

# A COMPLEX BALL UNIFORMIZATION OF THE MODULI SPACE OF CUBIC SURFACES VIA PERIODS OF $K3$ SURFACES

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**ABSTRACT.** In this paper we show that the moduli space of nodal cubic surfaces is isomorphic to a quotient of a 4-dimensional complex ball by an arithmetic subgroup of the unitary group. This complex ball uniformization uses the periods of certain  $K3$  surfaces which are naturally associated to cubic surfaces. A similar uniformization is given for the covers of the moduli space corresponding to geometric markings of the Picard group or to the choice of a line on the surface. We also give a detailed description of the boundary components corresponding to singular surfaces.

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## 1. INTRODUCTION

There are two main approaches to the construction of moduli spaces in algebraic geometry. One uses geometric invariant theory which allows one to construct the moduli space as a quotient of an open subset of an appropriate Hilbert scheme, the other one uses period maps to construct the moduli space as a quotient of an open subset of a Hermitian symmetric homogeneous domain by a discrete subgroup of its group of holomorphic automorphisms. Both approaches suggest a way to compactify the moduli space. In the algebraic approach one adds the equivalence classes of semi-stable points. In the transcendental approach one considers the whole domain together with its boundary.

There are several remarkable cases where both approaches work. Comparing the constructions gives a beautiful interplay between the algebraic theory of invariants and the theory of automorphic functions. The historically first example of such an interplay is of course the moduli space of elliptic curves which, on one hand, is the quotient of the space of binary forms of degree 4 by the group  $SL(2)$  and, on the other hand, is a natural quotient of the upper half-plane by the modular group. Similarly, binary forms of degree 5, 6, 8 and 12 give the moduli spaces of Del Pezzo surfaces of degree 4, and hyperelliptic curves of genus 2, 3 and 5, respectively. Using the theory of hypergeometric

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functions one can show that the corresponding domains are complex balls of dimension 2, 3, 5 and 9, respectively. Increasing the number of variables by one, one finds the ternary cubic forms which leads again to the moduli space of elliptic curves, the forms of degree 4 corresponding to the moduli space of non-hyperelliptic curves of genus 3 (in this case the domain is the Siegel upper half space of degree 3) and the forms of degree 6 corresponding to  $K3$  surfaces with degree 2 polarization (the domain is of type IV in Cartan's classification).

Using domains of type IV one can also give a uniformization of the moduli space of cubic and quartic forms in 4 variables. The case of forms of degree 3 (cubic surfaces) was treated in the work of K. Matsumoto, T. Sasaki and M. Yoshida [MSY], and degree 4 ( $K3$  surfaces with degree 4 polarization) much earlier by J. Shah [Sha]. Although cubic surfaces do not admit non-zero holomorphic 2-forms, so that the periods are not defined, there are identifications of this moduli space with other moduli spaces for which the period map is defined. In [MSY] one uses the moduli space of  $K3$  surfaces which have a certain primitive sublattice of rank 16 in the Picard group. Such a surface can be realized as a double cover of  $\mathbb{P}^2$  branched along the union of 6 lines in a general position. The blow-up of the dual set of 6 points in  $\mathbb{P}^2$  is a nonsingular cubic surface. Recent work of D. Allcock, J. Carlson and D. Toledo [ACT] gives a different uniformization of the moduli space of cubic surfaces where the domain of type IV is replaced by a complex ball. This ball quotient is the moduli space of principally polarized abelian varieties of dimension 5 with complex multiplication in the Eisenstein ring  $\mathbb{Z}[\zeta_3]$ . Each such variety can be realized as the intermediate Jacobian of the triple cyclic cover of  $\mathbb{P}^3$  branched over a nonsingular cubic surface. Independently this construction was found by the second author and B. Hunt. Subsequently, Allcock and Freitag [AF] found modular forms on the ball quotient which embed it into a nine dimensional projective space. Freitag [F] later proved that the ideal of the image is defined by cubic polynomials and that the quotient ring is the full ring of modular forms. The image variety turns out to be isomorphic to a compactification of the moduli space of marked cubic surfaces.

A similar approach works for Del Pezzo surfaces of degree 2 and 1 which can be realized as surfaces of degree 4 and 6 in weighted projective spaces  $\mathbb{P}(1, 1, 1, 3)$  and  $\mathbb{P}(1, 1, 2, 3)$ , respectively (see also [HL] for another approach to a complex ball uniformization of the moduli space of Del Pezzo surfaces of degree 1). All of this is based on the existence of an embedding of a complex ball into a Siegel domain. It is also known that a complex ball can be embedded into a type IV domain. For example a moduli space of lattice polarized  $K3$  surfaces admitting an automorphism of order 3 or 4 which acts non-trivially on the lattice of transcendental cycles is parametrized by an arithmetical quotient of an open subset of a complex ball. This observation was used by the third author [Ko1] and independently by the second author (unpublished) to construct a complex ball uniformization of the moduli space of Del Pezzo surfaces of degree 2. This moduli space is isomorphic to the moduli space of non-hyperelliptic curves of genus 3 via the map which associates to a Del Pezzo surface the fixed curve of the Geizer involution. The  $K3$  surface associated to such a surface is its double cover branched along this fixed curve. In [Ko2] a similar description of the moduli spaces of curves of genus 4 and of Del Pezzo surfaces of degree 1 is given.

In this paper we give a similar construction for the moduli space of cubic surfaces. To each stable cubic surface  $S$  we associate a  $K3$  surface  $X_S$  with an automorphism of order 3. Its periods are parametrized by a complex 4-ball and we do in fact recover most of the results from [ACT]. Our construction is also closely related to the work of K. Matsumoto and T. Terasoma [MT] who associate to a line on a cubic surface a certain curve  $C$  of genus 10 which admits an involution  $\sigma$  with two fixed

points such that the  $\text{Prym}(C, \sigma)$  is isomorphic to the intermediate Jacobian of the triple cover of  $\mathbb{P}^3$  branched along the cubic surface. The curve  $C$  also admits an automorphism  $\tau$  of order 6 such that  $\sigma = \tau^3$ . The  $K3$  surface associated to the cubic is the minimal nonsingular model of the quotient  $(C \times E)/\langle \tau \rangle$ , where  $E$  is an elliptic curve with an automorphism of order 6. The branching of the map  $C \rightarrow C/\langle \tau \rangle \cong \mathbb{P}^1$  is very special, we have 7 branch points, 5 of which have ramification index  $(3, 3)$  and two have index  $(6)$ . According to Deligne-Mostow [DM] the moduli space of such covers is isomorphic to an open subset of a complex ball quotient  $\mathcal{B}/\Gamma$ . We identify this moduli space with the moduli space of  $K3$  surfaces  $X_S$  and interpret the monodromy group  $\Gamma$  in terms of the orthogonal group of the lattice of transcendental cycles on the  $K3$  surfaces. We also give an interpretation of a compactification of the ball quotient in terms of  $K3$  surfaces.

Here is the review of the contents of the paper. In section 2 we study stable cubic surfaces. Since these have at most nodes as singularities we refer to them as nodal cubic surfaces. We define markings of these cubics and we introduce the moduli space of marked nodal cubic surfaces  $\mathcal{M}_{\text{ncub}}^m$ . The Weyl group  $W(E_6)$  acts on  $\mathcal{M}_{\text{ncub}}^m$  (the action can be described by Cremona transformations) and the quotient variety is  $\mathcal{M}_{\text{ncub}}$ , the moduli space of stable cubic surfaces. It has a natural compactification  $\overline{\mathcal{M}}_{\text{ncub}}$ , the moduli space of semi-stable cubic surfaces, which is obtained by adding one point. The moduli space  $\mathcal{M}_{\text{ncub}}^m$  also admits a natural compactification  $\overline{\mathcal{M}}_{\text{ncub}}^m$  which is obtained by adding 40 points. It admits a  $W(E_6)$ -equivariant embedding into  $\mathbb{P}^9$ . We discuss different constructions of the moduli space  $\overline{\mathcal{M}}_{\text{ncub}}^m$ .

For a nodal cubic surface and a line on it we define in section 3 a pair of binary forms, of degree 2 and 5, modulo the action of  $SL(2)$ . Using this, we prove that the moduli space of cubic surfaces together with a choice of a line on it is a rational variety.

In section 4 we define a  $K3$  surface  $X_{S,l}$  associated to a nodal cubic surface  $S$  together with the choice of a line  $l$  on  $S$ . The surface  $X_{S,l}$  admits a natural elliptic fibration as well as an automorphism of order three. We show that this  $K3$  surface depends only on  $S$  (and not on the choice of  $l$ ) by defining a  $K3$  surface  $X_{S,l,m}$ , where  $l$  and  $m$  are skew lines on  $S$ , which can be seen to be isomorphic to both  $X_{S,l}$  and  $X_{S,m}$ . We write  $X_S$  for the (isomorphism class of such a)  $K3$  surface associated to  $(S, l)$ . We relate  $X_S$  to the  $K3$  surface associated to a cubic fourfold with a plane, to the cubic threefold  $V$  associated to  $S$  by Allcock, Carlson and Toledo and to the ‘Matsumoto-Terasoma curve’  $C$ .

In section 5 we show that the Picard lattice of a generic  $X_S$  is isomorphic to the lattice  $M = U \oplus A_2^{\oplus 5}$ . The lattice of transcendental cycles is isomorphic to the lattice  $T = A_2(-1) \oplus A_2^{\oplus 4}$ . This follows from the fact that the elliptic fibration on the generic  $X_S$  has 5 singular fibres of type  $IV$  and 2 fibres of type  $II$  and some lattice theoretic considerations. We also compute the Picard lattices of the  $K3$  surfaces associated to general nodal cubic surfaces.

In section 6 we study the moduli space of  $M$ -polarized  $K3$  surfaces  $(X, \phi : M \rightarrow \text{Pic}(X))$ . If  $\phi(M) = \text{Pic}(X)$ , a  $M$ -marking  $\phi$  is equivalent to the data which consists of an elliptic fibration with a unique section, an order on the 5 reducible fibres of type  $IV$  or  $I_3$ , and an order on the set of irreducible components of each fibre which do not meet the section. An  $M$ -marking on the  $K3$  surface  $X_S$  associated to a smooth cubic surface  $S$  is equivalent to a marking on  $S$ , that is, an order on the set of 27 lines (or, equivalently, a choice of an ordered set of six skew lines). The  $M$ -polarized  $K3$  surfaces  $(X_S, \phi)$  are distinguished from general  $M$ -polarized  $K3$  surfaces by the property that there exists an automorphism  $\sigma$  of order 3 which is the identity on  $\phi(M)$  and coincides with some explicitly described isometry  $\rho$  on the orthogonal complement of  $\phi(M)$  in  $H^2(X_S, \mathbb{Z})$  for smooth  $S$ .

The isometry  $\rho$  fixes the period  $H^{2,0}(X_S)$  of  $X_S$  so that the image of the period map of the surfaces  $X_S$  lies in the fixed locus of a certain automorphism of order 3 on the period space of  $M$ -polarized  $K3$  surfaces. This fixed locus turns out to be isomorphic to a 4-dimensional complex ball  $\mathcal{B}$ . In this way we construct the moduli space  $\mathcal{K}3_{M,\rho}^m$  of  $(M, \rho)$ -marked  $K3$  surfaces as a quotient of  $\mathcal{B}$ . The Weyl group  $W(E_6)$  acts naturally on  $\mathcal{K}3_{M,\rho}^m$  by changing the markings.

In section 7 we establish a natural  $W(E_6)$ -equivariant isomorphism from the moduli space of marked nonsingular cubic surfaces  $\mathcal{M}_{\text{cub}}^m$  onto an open subset  $\mathcal{K}3_{M,\rho}^m \setminus \Delta^m$  of  $\mathcal{K}3_{M,\rho}^m$ . The moduli space of isomorphism classes of pairs  $(S, l)$  of cubic surfaces together with a choice of a line is isomorphic to the quotient of  $\mathcal{K}3_{M,\rho}^m \setminus \Delta^m$  by a subgroup of  $W(E_6)$  isomorphic to  $W(D_5)$ . In this way we obtain an interpretation of a line on a general cubic surface  $S$  as a choice, up to automorphisms of  $X_S$ , of an elliptic pencil with 5 fibres of type  $IV$  on the associated  $K3$  surface  $X_S$ .

In section 8 we study in detail the geometry of the discriminant locus  $\Delta^m$ . We show that each point  $[(X, \phi)] \in \Delta^m$  admits an automorphism  $\sigma$  of order 3 such that  $H^2(X, \mathbb{Z})^{\sigma^*}$  contains  $\phi(M) \oplus R$ , where  $R$  is spanned by all  $(-2)$ -vectors in  $\phi(M)^\perp \cap \text{Pic}(X)$ . The lattice  $R$  is isomorphic to  $r$  ( $\leq 4$ ) copies of the root lattice  $A_2$ . The marking  $\phi$  defines an elliptic fibration on  $X$  and we describe its possible singular fibres. We also prove that  $\Delta^m$  consists of 36 irreducible components on which  $W(E_6)$  acts transitively. The cubic surfaces with Eckardt points define another divisor in  $\mathcal{K}3_{M,\rho}^m$  and we prove that it consists of 45 irreducible components permuted transitively by  $W(E_6)$ . Finally we show that the Satake-Baily-Borel compactification of  $\mathcal{K}3_{M,\rho}^m$  contains 40 cusps, again transitively permuted by  $W(E_6)$ . This agrees with the results obtained in [ACT].

In section 9 we show that the  $W(E_6)$ -equivariant isomorphism from  $\mathcal{M}_{\text{cub}}^m$  onto  $\mathcal{K}3_{M,\rho}^m \setminus \Delta^m$  can be extended to an equivariant isomorphism from the moduli space of marked nodal cubics  $\mathcal{M}_{\text{ncub}}^m$  to  $\mathcal{K}3_{M,\rho}^m$ . We also show that the quotient  $\mathcal{K}3_{M,\rho}^m/W(D_5)$  and the moduli space of nodal cubic surfaces together with a choice of a line  $\mathcal{M}_{\text{ncub}}^1 = \mathcal{M}_{\text{ncub}}^m/W(D_5)$  are isomorphic. Moreover, the latter space is naturally isomorphic to the GIT-quotient  $P_1(2^5, 1, 1)/S_5 \times S_2 = (\mathbb{P}^1)^7//\text{SL}(2) \times (g)$ , where the linearization of  $\text{SL}(2)$  is defined by weighting the first five factors with weight 2 and the last two factors with weight 1. Here  $S_5$  acts by permutation of the first five factors and  $S_2$  acts by permutations of the last two factors.

The configuration space  $P_1(2^5, 1, 1)/(g) = (\mathbb{P}^1)^7//\text{SL}(2) \times (g)$  occurs in the work of Deligne and Mostow [DM] and we show that the group  $\Gamma$  is isogenous to the reflection group  $\Pi$  acting on  $\mathcal{B}$  which is generated by the reflection group  $\Pi'$  of the hypergeometric function defined by the multi-valued form  $\omega = z^{-1/6}[(z-1)(z-a_1)(z-a_2)(z-a_3)(z-a_4)]^{-1/3}dz$  and an involution  $g$ . Moreover, we match the types of degeneration of the elliptic fibration corresponding to the marking and the type of degeneration of a stable point set through this morphism.

Finally, in section 10, we compare the Hodge structure on the  $K3$  surface  $X_S$  with the principally polarized Hodge structure on  $H^1(P, \mathbb{Z})$ , where  $P$  is the intermediate Jacobian of a cubic threefold associated to the cubic surface  $S$ .

## 2. NODAL CUBIC SURFACES

**2.1. Nodal cubics and points in  $\mathbb{P}^2$ .** A *nodal cubic surface* is a surface of degree 3 in  $\mathbb{P}^3$  which has at most ordinary double points as singularities. Let  $S \subset \mathbb{P}^3$  be a nodal cubic surface with a node  $P = (0, 0, 0, 1)$ . Then its equation is of the form:

$$(2.1) \quad F_2(X_0, X_1, X_2)X_3 + F_3(X_0, X_1, X_2) = 0,$$

where the  $F_i$  are homogeneous of degree  $i$  and  $F_2 = 0$  defines a smooth conic. Projection from  $P$  is a birational isomorphism  $S \dashrightarrow \mathbb{P}^2$  with inverse given by:

$$\mathbb{P}^2 \dashrightarrow S, \quad x = (x_0 : x_1 : x_2) \longmapsto (F_2(x)x_0, F_2(x)x_1, F_2(x)x_2, -F_3(x)).$$

It is a rational map given by the linear system of cubics through  $B = (F_2 = 0) \cap (F_3 = 0)$ . The inverse image of a point in  $B$  is a line on  $S$ . There are at most two nodes on a line in  $S$  which implies that each point in  $B$  has multiplicity at most 2. In particular,  $S$  has at most 4 nodes.

Let  $S$  be a nodal cubic surface and let  $\tilde{S} \rightarrow S$  be the desingularization of  $S$ . The fibre over a node is a  $(-2)$ -curve, i.e. a smooth rational curve with selfintersection  $-2$ . The rational map  $S \dashrightarrow \mathbb{P}^2$  defines a morphism  $\pi : \tilde{S} \rightarrow \mathbb{P}^2$  which is the composition of birational morphisms

$$\pi : \tilde{S} = \tilde{S}_0 \rightarrow \tilde{S}_1 \rightarrow \cdots \rightarrow \tilde{S}_6 = \mathbb{P}^2,$$

where each  $\pi_i : \tilde{S}_{i-1} \rightarrow \tilde{S}_i, i = 1, \dots, 6$ , is the blow-down of an exceptional curve of the first kind (a  $(-1)$ -curve for short).

Let  $E'_i \subset \tilde{S}_i$  be the exceptional curve of  $\pi_i$  and put  $E_i = (\pi_{i-1} \circ \cdots \circ \pi_1)^{-1}(E'_i)$ . Let  $e_i$  be the divisor class of  $E_i$  and let  $e_0$  be the divisor class of the pre-image of a line  $l \subset \mathbb{P}^2$  under  $\pi$ . The classes  $e_0, e_1, \dots, e_6$  form an orthonormal basis in

$$H^2(\tilde{S}, \mathbb{Z}) = \text{Pic}(\tilde{S}) = \mathbb{Z}e_0 \oplus \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_6$$

in the sense that  $e_0^2 = 1, e_i^2 = -1, i \neq 0, (e_i, e_j) = 0, i \neq j$ . The canonical class  $K_{\tilde{S}}$  of  $\tilde{S}$  is equal to  $-3e_0 + e_1 + \cdots + e_6$ .

The anti-canonical map  $\tilde{S} \rightarrow \mathbb{P}^3$  maps  $\tilde{S}$  onto  $S$  and contracts the  $(-2)$ -curves to nodes. In particular,  $K_{\tilde{S}}$  is orthogonal to the class of each  $(-2)$ -curve. Such a class is, up to sign, one of the following 36:

$$e_i - e_j, \quad e_0 - e_i - e_j - e_k, \quad 2e_0 - e_1 - e_2 - \cdots - e_6,$$

with  $1 \leq i < j < k \leq 6$ . Let  $p_i = \pi(E_i) \in \mathbb{P}^2$ . Then  $e_i - e_j, i > j$ , is effective iff  $p_i$  and  $p_j$  coincide,  $e_0 - e_i - e_j - e_k$  is effective iff the points  $p_i, p_j$  and  $p_k$  are on a line and  $2e_0 - e_1 - e_2 - \cdots - e_6$  is effective if and only if the six points  $p_1, \dots, p_6$  are on a conic.

