AN ISOGENY OF $K3$ SURFACES

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Abstract. In a recent paper Ahlgren, Ono and Penniston described the L-series of $K3$ surfaces from a certain one parameter family in terms of those of a particular family of elliptic curves. The Tate conjecture predicts the existence of a correspondence between these $K3$ surfaces and certain Kummer surfaces related to these elliptic curves. A geometric construction of this correspondence is given here, using results of D. Morrison on Nikulin involutions.

1. The family

1.1. Recently, Ahlgren, Ono and Penniston [AOP] studied the $K3$ surfaces X_t which are the minimal resolutions of double covers of $\mathbf{P}^2$ branched over a union of 6 lines (hence over a sextic curve):

$$X_t: \quad y^2 = xz(x+1)(z+1)(x+zt).$$

Using an elaborate but elementary calculation with character sums, they determined the zeta function of $X_t/\mathbf{F}_p$. One way of interpreting their result is as follows.

For general $t \in \mathbf{Q}$ the Néron-Severi group of X_t has rank 19 (cf. Lemma 2.3 below). Hence there is an isomorphism of $G_{\mathbf{Q}} = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ representations:

$$H^2_{\text{ét}}(X_{t,\overline{\mathbf{Q}}}, \mathbf{Q}_\ell) \cong T_{t,\ell} \oplus \mathbf{Q}_\ell(-1)^{19}$$

for some ℓ-adic representation $T_{t,\ell}$ of dimension 3.

Consider the elliptic curve E_t and its quadratic twist $E_t^{(t+1)}$:

$$E_t: \quad y^2 = (x-1)(x^2 - \tfrac{1}{t+1}), \qquad E_t^{(t+1)}: \quad (t+1)y^2 = (x-1)(x^2 - \tfrac{1}{t+1}).$$

The Kummer surface $\mathrm{Km}(E_t \times E_t^{(t+1)})$ is by definition the smooth surface obtained by blowing up the 16 double points of the quotient $(E_t \times E_t^{(t+1)})/([-1]\times[-1])$. This Kummer surface is also a $K3$ surface. Since $E_t^{(t+1)}$ is a quadratic twist of E_t, we obtain another 3-dimensional $G_{\mathbf{Q}}$-representation

$$\mathrm{Sym}^2\left(H^1_{\text{ét}}(E_{t,\overline{\mathbf{Q}}}, \mathbf{Q}_\ell)\right)(\chi^{(t+1)}) \qquad (\subset H^2_{\text{ét}}(\mathrm{Km}(E_t \times E_t^{(t+1)})_{\overline{\mathbf{Q}}}, \mathbf{Q}_\ell)).$$

Here $\chi^{(t+1)}$ is the Dirichlet character of the quadratic extension $\mathbf{Q}(\sqrt{t+1})/\mathbf{Q}$ if $t+1$ is not a square in $\mathbf{Q}$, else it is trivial.

Proposition (Ahlgren, Ono, Penniston). *With notations as above, the two Galois representations $T_{t,\ell}$ and $\mathrm{Sym}^2\left(H^1_{\text{ét}}(E_{t,\overline{\mathbf{Q}}}, \mathbf{Q}_\ell)\right)(\chi^{(t+1)})$ are isomorphic.*

This isomorphism produces, via the Künneth formula and Poincaré duality, a Galois invariant class in $H^4_{\text{ét}}(X_{t,\overline{\mathbf{Q}}} \times \mathrm{Km}(E_t \times E_t^{(t+1)})_{\overline{\mathbf{Q}}}, \mathbf{Q}_\ell)$. The Tate conjecture asserts that for a variety Z, defined over $\mathbf{Q}$, the subspace of Galois invariants in $H^{2p}_{\text{ét}}(Z_{\overline{\mathbf{Q}}}, \mathbf{Q}_\ell)(p)$ is spanned by classes of codimension p cycles defined over $\mathbf{Q}$. Combined with the proposition above this suggested our main result:

1.2. **Theorem.** *For $t \in \mathbf{Q}$ there exists a correspondence*

$$\Gamma_t \ \subset \ X_t \times Km(E_t \times E_t^{(t+1)}),$$

defined over $\mathbf{Q}$, which induces an isomorphism of $G_{\mathbf{Q}}$-representations:

$$[\Gamma_t] : T_{t,\ell} \ \stackrel{\cong}{\longrightarrow} \ Sym^2\left(H^1_{\text{ét}}(E_{t,\overline{\mathbf{Q}}}, \mathbf{Q}_\ell)\right)(\chi^{(t+1)}).$$

In Remark 4.4 of the paper [AOP], the authors suggest finding a dominant rational map from X_t to $K_t = Km(E_t \times E_t)$. This is actually possible, but only over a finite extension of $\mathbf{Q}(t)$, and we do produce such a map.

1.3. **Proposition.** *Let K be a field of characteristic $\neq 2$ and take $t \neq 0, -1$ in K. Then there exists a dominant rational map from X_t to $K_t = Km(E_t \times E_t)$ over a finite extension of K.*

1.4. Such a geometric relation (at least, over the complex numbers) between the two families of $K3$ surfaces can also be shown to exist using their Picard-Fuchs differential equations. This has been worked out by Ling Long [L].

We now briefly outline the general facts we used and the strategy we followed to obtain our result.

1.5. **General results.** The general X_t has a Néron-Severi group $NS(X_t)$ of rank 19, and thus its transcendental lattice $T = NS(X_t)^\perp$ has rank 3, we computed (cf. Lemma 2.3) that

$$T \cong \langle 2 \rangle \oplus \langle 2 \rangle \oplus \langle -2 \rangle.$$

Recall that $\mathrm{Km}(A)$, the Kummer surface of an abelian surface A, is the $K3$ surface obtained by blowing up the 16 singular points of the quotient of A by the involution $a \mapsto -a$. A $K3$ surface S with rank $NS(S) = 19$ is a Kummer surface if and only if the (even) quadratic form $q : T_S = NS(S)^\perp \to 2\mathbf{Z}$ obtained from the intersection product on $H^2(S, \mathbf{Z})$ has values in $4\mathbf{Z}$ (cf. [Mo, Prop. 4.3]). In particular, the general X_t is not a Kummer surface.

