6 : f_2 \longmapsto f_3 := \frac{-\mu\rho(\lambda+\nu-\lambda\nu-\lambda\nu\rho-\lambda\mu\nu\rho+\lambda^2\mu\nu^2\rho^2)}{(\lambda\mu\rho-1)(\lambda\nu\rho-1)}$$

In the open subset $U = A(\Delta_0) = Spec(\mathbf{C}[\lambda,\mu,\nu,\rho])$, this function is zero on $\mu = 0$, which is $V((\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4)/2)\cap U$ (the zero locus in U of the i-th element in $\{\lambda,\mu,\nu,\rho\}$ is the divisor corresponding to the i-th vector of the dual basis). Thus we found one of the two effective components of the divisor of f_3. Note that $(\rho=0)\cap U$ lies in $V(\epsilon_1+\epsilon_2)$, which is not in the orbit of the 36 divisors and that the third factor of the numerator of f_3 defines the tritangent divisor labelled by $\bar{q}_1 = (25)$. Since s_1 permutes the indices 1 and 2 of an h_{ij}, we get $s_1^*D_{26} = D_{16}$ and $s_1^*D_{(15)} = D_{(25)}$. □

3.4. The labelling of these 36 divisors on $\tilde{T}'$ allows us to express various divisors in a convenient manner. For example (cf. 2.7):

$$(\lambda) = D_{13}+D_{26}+D_{136}+D_{246}+D_{256}+D_{345}-D_{12}-D_{36}-D_{126}-D_{346}-D_{356}-D_{245}$$

and similarly:

$$(\lambda-1) = D_{\rm w}+D_{23}-D_{12}-D_{36}-D_{126}-D_{346}-D_{356}-D_{245}.$$

4. The CAF-linear system.

4.1. To identify the linear system on the moduli space $\mathcal{M}$ introduced by Coble, Allcock and Freitag and to describe the $W(E_6)$-action on it, we consider two divisors with support in the boundary of the toric variety $\tilde{T}$.

4.2. **Definition.** Let $R,\ S\subset N$ be as in section 2.4. We define divisors in $\tilde{T}$ (cf. 2.6) by:

$$D_S := \sum_{\tau\in S} V(\tau), \qquad D_R := \sum_{\tau\in R} V(\tau).$$

4.3. **Lemma.** We have

$$H^0(\tilde{T},\mathcal{O}(D_S+2D_R)) = \langle\, f_0 = 1\,\rangle \oplus \langle\, f_\alpha \,:\, \alpha\in D_4\,\rangle,$$

in particular, $\dim H^0(\tilde{T},\mathcal{O}(D_S+2D_R)) = 25$. The divisor D_S+2D_R is very ample on $\tilde{T}$.

Proof. The space of global sections of the line bundle associated to a divisor $\sum n_\tau V(\tau)$ is spanned by certain f_α's:

$$H^0(\tilde{T},\mathcal{O}(\sum_\tau n_\tau V(\tau))) \;=\; \langle\, f_\alpha \,:\, \alpha\in M \text{ and } \langle\tau,\alpha\rangle \ge -n_\tau\,\rangle.$$

Thus we must find the $\alpha\in M$ with $\langle\tau,\alpha\rangle\ge -1$ for $\tau\in S$ and $\langle\tau,\alpha\rangle\ge -2$ for $\tau\in R$. Let $\alpha=\sum m_ie_i$ with $m_i\in\mathbf{Z}$. Taking $\tau=\pm\epsilon_i\in S$ we get $-1\le m_i\le 1$, taking $\tau=(\pm\epsilon_1\pm\ldots\pm\epsilon_4)/2 \in S$ we get $-2\le \pm m_1\pm m_2\pm m_3\pm m_4\le 2$, hence at most two of the m_i are non zero and thus $\alpha=0$, $\pm e_i$ or $\pm e_i\pm e_j$ with $i\ne j$. However $\pm e_i\notin M$ and therefore α is either zero or a root. All these α also satisfy $\langle\tau,\alpha\rangle\ge -2$ for $\tau\in R$.

The proof of the very ampleness is standard, cf. [Fu], and since we do not really need it, we omit the proof. □

4.4. **Divisors near the identity.** The functions $x_r := r-1$ with $r \in \{\lambda,\, \mu,\, \nu,\, \rho\}$ are local coordinates near the identity element $e = (1,1,1,1) \in T$. Any rational function f on T which is regular in e can be developed in a Taylor series:

$$f = f_d + f_{d+1} + \dots, \qquad \text{with} \quad f_k \in \mathbf{C}[x_\lambda,\, x_\mu,\, x_\nu,\, x_\rho]$$

with f_k homogeneous of degree k and $d \geq 0$. If the polynomial f_d is not identically zero we say that f vanishes to order d in $e \in T$ and we write $m_e(f) = d$, f_d is called the leading term of f.

For $\alpha = a(e_1 - e_2) + \ldots + d(e_2 - e_3) \in M - \{0\}$ we have:

$$f_\alpha - 1 = (x_\lambda + 1)^a (x_\mu + 1)^b (x_\nu + 1)^c (x_\rho + 1)^d - 1 = a x_\lambda + b x_\mu + c x_\nu + d x_\rho + \; H.O.T.$$

hence $f_\alpha - 1$ vanishes to order 1 at e and a product $\prod_{i=1}^m (f_{\alpha_i} - 1)$ of such functions vanishes to order m at e.

4.5. **Definition.** We define the vector space V of rational functions on $\tilde{T}$ to be the subspace of those global sections of $\mathcal{O}(D_S + 2D_R)$ which vanish to order at least 3 at $e \in \tilde{T}$:

$$V := \{\, f \in \; H^0(\tilde{T}, \mathcal{O}(D_S + 2D_R)) : \; m_e(f) \geq 3 \,\}.$$

4.6. **Lemma.** The dimension of V is 10. A basis for V, multiplied by $\lambda\mu\nu\rho^2$, is given in table 5.5.

Proof. Note that 10 is the expected dimension of V since the spaces of constant, linear and quadratic polynomials in 4 variables have dimension 1, 4, 10 respectively. Thus we only have to show that each monomial $x_\lambda^a x_\mu^b x_\nu^c x_\rho^d$ with $a+b+c+d \leq 2$ is the leading term of a function in $H^0(\tilde{T}, \mathcal{O}(D_S + 2D_R))$. Obviously we can use $f_0 = 1$ to get leading term 1 and the $r-1$ to get leading term x_r. For the roots $\alpha = \lambda\rho,\, \mu\rho,\, \nu\rho$ the leading term is a linear combination of the leading terms of the $r-1$'s which we already have. Subtracting these linear terms we get functions with the leading terms $x_\lambda x_\rho,\, x_\mu x_\rho,\, x_\nu x_\rho$. The Taylor series of $t-1$ with $t = \lambda\nu\rho,\, \mu\nu\rho,\, \lambda\mu\rho$, give us, modulo the leading terms we already found, the leading terms $x_\lambda x_\nu,\, x_\mu x_\nu,\, x_\lambda x_\nu$. To get the x_r^2 use that

$$r^{-1} = 1 - x_r + x_r^2 - \dots .$$

Thus we found all the 15 desired leading terms and we conclude that V has codimension 15 in $H^0(\tilde{T}, \mathcal{O}(D_S + 2D_R))$. $\square$

4.7. **Example.** The following function lies in V:

$$\nu^{-1}\rho^{-1}(\rho - 1)(\lambda\nu\rho - 1)(\mu\nu\rho - 1) = \lambda\mu\nu\rho^2 - \lambda\mu\nu\rho - \lambda\rho - \mu\rho + \lambda + \mu + \nu^{-1} - (\nu\rho)^{-1}.$$

The first expression shows it vanishes to order three in e, the second that it is a linear combination of roots, hence it lies in $H^0(\tilde{T}, \mathcal{O}(D_S + 2D_R))$.