It follows easily from considering equation (2.1) that other nodes of  $S$  appear only when the cubic defined by  $F_3$  is simply tangent to the conic defined by  $F_2$ . Equivalently,  $S$  can be obtained as the blow-up 6 points on a conic, where among the points there could be infinitely near points of order at most 2.

**2.2. Geometric markings.** A minimal resolution of a nodal cubic surface is a Del Pezzo surface of degree 3. Recall that a Del Pezzo surface of degree  $d$  is a smooth surface  $X$  with  $-K_X$  nef and  $K_X^2 = d > 0$ . For  $d \geq 3$ , the anti-canonical linear system  $| -K_X |$  maps  $X$  birationally to a surface of degree  $d$  in  $\mathbb{P}^d$  with at most rational double points as singularities. Notice that we do not assume that  $-K_X$  is ample, in that case one should call  $X$  a Fano surface. It is known that a Del Pezzo surface admits a birational morphism  $\pi : X \rightarrow \mathbb{P}^2$  as in 2.1. A choice of such  $\pi$  and its decomposition  $\pi = \pi_6 \circ \cdots \circ \pi_1$  is called a *geometric marking* of  $X$ . Two geometric markings  $X = X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_6 = \mathbb{P}^2$  and  $X' = X'_0 \rightarrow X'_1 \rightarrow \cdots \rightarrow X'_6 = \mathbb{P}^2$  are called *isomorphic* if there exist isomorphisms  $\phi_i : X_i \rightarrow X'_i, i = 0, \dots, 6$ , such that  $\pi'_{i+1} \circ \phi_i = \phi_{i+1} \circ \pi_{i+1}, i = 0, \dots, 5$ .

**2.3. Lattice markings.** The Picard lattice of a Del Pezzo surface  $X$  of degree  $d$  is isomorphic to

$$I_{1,9-d} = \langle 1 \rangle \oplus \langle -1 \rangle^{9-d},$$

the standard odd unimodular hyperbolic lattice with the standard orthonormal basis  $(\mathbf{e}_0, \dots, \mathbf{e}_{9-d})$ . Let  $k = -3\mathbf{e}_0 + \mathbf{e}_1 + \dots + \mathbf{e}_{9-d}$ . Let  $k^\perp$  be the orthogonal complement of  $\mathbb{Z}k$  in  $I_{1,9-d}$ . Assume  $d \leq 6$ . Then the sublattice  $k^\perp$  is isomorphic to  $Q(E_{9-d})$ , where  $E_{9-d}$  is the root lattice  $E_{9-d}$  if  $d = 1, 2, 3$ , the root lattice  $D_5$  if  $d = 4$ , the root lattice  $A_4$  if  $d = 5$ , and the root lattice  $A_2 + A_1$  if  $d = 6$ , formed by vectors  $\mathbf{e}_0 - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3, \mathbf{e}_1 - \mathbf{e}_2, \dots, \mathbf{e}_{9-d+1} - \mathbf{e}_{9-d}$ . A *lattice marking* of a Del Pezzo surface  $X$  is an isometry

$$\phi : I_{1,9-d} \longrightarrow \text{Pic}(X), \quad \text{such that } \phi(k) = K_X.$$

In particular, the restriction of  $\phi$  to  $k^\perp$  is an isometry  $k^\perp \rightarrow K_X^\perp$ .

A geometric marking defines a lattice marking by  $\phi(\mathbf{e}_i) = e_i$  with  $e_i$  as in 2.1.

Let  $W(X)$  be the subgroup of the orthogonal group of  $\text{Pic}(X)$  generated by reflections in the classes of the  $(-2)$ -curves on  $X$ . Two lattice markings  $\phi, \phi' : I_{1,9-d} \rightarrow \text{Pic}(X)$  are called *equivalent* if there exists an element  $\sigma \in W(X)$  such that  $\phi = \sigma \circ \phi'$ .

The proof of the following result can be found in [Lo].

**2.4. Proposition.** *Let  $X$  be a Del Pezzo surface. Then there is a natural bijection between the isomorphism classes of geometric markings and equivalence classes of lattice markings on  $X$ .*

**2.5. The moduli space of marked smooth cubics.** We denote by  $\mathcal{M}_{\text{cub}}^m$  the moduli space of marked smooth cubic surfaces. Its points correspond to isomorphism classes of pairs  $(S, \phi)$ , where  $S$  is a smooth cubic surface and  $\phi$  is a lattice marking of  $S$ . There is an isomorphism:

$$\mathcal{M}_{\text{cub}}^m \longrightarrow ((\mathbb{P}^2)^6 - \Delta) // \text{SL}(3), \quad (S, \phi) \longmapsto (p_1, \dots, p_6)$$

where the  $p_i \in \mathbb{P}^2$  are the images of the lines with classes  $\phi(\mathbf{e}_i) \in \text{Pic}(S)$  in the blow down  $\mathbb{P}^2$  of  $S$  and  $\Delta$  is the set of 6-tuples of points where either two points coincide, or three are on a line or all six are on a conic. The inverse image of a 6-tuple consists of the surface  $S$  obtained by blowing up the  $p_i$  and the marking is defined by putting  $\phi(\mathbf{e}_i)$  equal to the class of the exceptional divisor over  $p_i$ .

**2.6. The Cremona action on  $\mathcal{M}_{\text{cub}}^m$ .** The Weyl group  $W(E_6)$  is the subgroup of  $O(I_{1,6})$  which fixes the element  $k \in I_{1,6}$ . It acts naturally on  $\mathcal{M}_{\text{cub}}^m$  by composing a lattice marking with (the inverse of) an isometry in  $W(E_6)$ :

$$W(E_6) \longrightarrow \text{Aut}(\mathcal{M}_{\text{cub}}^m), \quad \sigma \longmapsto [(S, \phi) \longmapsto (S, \phi \circ \sigma^{-1})].$$

Equivalently,  $W(E_6)$  acts via the Cremona action on 6 ordered points in  $\mathbb{P}^2$  (see [DO]). We will simply identify  $W(E_6)$  with its image in  $\text{Aut}(\mathcal{M}_{\text{cub}}^m)$  from now on.

The quotient of  $\mathcal{M}_{\text{cub}}^m$  by  $W(E_6)$  is the moduli space of smooth cubic surfaces  $\mathcal{M}_{\text{cub}}$ . Let  $p_{\text{cub}}$  be this quotient map:

$$p_{\text{cub}} : \mathcal{M}_{\text{cub}}^m \longrightarrow \mathcal{M}_{\text{cub}}^m / W(E_6) \cong \mathcal{M}_{\text{cub}}.$$

**2.7. The GIT compactification.** Geometric Invariant Theory provides a natural compactification of the moduli space of cubic surfaces  $\mathcal{M}_{\text{cub}}$ :

$$\overline{\mathcal{M}}_{\text{cub}} = \mathbb{P}(H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)))^{\text{ss}} // \text{SL}(4).$$

The stable points in  $\mathbb{P}(H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)))$  are the isomorphism classes of nodal cubic surfaces. Points in

$$\mathcal{M}_{\text{ncub}} = \mathbb{P}(H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)))^s // \text{SL}(4)$$

are thus isomorphism classes of nodal cubics. The strictly semi-stable points all map to one point in  $\overline{\mathcal{M}}_{\text{cub}}$ . The complement of this point in  $\overline{\mathcal{M}}_{\text{cub}}$  is denoted by  $\mathcal{M}_{\text{ncub}}$ , the moduli space of nodal cubic surfaces.

The explicit computations of invariants of cubic quaternary forms, due to A. Cayley and G. Salmon [Sa1], gives an isomorphism

$$(2.2) \quad \overline{\mathcal{M}}_{\text{cub}} \cong \mathbb{P}(1, 2, 3, 4, 5).$$

The moduli space of nonsingular surfaces is isomorphic to the complement of a hypersurface of degree 4 defined by the discriminant. In particular,  $\mathcal{M}_{\text{cub}}$  is affine.

**2.8. Moduli of marked nodal cubics.** We can construct the moduli space of marked nodal cubic surfaces as follows. Let  $k(\mathcal{M}_{\text{cub}}^m)$  be the field of rational functions of  $\mathcal{M}_{\text{cub}}^m$ . It is an extension, with Galois group  $W(E_6)$ , of  $k(\mathcal{M}_{\text{cub}}) = k(\overline{\mathcal{M}}_{\text{cub}})$ . Now we define  $\overline{\mathcal{M}}_{\text{cub}}^m$  to be the normalisation of  $\overline{\mathcal{M}}_{\text{cub}}$  in the field  $k(\mathcal{M}_{\text{cub}}^m)$ .

By its definition,  $\overline{\mathcal{M}}_{\text{cub}}^m$  is a normal projective variety and, since  $\mathcal{M}_{\text{cub}}^m$  is smooth (see sections 2.9 and 2.11), we have

$$\mathcal{M}_{\text{cub}}^m \hookrightarrow \overline{\mathcal{M}}_{\text{cub}}^m,$$

the complement of  $\mathcal{M}_{\text{cub}}^m$  will be called the boundary of  $\overline{\mathcal{M}}_{\text{cub}}^m$ . By construction, the Weyl group  $W(E_6)$  acts on  $\overline{\mathcal{M}}_{\text{cub}}^m$  with quotient  $\overline{\mathcal{M}}_{\text{cub}}$ :

$$\bar{p}_{\text{cub}} : \overline{\mathcal{M}}_{\text{cub}}^m \longrightarrow \overline{\mathcal{M}}_{\text{cub}} = \overline{\mathcal{M}}_{\text{cub}}^m / W(E_6)$$

and  $\bar{p}_{\text{cub}} = p_{\text{cub}}$  on the subvariety  $\mathcal{M}_{\text{cub}}^m$ . Finally we define the moduli space of marked nodal cubic surfaces to be:

$$\mathcal{M}_{\text{ncub}}^m := \bar{p}^{-1}(\mathcal{M}_{\text{ncub}}).$$

This moduli space is the complement of a finite set of points, called the cusps, in  $\overline{\mathcal{M}}_{\text{cub}}^m$  and the cusps are all in one  $W(E_6)$ -orbit.

Despite its abstract definition, the variety  $\overline{\mathcal{M}}_{\text{cub}}^m$  is rather well-known. Below we present some other constructions of it, and we show that the points in  $\mathcal{M}_{\text{ncub}}^m$  correspond to isomorphism classes of marked nodal cubic surfaces. We do not know whether  $\mathcal{M}_{\text{ncub}}^m$  is the (coarse) moduli space of some functor.

**2.9. Naruki's model.** In [Nar], Naruki constructs a smooth, projective compactification of the moduli space  $\mathcal{M}_{\text{cub}}^m$  which he calls the cross-ratio variety. Its boundary contains 40 divisors which can be blown down to 40 singular points of a variety  $\mathcal{N}$ . The variety  $\mathcal{N}$  is normal (it is smooth outside the 40 singular points and  $\mathcal{N}$  is also normal in the singular points, where  $\mathcal{N}$  is locally a cone over a Segre embedding of  $(\mathbb{P}^1)^3$ ). Hence the isomorphism on the open subsets, isomorphic to  $\mathcal{M}_{\text{cub}}^m$ , of  $\mathcal{N}$  and  $\overline{\mathcal{M}}_{\text{cub}}^m$  extends to a birational morphism

$$\phi_{\mathcal{N}} : \mathcal{N} \longrightarrow \overline{\mathcal{M}}_{\text{cub}}^m.$$

Naruki also shows that the action of  $W(E_6)$  on  $\mathcal{M}_{\text{cub}}^m (\subset \mathcal{N})$  extends to a biregular action on  $\mathcal{N}$  with quotient  $\mathcal{N}/W(E_6) = \overline{\mathcal{M}}_{\text{cub}}$ . This implies that  $\phi_{\mathcal{N}}$  is a bijection. Hence, by Zariski's Main Theorem,  $\phi_{\mathcal{N}}$  is an isomorphism.

From Naruki's description of  $\mathcal{N}$ , see also [CvG], one obtains that the forty singular points of  $\mathcal{N}$  map to the cusps of  $\overline{\mathcal{M}}_{\text{cub}}^m$ . Moreover, the boundary of  $\overline{\mathcal{M}}_{\text{cub}}^m$  consists of 36 divisors, each of which is isomorphic to the Segre cubic threefold  $\mathcal{S}_3$ , best seen as a subvariety of  $\mathbb{P}^5$ :

$$\mathcal{S}_3 : \quad \sum_{i=1}^6 x_i = 0, \quad \sum_{i=1}^6 x_i^3 = 0.$$

The group  $W(E_6)$  acts transitively on the set of 36 boundary divisors.

**2.10. Boundary divisors.** The 36 boundary divisors are parametrized by the 36 positive simple roots of  $E_6$ . Let  $\alpha$  be a positive simple root, so  $\alpha = e_i - e_j$ ,  $e_0 - e_i - e_j - e_k$ ,  $2e_0 - e_1 - \dots - e_6$ , with  $i > j > k$ , note that indeed  $\alpha \in k^\perp \subset I_{1,6}$ . If  $(S, \phi)$  is a marked nodal cubic surface such that  $\phi(\alpha)$  is effective, then  $S$  has a node, cf. 2.1. To each  $\alpha$  we assign the divisor  $D_\alpha$  in  $\mathcal{M}_{\text{ncub}}^m$ , we write:

$$D_\alpha = \begin{cases} D_{ij} & \text{if } \alpha = e_i - e_j \\ D_{ijk} & \text{if } \alpha = e_0 - e_i - e_j - e_k \\ D_0 & \text{if } \alpha = 2e_0 - e_1 - e_2 - e_3 - e_4 - e_5 - e_6. \end{cases}$$

Each  $D_\alpha$  is isomorphic to the Segre cubic  $\mathcal{S}_3$  which is a projective model of the GIT-quotient  $(\mathbb{P}^1)^6 // \text{SL}(2)$  [DO]. This implies that the points of  $\mathcal{M}_{\text{ncub}}^m$  correspond to marked nodal cubics. In fact, for the points in  $\mathcal{M}_{\text{cub}}^m$  this is obvious, if a point  $x$  is in a boundary divisor  $D$  then it corresponds to a sextuple of points on  $\mathbb{P}^1$  which gives a geometrical marking of the corresponding nodal cubic, in fact the boundary divisor corresponds a  $(-2)$ -curve on the nodal cubic surface  $\bar{p}(x) \in \mathcal{M}_{\text{ncub}}$  and together with the sextuple of points gives a geometrical marking of the nodal cubic, see also the examples in 2.14.

In particular,  $D_\alpha$  parametrizes marked nodal cubic surfaces  $(S, \phi)$  for which  $\phi(\alpha)$  is effective. If  $\phi(\alpha)$  is effective and  $r_\alpha$  denotes the reflection in  $W(E_6)$  defined by the root  $\alpha$ , then the lattice marked nodal cubic surfaces  $(S, \phi)$  and  $(S, \phi \circ r_\alpha)$  are equivalent. This suggests that in the Cremona action of  $W(E_6)$  on  $\mathcal{M}_{\text{ncub}}^m$  the reflection  $r_\alpha$  acts identically on  $D_\alpha$ . This is in fact the case ([Nar], p. 22).

The Segre cubic has 10 nodes  $p_{1ij}$ , for example,  $p_{125} = (1 : 1 : -1 : -1 : 1 : -1)$ , corresponding to the minimal orbit of sextuples  $(p_1, \dots, p_6)$  of points on  $\mathbb{P}^1$  such that  $p_i = p_j = p_k$ ,  $p_l = p_m = p_n$ . Identifying  $\mathcal{S}_3$  with  $D_0$ , the nodes of  $\mathcal{S}_3$  are the cusps of  $\overline{\mathcal{M}}_{\text{cub}}^m$  lying on  $D_0$ .

A boundary divisor  $D$  is invariant with respect to a subgroup of  $W(E_6)$  isomorphic to  $S_6$ , under the isomorphism  $D \cong \mathcal{S}_3$  the  $S_6$  acts by permutation of the coordinates. On  $D_0$ , which parametrizes 6 points on a conic, this  $S_6$  acts by interchanging the 6 points.

The image  $\bar{p}(D)$  of a boundary divisor in  $\overline{\mathcal{M}}_{\text{cub}}$  is the locus of singular cubic surfaces. It is defined by the vanishing of the discriminant invariant on the space of cubic surfaces, which is of degree 32 in the coefficients of the cubic form. In the isomorphism (2.2) it corresponds to the hyperplane defined by the unknown with weight 4. Thus it is isomorphic to  $\mathbb{P}(1, 2, 3, 5)$ . It is also known that the quotient of the Segre cubic by  $S_6$  is isomorphic to  $\mathbb{P}(1, 2, 3, 5)$ . Hence the restriction of the map  $\bar{p} : \overline{\mathcal{M}}_{\text{cub}}^m \rightarrow \overline{\mathcal{M}}_{\text{cub}}$  to a boundary divisor  $D$  is the quotient map  $D \rightarrow D/S_6$ .

**2.11. A GIT model.** We sketch another construction of  $\overline{\mathcal{M}}_{\text{cub}}$ . First we recall the explicit construction of the GIT-quotient  $X = (\mathbb{P}^2)^6 // \text{SL}(3)$  given in [DO]. The graded ring of invariants

$$R = \bigoplus_{n=0}^{\infty} (H^0((\mathbb{P}^2)^6, \otimes_{i=1}^6 \pi_i^* \mathcal{O}_{\mathbb{P}^2}(n)))^{\text{SL}(3)}$$

is generated by elements  $t_0, t_1, t_2, t_3, t_4$  of degree 1 and one element  $t_5$  of degree 2. Here  $\pi_i$  is the  $i$ -th projection from  $(\mathbb{P}^2)^6$ . The relation between the generators is  $t_5^2 + F_4(t_0, t_1, t_2, t_3, t_4) = 0$ , where  $F_4$  is a homogeneous polynomial of degree 4. Thus  $X$  is isomorphic to a hypersurface of degree 4 in the weighted projective space  $\mathbb{P} = \mathbb{P}(1, 1, 1, 1, 1, 2)$ .

The quartic 3-fold  $V(F_4)$  in  $\mathbb{P}^4$  has 15 double lines  $l_{ij}$  corresponding to minimal semi-stable orbits of points sets  $(p_1, \dots, p_6)$  where  $p_i = p_j$ . Three lines  $l_{ij}, l_{kl}, l_{mn}$ , where  $\{1, 2, 3, 4, 5, 6\} = \{i, j\} \cup \{k, l\} \cup \{m, n\}$ , intersect at one point  $P_{ij,kl,mn}$ . It represents the orbit of the point set  $p_i = p_j, p_k = p_l, p_m = p_n$ . It follows from the explicit equation of  $F_4$  that its local equation at  $P_{ij,kl,mn}$  is given by  $w^2 + z_1 z_2 z_3 = 0$ , where  $w = z_i = z_j = 0$  is the local equation of one of the 3 double lines meeting at the point. This implies that  $X$  is given locally at the point  $P'_{ij,kl,mn} = (P_{ij,kl,mn}, 0)$  by the equation  $uv + xyz = 0$ .