The transcendental lattice T_S of any $K3$ surface S of rank 19 embeds into U^3 ([Mo, Cor. 2.6], where U is the hyperbolic plane ($\mathbf{Z}^2$ with quadratic form $q(x) = 2x_1x_2$). This gives an embedding of T_S in the $K3$ lattice $U^3 \oplus E_8(-1)^2$ which is unique up to isometry ([Mo, Cor. 2.10]). The Néron-Severi group $NS(X)$ of X thus contains $E_8(-1)^2$. Theorem 5.7 of [Mo] now implies that X has a Nikulin involution ι (that is, an involution which acts trivially on $H^{2,0}(X)$). The involution has 8 fixed points, blowing them up and taking the quotient we get a $K3$ surface V with $T_V \cong T(2)$ ([Mo, Thm. 5.7ii]). Hence V is a Kummer surface. The corresponding abelian surface A has transcendental lattice $T_A \cong T$ ([Mo, Prop. 4.3]). The following diagram summarizes the situation, it is called a Shioda-Inose structure for X.

$$\begin{array}{ccccc} X & & & & A \\ & \searrow & & \swarrow & \\ & & X/\iota \cong \mathrm{Km}(A) & & \end{array}$$

1.6. **Summary.** For the general X_t it is rather easy to find a sublattice $E_8(-1)^2$ of $NS(X_t)$, see section 3.1. Since the Nikulin involution exchanges the two copies of $E_8(-1)$ and we have an interpretation of the simple roots as nodal curves on the surface, we can make an educated guess as to what the involution should be. In section 3.2 we give the involution explicitly and we determine the quotient $K3$ surface V_t. In section 4, we show that V_t is isomorphic to a double cover W_t of $\mathbf{P}^2$ branched along 6 lines which are tangent to a conic. This shows that $V_t \cong \mathrm{Km}(JC_t)$ where JC_t is the Jacobian of the genus two curve C_t which is the double cover of the conic branched in the 6 points of tangency of the lines to the conic. The abelian surface JC_t is isogenous to a product of two elliptic curves $F_t \times F'_t$ which are quadratic twists of E_t (section 4.5). A main problem is that most isogenies and isomorphisms are not defined over $\mathbf{Q}$ (or $\mathbf{Q}(t)$). The varieties involved do have models over $\mathbf{Q}$, but one has to choose the right one (or twist a given one) so as to have a non-trivial correspondence defined over $\mathbf{Q}$. We conclude with some observations on the 'famous' $K3$ surface X_{-1}.

1.7. **Previous work.** In the literature, several results comparable to Proposition 1.3 can be found. However, we are not aware of any cases except the present one where an explanation is given how such isogenies may be constructed. We mention some examples here. Note that they are older than Morrison's paper which provides the basic technique for our construction. It may be interesting to study whether Long's method mentioned above can be used in the following examples as well to predict the existence of the isogenies involved.

In 1977, M. Mizukami [Mi] showed that the Kummer surface $\mathrm{Km}(E'_t \times E'_t)$ is isogenous to the $K3$ surface X'_t, for $t \neq \pm 1$ in $\mathbf{C}$, where

$$X'_t \;:\quad x_1^4 + x_2^4 + x_3^4 + x_4^4 + 2t(x_1^2x_2^2 + x_3^2x_4^2) = 0$$

and

$$E'_t \;:\quad y^2 = (x^2+1)(x^2+(1-t)/2).$$

This is proven by explicitly giving a rational $4:1$ map from $E'_t \times E'_t$ to X'_t.

Similarly, W. Hoyt [Hoyt] in 1984 presents an explicit rational dominant map from the product $E''_t \times E''_t$, where

$$E''_t \;:\quad y^2 = x^3 - (12-9t)x + 16 - 18t,$$

to the $K3$ surface X''_t corresponding to the equation

$$s^2 = x(x-1)(x-t)y(y-1)(y-x).$$

2. The $K3$ surfaces X_t

2.1. **Singularities of the branch curve.** For $t \neq 0$, the branch curve of the double cover defining X_t consists of 6 lines (including the line at infinity), from now on we assume $t \neq 0$.

For $t \neq -1$ these lines meet in 6 double points and 3 triple points. To obtain the corresponding $K3$ surface, one blows up the double and triple points. Over a triple point, one must next blow up the three intersection points of the strict transforms of the three lines and the exceptional divisor. We denote by E_P the inverse images in the $K3$ surface X_t of the strict transform of the fibre of the first blow up in P. Furthermore, by $E_P^{l=0}$ we denote the inverse image of the exceptional divisor over the point of intersection of E_P and the strict transform of the line $l = 0$. For symmetry reasons we write the point with coordinates $(x : z : 1) \in \mathbf{P}^2$ as $(x : z : -1)$, thus the exceptional divisor over the double point $(x, z) = (-1, 0)$ is denoted by E_{101}. All these curves, as well as the inverse images of the lines which make up the sextic (we denote these simply by $l = 0$ as in $\mathbf{P}^2$), are smooth rational curves, hence (-2)-curves, in the $K3$ surface. See [AOP], p. 363, figure 1, for a picture of the intersection graph of these -2-curves.

In the special case $t = -1$ there are 3 double points and 4 triple points. The $6+3+4\cdot 4 = 25$ rational curves in X_{-1} with self intersection -2 obtained as above are denoted in the same way.

2.2. **The case $t = -1$.** It is shown in [P, p. 298] (see also the proof of 2.3) that the $K3$ surface X_{-1} has transcendental lattice T_{-1} of rank 2 and discriminant 4, hence T_{-1} must be

$$T_{-1} = \langle 2\rangle \oplus \langle 2\rangle.$$

Vinberg [V] studied the (unique) $K3$ surface with this transcendental lattice and observed ([V, 2.1]) that its Picard lattice is isomorphic to the sublattice of $\mathbf{Z}^{20}$ (with quadratic form $x_1^2 - \sum_{i=2}^{20} x_i^2$) given by the vectors x with $\sum x_i \equiv 0 \pmod 2$. Shioda and Inose [SI] showed that X_{-1} is the desingularisation of the quotient of E_i^2, the self product of the elliptic curve $E_i = \mathbf{C}/\mathbf{Z}[i]$, by the automorphism ϕ of order 4 induced by $(z_1, z_2) \mapsto (iz_1, -iz_2)$ on $\mathbf{C}^2$, see also Section 5.