5. The action of $W(E_6)$ on the vector space V.

5.1. Naruki [N] defined a biregular action of $W(E_6)$ on $\mathcal{C}$ ([N], §5, p. 13). We show that this induces an action of $W(E_6)$ on the vector space V defined in 4.5. The vector space V may be identified, via pull-back

$$V \cong H^0(\tilde{T}', \mathcal{O}(2D_R + D_S - 3\mathbf{P}^3_{\mathrm{w}})),$$

where $\tilde{T}'$ is the blow up of $\tilde{T}$ in the identity element e and $\mathbf{P}^3_{\mathrm{w}}$ is the exceptional fiber.

The main problem is to find the images of the divisor $D_S - 3\mathbf{P}^3_{\mathrm{w}}$ under s_1, $s_6 \in W(E_6)$ and to show that the images are linearly equivalent to this divisor. For this we use the following rational function:

$$C_1 := \frac{(\lambda^2\mu\nu\rho^2-1)^3}{(\lambda-1)(\lambda\rho-1)(\lambda\nu\rho-1)(\lambda\mu\nu\rho^2-1)(\lambda\mu\nu\rho-1)(\lambda\mu\rho-1)}.$$

5.2. **Lemma.** The rational function C_1 on $\tilde{T}'$ has divisor

$$(C_1) = 3D_{\bar{\mathrm{x}}} - 3\mathbf{P}^3_{\mathrm{w}} + \sum_{\pm,i=2}^{4} V(\pm\epsilon_i) - D^1_\lambda - D^1_{\lambda\rho} - D^1_{\lambda\nu\rho} - D^1_{\lambda\mu\nu\rho^2} - D^1_{\lambda\mu\nu\rho} - D^1_{\lambda\mu\rho} + D$$

for some divisor D which is a combination of the divisors $V(\tau)$ with $\tau \in R$. Here $D_{\bar{\mathrm{x}}}$ is the tritangent divisor defined by the strict transform of the zero locus of $\lambda^2\mu\nu\rho^2 - 1$ in $\tilde{T}$.

Proof. The proof is straightforward using the formula from 2.6 and the examples in the proof of Lemma 3.3, for example

$$(\lambda^2\mu\nu\rho^2 - 1) = D_{\bar{\mathrm{x}}} + \mathbf{P}^3_{\mathrm{w}} - 2V(-\epsilon_1) - \sum_{\pm,\pm,\pm} V((-\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4)/2),$$

and the divisor of $\lambda - 1$, in $\tilde{T}$, was determined in 2.7. $\square$

5.3. For $f \in V$, the composition $f \circ s_1$ does not lie in V. However, we will show that the quotient $(f \circ s_1)/C_1$ does lie in V. To get an action of all of $W(E_6)$ however, the correct definition for the action of s_1 on V is $s_1(f) = -(f \circ s_1)/C_1$.

5.4. **Theorem.** The action of $W(E_6)$ on $\mathcal{C}$ defines an action of $W(E_6)$ on V by the following formulas:

$$s_i(f) := \begin{cases} -(f \circ s_1)/C_1 & \text{if } i = 1, \\ f \circ s_i & \text{if } 2 \le i \le 5, \\ -(f \circ s_6)/C_6 & \text{if } i = 6, \end{cases}$$

here the rational maps $s_i : T \to T$ are as defined by Naruki in [N], p. 13 and $C_6 = C_1 \circ \tau$ where $\tau(\lambda,\mu,\nu,\rho) = (\nu,\mu,\lambda,\rho)$.

The representation of $W(E_6)$ on V is its unique 10 dimensional irreducible representation and is denoted by 10_s in [Fr].

Proof. Recall that $D_S = \sum D_\alpha$ with $\alpha \in E_6 - D_4$ a positive root. Write:

$$D_S = D_S^{(0)} + D_S^{(1)}, \qquad D_S^{(0)} = \sum_\alpha D_\alpha$$

where we sum over the positive roots $\alpha \in E_6$, $\alpha \notin D_4$ which are fixed under s_1. Then $s_1^*D_S = D_S^{(0)} + s_1^*D_S^{(1)}$. Since $s_1^*D_{\mathrm{w}} = D_{\bar{\mathrm{x}}}$ (cf. the proof of 3.3), we get:

$$s_1^*(D_S - 3D_{\mathrm{w}}) = D_S^{(0)} + s_1^*D_S^{(1)} - 3D_{\bar{\mathrm{x}}}.$$

One verifies, using the tables 9.1 and 9.2 and the lemma above, that

$$(C_1) = 3D_{\bar{\mathrm{x}}} - 3D_{\mathrm{w}} + D_S^{(1)} - s_1^*D_S^{(1)}$$

hence $s_1^*(D_S - 3D_{\mathrm{w}}) + (C_1) = D_S - 3D_{\mathrm{w}}$. This suggests that $f \mapsto \pm(f \circ s_1)/C_1$ defines an endomorphism of V. To check this and to get a $W(E_6)$ representation on V, one computes

matrices and checks the defining relations for $W(E_6)$ (we used a computer, note this direct method avoids a detailed discussion of the divisor D_R and verifies that one has to put a '$-$' sign in the definition of s_1 and s_6). Since the only representations of $W(E_6)$ of dimension at most 10 are the trivial one, denoted by $1 = 1_p$, the 6 dimensional reflection representation 6_p, their tensor products with the determinant representation 1_n and 6_n, and 10_s, it suffices to compute the traces of a reflection s_i (which is 0) and of a product of two commuting reflections (which has trace 2) to prove that $V \cong 10_s$. $\square$

5.5. **Table of a basis of V.** To obtain functions in V, all entries have to be divided by $\lambda\mu\nu\rho^2$. All ten functions are in one $W(E_6)$-orbit.

$$\begin{aligned}
f_1 &= (\lambda\rho - 1)(\mu\rho - 1)(\nu\rho - 1)(\lambda\mu\nu\rho - 1),\\
g_1 &= (\rho - 1)(\lambda\mu\rho - 1)(\lambda\nu\rho - 1)(\mu\nu\rho - 1),\\
f_2 &= (\mu\rho - 1)(\nu\rho - 1)(1 - \lambda^2\mu\nu\rho^2),\\
g_2 &= (\rho - 1)(\mu\nu\rho - 1)(1 - \lambda^2\mu\nu\rho^2),\\
f_3 &= (\lambda\rho - 1)(\mu\rho - 1)(1 - \lambda\mu\nu^2\rho^2),\\
g_3 &= (\rho - 1)(\lambda\mu\rho - 1)(1 - \lambda\mu\nu^2\rho^2),\\
f_4 &= \rho(\mu\rho - 1)(\lambda + \nu - \lambda\nu - \lambda\nu\rho - \lambda\mu\nu\rho + \lambda^2\mu\nu^2\rho^2),\\
g_4 &= \rho(\mu - 1)(\lambda + \nu - \lambda\nu - \lambda\nu\rho - \lambda\mu\nu\rho + \lambda^2\mu\nu^2\rho^2),\\
f_5 &= (\lambda\mu\rho - 1)(\mu\nu\rho - 1)(1 - \lambda\nu\rho^2),\\
g_5 &= (\mu\rho - 1)(\lambda\mu\nu\rho - 1)(1 - \lambda\nu\rho^2).
\end{aligned}$$

5.6. **Crosses.** Allcock and Freitag construct a 10 dimensional space W of automorphic forms on the 4-ball ([AF], between 4.3 and 4.4) which defines the map $\mathcal{M} \hookrightarrow \mathbf{P}^9$. The vector space W is spanned by certain automorphic forms which, up to a scalar multiple, can be characterized by the fact that their divisors in the ball-quotient $\mathcal{M}$ are crosses ([AF], Theorem 4.6). A cross is defined to be a divisor

$$D_\alpha + D_\beta + D_\gamma + D_\delta + D_t$$

where t is a tritangent, defining a subroot system $t^\perp$ of type D_4 in E_6 (as in 1.8) and $\alpha, \ldots, \delta \in t^\perp \cap E_6^+$ are mutually perpendicular (cf. [AF], Definition 3.2). For each tritangent t, there are 3 crosses containing D_t, thus there are $45 \cdot 3 = 135$ crosses. For example, the crosses associated to $t = (16)$ have $\{\alpha, \ldots, \delta\}$ equal to one of the three sets:

$$\{h_{23}, h_{45}, h_{123}, h_{145}\}, \qquad \{h_{24}, h_{35}, h_{124}, h_{135}\}, \qquad \{h_{25}, h_{34}, h_{125}, h_{134}\}.$$

The following theorem identifies W with V (as spaces of global sections of a line bundle on $\mathcal{M}$).