Let  $Z$  be the singular locus of  $X$  and  $\mathcal{I}_Z$  its sheaf of ideals. One considers the linear system  $|\mathcal{I}_Z(3)| \subset R_3$ . A. Coble [Co] gives explicitly 40 elements of  $|\mathcal{I}_Z(3)|$  which span a  $\mathbb{P}V \cong \mathbb{P}^9$  and shows that the birational action of  $W(E_6)$  on  $X$  induces a linear action on  $V$ . We construct the moduli space of marked cubic surfaces as the image  $Y$  of  $X$  under the rational map given by the linear system  $\mathbb{P}V$ .

First we blow up the ambient space  $\mathbb{P}$  at the points  $P'_{ij,kl,mn}$ . Let  $E_{ij,kl,mn} \cong \mathbb{P}^4$  be the exceptional locus. The proper inverse transform  $X_1$  of  $X$  intersects each  $E_{ij,kl,mn}$  along the union of two hyperplanes  $H_{ij,kl,mn}, H'_{ij,kl,mn}$  corresponding to the tangent cone of the singular point. The proper inverse transforms of the lines  $l_{ij}$  are double curves  $C_{ij}$  on  $X_1$ . Each of the curves  $C_{ij}, C_{kl}, C_{mn}$  intersects  $E_{ij,kl,mn}$  at a point. The three points span the plane  $\Pi_{ij,kl,mn} = H_{ij,kl,mn} \cap H'_{ij,kl,mn}$ . Next we blow up the 15 singular curves  $C_{ab}$  to get a variety  $X_2$ . The proper inverse transform of the linear system  $\mathbb{P}V$  in  $X_2$  has base locus equal to the union of the proper transforms  $\bar{\Pi}_{ij,kl,mn}$  of the planes  $\Pi_{ij,kl,mn}$ . Each surface  $\bar{\Pi}_{ij,kl,mn}$  is isomorphic to the blow-up of 3 points on the plane. The proper transforms of the lines joining three pairs of points are double curves of  $X_2$ . Next we blow up the surfaces  $\bar{\Pi}_{ij,kl,mn}$  to get a nonsingular variety  $X_3$ . Now the proper inverse transforms of the hyperplanes  $H_{ij,kl,mn}, H'_{ij,kl,mn}$  become separated and the proper inverse transform of the linear system  $\mathbb{P}V$  has no base points.

Let  $Y \subset \mathbb{P}^9$  be the image of  $X_3$  under this linear system. Observe first that  $Y$  is a compactification of the geometric quotient  $\mathcal{M}_{\text{cub}}^m = U/\text{SL}(3)$ , where  $U = (\mathbb{P}^2)^6 - \Delta$  as in 2.5.

Next we shall see its complement. First of all we have 20 divisors  $D'_{ijk}$  in  $X$  representing 6-tuples of points where  $p_i, p_j, p_k$  are collinear. The sum of the two divisors  $D'_{ijk}$  and  $D'_{lmn}$ , where  $\{i, j, k\} \cup \{l, m, n\} = \{1, \dots, 6\}$ , is defined by a linear function  $L_{ijk} = L_{lmn} \in R_1$  (see [DO]). The corresponding hyperplane  $V(L_{ijk})$  cuts out the quartic  $V(F_4)$  along a nonsingular quadric  $Q_{ijk} = Q_{lmn}$ . The quadric contains 6 double lines  $l_{ij}, l_{ik}, l_{jk}, l_{lm}, l_{ln}, l_{mn}$ . Let  $D_{ijk}$  be the proper inverse transforms of  $D'_{ijk}$  in  $Y$ . Let  $D_{ij}$  be the proper inverse transforms in  $X_3$  of the pre-images of the curves  $C_{ij}$  in  $X_2$ . We have 15 such divisors. Finally, let  $D_0$  be the proper inverse transform of  $V(t_5) \cong V(F_4)$  in  $Y$ . It is easy to see that under the map  $X_3 \rightarrow Y$  the proper inverse transforms of the quadrics  $Q_{ijk}$  are blown down to points  $c_{ijk} = c_{lmn}$ . Also let  $c_{ij,kl,mn}, c'_{ij,kl,mn}$  be the images in  $Y$  of the hyperplanes  $H_{ij,kl,mn}, H'_{ij,kl,mn}$ . Altogether we have 40 points which we call the cusps. The

forty cusps is the set of singular points of the variety  $Y$ . So, we see that the complement of the image of  $U/\mathrm{SL}(3)$  in  $Y$  is equal to the union of 36 divisors  $D_{ijk}, D_{ij}, D_0$ .

The Weyl group  $W(E_6)$  acts on  $Y$  interchanging the boundary divisors. This makes them all isomorphic to each other. This is easy to check. The restriction of the linear system  $\mathbb{P}V$  to the quartic  $V(F_4)$  is the map given by the partials of  $F_4$ . It maps  $V(F_4)$  to the dual variety known to be isomorphic to the Segre cubic  $\mathcal{S}_3 \subset \mathbb{P}^4$ . This shows that  $D_0 \cong \mathcal{S}_3$ .

One can check that the variety  $Y$  is a normal proper  $W(E_6)$ -variety containing the  $W(E_6)$ -variety  $\mathcal{M}_{\mathrm{cub}}^m$  as an open subset. Thus there is a birational morphism  $f : Y \rightarrow \overline{\mathcal{M}}_{\mathrm{cub}}^m$ . We claim that  $f$  is an isomorphism. Let  $E$  be an irreducible component of the exceptional locus of  $f$ . It is contained in one of the 36 boundary divisors  $D$ . However  $D \cong \mathcal{S}_3$  has  $\mathrm{Pic}(D) \cong \mathbb{Z}$ . Nothing can be blown down on  $D$ . Thus we obtain that

$$Y \cong \overline{\mathcal{M}}_{\mathrm{cub}}^m.$$

**2.12. Remark.** In [ACT],  $\mathcal{M}_{\mathrm{cub}}^m$  is identified with an open subset of a smooth ball quotient. In [AF] Allcock and Freitag show, using modular forms constructed via a Borcherds lift, that this ball quotient embeds into a  $\mathbb{P}^9$  and that the closure of its image is isomorphic to the Satake compactification of the ball quotient, the boundary consists of 40 singular points. Freitag [F] proved that ideal of the image of the ball quotient is generated by explicitly given cubics and that it is a normal variety.

Coble, in [Co], defines a rational map  $(\mathbb{P}^2)^6 \rightarrow \mathbb{P}^9$  which is  $\mathrm{SL}(3)$ -invariant and hence factors over  $\mathcal{M}_{\mathrm{cub}}$ . It is easily seen to be a birational isomorphism between  $\mathcal{M}_{\mathrm{cub}}$  and its image. This map is moreover equivariant with respect to the Cremona action of  $W(E_6)$ . See also [Y] where in particular the restriction to a boundary divisor is worked out. It is easy to verify that the image of  $\mathcal{M}_{\mathrm{cub}}$  lies in the subvariety defined by the cubics.

In [vG2] the corresponding rational functions on Naruki's variety  $\mathcal{N} \cong \overline{\mathcal{M}}_{\mathrm{cub}}^m$  are explicitly identified, and also the 40 functions used by Coble are given.

Matsumoto and Terasoma [MT] showed how to get this embedding via an embedding of the complex ball into the Siegel space (of genus 5) followed by a map to  $\mathbb{P}^9$  given by explicitly determined theta constants.

**2.13. Moduli of  $i$ -nodal cubics.** The irreducible components of the locus of marked nodal cubics with  $i$  nodes are parametrized by unordered subsets of  $i$  orthogonal roots (up to sign) in  $E_6$ . We denote by  $D_{\alpha_1, \dots, \alpha_i}$  the intersection of the divisors  $D_{\alpha_1}, \dots, D_{\alpha_i}$  corresponding to  $i$  orthogonal roots  $\alpha_1, \dots, \alpha_i$ .

The stabilizer in  $W(E_6)$  of such a locus  $D_{\alpha_1, \dots, \alpha_i}$  is the product of the subgroup of order  $2^i$ , generated by the corresponding  $i$  roots (this subgroup acts trivially on the component), the permutations on  $i$  roots  $\alpha_1, \dots, \alpha_i$  ( $\simeq S_i$ ) and the subgroup generated by reflections in the roots orthogonal to the  $i$  simple roots. The stabilizer modulo the subgroup of order  $2^i$  is the group of permutations of geometric markings on  $S$ .

In case  $i = 1$ , the 30 roots  $e_i - e_j$  are all orthogonal to the root  $\alpha = 2e_0 - e_1 - \dots - e_6$ , so we see that  $\mathbb{Z}/2\mathbb{Z} \times W(A_5) \cong \mathbb{Z}/2\mathbb{Z} \times S_6$  acts on  $D_\alpha$ . Thus we recover the fact that  $W(A_5) \cong S - 6$  acts on a boundary divisor.

In case  $i = 2$ , there are 12 roots  $e_i - e_j$ , ( $3 \leq i, j \leq 6$ ) orthogonal to the two roots  $\alpha_1 = 2e_0 - e_1 - \dots - e_6$  and  $\alpha_2 = e_1 - e_2$ . They form a root system of type  $A_3$ . So  $(\mathbb{Z}/2\mathbb{Z})^2 \cdot S_2 \times W(A_3) \simeq (\mathbb{Z}/2\mathbb{Z})^2 \cdot S_2 \times S_4$  acts on  $D_{\alpha_1, \alpha_2}$ .

In case  $i = 3$ , there are two roots  $\pm(e_5 - e_6)$  orthogonal to the three roots  $\alpha_1 = 2e_0 - e_1 - \dots - e_6$ ,  $\alpha_2 = e_1 - e_2$  and  $\alpha_3 = e_3 - e_4$ . They form a root system of type  $A_1$ . So  $(\mathbb{Z}/2\mathbb{Z})^3 \cdot S_3 \times W(A_1) \simeq (\mathbb{Z}/2\mathbb{Z})^3 \cdot S_3 \times \mathbf{Z}/2\mathbf{Z}$  acts on  $D_{\alpha_1, \alpha_2, \alpha_3}$ .

In case  $i = 4$ , there are no roots orthogonal to the four roots  $\alpha_1 = 2e_0 - e_1 - \dots - e_6$ ,  $\alpha_2 = e_1 - e_2$ ,  $\alpha_3 = e_3 - e_4$  and  $\alpha_4 = e_5 - e_6$ . So  $(\mathbb{Z}/2\mathbb{Z})^4 \cdot S_4$  acts on  $D_{\alpha_1, \dots, \alpha_4}$ .

**2.14. Lines on a nodal cubic surface.** A nonsingular cubic surface contains 27 lines. They represent the classes  $e_0 - e_i - e_j$ ,  $1 \leq i < j \leq 6$ ,  $e_i$ ,  $2e_0 - e_1 - \dots - e_6 + e_i$ ,  $i = 1, \dots, 6$ .

Assume now that  $S$  has a node  $s_0$ . Projecting from  $s_0$ , we see that  $\tilde{S}$  admits a geometric marking  $\pi : \tilde{S} \rightarrow \mathbb{P}^2$  such that the images  $p_i$  of the  $E_i$  (as in 2.1) lie on an irreducible conic  $C$ . If  $S$  has no more nodes, the six points  $p_i$  are distinct. If there is one more node, we may assume without loss of generality that  $p_2$  is infinitely near to  $p_1$  (i.e.  $E_2 = E_1 + C$ , where  $C$  is a  $(-2)$ -curve and the point  $p_2$  corresponds to the tangent direction of  $C$  at  $p_1$ ). If  $S$  has three nodes we can further assume that  $p_4$  is infinitely near to  $p_3$  with the similar tangency condition. Finally if  $S$  has 4 nodes we can further assume that  $p_6$  is infinitely near to  $p_5$ . From this we easily deduce the following facts.

If  $S$  has one node, there are 21 lines on  $S$ . Six of them contain the node, and are represented by the exceptional curves  $E_i = \phi(e_i)$ , where  $\phi$  is the lattice marking corresponding to the geometric marking, we will simply omit  $\phi$  in what follows. The remaining 15 lines have the classes  $\phi(e_0 - e_i - e_j)$ . The  $(-2)$ -curve  $C$  has class  $\alpha_1 = 2e_0 - (e_1 + \dots + e_6)$  and the classes  $e_i + \alpha_1 = s_{\alpha_1}(e_i)$  also represent the lines on the node. So the lines on the nodes are limits of pairs of lines on a smooth cubic surface.

If  $S$  has 2 nodes, there are 16 lines on  $S$ . The  $(-2)$ -curves are  $\alpha_1 = 2e_0 - (e_1 + \dots + e_6)$  and  $e_2 - e_1$ , the orbits on the set of classes of 27 lines of the group generated by  $s_{\alpha_1}$  and  $s_{\alpha_2}$  correspond to the lines on  $S$ . One line connects the two nodes and represents the orbit  $\{e_1, e_2 = e_1 + \alpha_2, e_1 + \alpha_1, e_2 + \alpha_1\}$ . There are 4 lines passing through the node  $s_0$  which representing the orbits  $\{e_i, e_i + \alpha_1\}$ ,  $i = 3, 4, 5, 6$ . Another 4 lines pass through the second node. They represent the orbits  $\{e_0 - e_2 - e_i, e_0 - e_1 - e_i\}$ ,  $i = 3, 4, 5, 6$ . The remaining 7 lines do not contain nodes. They represent orbits with one element, given by the classes  $e_0 - e_i - e_j$ ,  $3 \leq i < j \leq 6$  and  $e_0 - e_1 - e_2$ .

If  $S$  has 3 nodes, there are 12 lines. There are 3 lines connecting pairs of nodes. They represent the classes  $e_1, e_3, e_0 - e_1 - e_3$ . There are 6 lines each containing one node. They represent the classes  $e_5, e_6, e_0 - e_1 - e_i, e_0 - e_3 - e_i$ ,  $i = 5, 6$ . The remaining 3 lines do not contain nodes. They represent the classes  $e_0 - e_1 - e_2, e_0 - e_3 - e_4, e_0 - e_5 - e_6$ .

If  $S$  has 4 nodes there are 9 lines. Six of them connect 6 pairs of nodes. They represent the classes  $e_1, e_3, e_5, e_0 - e_1 - e_3, e_0 - e_1 - e_5, e_0 - e_3 - e_5$ . The remaining three lines do not contain nodes and represent the classes  $e_0 - e_1 - e_2, e_0 - e_3 - e_4, e_0 - e_5 - e_6$ .

**2.15. Pencils of conics.** A conic on a nodal cubic surface  $S$  is cut out by a plane. The residual component of the plane section is a line. The pencil of planes through this line defines a pencil of conics. Thus the number of pencils of conics is equal to the number of lines. The preimage of the pencil on  $\tilde{S}$  is a conic bundle, i.e. a morphism  $f : \tilde{S} \rightarrow \mathbb{P}^1$  with general fibre isomorphic to  $\mathbb{P}^1$ . A standard computation shows that singular fibres of  $f$  are of the following three types:

Type I:  $F = E_1 + E_2$ , where  $E_1, E_2$  are two  $(-1)$ -curves and  $E_1 \cdot E_2 = 1$ .

Type II:  $F = E_1 + E_2 + R$ , where  $E_1, E_2$  are  $(-1)$ -curves,  $R$  is a  $(-2)$ -curve,  $E_1 \cdot E_2 = 0$ ,  $E_1 \cdot R = E_2 \cdot R = 1$ .

Type III:  $F = R_1 + R_2 + 2E$ , where  $R_1, R_2$  are  $(-2)$ -curves,  $E$  is a  $(-1)$ -curve,  $R_1 \cdot R_2 = 0$ ,  $R_1 \cdot E = R_2 \cdot E = 1$ .

The number of singular fibres is equal to 5 if we count the fibres of type II and III with multiplicity 2.

The pre-image of the line  $l$  corresponding to the pencil defines a bisection  $B$  of  $f$ . There are three possible cases:

No nodes on  $l$ :  $B$  is irreducible.

One node on  $l$ :  $B = B_0 + R$ , where  $B_0$  is a  $(-1)$ -curve,  $R$  is a  $(-2)$ -curve,  $B_0 \cdot R = 1$ . Each component of  $B$  is a section of  $f$ .

Two nodes on  $l$ :  $B = B_0 + R_1 + R_2$ , where  $B_0$  is a  $(-1)$ -curve,  $R_1, R_2$  are  $(-2)$ -curves,  $B_0 \cdot R_1 = B_0 \cdot R_2 = 1$ . The components  $R_1$  and  $R_2$  are sections of  $f$ . The component  $B_0$  is contained in a fibre.

Let  $p_1, \dots, p_s \in \mathbb{P}^1$  be the points such that the fibre  $f^{-1}(p_i)$  is singular. We assign to each point  $p_i$  the multiplicity  $m_i$  equal to 2 if the fibre is of type I and equal to 4 otherwise. The divisor  $D = \sum_{i=1}^s m_i p_i$  will be called the *discriminant* of the conic pencil. Let  $p_{s+1}, p_{s+2} \in \mathbb{P}^1$  be the points such that the bisection  $B$  ramifies over these points. If  $B$  is reducible, we assume that  $p_{s+1} = p_{s+2} = q$ , where  $B$  has a singular point over  $q$ . The divisor  $T = p_{s+1} + p_{s+2}$  will be called the *bisection branch divisor*. Let us write the divisor  $D + T = \sum_{i=1}^s m_i p_i + p_{s+1} + p_{s+2}$  as  $\sum_{i=1}^{s'} n_i p_i$ , where  $s' \leq s + 2$ . We order the points in such a way that  $n_1 \geq n_2 \geq \dots \geq n_{s'}$ . The vector  $\mathbf{t} = (n_1, \dots, n_{s'})$  will be called the *type vector* of the conic pencil.

The next table lists all possible type vectors. Also we indicate the total number  $r$  of nodes on  $S$ , the number  $e$  of Eckardt points on  $l$  (i.e. points where three lines meet).

The column ‘‘Kodaira fibres’’ will be explained later in section 4.3.

**2.16. Types of lines.** Let  $l$  be a line defining the pencil of conics

Case 1), 2), 3) on the above Table 1:  $l$  is any line.

Case 4):  $l$  is one of 6 lines containing the node.

Case 5), 6), 7):  $l$  is one of 15 lines not passing through the node.

Case 8):  $l$  is one of 8 lines through exactly one node.

Case 8\*):  $l$  is the unique line containing two nodes.

Case 9), 11):  $l$  is one of 6 lines not containing a node and not meeting the line of type 8\*).

Case 10), 12):  $l$  is the unique line not containing a node and meeting the line of type 8\*).

Case 13):  $l$  is one of 6 lines passing exactly through one node.

Case 13\*):  $l$  is one of 3 lines passing through two nodes.