The following lemma is not used in the proof of the main result, but it does show that X_t is not a Kummer surface, hence we cannot avoid the Nikulin involution.

2.3. **Lemma.** *The Néron-Severi group of the general X_t has rank 19 and it is generated by nodal curves defined over $\mathbf{Q}(t)$. The transcendental lattice T of the general X_t is given by*

$$T \cong \langle 2\rangle \oplus \langle 2\rangle \oplus \langle -2\rangle.$$

Proof. First we consider the special case $t = -1$. The sublattice of $H^2(X_{-1}, \mathbf{Z})$ generated by the following 20 nodal curves: $E_{011}, E_{010}^{x=0}, E_{111}, E_{111}^{x=-1}$ and the 16 curves which span the two copies of E_8 given in section 3.1 has rank 20 and determinant -4, as can be verified by a computation with their intersection matrix. Hence the determinant of the transcendental lattice T_{-1} is either 4 or 1, but since T is positive definite and even, $det(T_{-1}) = 4$ (and in fact $T_{-1} \cong \langle 2\rangle \oplus \langle 2\rangle$). It follows that the 20 curves are a $\mathbf{Z}$-basis of $NS(X_{-1})$. According to Nikulin, [N, Thm. 1.14.4], the embedding of T in the $K3$ lattice $U^3 \oplus E_8(-1)^2$ is unique up to isometry. We fix such an embedding and identify $NS(X_{-1})$ with $T^\perp$.

Since the $K3$ surfaces X_t depend on one parameter, $NS(X_t)$ has rank at most 19 for general t. For $t \neq -1$, there are three points $p_a = (-1, t^{-1}), p_b = (t, -1), p_c = (-1, -1)$ which specialize to the triple point $(x, z) = (-1, -1)$ as $t \to -1$. It is easy to check that the lattice spanned by the 24 nodal curves in X_t obtained from the desingularization of the branch locus is isomorphic to the sublattice of $NS(X_{-1})$ spanned by the corresponding curves in case $t = -1$ and where E_a maps to $E_{111}^{x=-1} + E_{111} + E_{111}^{x=-zt}$ et cetera (so the exceptional curve over the intersection of two lines $l = 0$ and $m = 0$ maps to the sum of the curves $E_{111}^{l=0} + E_{111} + E_{111}^{m=0}$). Now it is easy to see that the image of $NS(X_t)$ in $NS(X_{-1})$ is contained in the orthogonal complement L of E_{111}. Hence L is a primitive sublattice and it is actually generated by the 24 nodal curves (in fact, the 16 curves in the copies of E_8 and $E_{011}, E_{010}^{x=0}, E_c$ are a $\mathbf{Z}$-basis of L). Therefore $NS(X_t) \cong L$ for general t. A computation shows that $L \cong E_8(-1)^2 \oplus \langle -2 \rangle^2 \oplus \langle 2 \rangle$ and that $L^\perp$ is isomorphic to $\langle 2 \rangle \oplus \langle 2 \rangle \oplus \langle -2 \rangle$. □

3. The isogeny

3.1. **The sublattice $E_8(-1)^2$.** It is not hard to identify a sublattice of $NS(X_t)$ which is isomorphic to $E_8(-1)^2$. One copy of $E_8(-1)$ is spanned by the following eight -2-curves on X_t:

$$\begin{array}{ccccccccccccc} (x=0) & -- & E_{001}^{x=0} & -- & E_{001} & -- & E_{001}^{z=0} & -- & (z=0) & -- & E_{101} & -- & (x=-1) \\ & & & & | & & & & & & & & \\ & & & & E_{001}^{x=-zt} & & & & & & & & \end{array}$$

another copy of $E_8(-1)$, perpendicular to this one, is given by:

$$\begin{array}{ccccccccccccc} E_{010} & -- & E_{010}^{l_\infty} & -- & (l_\infty) & -- & E_{100}^{l_\infty} & -- & E_{100} & -- & E_{100}^{z=-1} & -- & (z=-1) \\ & & & & | & & & & & & & & \\ & & & & E_{110} & & & & & & & & \end{array}$$

By considering the effect of the symmetry on the nodal curves one is led to the following expression for the Nikulin involution.

3.2. **The Nikulin involution.** The pair of E_8's in $NS(X_t)$ described in 3.1 defines a Nikulin involution ι on X_t as in [Mo]. It is given by:

$$\iota = \iota_t : X_t \longrightarrow X_t, \qquad \iota(x, z, y) = \big(1/z, 1/x, -y/(x^2 z^2)\big)$$

(the minus sign assures that ι has only isolated fixed points).

The invariants under the action of ι in the function field of the surface X_t, are generated by

$$\xi_1 = x/z, \quad \xi_2 = x + 1/z \quad \text{ and } \quad \eta = y(xz-1)z^{-3}.$$

The desingularization of X_t/ι is a $K3$ surface denoted by V_t:

$$V_t : \qquad \eta^2 = \xi_1(\xi_1 + t)(\xi_1 + \xi_2 + 1)(\xi_2^2 - 4\xi_1).$$

Note that $\xi_1(\xi_1+t)(\xi_1+\xi_2+1)(\xi_2^2-4\xi_1)$ pulls back to $xz(x+1)(z+1)(x+zt)$ times $\big((xz-1)/z^3\big)^2$. As the isogeny is defined over $\mathbf{Q}(t)$ we obtain:

3.3. **Lemma.** *The desingularisation of the surface X_t/ι is the $K3$ surface V_t. The graph of the rational map $X_t \to V_t$ defines a correspondence, defined over $\mathbf{Q}(t)$, which induces an isomorphism on the transcendental parts of $H^2_{\acute{e}t}$.*

3.4. **An alternative description of the Nikulin involution.** Using an elliptic fibration on X_t given in [AOP], one obtains the following way to describe the involution ι.