5.7. **Theorem.** The rational map $\tilde{F}:\tilde{T}\longrightarrow \mathbf{P}^9$ defined by a basis of the vector space V defines a $W(E_6)$-equivariant morphism

$$F:\mathcal{C}\longrightarrow\mathcal{M}\ \subset\mathbf{P}^9$$

which blows down the 40 cusp divisors to the 40 cusps. The image of F is the moduli space $\mathcal{M}$ which is embedded into $\mathbf{P}^9$ via the map defined by Allcock and Freitag.

Proof. Using the results of [AF] and the $W(E_6)$-action on W and V, it suffices to show that there is a function $f\in V\cong H^0(\tilde{T}',\mathcal{O}(2D_R+D_S-3D_{\mathrm{w}}))$ such that the corresponding section has, modulo cusp divisors, a cross as zero divisor in $\tilde{T}'$. In fact, the exceptional divisors in the blow ups π' and π'' get blown down in the composition $\hat{T}\rightarrow\mathcal{C}\rightarrow\mathcal{M}$ and under push-pull via $\mathcal{M}\leftarrow\hat{T}\rightarrow\tilde{T}'$ crosses in $\mathcal{M}$ correspond to crosses in $\tilde{T}'$ and cusp divisors in $\tilde{T}'$ get contracted to points in $\mathcal{M}$.

Let $f=f_1$ in table 5.5, then the divisor in $\tilde{T}'$ of the corresponding function in V is:

$$\left(\frac{f_1}{\lambda\mu\nu\rho^2}\right)=4D_{\mathrm{w}}+D_{24}+D_{124}+D_{35}+D_{135}-D_S+D'$$

where D' is a divisor with support in D_R.

Thus the zero divisor on $\tilde{T}'$ of the section corresponding to f_1 is $D_{\mathrm{w}}+D_{24}+D_{124}+D_{35}+D_{135}+2D_R-D'$. Note that $\mathrm{w}=(16)$ and that the four roots h_{24}, h_{124}, h_{35}, h_{125} are in $D_4=(16)^{\perp}$ and are perpendicular. The remaining part, $2D_R-D'$, has support on cusp divisors.

We observe that using the explicit bases of V and the method of [AF] Corollary 7.3, one can also prove directly that F factors over $\mathcal{M}$ and embeds $\mathcal{M}$ into $\mathbf{P}^9$. □

5.8. **Cross ratios.** The basis of V given in 5.5 has the property that the quotients f_i/g_i are double ratios associated to tritangents (see the table 2 of [N]), and we have in fact one double ratio from each D_4-orbit:

$$r(\mathrm{w})=\frac{f_1}{g_1},\quad r(\bar{\mathrm{x}})=\frac{g_2}{f_2},\quad r(\bar{\mathrm{z}})=\frac{g_3}{f_3},\quad r(\bar{\mathrm{q}}_1)=\frac{g_4}{f_4},\quad r(\mathrm{y})=\frac{f_5}{g_5}.$$

(For completeness sake: $\mathrm{w}=(16)$, $\bar{\mathrm{x}}=(26)$, $\bar{\mathrm{z}}=(15)$, $\bar{\mathrm{q}}_1=(25)$, $\mathrm{y}=(16.23.45)$.) Note that the last factor in each function in 5.5 is the local equation of the associated tritangent.

The fact that we find one cross ratio from each D_4 orbit already implies that $\mathcal{C}$ is birationally isomorphic with $F(\mathcal{C})$ (use the argument of [N], § 5.5).

The involution $\gamma(t)\in W(E_6)$ associated to a tritangent t, see 1.10, has trace -6 on V (cf. [Fr], Table II), hence it has a 2 dimensional space of invariants V_t in V. There are, upto scalar multiple, 3 functions in V_t whose divisors are crosses (cf. [AF], Lemma 4.5). The pairs of functions f_i, g_i span such V_t's. The third function in $V_{(16)}$ is:

$$h_1:=f_1-g_1=\rho(\lambda-1)(\mu-1)(\nu-1)(\lambda\mu\nu\rho^2-1).$$

The stabilizer $W(F_4)$ of t acts on V_t through the action of a dihedral group with 12 elements; the subgroup $W(D_4)$ (generated by reflections in the long roots) acts a S_3 and the reflections in the short roots act as -1 on V_t. In fact, the elements σ_1, $\sigma_2\in W(F_4)$ given by Naruki in [N], §8, p. 16 act as -1 on $V_{(16)}$.

5.9. **Complex invariants.** In example 4.7 we considered the following function from V:

$$f = \nu^{-1}\rho^{-1}(\rho-1)(\lambda\nu\rho-1)(\mu\nu\rho-1).$$

Its divisor satisfies, modulo components with support in D_R:

$$(f) + D_S - 3D_{\mathrm{w}} = D_{16} + D_{34} + D_{25} + D_{125} + D_{256} + D_{136} + D_{146} + D_{234} + D_{345}.$$

The effective divisor on the right is the sum of the D_α where α runs over the positive roots of three mutually perpendicular A_2's:

$$\{h_{16},\ h_{125},\ h_{256}\}, \qquad \{h_{25},\ h_{234},\ h_{345}\}, \qquad \{h_{34},\ h_{136},\ h_{146}\}.$$

There are 40 such triples of orthogonal A_2's in E_6 which are permuted transitively by $W(E_6)$ ([H], 6.1.5.3; this particular triple is denoted by $[16, 25, 34]$). The corresponding 40 functions in V were considered by Coble who called them complex invariants (cf. [C], p. 340-341), see also [Y]. There are $80 = 2 \cdot 40$ functions in the $W(E_6)$-orbit of a complex invariant, the sign of a complex invariant is not well defined.

6. Images of divisors in $\mathcal{C}$

6.1. We can use Naruki's model $\mathcal{C}$ and the explicit basis of V to study the moduli space $\mathcal{M} \subset \mathbf{P}^9$. Here we consider various divisors in $\mathcal{M}$ as subvarieties of $\mathbf{P}^9$, in particular we find a nice parametrization of a tritangent divisor.

6.2. **The boundary divisors.** We consider the image in $\mathbf{P}^9$ of one of the 36 boundary divisors $D_\alpha \subset \mathcal{M}$ (1.6 and section 3). These parametrize cubic surfaces with at least one node. The divisor D_α is the fixed point set of the involution s_α. The trace of s_α on W is zero, hence W is the direct sum of two 5-dimensional eigenspaces of s_α. Since F is equivariant for $W(E_6)$, D_α will lie in a $\mathbf{P}^4$. The centralizer in $W(E_6)$ of the reflection s_α acts on the divisor D_α and on the eigenspaces of s_α. This subgroup is isomorphic to S_6. For example if $\alpha = h$, one obtains the 'standard' S_6 generated by all the s_i except s_2.