Case 14, 15):  $l$  is one of 3 lines not containing a node.

Case 16):  $l$  is one of 6 lines passing through two nodes.

Case 17):  $l$  is one of 3 lines not containing a node.

### 3. CUBIC SURFACES AND 2+5 POINTS ON THE LINE

**3.1. The forms  $(F_2, F_5)$ .** Let  $S$  be a nodal cubic surface and let  $l$  be a line on  $S$ . Consider the pencil of conics through the line  $l$ , cf. section 2.15. Let  $D = \sum_{i=1}^s m_i p_i$  be its discriminant divisor and let  $T = p_{s+1} + p_{s+2}$  be the bisection branch divisor. Let  $F_5(x_0, x_1)$  be a homogeneous form of degree 5 defining  $D$  and let  $F_2(x_0, x_1)$  be a homogeneous form of degree 2 defining  $T$ .

It follows from section 2.15 that the following properties are satisfied:

- (i)  $F_2 \neq 0$ ;
- (ii)  $F_5$  has at most double root;

	$\mathbf{t}$	Singular fibres	Kodaira fibres	$r$	$e$
1)	(2222211)	$5I$	$5IV, 2II$	0	0
2)	(322221)	$5I$	$I_0^*, 4IV, II$	0	1
3)	(33222)	$5I$	$2I_0^*, 3IV$	0	2
4)	(222222)	$5I$	$6IV$	1	0
5)	(422211)	$II, 3I$	$IV^*, 3IV, 2II$	1	0
6)	(43221)	$II, 3I$	$IV^*, I_0^*, 2IV, II$	1	1
7)	(4332)	$II, 3I$	$IV^*, 2I_0^*, IV$	1	2
8)	(42222)	$II, 3I$	$IV^*, 4IV$	2	0
8*)	(42222)	$5I$	$IV^*, 4IV$	2	0
9)	(44211)	$2II, I$	$2IV^*, IV, 2II$	2	0
10)	(52221)	$III, 3I$	$II^*, 3IV, II$	2	0
11)	(4431)	$2II, I$	$2IV^*, I_0^*, II$	2	1
12)	(5322)	$III, 3I$	$II^*, I_0^*, 2IV$	2	1
13)	(4422)	$2II, I$	$2IV^*, 2IV$	3	0
13*)	(4422)	$II, 3I$	$2IV^*, 2IV$	3	0
14)	(5421)	$III, II, I$	$II^*, IV^*, IV, II$	3	0
15)	(543)	$III, II, I$	$II^*, IV^*, I_0^*$	3	1
16)	(444)	$2II, I$	$3IV^*$	4	0
17)	(552)	$2III, I$	$2II^*, IV$	4	0

TABLE 1. Pencils of conics

(iii)  $F_2$  and  $F_5$  do not have common multiple roots.

A pair of binary forms  $(F_5, F_2)$  satisfying properties (i)-(iii) will be called a *stable pair*. Let  $V(d)$  be the space of binary forms of degree  $d$ . A pair of nonzero binary forms  $(F_5, F_2)$  defines a point in  $\mathbb{P}(V(5)) \times \mathbb{P}(V(2))$ .

**3.2. Proposition.** *A pair of nonzero binary forms  $(F_5, F_2)$  is stable if and only if it is a stable point with respect to the diagonal action of  $SL(2)$  and the linearization defined by the invertible sheaf  $\mathcal{O}_{\mathbb{P}(V(5))}(2) \boxtimes \mathcal{O}_{\mathbb{P}(V(2))}(1)$ . A semi-stable point corresponds to a pair of forms  $(F_5, F_2)$  such that either  $F_5$  has a root of multiplicity 3 or  $F_5$  and  $F_2$  share a double root.*

*Proof.* This easily follows from the Hilbert-Mumford numerical criterion of stability and is left to the reader.  $\square$

**3.3. Line marked cubic surfaces.** Let  $(S, \phi)$  be a nodal cubic surface with a geometric marking  $\phi$  on its minimal resolution and let  $l$  be a line on  $S$ . The Weyl group  $W(D_5)$ , a subgroup of  $W(E_6)$ , acts on markings of  $S$  stabilizing the line  $l$ . The quotient space

$$\mathcal{M}_{\text{ncub}}^l = \mathcal{M}_{\text{ncub}}^m / W(D_5)$$

is the moduli space of isomorphism classes of pairs  $(S, l)$ , where  $l$  is a line on  $S$ .

To a pair  $(S, l)$  we associate the binary forms  $F_2, F_5$  as in 3.1. This defines a map

$$(3.1) \quad \mathcal{M}_{\text{ncub}}^1 \longrightarrow (\mathbb{P}(V(2)) \times \mathbb{P}(V(5)))^s / \text{SL}(2), \quad (S, l) \longmapsto [(F_2, F_5)],$$

where  $(\mathbb{P}(V(2)) \times \mathbb{P}(V(5)))^s$  is the open subset corresponding to stable pairs of binary forms.

**3.4. Theorem.** *The map (3.1) is an isomorphism.*

*Proof.* We have to show how to reconstruct  $(S, l)$  from the  $\text{SL}(2)$ -orbit of a pair  $(F_5, F_2)$ . Let us first consider the case when  $S$  is nonsingular. We view the zeroes of binary forms as the tangent directions at a fixed point  $p \in \mathbb{P}^2$  and identify these with lines in  $\mathbb{P}^2$  containing  $p$ . Given  $(F_2, F_5)$ , fix a conic  $Q$  not containing  $p$  such that the lines through  $p$  defined by  $F_2$  are tangents of  $Q$ . Then a choice of 5 points  $p_1, \dots, p_5$  on the intersection of the lines defined by  $F_5$  with the conic, no two lying on the same line, defines uniquely (up to isomorphism) a cubic surface  $S$  with a line  $l$  corresponding to the conic. It is obtained by blowing up  $\mathbb{P}^2$  at the points  $p_1, \dots, p_5, p$ . Let  $p'_i$  be the point on  $Q$  such that  $p_i, p'_i, p$  are collinear. Let us show that replacing  $p_i$  with  $p'_i$  leads to an isomorphic pair  $(S', l')$ .

Note that changing  $(p_1, \dots, p_5)$  with  $(p'_1, \dots, p'_5)$  leads to the same surface because the points  $(p_1, \dots, p_5, p)$  and  $(p'_1, \dots, p'_5, p)$  are projectively equivalent. This can be easily seen by choosing projective coordinates such that  $p = (0, 0, 1)$  and  $Q = V(x_0x_1 - x_2^2)$ . Then  $p_i = (1, a_i^2, a_i)$  and  $p'_i = (1, a_i^2, -a_i)$ .

Now it is enough to show that changing a pair  $(p_i, p_j)$  with  $(p'_i, p'_j)$  defines an isomorphic surface. For this we consider the Cremona transformation with base points at  $p, p_i, p_j$ . It sends the conic  $Q$  through  $p_1, \dots, p_5$  to a conic  $Q'$ . Chose coordinates so that  $p = (0, 0, 1), p_i = (1, 0, 0), p_j = (0, 1, 0)$ . Then the sides of the triangle of base points are mapped to the opposite vertices. The transformation looks like  $(x_0, x_1, x_2) \mapsto (x_1x_2, x_0x_2, x_0x_1)$ . Now we know that the image of  $p'_i$  lying on the line  $\langle p_i, p \rangle$  is  $p_j$  and the image of  $p'_j$  is  $p_i$ . The transformation leaves invariant any line  $l$  through  $p$  not equal to a side of the triangle and induces a non-trivial involution on it. In particular,  $Q$  and  $Q'$  have the same tangent lines through  $p$  and have 2 common points  $p_i, p_j$ . It is easy to see that they must coincide. Hence the point  $p_k$  goes to  $p'_k$  for  $k \neq i, j$ . Composing with a projective transformation we map  $p_i$  to  $p'_j, p_j$  to  $p'_i$  and do not change  $p_k, k \neq i, j$ .

Thus we have constructed a map from  $(\mathbb{P}(V(2))' \times \mathbb{P}(V(5))^s) / \text{SL}(2)$  to  $\mathcal{M}_{\text{ncub}}^1$ . Let us show that it inverts the map (3.1). Choose a line  $m$  not intersecting  $l$  and 5 skew lines  $l'_1, \dots, l'_5$  intersecting  $l$  and  $m$ . Let  $l_1, \dots, l_5$  be the lines such that  $l, l_i, l'_i$  are coplanar. Then we can blow down the skew lines  $l_1, \dots, l_5, m$  to the points  $p_1, \dots, p_5, p$ , respectively. The image of  $l$  is a conic  $Q$  through the points  $p_1, \dots, p_5$ . The image of the line  $l'_i$  is the line  $\langle p_i, p \rangle$ . The pencil of planes through the line  $l$  corresponds to the pencil of lines through  $p$ . Thus the pair of binary forms  $(F_5, F_2)$  defined by  $(S, l)$  corresponds to the tangent directions at  $p$ . Clearly, the pair  $(S', l')$  reconstructed from  $(F_5, F_2)$  by the previous construction is isomorphic to  $(S, l)$ .

Now assume that  $S$  has nodes. Let us interpret the first column of Table 1 as the type of the pair  $(F_5, F_2)$  in the following way.

- (1)  $\mathfrak{t} = (2222211)$  corresponds to the case when  $F_5, F_2$  have no multiple roots and no roots in common;
- (2)  $\mathfrak{t} = (322221)$  corresponds to the case when  $F_5, F_2$  have no multiple roots and one common root;
- (3)  $\mathfrak{t} = (332222)$  corresponds to the case when  $F_5, F_2$  have no multiple roots and two common roots;

- (4)  $\mathbf{t} = (222222)$  corresponds to the case when  $F_5$  has no multiple roots,  $F_2$  has a double root which is not a root of  $F_5$ ;
- (5)  $\mathbf{t} = (422211)$  corresponds to the case when  $F_5$  has one double root,  $F_2$  has two roots which are not roots of  $F_5$ ;
- (6)  $\mathbf{t} = (43221)$  corresponds to the case when  $F_5$  has one double root,  $F_2$  has two roots, one of them is a root of  $F_5$ ;
- (7)  $\mathbf{t} = (4332)$  corresponds to the case when  $F_5$  has one double root,  $F_2$  has two roots which are roots of  $F_5$ ;
- (8)  $\mathbf{t} = (42222)$  corresponds to the case when  $F_5$  has one double root,  $F_2$  has a double root;
- (8\*)  $\mathbf{t} = (42222)$  corresponds to the case when  $F_2$  has a double root which is a simple root of  $F_5$ ;
- (9)  $\mathbf{t} = (44211)$  corresponds to the case when  $F_5$  has two double roots,  $F_2$  has two roots which are not roots of  $F_5$ ;
- (10)  $\mathbf{t} = (52221)$  corresponds to the case when  $F_5$  has one double root which is a simple root of  $F_2$ , the other root of  $F_2$  is not a root of  $F_5$ ;
- (11)  $\mathbf{t} = (4431)$  corresponds to the case when  $F_5$  has two double roots and its simple root is a root of  $F_2$ , the other root of  $F_2$  is not a root of  $F_5$ ;
- (12)  $\mathbf{t} = (5322)$  corresponds to the case when  $F_5$  has one double root which is a simple roots of  $F_2$  and another root of  $F_2$  is a simple root of  $F_5$ ;
- (13)  $\mathbf{t} = (4422)$  corresponds to the case when  $F_5$  has one double root,  $F_2$  has a double root which is a simple root of  $F_5$ ;
- (13\*)  $\mathbf{t} = (4422)$  corresponds to the case when  $F_5$  has two double roots,  $F_2$  has a double root which is not a root of  $F_5$ ;
- (14)  $\mathbf{t} = (5421)$  corresponds to the case when  $F_5$  has two double roots, one of which is a simple root of  $F_2$ , the other root of  $F_2$  is not a root of  $F_5$ ;
- (15)  $\mathbf{t} = (543)$  corresponds to the case when  $F_5$  has two double roots,  $F_2$  has two roots in common with  $F_5$ , one of them is a double root of  $F_5$ ;
- (16)  $\mathbf{t} = (444)$  corresponds to the case when  $F_5$  has two double roots and its simple root is a double root of  $F_2$ ;
- (17)  $\mathbf{t} = (552)$  corresponds to the case when  $F_5$  has two double roots which are simple roots of  $F_2$ .

We reconstruct the surface  $(S, l)$  in each case as the blow-up of a point  $p$  and 5 points  $p_1, \dots, p_5$  lying on the intersection of lines  $L_1, \dots, L_5$  with a conic  $Q$  through  $p_1, \dots, p_5$  not containing  $p$ . We may take two points  $p_i$  and  $p_j$  collinear with  $p$ . This is indicated by saying that  $L_i = L_j$ . The points  $p_i, p_j$  could be infinitely near in which case we indicate that the corresponding lines coincide and touch  $Q$ . The conic  $Q$  could also consist of two lines  $H_1$  and  $H_2$ .

- 1) A smooth conic  $Q$  and five lines  $L_1, \dots, L_5$  through  $p$ .
- 2) A line  $L_i$  is tangent to  $Q$ .
- 3) Two lines  $L_i, L_j$  are two tangent lines of  $Q$ .
- 4)  $Q$  splits into two lines  $H_1 + H_2$ , but none of the  $L_i$  contains the point  $H_1 \cap H_2$ .
- 5) Two lines  $L_i$  and  $L_j$  coincide.
- 6) A line  $L_i$  is tangent to  $Q$  and  $L_j = L_k$ .
- 7) Two lines  $L_i, L_j$  are two tangent lines of  $Q$  and  $L_k = L_l$ .
- 8)  $Q$  splits into two lines and two lines  $L_i, L_j$  coincide.
- 8\*)  $Q$  splits into two lines  $H_1 + H_2$  and a line  $L_i$  contains the point  $H_1 \cap H_2$ .

- 9)  $L_i = L_j$  and  $L_k = L_l$ .
- 10)  $L_i = L_j$  and this line is tangent to  $Q$ .
- 11) A line  $L_i$  is tangent to  $Q$  and  $L_j = L_k, L_l = L_m$ .
- 12)  $L_i = L_j$  and  $L_k$  are two tangent lines of  $Q$ .
- 13)  $Q$  splits into two lines  $H_1 + H_2$ , a line  $L_i$  contains the point  $H_1 \cap H_2$  and  $L_j = L_k$ .
- 13\*)  $Q$  splits into two lines  $H_1, H_2$ , no line contains the point  $H_1 \cap H_2$  and  $L_i = L_j, L_k = L_l$ .
- 14)  $L_i = L_j$  is tangent to  $Q$  and  $L_k = L_l$ .
- 15)  $L_i = L_j$  and  $L_k$  are tangent lines of  $Q$  and  $L_l = L_m$ .
- 16)  $Q$  splits into two lines  $H_1 + H_2$ ,  $L_i$  contains the point  $H_1 \cap H_2$  and  $L_j = L_k, L_l = L_m$ .
- 17)  $L_i = L_j$  and  $L_k = L_l$  are two tangent lines of  $Q$ .

As in the nonsingular case, we show that this defines the inverse map. □

3.5. Since the variety  $(\mathbb{P}(V(2)) \times \mathbb{P}(V(5)))/\mathrm{SL}(2)$  is obviously birationally isomorphic to the quotient  $\mathbb{P}(V(5))/\mathbb{C}^*$  (by fixing first a binary form of degree 2), we obtain the following:

3.6. **Corollary.** *The moduli space  $\mathcal{M}_{\mathrm{cub}}^1$  is isomorphic to an open subset of a toric variety. In particular, it is rational.*

3.7. **Remark.** It is well-known that the moduli space of cubic surfaces is rational. However, as far as we know, the rationality of the space  $\mathcal{M}_{\mathrm{cub}}^1$  was not known. Note also that the moduli space  $\mathcal{M}_{\mathrm{cub}}^1$  is birationally isomorphic to the universal surface over the moduli space of Del Pezzo surfaces of degree 4.

#### 4. THE $K3$ SURFACE ASSOCIATED TO A CUBIC SURFACE

4.1. In the previous section we associated a pair of binary forms  $(F_2, F_5)$  to a nodal cubic surface  $S$  with a line  $l$ . We now use these binary forms to define a  $K3$  surface  $X_{S,l}$ .

We will show that  $X_{S,l}$  depends only on the nodal cubic  $S$  and that the lines on a generic  $S$  correspond to certain ‘standard’ elliptic fibrations (cf. section 6.20, Corollary 7.6). Finally we relate  $X_{S,l}$  to  $S$  using a cubic fourfold.

4.2. **Definition.** Let  $S$  be a nodal cubic surface and let  $l$  be a line on  $S$ . Let  $F_2(x_0, x_1)$  be a homogeneous form of degree 2 and let  $F_5(x_0, x_1)$  be a homogeneous form of degree 5 associated to  $(S, l)$  as in 3.1.

To the pair  $(S, l)$  we associate a surface  $X_{S,l}$  which is a nonsingular minimal model of the double plane with the branch divisor

$$(4.1) \quad W : \quad x_2(F_2(x_0, x_1)x_2^3 + F_5(x_0, x_1)) = 0.$$

It is easy to check that the properties (i)-(iii) in 3.1 are equivalent to the property that any singular point of the curve  $W$  is analytically equivalent to a singularity  $f(x, y) = 0$  such that the surface singularity  $z^2 + f(x, y) = 0$  is a double rational point. This implies that  $X_{S,l}$  is a  $K3$  surface. The multiplication of  $x_2$  by a primitive cube root of unity induces an automorphism of  $X_{S,l}$  of order 3.

**4.3. Elliptic fibration.** Consider the pencil of lines

$$L(t_0, t_1) : t_1 x_0 - t_0 x_1 = 0$$

in  $\mathbb{P}^2$  passing through the point  $(0, 0, 1)$ . Since a general line  $L(\lambda, \mu)$  intersects  $W$  at four nonsingular points, we obtain that the pre-image of the pencil of lines on  $X_{S,l}$  is an elliptic pencil. Thus we have an elliptic fibration

$$f = f_l : X_{S,l} \longrightarrow \mathbb{P}^1.$$

The singular fibres correspond to lines  $L(t_0, t_1)$  such that  $F_5(t_0, t_1) = 0$  or  $F_2(t_0, t_1) = 0$ . The proper transform of  $W$  in the blow-up  $V \cong \mathbf{F}_1$  of the point  $(0, 0, 1)$  is a curve  $\bar{W}$  in the linear system  $|6f + 4e|$ , where  $e$  is the exceptional section and  $f$  is a fibre. The pre-image  $T$  of the line  $x_2 = 0$  is a component of  $\bar{W}$ . It is a section with the divisor class  $f + e$ . The pre-image of a line corresponding to a zero  $(x_0, x_1)$  of  $F_5$  is a fibre of  $V \rightarrow \mathbb{P}^1$  over  $(x_0, x_1)$  which intersects  $B = \bar{W} - T$  with multiplicity 3 at a point where  $B$  intersects  $T$ . A line corresponding to a zero of  $F_2$  is a fibre which intersects  $B$  with multiplicity 3 at a point where  $B$  intersects  $e$ . The surface  $X_{S,l}$  is isomorphic to a minimal resolution of the double cover of  $V$  branched along  $\bar{W}$ .