Consider the map

$$\pi:\ X_t \cdots \to \mathbf{P}^1 \quad (x,y,z) \mapsto \alpha := \frac{y}{z(x+tz)}.$$

This map in fact defines a morphism. The fibre over a general point $\alpha \in \mathbf{P}^1$ is the genus 1 curve D_α with equation

$$\alpha^2 z(x+zt) = x(x+1)(z+1).$$

Using the change of coordinates

$$\xi := t\alpha^2/x, \qquad \eta := (\xi + t\alpha^2)/z$$

one obtains for D_α the equation

$$\eta^2 + (1-\alpha^2)\xi\eta + t\alpha^2\eta = \xi^3 + t\alpha^2\xi^2.$$

In this way, $\pi:\ X_t \to \mathbf{P}^1$ is the elliptic surface $\pi:\ D_\alpha \to \mathbf{P}^1$ corresponding to $(\xi,\eta,\alpha) \mapsto \alpha$. Note that the fibre of this surface over α is the same as the fibre over $-\alpha$.

Let P_1 be the section of this surface over $\mathbf{P}^1$ given by $P_1(\alpha) = (0,0,\alpha)$. The Nikulin involution ι is then described as

$$\iota(\xi,\eta,\alpha) = (P_1(\alpha) - (\xi,\eta), -\alpha)\,,$$

where $P_1(\alpha) - (\xi,\eta)$ is interpreted in terms of the group law on the elliptic curve D_α.

3.5. **The branch locus of V_t.** The branch locus of V_t consists of four lines (including the line at infinity) and a conic. The line $\xi_1 + t = 0$ meets the conic transversely in two points, conjugate over the field $\mathbf{Q}(\sqrt{-t})$, whereas the other three lines are tangent to the conic and all contain the (triple) point $(0,1,0)$. Blowing up the singular points of the branch curve (in a point of tangency one must blow up twice, in the triple point four times (see 2.1)), one obtains a rational surface such that V_t is the double cover of this surface branched over six disjoint smooth rational curves (the strict transforms of the five irreducible components of the branch curve and the rational curve which maps to the triple point). In the next section we will see that one can blow down this rational surface to $\mathbf{P}^2$ in such a way that the images of these 6 rational curves are lines which are tangent to a conic.

3.6. **Remark.** From 2.3 and [Mo] we then have, for general t, that

$$T_{V_t} \cong T(2) = \langle 4\rangle \oplus \langle 4\rangle \oplus \langle -4\rangle.$$

This implies that the general V_t is not isomorphic to the Kummer surface of a product of two elliptic curves (consider the transcendental lattices!). It is not hard to check that for any elliptic curve E there is a subgroup $H \subset E \times E$, $H \cong (\mathbf{Z}/2\mathbf{Z})^2$ such that $(E\times E)/H$ has transcendental lattice $\langle 2\rangle \oplus \langle 2\rangle \oplus \langle -2\rangle = T$, hence the transcendental lattice of the Kummer variety of $(E\times E)/H$ is $T(2)$. We will not use this result explicitly since it does not guarantee the existence of a correspondence over $\mathbf{Q}(t)$.

3.7. **Five fold symmetry for $t=-1$.** It is amusing to observe that in the case $t=-1$ one finds 25 nodal curves on the $K3$ surface X_t which form a configuration already described by Vinberg.

In case $t=-1$, the 6 lines in $\mathbf{P}^2$, the 3 exceptional divisors over the double points $((1:1:0)$, $(1:0:1)$ and $(0:1:1))$ and the $4\cdot 4=16$ curves over the 4 triple points give a configuration of 25 -2-curves on X_{-1}. The graph of this configuration (vertices correspond to the nodal curves, edges are between vertices for which the corresponding nodal curves intersect) is given in [VK], p.195, figure 2, it has an obvious 5-fold symmetry! The vertices in that figure are numbered from 1 to 27 with exception of the numbers 19 and 24, the corresponding nodal curves can be chosen as: $1=E_{100}$, $4=E_{111}^{z=-1}$, $5=E_{111}$, $6=E_{111}^{x=-1}$, $9=(z=0)$, $11=E_{001}$, $15=(l_\infty)$, $17=E_{010}$, $23=(x=0)$, from this it is easy to find the curves corresponding to the other vertices.

Note that the two copies of E_8 given in 3.1 are exchanged by the symmetry of the graph given by reflection in the vertical axis (which contains the vertices 5, 27, 13 and 18). A similar symmetry exists for general t and is induced by the Nikulin involution.

4. V_t as a Kummer surface

4.1. **Lemma.** *Let W_t be the $K3$ surface defined by:*

$$W_t:\quad t(t+1)\eta_2^2=(\xi_7^2+t\xi_8^2)(4\xi_7-4t\xi_8-t-1)\left((\xi_7-2t\xi_8-t)^2+t(\xi_8+1)^2\right).$$

There exists an isomorphism

$$\phi: V_t\ \xrightarrow{\cong}\ W_t$$

which is defined over $\mathbf{Q}(t,\sqrt{-t})$. Let $\phi': V_t\to W_t$ be the $\mathrm{Gal}(\mathbf{Q}(\sqrt{-t})/\mathbf{Q}(t))$-conjugate of ϕ, and let Γ_ϕ, $\Gamma_{\phi'}$ be their graphs in $V_t\times W_t$. Then the correspondence $\Gamma_\phi+\Gamma_{\phi'}$, which is defined over $\mathbf{Q}$, induces an isomorphism between the part of $H^2_{\acute{e}t}(V_t)$ orthogonal to the 19 algebraically independent cycle classes and the corresponding part of $H^2_{\acute{e}t}(W_t)$.

Proof. To prove this, regard V_t as a double cover of the plane, with (affine) equation

$$\eta^2=\xi_1(\xi_1+t)(\xi_1+\xi_2+1)(\xi_2^2-4\xi_1).$$

We will explicitly describe two Cremona transformations of the plane whose composition induces the desired isomorphism ϕ.

The ramification locus consists of 4 lines (including the line at infinity) and a conic; note that 3 of these lines (the lines $\xi_1=0$, $\xi_1+\xi_2+1=0$ and the line at infinity) are tangent to the conic.