In particular we consider the image of $D_{345} = V(\epsilon_1)$ under F. This divisor is defined by $\lambda = 0$ on the open subset $A(\Delta_0) = Spec(\mathbf{C}[\lambda, \mu, \nu, \rho])$ of $\tilde{T}$. Since the 10 functions listed in table 5.5 are regular on $A(\Delta_0)$ and do not vanish simultaneously, we can simply take $\lambda = 0$ and determine (the closure of) the image. The image spans only a $\mathbf{P}^4$ since the following linear functions vanish on this divisor (in the notation of table 5.5):

$$f_1 - f_2, \qquad g_1 - g_2, \qquad f_3 - g_5, \qquad f_2 - f_4 - g_5, \qquad g_2 - g_3 - f_4 + g_4.$$

The image of $\mathcal{M}$ in $\mathbf{P}^9$ is defined by cubics (see [AF]), and one can show that the image of a boundary divisor is the Segre cubic hypersurface in this $\mathbf{P}^4$ (cf. [H], 3.2).

6.3. **The cusp divisors.** We consider one of the 40 cusp divisors in $\mathcal{C}$ (cf. 2.8), for example $V(\epsilon_1+\epsilon_2)$, note that $\epsilon_1+\epsilon_2 \in R$. This divisor is defined by $\rho = 0$ in $Spec(\mathbf{C}[\lambda, \mu, \nu, \rho])$. Putting $\rho = 0$ in the 10 functions in table 5.5 one finds that the image of this divisor is the point

$$(1 : 1 : 1 : 1 : 1 : 1 : 0 : 0 : 1 : 1).$$

6.4. **Tritangent divisors.** The tritangent divisor D_t is the fixed point set of the involution $\gamma(t) \in W(E_6)$. Each $\gamma(t)$ has trace -6 on V [Fr], hence it has two eigenspaces, of dimension 2 and 8, in V. Since the dimension of the divisor D_t is three we get $D_t \subset \mathbf{P}^7$.

The centralizer of $\gamma(t)$ is isomorphic to $W(F_4)$ and this group acts on both D_t and $\mathbf{P}^7$. We consider the case $t = (16) = \mathrm{w}$, hence D_t is birationally isomorphic to the exceptional fiber $\mathbf{P}^3_{\mathrm{w}}$ of the blow up of the torus T in the identity element e. Since e is fixed by $W(D_4)$, we get an induced action of $W(D_4)$ on $\mathbf{P}^3_{\mathrm{w}}$, and we will see that this action extends to a linear action of $W(F_4)$.

The root lattice $Q(F_4)$ of F_4 is the lattice in $\mathbf{R}^4$ generated by the 48 roots of F_4 which are (cf. [Hu], III 12.1) the 24 roots $\pm e_i \pm e_j$ of D_4 (these roots have length 2 and are called the long roots of F_4) and the 24 vectors $\pm e_i$, $(\pm e_1 \pm e_2 \pm e_3 \pm e_4)/2$ which have length 1, the short roots of F_4.

$$Q(F_4) = \langle \pm e_i \pm e_j,\ (\pm e_1 \pm e_2 \pm e_3 \pm e_4)/2 \rangle_{\mathbf{Z}} \qquad (\subset \mathbf{R}^4).$$

6.5. **Theorem.** Any tritangent divisor D_t is $W(F_4)$-equivariantly birationally isomorphic to $\mathbf{P}^3$ via the map

$$\mathbf{P}^3 = \mathbf{P}(Q(F_4) \otimes_{\mathbf{Z}} \mathbf{C}) \longrightarrow D_t \hookrightarrow \mathbf{P}^7$$

given by the linear system of cubics which are zero in the short roots of F_4.

Proof. Since $W(E_6)$ acts transitively on the tritangent divisors, it is sufficient to consider the case $t = (16)$. We show that the functions from V give the desired map $\mathbf{P}^3_{\mathrm{w}} \to D_{(16)}$.

The local coordinate functions $\lambda - 1, \ldots, \rho - 1$ near e induce projective coordinates $x_\lambda, \ldots, x_\rho$ on $\mathbf{P}^3_{\mathrm{w}}$. Since $\lambda = e_1 - e_2, \ldots, \rho = e_2 - e_3$ it is more convenient to use coordinates y_i with

$$(x_\lambda : x_\mu : x_\nu : x_\rho) = (y_1 - y_2 : y_3 + y_4 : y_3 - y_4, y_2 - y_3).$$

The group $W(F_4)$ is generated by the subgroup $W(D_4)$ and σ_1, σ_2 given in [N], §8. Using the explicit formulas for the σ_i one finds that these act on $\mathbf{P}^3_{\mathrm{w}}$ as reflection in the planes $y_4 = 0$ and $y_1 - y_2 - y_3 - y_4$ respectively. (For example σ_1 interchanges μ and ν and fixes the other roots, thus on $\mathbf{P}^3_{\mathrm{w}}$ it is the linear map which permutes $y_3 + y_4$ and $y_3 - y_4$ and fixes $y_1 - y_2$ and $y_2 - y_3$.) Thus these σ_i are reflections in the short roots. This implies that we may identify $\mathbf{P}^3_{\mathrm{w}}$ with $\mathbf{P}(Q(F_4) \otimes_{\mathbf{Z}} \mathbf{C})$.

All functions in V vanish to third order in e, but not all vanish to fourth order, hence restricted to $\mathbf{P}^3_{\mathrm{w}}$ the map F is given by the leading terms of third order. Note that f_1 and g_1 from table 5.5 vanish to order four at e, hence the image of $\mathbf{P}^3_{\mathrm{w}}$ spans at most a $\mathbf{P}^7$, as we observed earlier. The leading terms of the other 8 basis functions are cubics which all contain the 12 points:

$$(1:0:0:0)_y, \ldots, (0:0:0:1)_y \qquad \text{and} \quad (1:\pm 1:\pm 1:\pm 1)_y.$$

For example, f_2 from table 5.5 has leading term (up to sign):

$$(x_\mu + x_\rho)(x_\nu + x_\rho)(2x_\lambda + x_\mu + x_\nu + 2x_\rho) = (y_2 + y_4)(y_2 - y_4)(2y_1).$$

One can verify that the 8 leading cubics are independent and that these 12 points impose independent conditions on the (20 dimensional) space of cubics. Thus the map $\mathbf{P}^3_{\mathrm{w}} \to D_{(16)}$ is given by the subspace of cubics vanishing in these points and the image of $\mathbf{P}^3_{\mathrm{w}}$ spans a $\mathbf{P}^7$.

We observe that the twelve basepoints are in three D_4-orbits of length four. Two points in two distinct orbits determine a line on which there is a unique line from the third orbit. For

example the points $(0:0:0:1)_y$, $(1:1:1:1)_y$, $(1:1:1:-1)_y$ are on a line. In this way we get 16 lines, on each of these there are 3 base points. Actually, the two fourth-order leading terms are zero exactly on these 16 lines. $\square$

6.6. **Incidence of tritangent divisors.** The tritangent divisor $D_{(16)}$ is one of 45 such divisors, recall that $(16) = \mathrm{w} = \{a_1,\, b_6,\, c_{16}\}$. The remaining 44 tritangent divisors now divide into 4 groups, $44 = 4+4+4+2\cdot 16$ as follows. For each $l \in (16)$, there are 4 other tritangents containing l (for example, the $\{a_1,\, b_i,\, c_{1i}\}$ for $2 \le i \le 5$ are the other tritangents which also contain a_1).

The remaining 32 tritangents do not have a line in common with (16). These come in pairs as follows. Given one of these 32, say $\{l_1, l_2, l_3\}$, after a permutation of the indices one has that a_1 and l_1 meet (and a_1 does not meet l_2 and l_3) and thus there is a line m_1 such that $\{a_1, l_1, m_1\}$ is a tritangent. Similarly b_6 and l_2 determine a line m_2 and c_{16} and l_3 determine a line m_3. Now $\{m_1, m_2, m_3\}$ is another tritangent which has no line in common with (16). For example, $\{b_5, a_2, c_{25}\}$ determines $\{c_{15}, c_{26}, c_{34}\}$. (To see all this, consider a general cubic surface and the planes $V_{(16)}$ and V' spanned by the lines in (16) and the l_i respectively. These planes meet in a line which by assumption does not lie in the cubic surface. Thus this line meets the surface in 3 points and through each of these points there passes exactly one line from (16) and one line from the l_i.)