Now it is easy to describe the singular fibres of the elliptic fibration  $f : X_{S,l} \rightarrow \mathbb{P}^1$ . For example, in the case when  $F_5$  and  $F_2$  have no multiple roots and have no common roots, the fibres over the zeroes of  $F_2$  are cuspidal cubics. The fibres over the zeroes of  $F_5$  are reducible of type *IV* in Kodaira's notation. If  $F_2$  has a common zero with  $F_5$ , the fibre of  $V \rightarrow \mathbb{P}^1$  becomes an irreducible component of  $B$ . The corresponding fibre of  $f$  is of type  $I_0^*$ . If  $F_2$  has a double root which is not a root of  $F_5$ , then  $B$  acquires a cusp. Instead of two irreducible fibres of  $f$  we obtain one reducible fibre of type *IV*. If  $F_5$  has a double root which is not a root of  $F_2$ , then  $B$  acquires a cusp at the curve  $T$ . The corresponding fibre of  $f$  is of type  $IV^*$ . It is not difficult to describe the fibres in all possible cases. Their Kodaira types are given in Table 1. Note that the irreducible singular fibres correspond to zeroes of  $F_2$  which are not zeroes of  $F_5$ . Observe also that the pre-image of  $T$  in  $X_{S,l}$  is a section  $s$  of the elliptic fibration. The pre-image of  $e$  is a bisection  $b$ . If  $B$  acquires a cusp at the exceptional section  $e$  or has a fibre component, then  $b$  splits in two disjoint sections.

**4.4.** Let  $l$  be a line on a nodal cubic surface  $S$ , and let  $m$  be another line disjoint from  $l$ . Consider the rational map  $T : l \times m \rightarrow S$  defined by taking the third intersection point of the line through the points  $(p, q) \in l \times m$  with  $S$ . We denote by  $L$  and  $M$  the irreducible curves in  $l \times m$  which map onto the lines  $l$  and  $m$  in  $S$  respectively under  $T$ .

**4.5. Lemma.** *The rational map  $T$  extends to an isomorphism from the blow-up  $Z$  of  $l \times m$  along  $L \cup M$ , which is a set of 5 points (including infinitely near points) to a minimal resolution  $\tilde{S}$  of  $S$ . The curves  $L$  and  $M$  have bi-degree  $(2, 1)$  and  $(1, 2)$  respectively.*

*Proof.* This is just a straightforward computation. Choose coordinates on  $\mathbb{P}^3$  such that  $m : x_0 = x_1 = 0$  and  $l : x_2 = x_3 = 0$  so that the equation of  $S$  is given by

$$(4.2) \quad \sum_{i,j=0}^1 A_{ij}(x_2, x_3)x_i x_j + 2 \sum_{i=0}^1 B_i(x_2, x_3)x_i = 0,$$

where  $A_{ij}, B_i$  are homogeneous forms of degree 1 and 2, respectively. Let  $p = (a_0, a_1, 0, 0) \in l$ ,  $q = (0, 0, a_2, a_3) \in m$ . The line  $l'$  spanned by  $p, q$  has parametric equation  $(x_0, x_1, x_2, x_3) = (sa_0, sa_1$

$ta_2, ta_3$ ). Plugging it in equation (4.2), we obtain

$$st \left( s \sum_{i,j=0}^1 A_{ij}(a_2, a_3) a_i a_j + 2t \sum_{i=0}^1 B_i(a_2, a_3) a_i \right) = 0.$$

Thus the rational map  $T$  is given by the formula

$$(4.3) \quad T(p, q) = (Ma_0, Ma_1, La_2, La_3),$$

where

$$(4.4) \quad M(p, q) = -2 \sum_{i=0}^1 B_i(a_2, a_3) a_i, \quad L(p, q) = \sum_{i,j=0}^1 A_{ij}(a_2, a_3) a_i a_j.$$

It is easy to see that the base locus  $Z$  of the linear system of divisors of bi-degree (3,3) defining  $T$  is the complete intersection of the divisor  $M = 0$  of bi-degree (1,2) and  $L = 0$  of bi-degree (2,1). Local computations show that  $Z$  is reduced and consists of 5 points if and only if  $S$  is smooth. The rational map  $T$  is obviously birational, and defines a birational morphism  $T' : S' \rightarrow S$  of the blow-up  $S'$  of  $l \times m$  along  $Z$ . It is clear that the proper images under  $T$  of the divisors  $L = 0$  and  $M = 0$  are the lines  $l$  and  $m$ , respectively. Comparing the Betti numbers of  $S'$  and  $\tilde{S}$ , we see that they are equal. Thus  $T'$  defines an isomorphism from  $S'$  to  $\tilde{S}$ .  $\square$

**4.6. Remark.** Assume  $S$  is nonsingular. Then we obtain that  $S$  is isomorphic to the blow-up of 5 distinct points in  $\mathbb{P}^1 \times \mathbb{P}^1$ . The map  $S \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$  is the blowing down of 5 disjoint lines intersecting the lines  $l$  and  $m$ . This is of course well-known. Take any two skew lines on  $S$ . It is known that there are exactly five skew lines on  $S$  which intersect  $l, m$ . The easiest way to see it is to complete  $l, m$  to a set of six skew lines  $n_1 = l, n_2 = m, n_3, \dots, n_6$ , then consider the blow-down  $\pi : S \rightarrow \mathbb{P}^2$  of these lines to points  $p_1, \dots, p_6$  in the plane. The five skew lines are the proper inverse transforms of the line spanned by  $p_1, p_2$  and the four conics  $C_i$  passing through all  $p_j$ 's except  $p_i$  with  $3 \leq i \leq 6$ . Blowing down the five lines, we obtain  $\mathbb{P}^1 \times \mathbb{P}^1$ . The images of the lines  $l, m$  are the curves of bi-degree (2, 1) and (1, 2). The blowing down morphism  $S \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$  which inverts  $T$  is the Cartesian product of the linear projections from the lines  $l$  and  $m$ .

**4.7. The surface  $X_{S,l,m}$ .** The divisor  $W' = L + M$  on  $l \times m = \mathbb{P}^1 \times \mathbb{P}^1$  is of bi-degree (3,3). Let us consider the cyclic triple cover  $Y \rightarrow l \times m$  branched along  $W'$ . It has singular points over the singular locus of  $W'$ . If  $L$  intersects  $M$  transversally,  $Y$  has 5 double rational points of type  $A_2$ . Let  $X_{S,l,m}$  be a nonsingular minimal model of  $Y$ .

**4.8. Lemma.** *Let*

$$f = f_{l,m} : X_{S,l,m} \longrightarrow m \cong \mathbb{P}^1$$

*be the composition of the blow down map  $X_{S,l,m} \rightarrow Y$ , the triple covering  $Y \rightarrow l \times m$  and the second projection  $l \times m \rightarrow m$ . Then  $f$  is an elliptic fibration with a section whose Weierstrass form is given by:*

$$(4.5) \quad y^2 + x^3 + F_5(t_0, t_1)^2 F_2(t_0, t_1) = 0,$$

*where the binary forms  $F_2(t_0, t_1)$  and  $F_5(t_0, t_1)$  coincide with the binary forms  $F_2$  and  $F_5$  associated to  $(S, l)$  in section 3.1.*

*Proof.* For any general point  $(t_0, t_1) \in \mathbb{P}^1$ , the fibre of  $f$  over this point is isomorphic to a plane cubic curve with the equation

$$(4.6) \quad x_2^3 + (B_0(t_0, t_1)x_0 + B_1(t_0, t_1)x_1)(A_{00}(t_0, t_1)x_0^2 + 2A_{01}(t_0, t_1)x_0x_1 + A_{11}(t_0, t_1)x_1^2) = 0.$$

The cubic curve has an obvious automorphism of order 3 defined by multiplying  $x_2$  by the third roots of unity. As is well-known such a cubic can be reduced by a projective transformation to the Weierstrass form

$$y^2t + x^3 + bt^3 = 0.$$

The coefficient  $b$  is the value of a certain  $\mathrm{SL}(3)$ -invariant  $T$  on the space of homogeneous polynomials of degree 3 in 3 variables. Using the explicit formula for  $T$  (see [Sa2], p. 192), a direct computation shows that

$$(4.7) \quad b = F_5(t_0, t_1)^2 F_2(t_0, t_1),$$

where

$$\begin{aligned} F_5 &= B_0(t_0, t_1)^2 A_{11}(t_0, t_1) + B_1(t_0, t_1)^2 A_{00}(t_0, t_1) - 2A_{01}(t_0, t_1)B_0(t_0, t_1)B_1(t_0, t_1), \\ F_2 &= A_{00}(t_0, t_1)A_{11}(t_0, t_1) - A_{01}(t_0, t_1)^2. \end{aligned}$$

Let  $t_1x_2 - t_0x_3 = 0$  be the pencil of planes through the line  $l : x_2 = x_3 = 0$ . Using the equation (4.2) of  $S$  we find that the pencil of conics defined by the line  $l$  has the equation

$$(4.8) \quad A_{00}(t_0, t_1)x_0^2 + 2A_{01}(t_0, t_1)x_0x_1 + A_{11}(t_0, t_1)x_1^2 + 2B_0(t_0, t_1)x_2x_0 + 2B_1(t_0, t_1)x_2x_1 = 0.$$

Its discriminant is equal to

$$\det \begin{pmatrix} A_{00} & A_{01} & B_0 \\ A_{01} & A_{11} & B_1 \\ B_0 & B_1 & 0 \end{pmatrix} = -F_5(t_0, t_1).$$

The restriction of the member of the pencil corresponding to the parameters  $(t_0, t_1)$  to the line  $l$  is given by the binary form

$$(4.9) \quad A_{00}(t_0, t_1)x_0^2 + 2A_{01}(t_0, t_1)x_0x_1 + A_{11}(t_0, t_1)x_1^2 = 0.$$

The discriminant of this binary form is equal to

$$\det \begin{pmatrix} A_{00} & A_{01} \\ A_{01} & A_{11} \end{pmatrix} = F_2(t_0, t_1).$$

If  $l$  does not contain nodes, the equation (4.9) defines a base-point free pencil of divisors of degree 2 on  $l$ , and we see that  $F_2 = 0$  describes the locus of points in the parameter space of the pencil of conics where the bisection  $l$  ramifies. If  $l$  contains a node, we may assume that its coordinates are  $(1, 0, 0, 0)$ . Then  $A_{11} = 0$  and we get a pencil of divisors of degree 1 on  $l$  with one base point. The discriminant is equal to  $-A_{01}^2$  and describes one point with multiplicity 1 corresponding to the singular point of the bisection  $B$  defined by  $l$ . Finally, if  $l$  contains two nodes, we may assume that  $A_{11} = A_{00} = 0$ . Then the pencil (4.9) cuts out the fixed divisor with equation  $A_{01}(t_0, t_1)x_0x_1 = 0$ . It is equal to zero when  $A_{01}(t_0, t_1) = 0$ . These points correspond to fibre components of the bisection  $B$  of the conic bundle. The discriminant is again  $-A_{01}(t_0, t_1)^2$ .  $\square$

**4.9. Theorem.** *Let  $S$  be a nodal cubic surface and let  $l$  be a line on  $S$ . Then the isomorphism class of the  $K3$  surface  $X_{S,l}$  associated to a pair  $(S, l)$  is independent on the choice of the line  $l$ .*

*Proof.* We compare the elliptic fibration  $f_l$  on  $X_{S,l}$  obtained from the pencil of lines through the singular point  $(0, 0, 1)$  of the branch curve  $W$  and the elliptic fibration  $f_{l,m}$  on the triple cover  $X_{S,l,m}$ , where  $m$  is a line disjoint from  $l$ . The fibre of  $f_l$  corresponding to a general line  $t_1x_0 - t_0x_1 = 0$ , with  $t_0 = 1$ , passing through the point  $(0, 0, 1)$  is birationally isomorphic to the curve

$$z^2 + x_2x_0^2(F_2(1, t_1)x_2^3 + F_5(1, t_1)x_0^3) = 0.$$

After the change of variables  $y = F_5z/x_0x_2^2$ ,  $x = F_5x_0/x_2$  we reduce this equation to the Weierstrass form (4.5) from Lemma 4.8. This shows that the surfaces  $X_{S,l}$  and  $X_{S,l,m}$  have isomorphic elliptic pencils. Hence  $X_{S,l} \cong X_{S,l,m}$ . Switching the roles of  $l$  and  $m$ , we see that  $X_{S,l} \cong X_{S,m}$ . It is easy to see that if two lines  $l, m$  on  $S$  are not skew, then there exists a third line  $n$  which is disjoint from  $l$  and  $m$ , so again  $X_{S,l} \cong X_{S,n} \cong X_{S,m}$ . We conclude that  $X_{S,l}$  does not depend on a choice of a line  $l$ .  $\square$

**4.10. Definition.** Let  $S$  be a nodal cubic surface. A  $K3$  surface associated to  $S$  is a  $K3$  surface  $X_S$  isomorphic to the surface  $X_{S,l}$  associated to a pair  $(S, l)$ , where  $l$  is a line on  $S$  defined in section 4.2 or the surface  $X_{S,l,m}$  associated to a triple  $(S, l, m)$ , where  $l, m$  is a pair of skew lines on  $S$  defined in 4.7.

As a corollary of the results above and those of the previous section we have:

**4.11. Corollary.** *The moduli space  $\mathcal{M}_{ncub}^1$  is isomorphic to the moduli space of elliptic  $K3$  surfaces with the Weierstrass form*

$$y^2 + x^3 + F_5(t_0, t_1)^2 F_2(t_0, t_1) = 0,$$

where  $(F_5, F_2)$  is a stable pair of binary forms of degrees 5 and 2.

**4.12. Cubic fourfolds.** Let  $Y \subset \mathbb{P}^5$  be a cubic hypersurface containing a plane  $\Pi$ . Then the projection from a plane defines a structure of a quadric bundle on the blow-up of  $Y$  along the plane  $\Pi$ . The discriminant curve of the quadric bundle is of degree 6. The double cover of  $\mathbb{P}^2$  branched along the discriminant parametrizes the rulings of quadrics. If  $Y$  is general enough, a minimal nonsingular model of the double cover is a  $K3$  surface. All of this is well-known and can be found for example in [Voi]. Let  $F(x_0, x_1, x_2, x_3) = 0$  be a nodal cubic surface. Consider the cubic fourfold  $Y$  defined by the equation

$$(4.10) \quad F(x_0, x_1, x_2, x_3) + x_4x_5(x_4 + x_5) = 0.$$

Let  $l \subset \{x_4 = x_5 = 0\}$  be a line on  $S$  and  $\Pi$  the plane spanned by  $l$  and a point  $(0, 0, 0, 0, 0, 1)$ . Then  $\Pi$  is contained in  $Y$ , and the direct computation shows that the corresponding  $K3$  surface is isomorphic to the  $K3$  surface  $X_{S,l}$ .

Let  $m$  be a line on  $S$  disjoint from  $l$  and let  $\Pi'$  be the plane contained in  $Y$  spanned by  $m \subset \{x_4 = x_5 = 0\}$  and the point  $(0, 0, 0, 0, 1, 0)$ . Consider the rational map  $T : \Pi \times \Pi' \dashrightarrow Y$  defined by taking the third intersection point of the line through the points  $(p, q) \in \Pi \times \Pi'$  with  $Y$ . A straightforward computation shows that the fundamental locus  $B$  of  $T$  is a complete intersection of divisors of bi-degree  $(2, 1)$  and  $(1, 2)$  and its minimal nonsingular model is isomorphic to the  $K3$  surface  $X_S$ . The surface  $B$  is nonsingular if  $S$  is nonsingular.

The lattice of transcendental cycles of  $X_S$  and that of the cubic fourfold  $Y$  are isomorphic. In fact, the blow-up  $Y'$  of  $Y$  along the union of two disjoint planes is isomorphic to the blow-up of  $\mathbb{P}^2 \times \mathbb{P}^2$  along the  $K3$  surface  $X \cong X_{l,m}$ . This gives an isomorphism of Hodge structures

$$H^4(Y', \mathbb{Z}) \cong H^4(\mathbb{P}^2 \times \mathbb{P}^2, \mathbb{Z}) \oplus H^2(X, \mathbb{Z})[1].$$

This isomorphism is compatible with the cup-product such that the two summands become orthogonal. Here  $H^2(X, \mathbb{Z})[1]$  is identified with  $\xi \cdot \pi^*(H^2(X, \mathbb{Z}))$ , where  $\pi : Y' \rightarrow \mathbb{P}^2 \times \mathbb{P}^2$  is the natural morphism of the blow-up and  $\xi$  is a cohomology class from  $H^2(Y', \mathbb{Z})$  which cuts out the tautological class of the exceptional divisor isomorphic to the projectivization of the normal bundle of  $X$ . This implies that the sublattice consisting of algebraic cycles in  $H^4(Y', \mathbb{Z})$  is isomorphic to  $H^4(\mathbb{P}^2 \times \mathbb{P}^2, \mathbb{Z}) \oplus \text{Pic}(X)[1]$ . Passing to the orthogonal complements we get the result.

**4.13. Cubic threefolds.** We relate the  $K3$  surface  $X_S$  to the Matsumoto-Terasoma curve associated to  $(S, l)$ . Given a smooth cubic surface  $S$  in  $\mathbb{P}^3$ , we define, following [ACT], the cubic threefold  $V \subset \mathbb{P}^4$  to be the triple cover of  $\mathbb{P}^3$  branched along  $S$ . So if

$$S : F(x_0, x_1, x_2, x_3) = 0,$$

then

$$V : F(x_0, x_1, x_2, x_3) + x_4^3 = 0.$$

Note that  $S \subset V$  (the points of  $V$  with  $x_4 = 0$ ), hence a line  $l \subset S$  defines a line, also denoted by  $l$ , in  $V$ . The projection of a cubic threefold away from a line on  $\mathbb{P}^2$  defines the structure of a conic bundle on the blow-up of  $V$  along the line. The associated discriminant curve in  $\mathbb{P}^2$  is a plane quintic. A straightforward computation shows that the discriminant curve is a plane quintic with the equation

$$W' : F_5(t_0, t_1) + t_2^3 F_2(t_0, t_1) = 0,$$

where the  $F_i$  are as in 3.1, so  $W'$  is a component of  $W$ .

**4.14. Remark.** Each smooth point  $t$  of the plane quintic  $W'$  defines two lines (the components of the singular conic in the fibre of  $V \rightarrow \mathbb{P}^2$  over  $t$ ). Thus there is a natural double cover  $C' \rightarrow W'$ . This double cover was studied by Matsumoto and Terasoma in [MT], the corresponding double cover  $C \rightarrow \bar{W}'$  of the normalizations of these curves is ramified in two points, which are identified in  $C'$ . The curve  $C$  is isomorphic to the affine curve ([MT], (3.1)):

$$v^3 - xf(x^2) = 0,$$

where  $f$  is a polynomial of degree 5. The Prym variety of the double cover  $C \rightarrow \bar{W}'$  is a 5-dimensional principally polarized abelian variety which is isomorphic to the intermediate Jacobian variety  $P$  of the cubic threefold  $V$  (cf. [MT]). The Matsumoto-Terasoma curve  $C$  has the following property.