We first apply the Cremona transformation which blows up these three points of tangency and blows down the three lines connecting them. In explicit (affine) coordinates, this map can be described by

$$(\xi_1,\xi_2)\longmapsto(\xi_3,\xi_4):=\left(\xi_1(\xi_2+2)/(\xi_2^2-4\xi_1),(\xi_2+2\xi_1)/(\xi_2^2-4\xi_1)\right).$$

It transforms the three lines tangent to the conic and the conic itself into four lines, the remaining line (given by $\xi_1 + t = 0$) is mapped onto a conic. One computes (the factors below correspond to the equations of the resulting lines and conic)

$$\eta_1^2 = \xi_3\xi_4(\xi_3 + \xi_4 + 1)(2\xi_3^2 + 2t\xi_4^2 + \xi_3 + t\xi_4),$$

where $\eta_1 := \eta\xi_2(\xi_2 + 2)(2\xi_1 + \xi_2)(\xi_2^2 - 4\xi_1)^{-3}$.

In the coordinates ξ_3, ξ_4, η_1, this surface is again described as a double cover of the plane ramified over a conic and 4 lines, one of which is the line at infinity. Two of the lines intersect in the point $(\xi_3, \xi_4) = (0, 0)$ which is on the conic (hence the configuration has one triple point), the other intersection points are ordinary double points.

Next, apply the Cremona transformation whose base points are this triple point $(0, 0)$, the point $(-t/(t+1), -1/(t+1))$ in the intersection of the line $\xi_3 + \xi_4 + 1 = 0$ and the conic, and a (nonrational) point $(s, 1, 0)$ where the line at infinity and the conic intersect (note that $s^2 = -t$). This transformation has the property that each of the 5 components of the branch locus has a line as image.

Explicitly, this second transformation can be given as

$$(\xi_3, \xi_4) \mapsto (\xi_5, \xi_6) := \left((s-1)\frac{\xi_3^2 - s^3\xi_4^2 + (s^2 - s)\xi_3\xi_4}{\xi_3 + s^2\xi_4}, (s^2 + s)\frac{(s - s^2)\xi_4^2 + (s-1)\xi_3\xi_4 + s\xi_4}{\xi_3 + s^2\xi_4} \right).$$

It lifts to a birational map from our surface to the one given by

$$\begin{gathered} t(t+1)\eta_2^2 = \\ \xi_5\xi_6\left((1+s)\xi_5 - s\xi_6 + s^2 + s\right)\left((1-s)\xi_6 + s\xi_5 + s^2 - s\right)\left((2+2s)\xi_5 + (2-2s)\xi_6 + s^2 - 1\right), \end{gathered}$$

with $\eta_2 = \eta_1(1-s)(\xi_3 - s\xi_4)\left((st + s)\xi_4 - (t+1)\xi_3 - t + s\right)(\xi_3 - t\xi_4)^{-2}$.

Finally, put

$$\xi_7 := (\xi_5 + \xi_6)/2, \quad \xi_8 := (\xi_5 - \xi_6)/2s \qquad \text{so} \quad \xi_5 = \xi_7 + s\xi_8, \quad \xi_6 = \xi_7 - s\xi_8.$$

With these coordinates, the equation is

$$t(t+1)\eta_2^2 = (\xi_7^2 + t\xi_8^2)(4\xi_7 - 4t\xi_8 - t - 1)\left((\xi_7 - 2t\xi_8 - t)^2 + t(\xi_8 + 1)^2\right),$$

thus it defines a $K3$ surface, W_t, which is defined over $\mathbf{Q}(t)$. The composition of the birational maps described so far yields the isomorphism $\phi : V_t \xrightarrow{\cong} W_t$, defined over $\mathbf{Q}(\sqrt{-t})$. Let ϕ' be the conjugate isomorphism (defined by the same formulas as ϕ but with $-s$ for s). A generator of $H^{2,0}(W_t)$ is given in local coordinates by the regular 2-form $\omega_W := \mathrm{d}\xi_7 \wedge \mathrm{d}\xi_8/\eta_2$. A direct calculation shows that

$$\phi^*\omega_W + (\phi')^*\omega_W \;=\; \mathrm{d}\xi_1 \wedge \mathrm{d}\xi_2/\eta \neq 0.$$

Hence the correspondence on $V_t \times W_t$, defined over $\mathbf{Q}$, which is the sum of the graphs $\Gamma_\phi + \Gamma_{\phi'}$ defines a nonzero map $H^{2,0}(W_t) \to H^{2,0}(V_t)$. Thus it must induce an isomorphism on the transcendental lattices of W_t and V_t. The comparison theorem for complex and ℓ-adic cohomology implies that the same is true for the corresponding Galois representations.

This proves the lemma. $\square$

4.2. **The $K3$ surface W_t.** The branch curve of the double cover $W_t \to \mathbf{P}^2$ as described in Lemma 4.1 consists of 6 lines (defined over $\mathbf{Q}(t, \sqrt{-t})$), including the line at infinity. The smooth conic defined by $4\xi_7 + 4t\xi_8^2 - 1 = 0$ is tangent to each of these lines. In particular, W_t is the Kummer surface of the Jacobian of the genus two curve C_t which is the double cover of the conic branched over the 6 points of tangency with the lines (see [Beau, Exc. VIII.6], [C-F, §3.10]). We briefly recall some of these classical results.

4.3. **Kummer surfaces and genus 2 curves.** Let K be a field of characteristic $\neq 2$. Suppose C/K is given by $y^2 = f(x)$ for some separable polynomial $f \in K[x]$ of degree 5 or 6. Over some extension field of K we write $f(x) = \prod(x - a_j)$.

The Jacobian JC of C is birational to the symmetric product $(C \times C)/S_2$, hence its function field is the subfield of $K(x_1, x_2, y_1, y_2)$ (with the relations $y_i^2 = f(x_i)$) of elements fixed under the involution σ given by $\sigma(x_1) = x_2$ and $\sigma(y_1) = y_2$. The Kummer surface $\mathrm{Km}(JC)$ is birational to the quotient of JC under the $[-1]$-map, hence its function field is the subfield of $K(x_1, x_2, y_1, y_2)$ of elements fixed under the two involutions σ and ι, with $\iota(x_i) = x_i$ and $\iota(y_i) = -y_i$.