If the lines in (16) all pass through an Eckart point P_{w} and similarly the lines l_i all pass through an Eckart point P', the line L spanned by P_{w} and P' meets the cubic surface in a third point P'' which is an Eckart point, being the intersection of the m_i. (To see that each m_i passes through P'', consider the plane spanned by, say, a_1 and l_1; it cuts out m_1 and contains the line L, hence m_1 meets L in P'', similarly for the other pairs of lines.)

As a consequence, a point in the intersection of two tritangent divisors without a common line will lie in a third tritangent divisor.

6.7. **Tritangent divisors and $\mathbf{P}^3_{\mathrm{w}}$.** The intersections of $\mathbf{P}^3_{\mathrm{w}}$ with the other 44 tritangent divisors are given by the leading terms of their equations. Those tritangent divisors which have a line in common with $\mathrm{w} = \{a_1,\, b_6,\, c_{16}\}$ have a linear leading term, in fact one finds the following 12 linear terms:

$$y_i \quad (1 \le i \le 4), \qquad \text{and} \quad y_1 \pm y_2 \pm y_3 \pm y_3$$

where the last 8 come in two $W(D_4)$-orbits distinguished by the parity of the number of minus signs. For example, the tritangent $(15) = \bar{\mathrm{z}}$ defined by $\lambda\mu\nu^2\rho^2 - 1$ has leading term $y_1 + y_2 + y_3 - y_4$.

The tritangents which do not have a line in common with w have a leading term of degree two, in fact the two tritangents in a pair have the same leading term (as they should, see the last part of 6.6). These quadrics correspond to the 16 lines in $\mathbf{P}^3_{\mathrm{w}}$ containing 3 of the 12 base points of F (see 6.4); each line determines a unique quadric by the condition that it contains the other 9 base points (and this quadric will not contain any of the 3 points on the line). For example, the tritangent divisor $\bar{q}_1 = \{a_2, b_5, c_{25}\}$ is defined by $\lambda + \nu - \lambda\nu - \lambda\nu\rho - \lambda\mu\nu\rho + \lambda^2\mu\nu^2\rho^2$ has leading term

$$Q_{25} := y_1^2 + y_2 y_3 - y_2 y_4 - y_3 y_4$$

which does not contain the 3 colinear points $(1:0:0:0)_y$, $(1:-1:-1:1)_y$ and $(1:1:1:-1)_y$. The other tritangent divisor having the same leading term is $\bar{q}=\{c_{15},c_{26},c_{34}\}$ which is defined by $1-\lambda\nu\rho-\lambda\mu\nu\rho-\lambda\mu\nu\rho^2+\lambda^2\mu\nu\rho^2+\lambda\mu\nu^2\rho^2$.

The other 15 quadrics can be obtained from Q_{25} by the action of $W(D_4)$, that is, by permuting the coordinates and changing the signs of an even number of the y_i. These quadrics are smooth and hence are isomorphic to $\mathbf{P}^1\times\mathbf{P}^1$.

7. Equations for the moduli space.

7.1. The universal marked cubic surface is embedded in a $\mathbf{P}^3$-bundle over $\mathcal{C}$. Over the moduli space $\mathcal{M}^0$ of smooth marked cubic surfaces, this bundle is the projectivization of the tangent bundle ([ACT] § 10). In Naruki's paper [N] one finds an explicit cubic polynomial in $R[X,Y,Z,T]$, with $R:=\mathbf{C}[\lambda,\mu,\nu,\rho]$, which defines the universal family over an open part of $\mathcal{M}$. He also gives 45 linear forms in $R[X,Y,Z,T]$ which define the tritangent planes.

We will verify that there are linear relations between these, suitably normalized, linear forms with coefficients which are elements from V (note that elements from V are rational functions on T and thus are in the field of fractions of R). This allows us to recover the cubic equations found by Allcock and Freitag which define $\mathcal{M}$. We also find a six dimensional vector space of quintic polynomials, on which $W(E_6)$ acts via its standard representation, which define a variety $X\subset\mathbf{P}^9$ whose singular locus contains $\mathcal{M}$.

7.2. Consider two tritangents which contain a common line. For any point in the interior of $\mathcal{C}$, the corresponding planes are distinct. However over one of the 36 boundary divisors the planes may coincide. Over a boundary divisor D_α the 6 pairs of lines in the double six corresponding to α (cf. [H]) on the universal marked surface specialize to the six lines through the node of the universal surface over D_α. The reflection s_α in $W(E_6)$ interchanges the lines in each of the six pairs and fixes the other 15 lines. Thus if s_α maps one tritangent set to another, then the lines in the planes and thus the planes themselves will coincide over D_α.

7.3. **Lemma.** Let t_1, t_2 be two distinct tritangent sets which have a line in common. Then there are exactly two reflections in $W(E_6)$ which map t_1 to t_2. The corresponding roots in E_6 are perpendicular.

Proof. Since $W(E_6)$ acts transitively on the set of lines, we may assume that the common line is b_6. Then the t_i are of type $\{a_i,b_6,c_{i6}\}$ with $1\le i\le 5$ and applying a suitable element of $W(E_6)$ we may assume that $t_1=\{a_1,b_6,c_{16}\}$, $t_2=\{a_2,b_6,c_{26}\}$. By inspection of the lists of double sixes in [H] one finds exactly one double six which contains the pairs (a_1,a_2) and (c_{16},c_{26}) (it is N_{12}) and one which contains the pairs (a_1,c_{26}) and (c_{16},a_2) (it is N_{345}). Thus only reflections in h_{12} (which permutes the indices 1 and 2) and in h_{345} (which interchanges $a_1\leftrightarrow c_{26}$ and $a_2\leftrightarrow c_{16}$) permute these two tritangent sets. It is easy to verify that h_{12} and h_{345} are perpendicular. $\square$

7.4. Given three linear forms K, L, $M\in R[X,Y,Z,T]$ which define tritangent planes to the universal cubic surface having a line in common, there is a linear relation, with coefficients in R,

$$AK+BL+CM=0.$$

The next proposition shows that three tritangent planes with a line in common define three crosses. Recall that a cross is a divisor in $\mathcal{M}$ determined by the choice of a tritangent set t and on 4 perpendicular roots in $t^{\perp} \cong D_4$. In the example below we then verify that these crosses are the divisors of the coefficients in the linear relation.

7.5. **Proposition.** Let t_1, t_2 and t_3 be tritangent sets with a line in common. Then there are crosses X_i determined by the tritangent sets t_i, the pair of roots whose reflections interchange t_j and t_k (with $\{i, j, k\} = \{1, 2, 3\}$) and the pair of roots which is perpendicular to all the weights in the union of these three tritangent sets.

Proof. Again we use the $W(E_6)$ action, and so we may assume that $t_i = \{a_i, b_6, c_{i6}\}$. These span the subspace $\langle x_1, x_2, x_5, x_6\rangle$ ([H], 6.1.3) hence only the roots $h_{45} = -x_3 + x_4$ and $h_{145} = x_3 + x_4$ are perpendicular to this subspace. The two roots whose reflections interchange t_1 and t_2 are h_{12}, h_{345}. The roots h_{12}, h_{345}, h_{45} and h_{145} are orthogonal and lie in the D_4 perpendicular to the weights in t_3. Therefore there is a cross X_3 which is the sum of the tritangent divisor corresponding to t_3 and the four boundary divisors corresponding to these four roots. Similarly one finds crosses X_1 and X_2. ☐

7.6. **Example.** We consider the tritangents which contain the line b_6. They are:

set	label	local equation	linear form
$\{a_1, b_6, c_{16}\}$	$(16) = \mathrm{w}$	1	W
$\{a_2, b_6, c_{26}\}$	$(26) = \bar{\mathrm{x}}$	$\lambda^2\mu\nu\rho^2 - 1$	$\lambda X - (\lambda\rho - 1)(\lambda\mu\nu\rho - 1)W$
$\{a_3, b_6, c_{36}\}$	$(36) = \mathrm{x}$	$\mu\nu\rho^2 - 1$	$-X + (\rho - 1)(\mu\nu\rho - 1)W$
$\{a_4, b_6, c_{46}\}$	$(46) = x$	$-\rho(\mu\nu - 1)$	X
$\{a_5, b_6, c_{56}\}$	$(56) = \xi$	$\mu - \nu$	$X + \rho(\mu - 1)(\nu - 1)W$

The conversion of the labels is given in [Se], the equation of the planes is given in [N], Table 1, but we changed the sign of (36) and we multiplied the local equation of (46) by a unit.