**4.15. Proposition.** *Let  $f : X_{S,l} \rightarrow \mathbb{P}^1$  be the elliptic fibration as in the subsection 4.3. The pull-back of  $X_{S,l}$  along the base change  $C \rightarrow \mathbb{P}^1$ ,  $(v, x) \mapsto x$ , is birationally equivalent to the product  $C \times E$  where  $E$  is the elliptic curve with  $j = 0$ :  $E \cong \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\zeta_3)$ .*

*Proof.* In [MT] it is proved that that  $W = C/\iota$  where  $\iota$  is the (Clemens-Griffiths) involution  $\iota : (v, x) \mapsto (-v, -x)$ . Therefore the quotient curve is given by  $y^3 = u^2 f(u)$  where  $u = x^2$  and  $y = xv$ . This curve is birationally equivalent to  $W'$ . In fact, choosing coordinates such that  $F_2(y_0, y_1) = y_0 y_1$  the equation of  $W'$  is  $y_2^3 y_0 y_1 + F_5(y_0, y_1)$ , hence  $y_2^3 y_1 + F_5(1, y_1)$  is an affine equation. Putting  $v = -y_1 y_2$ ,  $u = y_1$  we find the birational isomorphism with  $f(u) = F_5(1, u)$ .

The function field of  $X_{S,l}$  is defined by  $s^2 = y_0y_1 + F_5(y_0, y_1)$ . The elliptic fibration is given by the rational function  $t = y_1/y_0$ . Rewriting the equation we get:  $(s/y_0)^2 = t + y_0^3F_5(1, t)$ , equivalently, since  $F_5(1, t) = f(t)$ :

$$Y^2 = X^3 + tf(t)^2 \quad (X = y_0f(t), Y = sf(t)/y_0).$$

Since on  $C$  we have  $v^6 = tf(t)^2$  we can write this as  $(sf(t)/y_0v^3)^2 = (y_0f(t)/v^2)^3 + 1$ , which is the equation  $Y^2 = X^3 + 1$  of the curve  $E$ .  $\square$

**4.16. Remark.** According to Donagi and Smith [DS], the Prym map  $\mathcal{R}_6 \rightarrow \mathcal{A}_5$  has degree 27 with the Galois group  $W(E_6)$ . Identifying the branch points on  $W$  and the ramification points on  $C$ , we obtain the admissible double cover  $C' \rightarrow W'$  in  $\mathcal{R}_6$ . Thus we get 27 ‘natural’ pre-images of  $P$  under the Prym map. However, the Prym map has 2-dimensional fibre over the intermediate Jacobian of a cubic threefold, in fact any line in the threefold defines an admissible double cover in  $\mathcal{R}_6$ .

## 5. THE PICARD LATTICE

In this section we compute the Picard lattice  $\text{Pic}(X_S) \subset H^2(X_S, \mathbb{Z})$  of the  $K3$  surface  $X_S$  associated to a nodal cubic surface and its orthogonal complement, the lattice of transcendental cycles  $T_{X_S} := \text{Pic}(X_S)^\perp$ .

**5.1. Lattices.** Recall the following two lattices:

$$U = \left( \mathbb{Z}^2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right), \quad A_2 = \left( \mathbb{Z}^2, \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \right).$$

The second cohomology group  $H^2(X, \mathbb{Z})$  equipped with the quadratic form defined by the cup-product is an even unimodular lattice of signature (3,19). It is isomorphic to the  $K3$  lattice

$$L = U^{\oplus 3} \oplus E_8^{\oplus 2},$$

where  $E_8 = \mathbb{Z}^8$  with the quadratic form defined by the opposite of the Cartan matrix of the root system of type  $E_8$ . In general,  $A_m, D_n, E_k$  denote the root lattices of the simple root systems of the corresponding symbol (with the Cartan matrix multiplied by  $-1$ ).

For any lattice  $M$  we denote by  $M(n)$  the lattice  $M$  with the quadratic form multiplied by  $n$ . Let  $M$  be a nondegenerate even lattice. The dual abelian group  $M^*$  contains  $M$  as a subgroup of finite index, the quotient group  $D(M) = M^*/M$  is called the *discriminant group* of  $M$ . It is equipped with a quadratic form

$$q : D(M) \rightarrow \mathbb{Q}/2\mathbb{Z}, \quad q(m^* + M) = t^{-2}(tm^*, tm^*) + 2\mathbb{Z},$$

where  $t \in \mathbb{Z}$  is such that  $tm^* \in M$ . We use the notation  $\text{O}(M)$  (resp.  $\text{O}(D)$ ) to denote the group of automorphisms of  $M$  (resp.  $D(M)$ ) preserving the quadratic form. If  $M$  is a primitive sublattice of a unimodular lattice there is a natural isomorphism  $D(M) \cong D(M^\perp)$ .

**5.2. Lattices  $M(\mathbf{t})$  and  $T(\mathbf{t})$ .** Recall that a choice of a line on a nodal cubic surface  $S$  defines an elliptic pencil on  $f : X_S \rightarrow \mathbb{P}^1$ . Its type is determined by the type vector  $\mathbf{t}$  of the conic bundle on  $S$  corresponding to  $l$ , cf. 2.15. We call it the type vector of  $(S, l)$  and the type vector of the elliptic fibration. We will explain later that for any possible type vector  $\mathbf{t}$  there exists a pair  $(S, l)$  of type  $\mathbf{t}$  such that the Picard lattice of the  $K3$  surface  $X_S$  is of rank  $12 + 2r + 2e$ , where  $r$  is the number of nodes on  $S$  and  $e$  is the number of Eckardt points on  $l$ . We denote by  $M(\mathbf{t})$  the smallest primitive sublattice of  $H^2(X_S, \mathbb{Z})$  containing the sections and components of fibres of the elliptic fibration defined by the

line  $l$ . Note that  $\text{Pic}(X_S) \cong M(\mathbf{t})$ . We will compute the lattice  $M(\mathbf{t})$  and its orthogonal complement  $T(\mathbf{t})$  in  $H^2(X_S, \mathbb{Z})$ .

**5.3. Proposition.** *Assume that the Mordell-Weil group  $MW(f)$  is finite. Then the lattices  $M(\mathbf{t})$  and  $T(\mathbf{t})$  are given in the following Table 2:*

$\mathbf{t}$	$M(\mathbf{t})$	$T(\mathbf{t})$
1) (2222211)	$U \oplus A_2^{\oplus 5}$	$A_2(-1) \oplus A_2^{\oplus 4}$
2) (322221)	$U \oplus D_4 \oplus A_2^{\oplus 4}$	$A_2(-2) \oplus A_2^{\oplus 3}$
3) (33222)	$U \oplus D_4^{\oplus 2} \oplus A_2^{\oplus 3}$	$A_2(-1) \oplus A_2(2)^{\oplus 2}$
4) (222222)	$U \oplus E_6 \oplus A_2^{\oplus 3}$	$A_2(-1) \oplus A_2^{\oplus 3}$
5) (422211)	$U \oplus E_6 \oplus A_2^{\oplus 3}$	$A_2(-1) \oplus A_2^{\oplus 3}$
6) (43221)	$U \oplus D_4 \oplus E_6 \oplus A_2^{\oplus 2}$	$A_2(-2) \oplus A_2^{\oplus 2}$
7) (4332)	$U \oplus D_4^{\oplus 2} \oplus E_6 \oplus A_2$	$A_2(-2) \oplus A_2(2)$
8) (42222)	$U \oplus E_6^{\oplus 2} \oplus A_2$	$A_2(-1) \oplus A_2^{\oplus 2}$
9) (44211)	$U \oplus E_6^{\oplus 2} \oplus A_2$	$A_2(-1) \oplus A_2^{\oplus 2}$
10) (52221)	$U \oplus E_8 \oplus A_2^{\oplus 3}$	$A_2(-1) \oplus A_2^{\oplus 2}$
11) (4431)	$U \oplus E_6^{\oplus 2} \oplus D_4$	$A_2(-2) \oplus A_2$
12) (5322)	$U \oplus E_8 \oplus D_4 \oplus A_2^{\oplus 2}$	$A_2(-2) \oplus A_2$
13) (4422)	$U \oplus E_8 \oplus E_6 \oplus A_2$	$A_2(-1) \oplus A_2$
14) (5421)	$U \oplus E_8 \oplus E_6 \oplus A_2$	$A_2(-1) \oplus A_2$
15) (543)	$U \oplus E_8 \oplus E_6 \oplus D_4$	$A_2(-2)$
16) (444)	$U \oplus E_8^{\oplus 2} \oplus A_2$	$A_2(-1)$
17) (552)	$U \oplus E_8^{\oplus 2} \oplus A_2$	$A_2(-1)$

TABLE 2. The Picard lattices

*Proof.* We will consider only the first two cases. Let  $f : X_S \rightarrow \mathbb{P}^1$  be the elliptic fibration of type  $\mathbf{t} = (2222211)$  with Picard lattice  $\text{Pic}(X_S) \cong M(\mathbf{t})$ . It follows from 4.3 that it has 5 reducible fibres of type *IV* and a section  $s$  defined by the line  $x_2 = 0$ . We will use the Shioda-Tate formula [Shi]:

$$(5.1) \quad (\#MW)^2 \cdot D(M(\mathbf{t})) = d_1 \dots d_k,$$

where  $MW$  is the Mordell-Weil group and  $d_1, \dots, d_k$  are the discriminants of the lattices generated by components of reducible fibres not intersecting the zero section. It follows from (5.1) that the Mordell-Weil group  $MW$  is a torsion group of order  $3^l$ . We claim that it is trivial. Assume  $MW$  is not trivial. Then the translation by a nontrivial section defines an automorphism of  $X_S$  of order 3 which acts trivially on  $\text{Pic}(X_S)^\perp$  and has at least 7 isolated fixed points on  $X_S$  (= the singular points of the 7 singular fibres). This contradicts the fact that the fixed locus of any symplectic automorphism of  $K3$  surface of order 3 is exactly 6 isolated points (Nikulin [N3], §5). Thus  $f$  has a unique section  $s$ . Now we use (5.1) again and find that the discriminant of  $M$  is equal to  $3^5$ . Since  $M = M(\mathbf{t})$  obviously contains the sublattice  $U \oplus A_2^{\oplus 5}$  of the same rank and discriminant (it is spanned by the class of a fibre, the section, and irreducible components of reducible fibres), it must coincide with it. The discriminant

group is then easy to compute. Let  $q_T$  be the discriminant form of  $T$ , then  $q_T = -q_M$  ([N1], Prop. 1.6.1). We can easily see that  $T$  and  $A_2(-1) \oplus A_2^4$  have the same discriminant form. It now follows from Nikulin [N1], Cor. 1.13.3 that  $T \cong A_2(-1) \oplus A_2^4$ .

Assume that the fibration is of type (322221). The product  $d_1 \dots d_k$  is equal to  $2^2 3^4$ . The Shioda-Tate formula gives that either  $\#\text{MW} = 1, 3$ , or  $3^2$ , or  $6$ . Assume that MW contains a nontrivial section  $s'$ . If  $s'$  is of order 2, it must leave invariant one component in each fibre of type  $IV$  and has one fixed point in it different from the singular point of the fibre. Altogether this gives 8 fixed points on reducible fibres and 1 fixed point on the irreducible fibre. However, a symplectic involution on a  $K3$  surface has exactly 8 fixed points. Assume  $s'$  is of order 3. Then it leaves the multiple component and non-multiple component of the fibre of type  $I_0^*$  invariant. This gives  $\geq 3$  isolated fixed points on this fibre. Together with singular points of other fibres we get  $\geq 8$  isolated fixed points. This is impossible. So, the Shioda-Tate formula tells us that  $D(M(\mathfrak{t}))$  is of order  $2^2 3^4$ . The remaining arguments are similar to the previous case.  $\square$

**5.4. The lattices  $M, T$ .** We set

$$M := U \oplus A_2^{\oplus 5}, \quad T := A_2(-1) \oplus A_2^{\oplus 4}.$$

Since their discriminant groups are isomorphic and the quadratic forms are the negative of each other, they are orthogonal complements of each other in the unimodular lattice  $L$  (see [N1]). We set

$$D = D(M) \cong D(T).$$

These lattices correspond to the type  $\mathfrak{t} = (2222211)$ .

**5.5. An automorphism of order 3.** As in section 4.7, we choose two skew lines on a nodal cubic surface  $S$  and consider the associated  $K3$  surface  $X = X_S \cong X_{S,l,m}$ . Recall that it is obtained as a minimal resolution of the triple cyclic cover  $Y$  of  $\mathbb{P}^1 \times \mathbb{P}^1$  branched along the union of two divisors  $L$  and  $M$  of bidegree  $(1, 2)$  and  $(2, 1)$ . It is easy to describe the set of fixed points of the automorphism  $\sigma$  of  $X$  defined by the triple cover. We do it only in the case when  $S$  is a nonsingular surface. Let  $q_1, \dots, q_5$  be the intersection points of  $L$  and  $M$ . The cubic surface  $S$  is obtained by blowing up the points  $q_i$ 's. The surface  $S$  is nonsingular if and only if no two points lie on a ruling, and no four points lie on a plane section. An Eckardt point on the line  $l$  corresponds to a ruling which is tangent to  $L$  at some point  $q_i$ .

Assume that there are no Eckardt points on  $l$ . Consider the elliptic fibration on  $f : X \rightarrow \mathbb{P}^1$  corresponding to the projection  $\mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$  such that  $L$  is a section. Its reducible singular fibres corresponds to the ruling passing through the points  $q_i$ . Each fibre is of type  $IV$ . Two components are the exceptional curves of the resolution  $X \rightarrow Y$  of a singular point of type  $A_2$ . The third component is the proper transform of the ruling passing through the corresponding point  $q_i$ . The bisection  $b$  intersects the latter component and one of the first two components. The section  $s$  intersects the other component coming from the resolution of singularities. The set of fixed points of  $\sigma$  is equal to the union of the section  $s$ , the bisection  $b$  and the singular points of the reducible fibres.

In the case when  $l$  contains one Eckardt point, the elliptic fibration acquires one reducible fibre of type  $I_0^*$ . Other reducible fibres are of type  $IV$ . The bisection  $b$  intersects the multiple component  $E_0$  of this fibre. The section  $s$  intersects a reduced component  $E_1$ . The fixed points of the involution  $\sigma$  is the union of the section  $s$ , the bisection  $b$ , the point  $E_0 \cap E_1$ , and the singular points of fibres of type  $IV$ . If  $l$  has two Eckardt points, we have two reducible fibres of type  $IV$  and the set of fixed points is described similarly to the previous case.

**5.6. The involution  $\tau$ .** Let  $f : X \rightarrow \mathbb{P}^1$  be the elliptic fibration with a section  $s$  as in section 5.5. Let  $\tau$  be the involution of  $X$  defined by the inversion  $x \mapsto -x$  of each fibre. Then  $\tau$  switches the two components of each singular fibre of type  $IV$  which do not meet  $s$  and preserves each component of any singular fibre of type  $I_0^*$ .

If  $f$  has five singular fibres of type  $IV$  and two singular fibres of type  $II$ , then the fixed locus of  $\tau$  is the union of  $s$  and a smooth curve  $C$  of genus 5 which passes through the singular point of each singular fibre. If  $f$  has four singular fibres of type  $IV$ , one of type  $I_0^*$  and one of type  $II$ , then the fixed locus of  $\tau$  is the union of  $s$ , the multiple component of the fibre of type  $I_0^*$  and a smooth curve of genus 3. If  $f$  has three singular fibres of type  $IV$  and two fibres of type  $I_0^*$ , then the fixed locus of  $\tau$  is the union of  $s$ , two multiple components of singular fibres of type  $I_0^*$  and a smooth elliptic curve.

**5.7. Remark.** The automorphism group of the  $K3$  surface  $X$  is infinite. For example, consider the divisor consisting of the 2-section and the two components of a reducible singular fibre of  $f$  not meeting the section. It defines an elliptic fibration on  $X$  with a section which has two reducible singular fibres, one is of type  $I_3$  and another type  $I_0^*$ . This elliptic fibration has a Mordell-Weil group of rank 4. Considering translations by the sections of infinite order we see that  $\text{Aut}(X)$  is an infinite group.

**5.8. Lemma.** *Assume  $S$  is nonsingular. Then*

$$H^2(X, \mathbb{Z})^{\sigma^*} \subset \text{Pic}(X), \quad H^2(X, \mathbb{Z})^{\sigma^*} \cong M.$$

*The automorphism  $\sigma$  acts trivially on the discriminant lattice  $D(H^2(X, \mathbb{Z})^{\sigma^*}) \cong D(M)$ .*

*Proof.* Consider the elliptic fibration on  $X$  defined in 4.3. From 5.5 we know the description of fixed points of  $\sigma$ . Assume first that all reducible fibres are of type  $IV$ . Let  $P$  be the sublattice of  $\text{Pic}(X)$  spanned by the divisor classes of a fibre, of the section  $s$  and of the irreducible components of fibres which do not intersect  $s$ . It is immediate that  $P \cong M$  and  $\sigma$  acts identically on  $P$ . The fixed locus  $X^\sigma$  of the automorphism  $\sigma$  consists of 5 isolated fixed points (the singular points of the reducible fibres) and two smooth rational curves (the section  $s$  and the bisection  $b$ ). Applying the Lefschetz fixed point formula we obtain that the trace of  $\sigma^*$  on  $H^2(X, \mathbb{Z})$  is equal to 7. Thus the trace of  $\sigma^*$  on  $P^\perp$  is equal to  $7 - 12 = -5$ . This easily implies that the characteristic polynomial of  $\sigma^*$  on  $P^\perp \otimes \mathbb{C}$  is equal to  $(t^2 + t + 1)^5$ . Therefore  $P^\perp \otimes \mathbb{C}$  does not contain non-zero  $\sigma^*$ -invariant elements, so  $H^2(X, \mathbb{Z})^{\sigma^*} = P \cong M$ . Since  $\sigma^*$  acts trivially on  $P \cong M$ , it also acts trivially on  $D(P) \cong D(M)$ .

Suppose now that  $f$  contains a fibre  $F = 2E_0 + E_1 + E_2 + E_3 + E_4$  of type  $I_0^*$ . Assume that  $E_1$  intersects the section  $s$ . Then the divisor classes  $E_0 + E_2 + E_3 + E_4$  and  $E_0$  are  $\sigma$ -invariant and span a lattice of type  $A_2$ . We define the lattice  $P$  similar to the above by using this contribution from a fibre of type  $I_0^*$ . The remaining arguments are the same.  $\square$

## 6. THE MODULI SPACE OF $K3$ SURFACES ASSOCIATED TO A CUBIC SURFACE

**6.1.** We first recall the basic facts about moduli of  $K3$  surfaces. In the subsections before 6.5,  $M$  will be any even non-degenerate sublattice of signature  $(1, t)$ .