The latter subfield is generated over K by the functions $\eta := y_1 y_2$ and $\xi := x_1 x_2$ and $\zeta := x_1 + x_2$. They satisfy a relation

$$\eta^2 = F(\xi, \zeta)$$

with F the unique polynomial such that $f(x_1)f(x_2) = F(x_1x_2, x_1 + x_2)$.

Observe that over an extension of K one has

$$f(x_1)f(x_2) = \prod((x_1 - a_j)(x_2 - a_j)) = \prod(\xi - a_j\zeta + a_j^2).$$

Hence one concludes that $\mathrm{Km}(JC)$ is birational over K to a double cover of the plane, ramified over six lines (including the line at infinity in the case that the degree of f is 5). The points $P_j := (\xi = a_j^2, \zeta = 2a_j)$ correspond to the pairs of Weierstrass points $(T, T) \in C \times C$, with $T = (a_j, 0) \in C$. Note that the P_j are also on the conic defined by $\zeta^2 = 4\xi$, and in fact the line $\xi - a_j\zeta + a_j^2 = 0$ is tangent to this conic in P_j. The same is true for the line at infinity (the point of tangency comes from the point at infinity on C in the case where the degree of f is 5). This shows that seen as a double cover of the plane, $\mathrm{Km}(JC)$ is ramified over six lines which are tangent to a given conic. The inverse image of this conic is given by the two equations $\zeta^2 = 4\xi$ and $\eta^2 = \prod(\zeta/2 - a_j)^2$. Hence it consists of two irreducible components, both defined over K. Moreover, we can recover the Weierstrass points of C (and hence C itself up to a quadratic twist) from the six points of tangency of the lines with the conic.

4.4. **Lemma.** *The $K3$ surface W_t studied in 4.1 and 4.2 is isomorphic to $\mathrm{Km}(JC_t)$, where the genus two curve C_t is defined by*

$$C_t: \quad y^2 = x(x^2 - 4x + 4 + 4t)(x^2 + 4x + 4 + 4t).$$

This isomorphism is defined over $\mathbf{Q}(t, \sqrt{-t}, \sqrt{-t-1})$.

Given an equation $\eta^2 = F(\xi, \zeta)$ for $\mathrm{Km}(JC_t)$ as above, let $\mathrm{Km}(JC_t)^{(-t-1)}$ be the 'twist' defined by $(-t-1)\eta^2 = F(\xi, \zeta)$.

Then there is a correspondence on $W_t \times \mathrm{Km}(JC_t)^{(-t-1)}$, defined over $\mathbf{Q}(t)$, which induces an isomorphism of $G_{\mathbf{Q}(t)}$-representations between the transcendental parts of the $H^2_{ét}$'s.

Proof. As before, write $s^2 = -t$. We will use new coordinates to describe W_t, namely ξ_9 and ξ_{10} given by

$$\xi_8 = \frac{\xi_{10}}{8s} - \frac{1}{2}$$

and

$$\xi_7 = (\xi_9 - 2s\xi_{10} - 4t + 4)/16.$$

In these coordinates, the conic $4\xi_7 + 4t\xi_8^2 - 1 = 0$ becomes $\xi_{10}^2 = 4\xi_9$. The 6 lines over which $W_t \to \mathbb{P}^2$ ramifies become the line at infinity and five lines $\xi_9 - b_j\xi_{10} + b_j^2 = 0$, with

$$\{b_1, b_2, b_3, b_4, b_5\} \;=\; \{0, 2+2s, 2-2s, -2+2s, -2-2s\}.$$

The equation for W_t in the new coordinates is

$$W_t: \quad 2^{18}t(t+1)\eta_2^2 = \prod_{j=1}^{5}(\xi_9 - b_j\xi_{10} + b_j^2),$$

which is in fact an equation over $\mathbf{Q}(t)$.

The discussion in 4.3 above shows that provided we have a square root of $t(t+1)$ available, this defines a birational model of the Kummer surface $\mathrm{Km}(JC_t)$ where C_t is the hyperelliptic curve with Weierstrass points over infinity and over the b_j's, so the equation defining C_t is the one given in the lemma:

$$y^2 = \prod_{j=1}^{5}(x - b_j) = x(x^2 - 4x + 4 + 4t)(x^2 + 4x + 4 + 4t).$$

To show the second part, put

$$\mathrm{Km}(JC_t)^{(-t-1)}: \quad (-t-1)\eta^2 = \prod_{j=1}^{5}(\xi - b_j\zeta + b_j^2).$$

A birational map $\psi : \mathrm{Km}(JC_t)^{(-t-1)} \to W_t$ is given by

$$\psi(\eta, \xi, \zeta) := \left(\eta_2 = \frac{2^{-9}\eta}{s}, \xi_7 = \frac{(\xi - 2s\zeta - 4t + 4)}{16}, \xi_8 = \frac{\zeta}{8s} - \frac{1}{2}\right).$$

One obtains the 'conjugate' ψ' by replacing all occurrences of s by $-s$ in this description. A direct calculation reveals that

$$\psi^* \frac{\mathrm{d}\xi_7 \wedge \mathrm{d}\xi_8}{\eta_2} = \psi'^* \frac{\mathrm{d}\xi_7 \wedge \mathrm{d}\xi_8}{\eta_2} = 4\frac{\mathrm{d}\xi \wedge \mathrm{d}\zeta}{\eta},$$

from which the lemma follows by the same argument as in the proof of Lemma 4.1. $\square$

4.5. **The product of elliptic curves.** The curve C_t has, besides the hyperelliptic involution, another involution:

$$\varphi = \varphi_t : C_t \longrightarrow C_t, \qquad \varphi(x,y) := (r^2/x, r^3y/x^3) \qquad (r^2 = 4+4t).$$