We write $t_i := \{a_i, b_6, c_{16}\}$. Then $t_2 = s_1(t_2)$, $t_3 = s_3(t_2)$, $t_4 = s_4(t_3)$ and $t_5 = s_5(t_4)$ where s_i is the reflection in α_i. The two roots perpendicular to the span of the sets t_1, t_2 and t_3 are h_{45} and h_{123}. The cross X_1 is then:

$$X_1 = D_{23} + D_{145} + D_{45} + D_{123} + D_{(16)}$$

and $X_2 = s_1(X_1)$, $X_3 = s_3(X_2)$.

Note that X_1 is the divisor of the section corresponding to

$$A_1 = \rho(-1 + \lambda)(-1 + \mu)(-1 + \nu)(-1 + \lambda\mu\nu\rho^2)(\lambda\mu\nu\rho^2)^{-1} \qquad (\in V),$$

and that $A_1 = h_1$ in 5.8. Similarly we define $A_2 = s_1(A_1)$, $A_3 = s_3(A_2) \in V$.

We define $L_{i6} \in \mathbf{C}(\lambda, \dots, \rho)[X, W]$ to be the quotient of the linear form defining the tritangent plane $(i6)$ by the local equation of the tritangent divisor $D_{(i6)}$ as listed in the table. One can then verify the following linear relation:

$$A_1 L_{16} + A_2 L_{26} + A_3 L_{36} = 0.$$

7.7. **Proposition.** Let A_i and L_{ij} be as in Example 7.6. Define functions $B_i, \dots, F_i \in V$ by:

$$B_i = s_4(A_i), \quad C_i = s_3(B_i), \quad D_i = s_1(C_i), \quad E_i = s_5(D_i), \quad F_i = s_5(B_i).$$

Then we have $Mv = 0$ where

$$M = \begin{pmatrix} A_1 & A_2 & A_3 & 0 & 0 \\ B_1 & B_2 & 0 & B_3 & 0 \\ C_1 & 0 & -C_2 & C_3 & 0 \\ 0 & D_1 & D_2 & -D_3 & 0 \\ 0 & E_1 & E_2 & 0 & E_3 \\ F_1 & F_2 & 0 & 0 & -F_3 \end{pmatrix}, \qquad v = \begin{pmatrix} L_{16} \\ L_{26} \\ L_{36} \\ L_{46} \\ L_{56} \end{pmatrix}.$$

In particular, M has rank at most three.

Proof. Applying the reflection s_4 in $\alpha_4 = h_{34}$ (which permutes the indices 3 and 4) to the linear relation from Example 7.6 we obtain a relation between the linear forms defining the tritangents corresponding to $t_1 = s_4(t_1)$, $t_2 = s_4(t_2)$ and $t_4 = s_4(t_3)$. One verifies that this is $B_1L_{16} + B_2L_{26} + B_3L_{46} = 0$ with coefficients $B_i = s_4(A_i)$. Similarly, one verifies the other relations. Since each entry of v is of the form $a_iX + b_iW$ we see that $\ker(M)$ contains the two vectors $a = (a_1, \dots, a_5)$ and $b = (b_1, \dots, b_5)$. Thus the rank of M is at most $5 - 2 = 3$. $\square$

7.8. **Equations.** To obtain equations for $\mathcal{M} \subset \mathbf{P}^9$ from this proposition, one chooses a basis $X_0, \dots, X_9$ of V. Then each function in V is a linear form in the X_i with coefficients in $\mathbf{C}$. Thus each entry of the matrix M is a linear form in the X_i. Since the rank of M is at most 3, the determinant of each 4×4 submatrix of M, which is a degree 4 polynomial in the X_i, is identically zero as function on $\mathcal{M}$. Therefore each such determinant gives a, possibly trivial, quartic polynomial in the ideal of $\mathcal{M}$.

7.9. **Cubics.** To get cubic equations we consider the following submatrix of M:

$$N = \begin{pmatrix} A_1 & A_2 & A_3 & 0 \\ B_1 & B_2 & 0 & B_4 \\ C_1 & 0 & -C_3 & C_4 \end{pmatrix}.$$

The matrix N has rank at most two since $Nw = 0$, where $w = (L_{16}, \dots, L_{46})$, gives two vectors in $\ker N$ (put $X = 1$, $W = 0$ and $X = 0$, $W = 1$ in w). In particular,

$$\det \begin{pmatrix} A_2 & A_3 & 0 \\ B_2 & 0 & B_4 \\ 0 & C_3 & -C_4 \end{pmatrix} = -A_2B_4C_3 + A_3B_2C_4 = 0.$$

The corresponding cubic polynomial in the X_i is not identically zero in $\mathbf{C}[\dots, X_i, \dots]$ and is one of those found in [AF] Lemma 6.3. Theorem 6.4 of that paper implies that $\mathcal{M}$ is defined by the $W(E_6)$-orbit of this cubic equation.

7.10. **Quintics.** One verifies that the determinant of the following submatrix of M is a degree 5 polynomial in the X_i which is not identically zero:

$$M_2 = \begin{pmatrix} A_1 & A_2 & A_3 & 0 & 0 \\ C_1 & 0 & -C_3 & C_4 & 0 \\ 0 & D_2 & D_3 & -D_4 & 0 \\ 0 & E_2 & E_3 & 0 & E_5 \\ F_1 & F_2 & 0 & 0 & -F_5 \end{pmatrix}.$$

By Proposition 7.7 the rank of M_2 is at most 3. Therefore the determinant of any 4×4 submatrix of M_2 is zero on $\mathcal{M}$. Since the partial deriviatives of $\det(M)$ with respect to the X_i are linear combinations of determinants of such submatrices, we conclude that the quintic hypersurface X in $\mathbf{P}V$ defined by $\det(M)$ is singular along moduli space of marked cubic surfaces $\mathcal{M} \subset \mathbf{P}^9$.

Using the 10×5 matrix obtained from all $\binom{5}{3} = 10$ linear relations between 3 of the 5 tritangent planes containing the line b_6, we get $\binom{10}{5}$ quintics, but they are either 0 or the same as $\det(M)$ up to sign. It can be checked that the $W(E_6)$-orbit of such a quintic has 27 elements and that these quintics span a copy of the standard 6-dimensional representation 6_p of $W(E_6)$.

8. Hunt's quintic.

8.1. **Supercrosses.** We show how to construct 27 quintic polynomials, which we call supercrosses, on V which are permuted, up to sign, as the 27 lines on the cubic surface under the action of $W(E_6)$. We show that the supercrosses span a 6-dimensional vector space on which $W(E_6)$ acts as 6_n and that they define a rational map

$$\Sigma : \mathcal{M} \longrightarrow \mathbf{P}^5$$

which maps the moduli space onto the the unique $W(E_6)$-invariant hypersurface of degree 5 in $\mathbf{P}^5$. This hypersurface was investigated by Hunt in [H].