**6.2. Markings.** We recall the definition of a  $M$ -marking of a  $K3$  surface  $X$  (see [Do]). Fix a connected component  $V(M)^+$  of the cone  $V(M) = \{x \in M \otimes \mathbb{R} : (x, x) > 0\}$  and a subset  $\Delta(M)^+$  of the set  $\Delta(M) = \{\delta \in M : (\delta, \delta) = -2\}$  such that

- $\Delta(M) = \Delta(M)^+ \amalg \Delta(M)^-$ , where  $\Delta(M)^- = \{-\delta : \delta \in \Delta(M)^+\}$ ,

- any  $\delta \in \Delta(M)$  which can be written as a nonnegative linear combination of elements from  $\Delta(M)^+$  belongs to  $\Delta(M)^+$ .

With these choices, we define the subset

$$C(M)^+ = \{h \in V(M)^+ : (h, \delta) > 0 \text{ for all } \delta \in \Delta(M)^+\}$$

and we define  $C(M)$  to be the closure of  $C(M)^+$  in  $M \otimes \mathbb{R}$ .

Now we define a  $M$ -marking of  $X$  as a primitive lattice embedding  $\phi : M \rightarrow \text{Pic}(X)$  such that  $C(X)^+ \cap \phi(M \otimes \mathbb{R}) \subset \phi(C(M)^+)$ , where  $C(X)^+$  is the cone in  $\text{Pic}(X) \otimes \mathbb{R}$  spanned by the pseudo-ample (i.e. nef and big) divisor classes of  $X$ .

Note that the closure of  $C(X)^+$  is the nef cone  $C(X)$ . The closure  $C(M)$  of  $C(M)^+$  is the subset of the closure of  $V(M)^+$  which consists of vectors  $v$  such that  $(v, \delta) \geq 0$  for any  $\delta \in \Delta(M)^+$ . The marking  $\phi$  embeds  $C(X) \cap \phi(M \otimes \mathbb{R})$  in  $\phi(C(M))$ . For any  $\delta \in \Delta(M)^+$  the image  $\phi(\delta)$  is a divisor class  $R$  with  $R^2 = -2$ . For any  $v \in C(M)$  the image  $\phi(v)$  is a pseudo-ample divisor  $D$  with  $D^2 \geq 0$ . Since  $R \cdot D = (\delta, v) > 0$ , it follows from Riemann-Roch that  $R$  is effective. Note that  $R$  is not necessary the divisor class of an irreducible curve (a  $(-2)$ -curve).

The marking is called *ample* if  $\phi(C(M)^+) \cap \text{Pic}(X)^+ \neq \emptyset$ , where  $\text{Pic}(X)^+$  is the ample cone of  $X$ . It is easy to see that a marking  $\phi$  is ample if and only if the orthogonal complement of  $\phi(M)$  in  $\text{Pic}(X)$  does not contain the divisor classes of  $(-2)$ -curves. In particular, any marking with  $\phi(M) = \text{Pic}(X)$  is ample.

A pair  $(X, \phi)$ , where  $\phi$  is a  $M$ -marking (resp. an ample  $M$ -marking), is called a  *$M$ -polarized  $K3$  surface* (resp. *ample  $M$ -polarized  $K3$  surface*). Two  $M$ -polarized  $K3$  surfaces  $(X, \phi)$  and  $(X', \phi')$  are called isomorphic if there exists an isomorphism  $f : X \rightarrow X'$  such that  $\phi = f^* \circ \phi'$ .

**6.3. Moduli of  $M$ -polarized surfaces.** It is known (see [Do]) that there exists a coarse moduli space  $\mathcal{M}_{K3, M}$  of isomorphism classes of  $M$ -polarized  $K3$  surfaces. Let us assume that  $M$  admits a unique (up to an isometry) embedding into the  $K3$  lattice  $L = U^{\oplus 3} \oplus E_8^{\oplus 2}$ . Fix such an embedding. Let  $T$  be the orthogonal complement of  $M$  in  $L$ . Any  $M$ -marking  $\phi$  of a  $K3$  surface  $X$  extends to an isometry  $\tilde{\phi} : L \rightarrow H^2(X, \mathbb{Z})$  (a cohomology marking of  $X$ ). Extending  $\tilde{\phi}$   $\mathbb{C}$ -linearly, we get a one dimensional subspace  $\tilde{\phi}^{-1}(H^{2,0}(X)) \subset T \otimes \mathbb{C}$  which is called the *period* of  $(X, \tilde{\phi})$ .

$$\begin{array}{ccc} M & \subset & L \\ \phi \downarrow & & \downarrow \tilde{\phi} \\ \text{Pic}(X) & \hookrightarrow & H^2(X, \mathbb{Z}) \end{array}$$

The moduli space  $\mathcal{M}_{K3, M}$  is isomorphic to the quotient  $\mathcal{D}_M / \Gamma_M$ , where  $\mathcal{D}_M$  is the union of two copies of a Hermitian symmetric domain of type IV corresponding to the inner product vector space  $T \otimes \mathbb{R}$  of signature  $(2, 20 - t)$ ,  $\mathcal{D}_M$  is a subset of the projective space  $\mathbb{P}(T \otimes \mathbb{C})$ . The group  $\Gamma_M$  is the subgroup of the orthogonal group  $O(L)$  of  $L$  which leaves  $M$  pointwise fixed. It is also isomorphic to the subgroup of  $O(T)$  which acts identically on the discriminant group  $D(T) = T^*/T$ .

The isomorphism classes of ample  $M$ -polarized  $K3$  surfaces are parametrized by an open subset of  $\mathcal{M}_{K3, M}$  whose complement is the image in  $\mathcal{M}_{K3, M}$  of the union of hypersurfaces in  $\mathcal{D}_M$  defined by lines in  $T \otimes \mathbb{C}$  orthogonal to vectors  $r \in T$  with  $r^2 = -2$ .

**6.4. The group  $W(M)$ .** For any  $\delta \in \Delta(M)$  we can define a reflection  $s_\delta \in O(M)$  associated to  $\delta$  by  $s_\delta : v \mapsto v + (v, \delta)\delta$ . Let  $W(M)$  be the subgroup of  $O(M)$  generated by all  $s_\delta$ 's. The set  $C(M)$  is a fundamental domain for  $W(M)$  in the closure of  $V(M)^+$ . Thus for any  $v \in M$  with  $v^2 \geq 0$  there exists a  $w \in W(M)$  such that  $(w(v), \delta) \geq 0$ , for any  $\delta \in \Delta(M)^+$ .

Let  $(X, \phi)$  be a  $M$ -polarized  $K3$  surface. Then for any  $v \in M$  with  $v^2 \geq 0$  there is a  $w \in W(M)$  such that  $\phi(w(v)) \in C(M)$ . In particular, for any given embedding  $\phi : M \rightarrow \text{Pic}(X)$ , there is a  $w \in W(M)$  such that  $C(X)^+ \cap \phi(M \otimes \mathbb{R}) \subset (\phi \circ w)(C(M)^+)$ , i.e.,  $\phi \circ w$  is a  $M$ -marking.

**6.5. Fixing  $V(M)^+$  and  $\Delta(M)^+$ .** The lattice  $M$  from 5.4 has a unique (up to an isometry) primitive embedding in the  $K3$  lattice  $L$  [N1] and we identify  $M$  with a primitive sublattice of  $L$  from now on. We fix a basis in  $U$  formed by two isotropic vectors  $f_1, f_2$  with  $(f_1, f_2) = 1$  and a simple root basis  $r_1, r_2$  in  $A_2$ , i.e.,  $(r_1)^2 = (r_2)^2 = -2$  with  $(r_1, r_2) = 1$ . We define a basis of  $M$  by taking  $f_1, f_2$  in  $U$  and  $r_1, r_2$  in each copy of  $A_2$ .

We define  $V(M)^+$  by requiring that  $f_1 + f_2 \in V(M)^+$ . We define  $\Delta(M)^+$  as follows. Firstly,  $(-2)$ -vectors with  $(f_1 + f_2, v) > 0$  belong to it. Secondly, if  $(f_1 + f_2, v) = 0$ , then  $v \in \Delta(M)^+$  if and only if it is a nonnegative combination of  $f_2 - f_1$  and the  $r_i$ 's in each copy of  $A_2$ .

**6.6. Automorphisms of  $L$ .** Let  $\rho_o$  be the isometry of  $A_2$  defined by

$$\rho_o(r_1) = r_2, \quad \rho_o(r_2) = -r_1 - r_2.$$

Obviously  $\rho_o$  is of order 3, has no non-zero fixed vectors and acts trivially on  $D(A_2) = (A_2)^*/A_2$ . Let  $\rho$  be the isometry of  $T = A_2(-1) \oplus A_2^{\oplus 4}$  defined by  $\rho = (\rho_o)^{\oplus 5}$ . Then  $\rho$  is of order 3, has no non-zero fixed vectors and acts trivially on  $D(T)$ . Thus the isometry  $(1_M, \rho)$  of  $M \oplus T$  can be extended to the one of the  $K3$  lattice  $L$  (Nikulin [N1], Corollary 1.5.2). For simplicity we denote this isometry of  $L$  by the same letter  $\rho$ .

**6.7. Period domains.** The period domain for  $M$ -polarized  $K3$  surfaces is

$$\mathcal{D}_M = \{\omega \in \mathbb{P}(T \otimes_{\mathbb{Z}} \mathbb{C}) : (\omega, \omega) = 0, \quad (\omega, \bar{\omega}) > 0\}.$$

Let  $\rho$  be the isometry of  $T$  defined in 6.6. Let

$$T \otimes \mathbb{C} = V_+ \oplus V_-$$

be the decomposition of  $T \otimes \mathbb{C}$  into the two 5-dimensional eigenspaces of  $\rho$  with eigenvalues  $\zeta_3 = e^{2\pi i/3}$  and  $\zeta_3^{-1}$ , respectively. Since

$$(\omega, \omega) = (\rho(\omega), \rho(\omega)) = \zeta^2(\omega, \omega),$$

we see that  $(\omega, \omega) = 0$  for all  $\omega \in V_+$ , and similarly for  $V_-$ . Let

$$\mathcal{B} = \{\omega \in \mathbb{P}(V_+) : (\omega, \bar{\omega}) > 0\} = \mathcal{D}_M \cap \mathbb{P}(V_+).$$

In a suitable basis of  $V_+$  we have  $(\omega, \bar{\omega}) = x_0 \bar{x}_0 - (x_1 \bar{x}_1 + \dots + x_4 \bar{x}_4)$ . Thus, if  $(\omega, \bar{\omega}) > 0$ , then  $x_0 \neq 0$  and we can normalize  $x_0 = 1$ , hence  $\mathcal{B}$  is a 4-dimensional complex ball:

$$\mathcal{B} \cong \{x = (x_1, \dots, x_4) \in \mathbb{C}^4 : \sum_i x_i \bar{x}_i < 1\}.$$

The 4-ball is a bounded symmetric domain of type  $I_{1,4}$ .

**6.8. Discrete groups.** We define the following four groups using the notation from 6.6:

$$\begin{aligned}\Gamma_M &= \{g \in \mathbf{O}(L) : g(m) = m, \quad \forall m \in M\}, \\ \tilde{\Gamma}_\rho &= \{g \in \mathbf{O}(L) : g \circ \rho = \rho \circ g\}, \\ \Gamma_\rho &= \{g \in \mathbf{O}(T) : g \circ \rho = \rho \circ g\}, \\ \Gamma_{M,\rho} &= \mathbf{Ker}(\Gamma_\rho \rightarrow \mathbf{O}(D)).\end{aligned}$$

**6.9. The Hermitian module.** The isometry  $\rho$  of  $T$  gives  $T$  the structure of a free module  $\Lambda$  of rank 5 over the ring of Eisenstein integers  $\mathbb{Z}[\zeta_3]$ : for any  $a + b\zeta_3 \in \mathbb{Z}[\zeta_3]$  and any  $x \in T$  we have

$$(a + b\zeta_3) \cdot x = (a1_T + b\rho)(x).$$

If  $r_i, r'_i$  is the simple root basis of the  $i$ -th copy of  $A_2$  with  $\rho(r_i) = r'_i$ , then  $\zeta_3 r_i = r'_i$  and any element in this  $A_2$  can be written as  $r = zr_i$  with  $z = a + b\zeta_3 \in \mathbb{Z}[\zeta_3]$ . Note that:

$$z\bar{z} = (a + b\zeta_3)(a + b\zeta_3^{-1}) = a^2 - ab + b^2 = -(r, r)/2.$$

Therefore the quadratic form on  $T$  is twice the real part of the  $\mathbb{Z}[\zeta_3]$ -valued Hermitian form  $H$ , of signature  $(1, 4)$ , on the Eisenstein lattice  $T$  with

$$H(z, w) = z_0\bar{w}_0 - (z_1\bar{w}_1 + \dots + z_4\bar{w}_4).$$

The group  $\Gamma_\rho$  is the unitary group  $U(T)$  of  $T$  considered as a Hermitian lattice over the ring of Eisenstein integers (see [ACT], [AF]).

**6.10. The discriminant group.** The residue field  $\mathbb{Z}[\zeta_3]/\sqrt{-3}\mathbb{Z}[\zeta_3]$  is isomorphic to  $\mathbb{F}_3$  and  $\zeta_3$  maps to 1 mod 3. Thus  $V = \Lambda/\sqrt{-3}\Lambda$  acquires a natural structure of a 5-dimensional vector space over  $\mathbb{F}_3$  equipped with a non-degenerate quadratic form. We show that the discriminant group  $D(T)$  is isomorphic to  $V$ . Define a  $\mathbb{Z}$ -linear homomorphism

$$(6.1) \quad h : \Lambda \longrightarrow T^*, \quad h(x) = (x + 2\rho(x))/3,$$

where we identify  $\Lambda$  with  $T$  as a  $\mathbb{Z}$ -module. Then  $h(\sqrt{-3}x) = h((1 + 2\zeta_3)x) = (1 + 2\rho)^2 x/3 = -x \in T$ . This shows that  $h$  factors through an isomorphism

$$V = \Lambda/\sqrt{-3}\Lambda \longrightarrow D(T) = T^*/T.$$

The basis  $(r_1, \dots, r_5)$  of  $\Lambda$  (as  $\mathbb{Z}[\zeta_3]$ -module) is an orthonormal basis with respect to  $H$ . Since  $h(r_i)^2 = (r_i + 2r'_i)^2/9 = -\frac{2}{3}$ ,  $(h(r_i), h(r_j)) = 0, i \neq j$ , we obtain that

$$h(x)^2 = -\frac{2}{3}x^2.$$

In particular, if we identify  $D(T)$  with  $V$ , then the quadratic form on  $D(T)$  obtained from the quadratic form on  $V$  by multiplying it by  $-\frac{2}{3}$ .

If  $Q$  is the root lattice of type  $E_6$ , then  $Q/3Q$  inherits a non-degenerate quadratic form such that  $Q$  is isomorphic to  $V$  as quadratic spaces over  $\mathbb{F}_3$ . This defines an isomorphism of groups

$$(6.2) \quad \begin{aligned}W(E_6) &\cong \mathbf{SO}(V), \\ \mathbf{O}(D(T)) &\cong \mathbf{O}(V) \cong \{1, -1\} \times \mathbf{SO}(V).\end{aligned}$$

All of this is well-known and can be found, for example, in [Bo], Chapter 6, §4, exercise 2.

**6.11. Proposition.** *Each of the natural maps*

$$\tilde{\Gamma}_\rho \longrightarrow \Gamma_\rho \longrightarrow \mathcal{O}(D(T))$$

*is surjective. In particular,*

$$\Gamma_\rho/\Gamma_{M,\rho} \cong \mathcal{O}(D(T)) \cong \{\pm 1\} \times W(E_6).$$

*Moreover, any isometry in  $\Gamma_{M,\rho}$  can be extended to an isometry of  $L$  which acts trivially on  $M$  defining an injective homomorphism of groups*

$$\Gamma_{M,\rho} \hookrightarrow \Gamma_M.$$

*Proof.* For the surjectivity of the map  $\Gamma_\rho \rightarrow \mathcal{O}(D(T))$  see [ACT], Lemma 4.5. It is proven in Nikulin [N1], Theorem 1.14.2 that the natural map  $\mathcal{O}(M) \rightarrow \mathcal{O}(D(M))$  is surjective. By Corollary 1.5.2 of loc. cit. this implies that the map  $\tilde{\Gamma}_\rho \rightarrow \Gamma_\rho$  is surjective. The inclusion  $\Gamma_{M,\rho} \rightarrow \Gamma_M$  follows from (Nikulin [N1], Corollary 1.5.2).  $\square$

**6.12. Definition.** An (ample)  $(M, \rho)$ -polarized  $K3$  surface is an (ample)  $M$ -polarized  $K3$  surface  $(X, \phi)$  such that there is an extension  $\tilde{\phi} : L \rightarrow H^2(X, \mathbb{Z})$  of  $\phi$  which satisfies

$$\tilde{\phi}^{-1}(H^{2,0}(X)) \in \mathcal{B} \quad (\subset \mathbb{P}(T \otimes \mathbb{C})).$$

Two  $(M, \rho)$ -polarized  $K3$  surfaces  $(X, \phi)$  and  $(X', \phi')$  are said to be isomorphic if there is an isomorphism  $f : X \rightarrow X'$  such that  $\phi = f^* \circ \phi'$  and  $\tilde{\phi}^{-1} \circ f^* \circ \tilde{\phi}' \in \mathcal{O}(L)$  commutes with  $\rho \in \mathcal{O}(L)$ .