The quotient by this involution is an elliptic curve. In fact, the invariant functions on C_t are generated by $\eta := y(x+r)/x^2$ and $\xi := -x/(2r) - r/(2x)$ and the quotient curve F_t is defined by

$$F_t := C_t/\varphi : \quad \eta^2 = -8r^3(\xi - 1)(\xi^2 - \tfrac{1}{t+1}).$$

Replacing r by $-r$ yields yet another involution (namely, the composition of the previous one and the hyperelliptic involution τ) and hence a second elliptic curve

$$F_t' := C_t/(\varphi \circ \tau) : \quad \eta^2 = 8r^3(\xi - 1)(\xi^2 - \tfrac{1}{t+1}).$$

By considering the pull back to C_t of the invariant differentials on these elliptic curves one concludes that JC_t is isogenous (over $\mathbf{Q}(t,r) = \mathbf{Q}(t, \sqrt{t+1})$) to the product $F_t \times F_t'$ of these two elliptic curves. In explicit form, this isogeny is obtained from the two quotient maps $\alpha : C_t \to F_t$ and $\alpha' : C_t \to F_t'$ using

$$C_t \times C_t \longrightarrow F_t \times F_t' \quad (P,Q) \longmapsto (\alpha(P) + \alpha(Q), \alpha'(P) + \alpha'(Q)).$$

The associated Kummer surfaces are isogenous (again, over $\mathbf{Q}(t,r)$ and not necessarily over $\mathbf{Q}(t)$!) as well. It is easily seen that the Kummer surface of $F_t \times F_t'$ is birational over $\mathbf{Q}(t,r)$ to the surface with equation

$$-y^2 = (x_1 - 1)(x_1^2 - \tfrac{1}{t+1})(x_2 - 1)(x_2^2 - \tfrac{1}{t+1}).$$

By twisting, this also gives a rational map, defined over $\mathbf{Q}(t,r)$, from $\mathrm{Km}(JC_t)^{(-t-1)}$ to the surface defined by $-(-t-1)y^2 = (x_1-1)(x_1^2 - \frac{1}{t+1})(x_2-1)(x_2^2 - \frac{1}{t+1})$. Note that the latter equation in fact defines the Kummer surface of $E_t \times E_t^{(t+1)}$ over $\mathbf{Q}(t)$.

Now we show that this rational map together with its $\mathbf{Q}(t,r)/\mathbf{Q}(t)$-conjugate yields a correspondence defined over $\mathbf{Q}(t)$ between $\mathrm{Km}(JC_t)^{(-t-1)}$ and $\mathrm{Km}(E_t \times E_t^{(t+1)})$ with the property that it is nonzero on the transcendental part of $H^2_{\text{ét}}$'s.

4.6. **Genus 2 curves with non-simple Jacobians.** Suppose k is a field of characteristic $\neq 2$. Let $f \in k[x]$ be a separable polynomial of degree 5 and $C: \; y^2 = f(x)$. The regular differentials on C form a k-vector space with a basis $\frac{dx}{y}$, $x\frac{dx}{y}$. Assume that for $i = 1,2$ an elliptic curve E_i over k is given, with a nonzero regular differential ω_i on E_i and a morphism

$$\alpha_i : \; C \longrightarrow E_i$$

having the property that $\alpha_1^*\omega_1$ and $\alpha_2^*\omega_2$ are linearly independent. Moreover we assume that α_i sends the point at infinity on C to the zero on E_i. Write $\alpha_i^*\omega_i = (a_ix + b_i)\frac{dx}{y}$. The independence of the pull backs can be phrased by saying that

$$d := a_1b_2 - a_2b_1 \neq 0.$$

Consider the commutative diagram of rational maps

$$\begin{array}{ccc} C\times C & \cdots\cdots\longrightarrow & \mathrm{Km}(JC) \\ \big\downarrow\psi & & \big\downarrow \\ E_1\times E_2 & \cdots\cdots\longrightarrow & \mathrm{Km}(E_1\times E_2). \end{array}$$

Here ψ is the morphism $\psi:\ (P,Q)\mapsto(\alpha_1(P)+\alpha_1(Q),\alpha_2(P)+\alpha_2(Q))$.

Note that $\omega_1\wedge\omega_2$ can be regarded both as a regular 2-form on $\mathrm{Km}(E_1\times E_2)$ and as the regular 2-form on $E_1\times E_2$ obtained as the pull back of the one on the Kummer. One computes that

$$\psi^*(\omega_1\wedge\omega_2)=d(x_1-x_2)\frac{\mathrm{d}x_1\wedge\mathrm{d}x_2}{y_1y_2}$$

using coordinates x_1,y_1,x_2,y_2 on $C\times C$ which satisfy $y_i^2=f(x_i)$.

As is explained in (4.3) above, one has coordinates $\eta=y_1y_2$ and $\xi=x_1x_2$ and $\zeta=x_1+x_2$ on $\mathrm{Km}(JC)$. The regular 2-form $\eta^{-1}\mathrm{d}\xi\wedge\mathrm{d}\zeta$ on $\mathrm{Km}(JC)$ pulls back under the horizontal rational map at the top of the diagram above to $(y_1y_2)^{-1}\mathrm{d}(x_1x_2)\wedge\mathrm{d}(x_1+x_2)=(x_2-x_1)(y_1y_2)^{-1}\mathrm{d}x_1\wedge\mathrm{d}x_2$.

Combining the above pull backs, one concludes that using the vertical arrow on the right of our diagram, $\omega_1\wedge\omega_2$ pulls back to $-d\eta^{-1}\mathrm{d}\xi\wedge\mathrm{d}\zeta$ on $\mathrm{Km}(JC)$.

4.7. We now apply this to the situation described in (4.5). Here we have

$$\rho:\ \mathrm{Km}(JC_t)\longrightarrow\mathrm{Km}(F_t\times F_t')\cong V_t$$

where V_t is defined by $-y^2=(x_1-1)(x_1^2-\frac{1}{t+1})(x_2-1)(x_2^2-\frac{1}{t+1})$. The surface $\mathrm{Km}(F_t\times F_t')$ corresponds to the equation $-\tilde{y}^2=64r^6(x_1-1)(x_1^2-\frac{1}{t+1})(x_2-1)(x_2^2-\frac{1}{t+1})$. An isomorphism between V_t and this surface is described by $\tilde{y}=8r^3y$. Hence the 2-form $y^{-1}\mathrm{d}x_1\wedge\mathrm{d}x_2$ on V_t pulls back to $(8r^3\tilde{y})^{-1}\mathrm{d}x_1\wedge\mathrm{d}x_2$ on $\mathrm{Km}(F_t\times F_t')$.