8.2. The line a_1 on a marked cubic surface defines a weight of E_6. The roots $\alpha_2, \dots, \alpha_6$ are perpendicular to this weight and span a root system, of type D_5, consisting of $2\cdot 20 = 40$ roots. In the notation of [H], this system is 'in standard form'

$$a_1^\perp = \{\pm x_j \pm x_k \,:\, 1 \le j < k \le 5\} \cong D_5.$$

Any line on a cubic surface lies in 5 tritangent planes. The tritangent planes containing a_1 are the $(1i) = \{a_1, b_j, c_{1j}\}$, $2 \le j \le 6$. The three weights corresponding to the three lines in a tritangent are linearly dependent, hence span a line, and the orthogonal complement of the line is a root system of type D_4, in fact $a_1 = -(2/3)x_6$, $b_j = x_{j-1} + (1/3)x_6$, thus

$$\{a_1, b_j, c_{1j}\}^\perp = \{\, x_{j-1},\, x_6\}^\perp = \{\pm x_i \pm x_k :\; i<k,\; i,\,k \in \{1, \dots, \widehat{j-1}, \dots, 5\}\,\} \cong D_4.$$

Now the main point is that the 20 positive roots which are perpendicular to a_1 split in 5 sets of 4 perpendicular roots such that each of the 5 sets is also perpendicular to the weights corresponding to the lines in a tritangent plane containing a_1. Thus each line l determines 5

crosses. In the notation of [H]:

$$\begin{array}{llll}
(12) = \langle a_1,\, b_2,\, c_{12}\rangle = \langle x_1,\, x_6\rangle & \{\pm x_2 + x_3,\ \pm x_4 + x_5\} = & \{h_{34},\, h_{56},\, h_{134},\, h_{156}\} \\
(13) = \langle a_1,\, b_3,\, c_{13}\rangle = \langle x_2,\, x_6\rangle & \{\pm x_1 + x_4,\ \pm x_3 + x_5\} = & \{h_{25},\, h_{46},\, h_{125},\, h_{146}\} \\
(14) = \langle a_1,\, b_4,\, c_{14}\rangle = \langle x_3,\, x_6\rangle & \{\pm x_1 + x_5,\ \pm x_2 + x_4\} = & \{h_{26},\, h_{35},\, h_{126},\, h_{135}\} \\
(15) = \langle a_1,\, b_5,\, c_{15}\rangle = \langle x_4,\, x_6\rangle & \{\pm x_1 + x_3,\ \pm x_2 + x_5\} = & \{h_{24},\, h_{36},\, h_{124},\, h_{136}\} \\
(16) = \langle a_1,\, b_6,\, c_{16}\rangle = \langle x_5,\, x_6\rangle & \{\pm x_1 + x_2,\ \pm x_3 + x_4\} = & \{h_{23},\, h_{45},\, h_{123},\, h_{145}\}
\end{array}$$

8.3. **The functions F_l.** To each cross corresponds a function, up to scalar multiple, in V. Fixing one such function and applying $W(E_6)$ we find other fuctions, unique up to sign, whose divisors are crosses. Fix a line l, then we can associate to it the function F_l, unique up to sign, which is the product of the 5 functions in V corresponding to the 5 crosses associated to l. The divisor of F_l is then essentially the sum of the 5 tritangent divisors D_t with $l \in t$ and the 20 boundary divisors D_α with $\alpha \in l^\perp \cap E_6^+$. If m is a line and $m = \sigma(l)$ for some $\sigma \in W(E_6)$, we define $F_m := \det(\sigma)\sigma(F_l)$ where $\det(\sigma)$ is the determinant of σ in the 6-dimensional reflection representation. The F_m's will be called a supercrosses, they are uniquely determined by F_l.

8.4. **Proposition.** The 27 functions F_l on Naruki's cross ratio variety span a 6 dimensional vector space. The Weylgroup $W(E_6)$ acts on this vector space as 6_n, the tensor product of the standard 6 dimensional representation with its determinant.

Proof. The functions F_l, with scalar factors suitably normalized, satisfy the linear relations $F_l \pm F_m \pm F_n = 0$ whenever the lines l, m, n are in a tritangent plane. From this one concludes that they span a space of dimension 6 on which $W(E_6)$ acts (the relations $F_{a_i} \pm F_{b_j} \pm F_{c_{ij}} = 0$ imply one can can express the $F_{c_{ij}}$ in terms of the F_{a_i} and F_{b_j}, now use the relations $F_{c_{ij}} \pm F_{c_{kl}} \pm F_{c_{mn}}$ to eliminate the F_{b_j}).

Since reflections in the stabilizer of an F_l act by as multiplication by -1 on F_l, the representation is the twist of the standard representation. □

8.5. The theorem provides us with a $W(E_6)$-equivariant rational map

$$\Sigma : \mathcal{M} \longrightarrow \mathbf{P}^5.$$

By computing the differential of Σ in some point of $\mathcal{M}$ we found that it has maximal rank. Hence the (closure of the) image of Σ is a $W(E_6)$-invariant hypersurface in $\mathbf{P}^5$.

8.6. **Theorem.** The hypersurface $\Sigma(\mathcal{M}) \subset \mathbf{P}^5$ is Hunt's quintic, the unique quintic hypersurface which is $W(E_6)$-invariant. It is defined by:

$$I_5 := \sum_l \lambda_l^5 = 0$$

where λ_l is the linear form on $\mathbf{P}^5$ defined by the E_6-weight which corresponds to the line l.

Proof. We will show that the following sextic relation holds:

$$\prod_{l \in A} F_l = \prod_{l \in B} F_l$$

where $A = \{a_1, \ldots, a_6\}$ and $B = \{b_1, \ldots, b_6\}$ form a double six of lines. As observed by Naruki (see [H], p.235), this equation is reducible, being the product of I_5 and a linear factor which

is the linear form defined by the root corresponding to the double six given by A and B. The $W(E_6)$-invariance of the image implies that the image is defined by I_5.

The divisors of both sides of the equation are the sum of the $6 \cdot 5 = 30$ tritangent divisors $D_{(ij)}$ as well as the sum of $6 \cdot 20 = 120$ boundary divisors. We already determined the positive roots in $a_1^\perp$ above, those in $b_1^\perp$ are:

$$b_1^\perp = \{h_{jk} = -x_{j-1} + x_{k-1} :\ 2 \leq j < k \leq 6\} \cup \{h_{pqr} :\ 2 \leq p < q < r \leq 6\ \}.$$

Thus each h_{ij} occurs 4 times whereas each h_{pqr} occurs 3 times in both the left and the right hand side, note that $4 \cdot 15 + 3 \cdot 20 = 120$). Thus, upto scalar multiple, the left and right hand side coincide. Using the reflection s in the root h (note $s(a_i) = b_i$) one finds the equality. □

8.7. Direct computations show that the images of the 36 divisors are 36 points in $\mathbf{P}^5$, these are the roots of E_6. The images of the 45 tritangent divisors are the 45 $\mathbf{P}^3$'s in Hunt's quintic (see the proof of the theorem below).

8.8. **Theorem.** The rational map

$$\Sigma : \mathcal{M} \longrightarrow I_5$$

has generic degree at least 10.

Proof. We verified by machine computation that Σ has maximal rank at the point $(\lambda, \mu, \nu, \rho) = (-1, -1, 2, 3) \in T$. This point lies in the intersection of the two tritangent divisors $(12) = \zeta$ defined by $\lambda = \mu$ and $(13) = z$ defined by $\lambda\mu = 1$ (cf. [N] Table 3). These tritangents have the line a_1 in common. Since Σ is $W(E_6)$-equivariant we conclude that Σ has maximal rank at the general point in the intersection of any two tritangent divisors with a line in common.

We consider the restriction of Σ to the intersection of the tritangent divisors $D_{\mathrm{w}} = D_{(16)}$ and $D_{(26)}$ which have the line b_6 in common. The divisor D_{w} is birationally isomorphic to $\mathbf{P}^3_{\mathrm{w}}$, the exceptional fiber of the blow up of T in e, and we consider the map induced by Σ on this $\mathbf{P}^3$. The local equation of $(26) = \bar{\mathrm{x}}$ is $\lambda^2\mu\nu\rho^2 = 1$ and its intersection with $\mathbf{P}^3_{\mathrm{w}}$ is given by $y_1 = 0$. (cf. 6.6, 6.7). Note that Σ has maximal rank in a general point of $\mathbf{P}^3_{\mathrm{w}} \cap (y_1 = 0)$.