Since $\alpha^*\frac{\mathrm{d}\xi}{\eta}=\left(\frac{rx}{8t+8}+\frac{1}{2}\right)\frac{\mathrm{d}x}{y}$ and $\alpha'^*\frac{\mathrm{d}\xi}{\eta}=\left(\frac{-rx}{8t+8}+\frac{1}{2}\right)\frac{\mathrm{d}x}{y}$, it follows that

$$d=\frac{r}{8t+8}.$$

Hence, the pull back of $\frac{\mathrm{d}x_1\wedge\mathrm{d}x_2}{8r^3\tilde{y}}$ to $\mathrm{Km}(JC_t)$ is $-\frac{\mathrm{d}\xi\wedge\mathrm{d}\zeta}{(16t+16)^2\eta}$.

One concludes that

$$\rho^*\left(\frac{\mathrm{d}x_1\wedge\mathrm{d}x_2}{y}\right)=-\frac{\mathrm{d}\xi\wedge\mathrm{d}\zeta}{(16t+16)^2\eta}.$$

Denoting by ρ' the $\mathrm{Gal}(\mathbf{Q}(t,r)/\mathbf{Q}(t))$-conjugate of ρ, it follows as before that the sum of the graphs $\Gamma_\rho+\Gamma_{\rho'}$ defines a correspondence over $\mathbf{Q}(t)$ which is nonzero on the transcendental parts of $H^2_{\text{ét}}$'s.

Twisting all surfaces over $\mathbf{Q}(t,\sqrt{-t-1})$ one obtains the same conclusion for the surfaces $\mathrm{Km}(JC_t)^{(-t-1)}$ and $\mathrm{Km}(E_t\times E_t^{(t+1)})$, hence we proved:

4.8. **Lemma.** There is a correspondence on $\mathrm{Km}(JC_t)^{(-t-1)}\times Km(E_t\times E_t^{(t+1)})$, defined over $\mathbf{Q}(t)$, which induces an isomorphism of $G_{\mathbf{Q}(t)}$-representations between the transcendental parts of the $H^2_{\text{ét}}$'s.

4.9. **Conclusion.** Putting together the various correspondences we have constructed, one obtains the desired correspondence defined over $\mathbf{Q}(t)$ on the product of X_t and $\mathrm{Km}(E_t \times E_t^{(t+1)})$. This finishes the proof of Theorem 1.2.

The maps we constructed, over finite extensions of $\mathbf{Q}(t)$, compose (eventually after a further field extension to undo twists) to give a dominant rational map:

$$X_t \longrightarrow V_t \longrightarrow W_t \longrightarrow K(JC_t) \longrightarrow K(E_t \times E_t),$$

hence also Proposition 1.3 follows.

5. The fibre at $t = -1$

5.1. We conclude this paper with some remarks on various models of the famous $K3$ surface X_{-1}. As observed in 2.2, the $K3$ surface X_{-1} is the desingularisation of the quotient of $E_i \times E_i$, the self product of the elliptic curve $E_i = \mathbf{C}/\mathbf{Z}[i]$, by the automorphism ϕ of order 4 induced by $(z_1, z_2) \mapsto (iz_1, -iz_2)$ on $\mathbf{C}^2$, [SI]. Below we show how to obtain this isomorphism directly from the equations defining X_{-1} and E_i. It is convenient to use projective coordinates $(u : v : w) = (x : z : -1)$, so the equation for X_{-1} becomes:

$$X_{-1} : \qquad \sigma^2 = uvw(u-w)(v-w)(u-v).$$

5.2. The elliptic curve E_i is isomorphic to $E:\ t^2 = s(s^2-1)$ and also to $E':\ y^2 = x(1-x^2)$, hence $E_i \times E_i \cong E \times E'$ and ϕ may be given by $\phi((s,t),(x,y)) = ((-s, it), (-x, -iy))$. The quotient map $E \times E' \to X_{-1}$ is given by

$$(\sigma : u : v : w) = (xs(xs+1)(x+s)ty :\ xs^2 - x :\ xs^2 + s :\ xs^2 + s^2x)$$

in fact, a direct calculation shows that

$$uvw(u-v)(u-w)(v-w) = (xs(xs+1)(x+s))^2 x(1-x^2)s(s^2-1).$$

This map was found from the results below.

5.3. **Vinberg's model.** The surface X_{-1} has a projective model Y which is a singular quartic surface in $\mathbf{P}^3$ (see [V], Theorem 2.5, but we replaced X_0 there by ζX_0 for a $\zeta \in k$ with $\zeta^4 = -1$):

$$Y : \quad X_0^4 = X_1X_2X_3(X_1 + X_2 + X_3).$$

The elliptic curve E_i is isomorphic to $E:\ t^2 = s^4 - 1$ and also to $E':\ y^2 = x^4 + 1$, hence $E_i \times E_i \cong E \times E'$ and ϕ may be given by $\phi((s,t),(x,y)) = ((is, t), (-ix, y))$. The quotient map $E \times E' \to Y$ is given by

$$(X_0 : X_1 : X_2 : X_3) = (sx : y - 1 : 1 + t : 1 - t),$$

it is easy to see that this map has degree 4 and is invariant under ϕ. This map was found by studying the pencil of curves on Y defined by $X_3 = \lambda X_2$.

5.4. An isomorphism $X_{-1} \to Y$ is given by

$$(X_0 : X_1 : X_2 : X_3) = (\sigma : vw(v-w) : -uw(u-w) : uv(u-v)).$$

Note that $X_1 + X_2 + X_3 = (u-v)(u-w)(v-w)$ and thus the equation for Y pulls back to $\sigma^4 = (uvw(u-v)(u-w)(v-w))^2$.

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