On $\mathbf{P}^3_{\mathrm{w}}$ the leading terms of the F_l are of degree 15 or 16 (only for F_{a_1}, F_{b_6} and $F_{c_{16}}$), hence the restriction of Σ is given by homogeneous polynomials of degree 15 and the image of $\mathbf{P}^3_{\mathrm{w}}$ under Σ lies in the intersection of the hyperplanes defined by a_1, b_6 and c_{16} which is a $\mathbf{P}^3$. After omitting leading terms which are multiples of y_1 and dividing the remaining ones by their common factor $y_2y_3y_4$, we found that Σ restricts to $\mathbf{P}^3_{\mathrm{w}} \cap (y_1 = 0)$ to give a map

$$\Sigma_r : \mathbf{P}^2 \longrightarrow \mathbf{P}^2$$

defined by homogeneous polynomials of degree 12. One coordinate function is

$$F_2 := y_3y_4(y_3 - y_4)(y_3 + y_4)(y_2^2 - y_3y_4)^2(y_2^2 + y_3y_4)^2,$$

the other two, F_3 and F_4, are obtained by permuting the coordinates cyclically. All these functions satisfy

$$F(y_2, y_3, y_4) = -(y_2y_3y_4)^8 F(y_2^{-1}, y_3^{-1}, y_4^{-1})$$

hence the map Σ_r has degree at least 2.

The inverse image of a general point $(x_2 : x_3 : x_4) \in \mathbf{P}^2$ under Σ_r is defined by the two equations, each homogeneous of degree 12:

$$G_1 := x_3F_2 - x_2F_3 = 0, \qquad G_2 := x_4F_2 - x_2F_4 = 0.$$

The 0-cycle defined by these equations has degree $12^2 = 144$, but the linear system defined by the F_i has base points. Below we list the base points and their contribution to the intersection multiplicities (determined with computer). Here ω is a primitive cube root of unity.

$$\begin{array}{ll} (0:0:1), \quad (0:1:0), \quad (1:0:0), & m_P = 20, \\ (0:1:\pm 1), \quad (1:0:\pm 1), \quad (1:\pm 1:0), & m_P = 1, \\ (1:1:-1), \quad (1:-1:1), \quad (-1:1:1), & m_P = 9, \\ (1:1:1), & m_P = 9, \\ (1:\omega:\pm\omega^2), \quad (1:\omega^2:\pm\omega), \quad (1:-\omega:\pm\omega^2), \quad (1:-\omega^2:\pm\omega) & m_P = 4. \end{array}$$

Thus we find that the base points contribute

$$3\cdot 20 + 6\cdot 1 + 3\cdot 9 + 1\cdot 9 + 8\cdot 4 = 134$$

to the intersection, so there remain 10 points unaccounted for. Since Σ has maximal rank in a general point of this $\mathbf{P}^2$, we conclude that the degree of Σ is at least 10. $\square$

9. Tables.

The following tables identify the 36 positive roots of E_6, in the notation of Hunt [H], with the 12 positive D_4 roots, in the notation of Naruki [N], and 24 D_4-weights. We also list the functions f_α on T corresponding to the positive roots $\alpha \in D_4$.

9.1.

roots of D_4	f_α	roots of E_6	roots of D_4	f_α	roots of E_6
$e_1 - e_2$	λ	$h_{23} = -x_1 + x_2$	$e_1 + e_2$	$\lambda\mu\nu\rho^2$	$h_{145} = x_3 + x_4$
$e_1 - e_3$	$\lambda\rho$	$h_{24} = -x_1 + x_3$	$e_1 + e_3$	$\lambda\mu\nu\rho$	$h_{135} = x_2 + x_4$
$e_1 - e_4$	$\lambda\nu\rho$	$h_{25} = -x_1 + x_4$	$e_1 + e_4$	$\lambda\mu\rho$	$h_{134} = x_2 + x_3$
$e_2 - e_3$	ρ	$h_{34} = -x_2 + x_3$	$e_2 + e_3$	$\mu\nu\rho$	$h_{125} = x_1 + x_4$
$e_2 - e_4$	$\nu\rho$	$h_{35} = -x_2 + x_4$	$e_2 + e_4$	$\mu\rho$	$h_{124} = x_1 + x_3$
$e_3 - e_4$	ν	$h_{45} = -x_3 + x_4$	$e_3 + e_4$	μ	$h_{123} = x_1 + x_2$

9.2.

D_4−weight	E_6−root	D_4−weight	E_6−root	D_4−weight	E_6−root
ϵ_1	h_{345}	$(\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4)/2$	h_{16}	$(-\epsilon_1-\epsilon_2-\epsilon_3+\epsilon_4)/2$	h_{56}
ϵ_2	h_{245}	$(\epsilon_1+\epsilon_2-\epsilon_3-\epsilon_4)/2$	h_{236}	$(-\epsilon_1-\epsilon_2+\epsilon_3-\epsilon_4)/2$	h_{46}
ϵ_3	h_{235}	$(\epsilon_1-\epsilon_2+\epsilon_3-\epsilon_4)/2$	h_{246}	$(-\epsilon_1+\epsilon_2-\epsilon_3-\epsilon_4)/2$	h_{36}
ϵ_4	h_{234}	$(\epsilon_1-\epsilon_2-\epsilon_3+\epsilon_4)/2$	h_{256}	$(\epsilon_1-\epsilon_2-\epsilon_3-\epsilon_4)/2$	h_{26}
$-\epsilon_1$	h_{12}	$(-\epsilon_1+\epsilon_2+\epsilon_3-\epsilon_4)/2$	h_{346}	$(-\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4)/2$	h_{126}
$-\epsilon_2$	h_{13}	$(-\epsilon_1+\epsilon_2-\epsilon_3+\epsilon_4)/2$	h_{356}	$(\epsilon_1-\epsilon_2+\epsilon_3+\epsilon_4)/2$	h_{136}
$-\epsilon_3$	h_{14}	$(-\epsilon_1-\epsilon_2+\epsilon_3+\epsilon_4)/2$	h_{456}	$(\epsilon_1+\epsilon_2-\epsilon_3+\epsilon_4)/2$	h_{146}
$-\epsilon_4$	h_{15}	$(-\epsilon_1-\epsilon_2-\epsilon_3-\epsilon_4)/2$	h	$(\epsilon_1+\epsilon_2+\epsilon_3-\epsilon_4)/2$	h_{156}

9.3. $W(E_6)$**-representations.** In the notation of Frame [Fr], the (unique) 10 dimensional representation V of $W(E_6)$ is denoted by 10_s. One has:

$$\begin{array}{rcl} Sym^2(10_s) & = & 1 + 15_m + 15_q + 24_p, \\ Sym^3(10_s) & = & 20_s + 2\cdot 30_m + 2\cdot 30_p + 80_s, \\ Sym^4(10_s) & = & 2\cdot 1 + 1_n + 3\cdot 15_m + 4\cdot 15_q + 20_p + 20_s + \ldots\,, \\ Sym^5(10_s) & = & 2\cdot 6_p + 2\cdot 6_n + 15_p + 15_q + 7\cdot 30_m + 7\cdot 30_p + \ldots\,, \\ Sym^6(10_s) & = & 5\cdot 1 + 3\cdot 1_n + 11\cdot 15_m + 14\cdot 15_q + \ldots\,, \end{array}$$

here 6_p is the standard 6-dimensional representation and 6_n is the tensor product of 6_p with its determinant. On $\mathbf{P}^5$ the representations 6_p and 6_n are the same. In particular, there are two 1-dimensional families of 6-dimensional representations in S^5V.

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