**6.13. Lemma.** *Let  $(X, \phi)$  be an ample  $(M, \rho)$ -polarized  $K3$  surface. Then  $X$  has an automorphism  $\sigma$  of order 3 such that  $\sigma^* = \tilde{\phi} \circ \rho \circ \tilde{\phi}^{-1}$  for an extension  $\tilde{\phi} : L \rightarrow H^2(X, \mathbb{Z})$  of  $\phi$ . In particular,  $\sigma$  acts trivially on  $\phi(M) (\subset \text{Pic}(X))$ .*

*Proof.* Choosing  $\tilde{\phi}$  as in the definition of  $(M, \rho)$ -polarization, the period of  $X$  is fixed by  $\rho$ . Since  $(X, \phi)$  is amply polarized,  $\text{Pic}(X) \cap M^\perp$  contains no  $(-2)$ -vectors. Moreover, the  $M$ -polarization of  $X$  is ample and  $\rho$  acts trivially on  $M$ . Therefore [Na], Theorem 3.10 shows that  $X$  has an automorphism  $\sigma$  with  $\sigma^* = \tilde{\phi} \circ \rho \circ \tilde{\phi}^{-1}$ .  $\square$

**6.14. The moduli spaces  $\mathcal{K}3_{M,\rho}^m$  and  $\mathcal{K}3_{M,\rho}$ .** We know from section 6.3 that the moduli space of  $M$ -polarized  $K3$  surfaces is isomorphic to  $\mathcal{D}/\Gamma_M$ . The isometry  $\rho$  acts naturally on  $T_{\mathbb{C}}$  as is described in 6.7 and induces an automorphism of order 3 of the domain  $\mathcal{D}_M \subset \mathbb{P}(T_{\mathbb{C}})$ . It defines the union of two balls  $\mathcal{B}_\pm = \mathcal{D}_M \cap \mathbb{P}(V_\pm)$ . Complex conjugation switches the two balls  $\mathcal{B}_\pm$ . Obviously the group  $\Gamma_\rho$  is the stabilizer subgroup of  $\mathcal{B} = \mathcal{B}_+$  in  $\Gamma_M$ . We set

$$\mathcal{K}3_{M,\rho}^m = \mathcal{B}/\Gamma_{M,\rho}, \quad \mathcal{K}3_{M,\rho} = \mathcal{B}/\Gamma_\rho.$$

The element  $-I \in \Gamma_\rho$  acts trivially on  $\mathbb{P}(T \otimes \mathbb{C})$  and thus on  $\mathcal{B}$ , and  $-I$  maps to  $-1 \in \mathcal{O}(D)$ . Thus  $\mathcal{O}(D)/\{\pm 1\} \cong W(E_6)$  acts on  $\mathcal{K}3_{M,\rho}^m$  and there is a natural map:

$$\pi_M : \mathcal{K}3_{M,\rho}^m \longrightarrow \mathcal{K}3_{M,\rho} \cong \mathcal{K}3_{M,\rho}^m/W(E_6).$$

For  $r \in L$ , let  $r^\perp$  be the hyperplane in  $\mathbb{P}(V_+)$  of lines orthogonal to  $r$ , and let  $H(r)$  be its intersection with  $\mathcal{B}$ . The *discriminant locus* is the subset  $\mathcal{H} \subset \mathcal{B}$  defined by:

$$\mathcal{H} = \bigcup_r H(r),$$

where  $r$  varies over the set of all  $(-2)$ -vectors in  $T = M^\perp$ . The image of  $\mathcal{H}$  in  $\mathcal{K}3_{M,\rho}^m$  (resp.  $\mathcal{K}3_{M,\rho}$ ) will be denoted by  $\Delta^m$  (resp.  $\Delta$ ).

It follows from Lemma 6.13 that the quasi-projective variety  $\mathcal{K}3_{M,\rho}^m \setminus \Delta^m$  is the coarse moduli space of ample  $(M, \rho)$ -polarized  $K3$  surfaces. We will refer to  $\mathcal{K}3_{M,\rho}^m$  as the moduli space of  $(M, \rho)$ -polarized  $K3$  surfaces.

**6.15. Remark.** If  $[(X, \phi)], [(X', \phi')] \in \mathcal{K}3_{M,\rho}^m$  are in the same fibre of  $\pi_M$ , then the  $K3$  surfaces  $X$  and  $X'$  are isomorphic. This follows from the surjectivity of the map  $\tilde{\Gamma}_\rho \rightarrow \Gamma_\rho$  and the Torelli theorem for  $K3$  surfaces. Let  $\alpha \in \mathcal{O}(D(M))$ . As we already noticed in the proof of Proposition 6.11, we can lift  $\alpha$  to an isometry  $\tilde{\alpha}$  of  $M$ . Composing it with some element of  $W(M)$  which acts identically on  $D(M)$ , we may assume that  $\tilde{\alpha}$  leaves  $\Delta(M)^+$  invariant. Now  $\alpha$  acts on  $[(X, \phi)] \in \mathcal{K}3_{M,\rho}^m$  by  $[(X, \phi)] \mapsto [(X, \phi \circ \tilde{\alpha}^{-1})]$ . This describes the action of  $\mathcal{O}(D(M))$  on  $\mathcal{K}3_{M,\rho}^m$ . If  $\phi(M) = \text{Pic}(X)$ , then  $\mathcal{O}(D(M))$  acts transitively on the markings of  $X$ . Thus we can interpret a general point of  $\mathcal{K}3_{M,\rho}$  as the isomorphism class of a  $K3$  surface which admits an ample  $(M, \rho)$ -marking.

**6.16.** Recall that the subspaces  $V_+$  and  $V_-$  (see 6.7) are defined over  $\mathbb{Q}(\zeta_3)$  where  $\zeta_3$  is a primitive cube root of unity. Let  $K$  be the extension field of  $\mathbb{Q}(\zeta)$  obtained by adjoining all primitive  $6l$ -th roots of unity for which the value of the Euler function satisfies  $\varphi(6l) \leq 10 = \text{rank}(T)$ . The only possible values of  $l$  are as follows:  $l = 1, 2, 3, 4, 5$ . We consider the union  $\mathcal{W}$  of hyperplanes of  $\mathbb{P}(V_+)$  defined over  $K$ . A non-singular cubic surface  $S$  is called *generic* if the period of the associated  $K3$  surface  $X_S$  is contained in the complement of  $\mathcal{W}$ . For example, a cubic surface with an Eckardt point is not generic (we shall show in 8.9 that the period of  $X_S$  is contained in the hyperplane orthogonal to some vector  $r \in T$ ).

**6.17. Lemma.** *Assume that  $S$  is a generic cubic surface and let  $X_S$  be the associated  $K3$  surface. Then the image of the natural map*

$$\text{Aut}(X_S) \longrightarrow \mathcal{O}(T)$$

*is a cyclic group of order 6 generated by  $\tau$  and  $\sigma$  (For  $\tau, \sigma$ , see 5.5, 5.6). In particular the image of the natural map*

$$\text{Aut}(X_S) \longrightarrow \mathcal{O}(D(T))$$

*is  $\{\pm 1\}$ .*

*Proof.* The proof is similar to the one given in [BP], Lemma 2.9. It is known that the image  $G$  is a cyclic group (Nikulin [N3], Theorem 3.1). Let  $m$  be the order of  $G$ . If  $g \in \text{Aut}(X_S)$  is a generator of  $G$ , then  $g^*\omega_X = \zeta_m \cdot \omega_X$  where  $\omega_X$  is a nowhere vanishing holomorphic 2-form on  $X = X_S$  and  $\zeta_m$  is a primitive  $m$ -th root of unity. Since  $\tau^*\omega_X = -\omega_X$  and  $\sigma^*\omega_X = \zeta_3\omega_X$ ,  $m$  is divisible by 6. Since  $g^*$  is defined over  $\mathbb{Q}$ , the eigenspaces of  $g^*$  are defined over  $\mathbb{Q}(\zeta_m)$ . If  $m > 6$ , then an eigenspace is a non-trivial subspace of  $V_+$ . This contradicts the assumption of genericity of  $S$ .  $\sigma^*$  acts trivially on  $D(T)$  and  $\tau^*$  acts as  $-1$ . Hence the second assertion follows.  $\square$

**6.18. Corollary.** *The map  $\pi_M : \mathcal{K}3_{M,\rho}^m \rightarrow \mathcal{K}3_{M,\rho}$  is a Galois cover with the Galois group isomorphic to  $W(E_6)$ .*

*Proof.* As we explained in 6.14 the group  $O(D(T))/\{\pm 1\} \cong W(E_6)$  acts on  $\mathcal{K}3_{M,\rho}$  with quotient isomorphic to  $\mathcal{K}3_{M,\rho}$ . The isotropy subgroup of  $[(X, \phi)]$  is isomorphic to the image of  $\text{Aut}(X)$  in  $D(\phi(M)^\perp)/\{\pm 1\}$ . By the previous lemma it is trivial for a generic surface  $X$ .  $\square$

**6.19. Nef divisors.** Let  $(X, \phi)$  be an ample  $M$ -polarized  $K3$  surface. Then  $X$  has an automorphism  $\sigma$  of order 3 (6.13). For any  $v \in M$  with  $v^2 \geq 0$  there is a  $w \in W(M)$  such that  $\phi(w(v)) \in C(M)$ . If  $\phi(w(v))$  is not nef, then there is a smooth rational curve  $R$  with  $(R, \phi(w(v))) < 0$ . Since  $\phi(M)^\perp \cap \text{Pic}(X)$  does not contain  $(-2)$ -vectors,  $R = r + r'$  where  $r \in M^*$ ,  $r' \in T^*$  and  $r^2 < 0$ ,  $(r')^2 < 0$ . Since  $r^2 + (r')^2 = R^2 = -2$ ,  $r^2 = -2/3$  or  $-4/3$ . Since  $\sigma$  is an automorphism,  $(R, \sigma(R)) \geq 0$ . Hence  $(3r)^2 = (R + \sigma(R) + \sigma^2(R))^2 \geq -6$ . Thus  $r^2 = -2/3$ . Then  $r$  defines a reflection

$$s_r : x \longmapsto x + 3(x, r)r$$

which acts trivially on  $T$ . Obviously  $(R, \phi(s_r(w(v)))) > 0$ . If necessary, by using these reflections successively, we may assume that  $\phi(w(v)) \in C(X)$ , i.e.,  $\phi(w(v))$  is nef. In particular, any primitive isotropic vector  $f$  in  $M$  defines, uniquely, a nef divisor in  $\text{Pic}(X)$ . As is well-known a primitive nef divisor  $F$  with  $F^2 = 0$  defines an elliptic fibration with the cohomology class of a fibre equal to  $F$  ([PS], §3, Cor.3).

**6.20. Elliptic fibrations.** Let  $(X, \phi)$  be an ample  $M$ -polarized  $K3$  surface. With the definitions from 6.5, we have  $f_1 \in C(M)$  and  $f_1$  is obviously isotropic and primitive. Therefore,  $\phi(f_1) \in \text{Pic}(X)$  defines an elliptic fibration on  $V$  (cf. 6.19) which we denote by

$$\Phi_\phi : X \longrightarrow \mathbb{P}^1$$

and we call it the *standard elliptic fibration*. Since  $\phi(f_2 - f_1) \cdot \phi(f_1) = (f_2 - f_1, f_1) = 1$ , the divisor class  $\phi(f_2 - f_1)$  is an effective class with  $D^2 = -2$ . Let  $D$  be the effective representative of this class written as a sum  $\sum n_i R_i$ , where  $R_i$  are irreducible curves. Since  $D$  intersects any fibre  $F$  with multiplicity 1, we see that one of the components  $R_i$ , say  $R_1$ , is a section of the fibration. We also have  $n_1 = 1$  and  $R_i \cdot F = 0$  for  $i > 1$ . By the Hodge Index Theorem,  $R_i^2 < 0$  for  $i > 1$ . By the adjunction formula, all  $R_i$ 's are  $(-2)$ -curves and the  $R_i$ 's,  $i \neq 1$ , are contained in fibres of the fibration. This easily implies that  $R_1$  is determined uniquely by  $\phi(f_2 - f_1)$ . We shall denote the section corresponding to  $R_1$  by  $s$ . We remark that  $R_1$  is obtained from  $D$  by applying suitable reflections corresponding to  $R_i$  ( $i > 1$ ). Thus, up to isometries, we may assume that the classes  $f_1$  and  $f_2 - f_1$  define an elliptic fibration  $\Phi_\phi$  with a section  $s$ .

The images under  $\phi$  of the simple root bases  $\{r_i, r'_i\}$ ,  $i = 1, \dots, 5$ , of each copy of  $A_2$  are effective divisor classes  $R_i, R'_i$  on  $X$  which are orthogonal to  $F$  and to the section  $s$ . As above we can show that each such divisor class is a sum of  $(-2)$ -curves contained in a fibre. Thus  $X$  has at least 10 smooth rational curves contained in fibres of  $\Phi_\phi$ .

**6.21. Lemma.** *Let  $(X, \phi)$  be an ample  $(M, \rho)$ -polarized  $K3$  surface, let  $\sigma$  be an automorphism of order three as in 6.13 and let  $\Phi_\phi$  be the standard elliptic fibration on  $X$ .*

*Then  $\sigma$  preserves  $\Phi_\phi$  and fixes pointwisely its section  $s$  and a smooth bisection  $b$ . Moreover, the singular fibres of  $\Phi_\phi$  are of the following types:*

$$(II, II, IV, IV, IV, IV, IV), \quad (II, IV, IV, IV, IV, I_0^*), \quad (IV, IV, IV, I_0^*, I_0^*).$$

*In each case the fibration has exactly 5 reducible fibres.*

*Proof.* Let  $X^\sigma$  be the fixed locus of the automorphism  $\sigma$ . Since  $\sigma$  can be locally linearized,  $X^\sigma$  is a smooth closed subset of  $X$ . It is easy to see that the trace of  $\rho$  in its action on  $L \cong H^2(X, \mathbb{Z})$  is equal to 7. Applying the Lefschetz fixed point formula, we obtain that the Euler characteristic of  $X^\sigma$  is equal to 9. Since  $\sigma$  acts identically on  $\phi(M)$ , it preserves the section  $s$  and the divisor class of a fibre of  $\Phi_\phi$ . Let us show that  $\sigma$  fixes the section  $s$  pointwisely, or, equivalently, leaves invariant each fibre of  $\Phi_\phi$ . Assuming otherwise, we obtain that  $X^\sigma$  is contained in fibres of  $\Phi_\phi$ . Thus any irreducible one-dimensional component of  $X^\sigma$  has the Euler characteristic equal to 0 (if it is nonsingular fibre) or 2 (if it is a component of a reducible fibre), the smoothness of the fixed point set excludes nodal cubics. Let  $l$  be the number of irreducible one-dimensional components of  $X^\sigma$  different from a fibre, and let  $k$  be the number of isolated fixed points. Then  $2l + k = \chi(X^\sigma) = 9$ . Since  $\sigma$  has exactly two fixed points on  $s$ , it leaves invariant the two fibres  $F_1, F_2$  passing through these points. Obviously the curves  $R_i, R'_i$  (see 6.20) are contained in the union  $F_1 \cup F_2$ . In particular, the number of irreducible components of the divisor  $F_1 + F_2$  is greater than or equal to 12. Since a Dynkin diagram of type ADE admits a non-trivial automorphism of order 3 only in the case  $D_4$ , the automorphism  $\sigma$  acts identically on the set of irreducible components of a fibre  $F_i$  unless it is of type  $I_0^*$ . Note that either  $F_1$  or  $F_2$  is not of type  $I_0^*$  because  $F_1 + F_2$  has at least 12 components. Assume that both of the  $F_i$ 's are not of this type. We apply the Lefschetz fixed point formula to the cell complex  $F_i$ . Let  $n_i$  be the number of irreducible components of  $F_i$ . The Lefschetz number of  $\sigma|_{F_i}$  is equal to  $n_i$  if  $F_i$  is of type  $I_n$  and to  $n_i + 1$  otherwise. Let  $l_i$  be the number of one-dimensional rational components of  $X^\sigma$  contained in  $F_i$  and let  $k_i$  be the number of isolated fixed points of  $\sigma$  contained in  $F_i$ . We have  $2l_i + k_i \geq n_i$ , hence  $9 = 2l + k \geq 2l_1 + k_1 + 2l_2 + k_2 \geq n_1 + n_2 \geq 12$ , a contradiction. Assume that one of the fibres, say  $F_1$  is of type  $I_0^*$ . Then  $2l_2 + k_2 \geq n_2 \geq 12 - 5 = 7$ . The automorphism  $\sigma$  has a fixed point on the non-multiple component  $E$  of  $F_1$  which is intersected by  $s$ . The multiple component  $E_0$  of  $F_1$  is  $\sigma$ -invariant. If  $\sigma$  is the identity on  $E_0$ , then  $l_1, k_1 \geq 1$ , and  $2l_1 + k_1 \geq 3$ . If  $\sigma$  does not acts identically on  $E_0$ , it has 2 fixed points on it. In both cases it is easy to see that  $2l_1 + k_1 \geq 3$  again. Thus we get  $2l_1 + k_1 + 2l_2 + k_2 \geq 3 + n_2 \geq 3 + 7 = 10$ , again a contradiction.

Now we know that  $\sigma$  preserves every fibre of  $\Phi_\phi$ , so that the general fibre has a non-trivial automorphism of order 3 over the function field of the base. This implies that the  $j$ -function of the fibration is constant. In particular, the singular fibres must be of additive type  $II, III, IV, IV^*, II^*, III^*, I_n^*$ . Each nonsingular fibre has exactly 3 fixed points of  $\sigma$ , one lies on the section  $s$ , and the pairs of others lie on a bisection  $b$  (which could be the union of two sections). The bisection  $b$  is a part of  $X^\sigma$  and hence smooth.

Let  $\pi : X' \rightarrow X$  be the blow-up of the 0-dimensional part of  $X^\sigma$ . We know that  $\sigma$  is not symplectic (i.e. does not leave invariant a non-zero holomorphic 2-form on  $X$ ). This easily shows that it lifts to an automorphism  $\sigma'$  of  $X'$  with  $X'^{\sigma'}$  purely one-dimensional. Let  $\bar{X}'$  be the quotient surface  $X'/(\sigma')$ . It is a smooth surface. Let  $C$  be a smooth rational curve on  $X$  such that  $\sigma(C) = C$  but  $\sigma|_C$  is not the identity. Then  $\sigma$  has two fixed points  $p, q$  on  $C$ . If  $p, q$  are isolated fixed points of  $\sigma$  on  $X$ , then the proper inverse transform  $C'$  on  $X'$  has self-intersection  $-4$ . Since  $C'$  is equal to the pre-image of some curve on  $\bar{X}'$  and  $-4$  is not divisible by 3, we get a contradiction. Similarly, if  $p, q$  belong to one-dimensional part of  $X^\sigma$ , we get  $C'^2 = -2$  and again get a contradiction. Thus, one fixed point is an isolated fixed point of  $\sigma$  and another one belongs to the one-dimensional part of  $X^\sigma$ .

As we have already observed before,  $\sigma$  acts identically on the set of irreducible components of any fibre, unless it is of type  $I_0^*$ . In particular, all intersection points of components are fixed. The previous

observation about the intersection of  $X^\sigma$  with fibres easily excludes the possibility for a reducible fibre of  $\Phi_\phi$  to be of type  $III$ ,  $I_n^*$  ( $n \neq 0$ ),  $III^*$ . In the case of  $I_0^*$ ,  $\sigma$  preserves the multiple component  $E$  and permutes the three simple components  $E_1, E_2, E_3$  not meeting the section. Notice that any  $\sigma$ -invariant irreducible component of a fibre not intersecting the section  $s$  must belong to  $\phi(M) \cap \phi(U)^\perp = \phi(A_2^5)$ . The fixed part of  $D_4 = \langle E, E_1, E_2, E_3 \rangle$  under  $\sigma^*$  is  $\langle E, E + E_1 + E_2 + E_3 \rangle \cong A_2$ . Since  $E_6$  and  $E_8$  can not be embedded into  $A_2^5$ , singular fibres of type  $IV^*$ ,  $II^*$  do not appear.

Using that the Euler characteristics of the fibres add up to 24, it remains to show that we have exactly 5 reducible fibres. Since a fibre of type  $I_0^*$  or  $IV$  contributes one copy of  $A_2$  in  $A_2^5 \cong \phi(M) \cap \phi(U)^\perp$ , there must be five of them. The lemma is now proven.  $\square$

## 7. A COMPLEX BALL UNIFORMIZATION

**7.1. From  $K3$ 's to cubics.** We are going to construct a map

$$G : \mathcal{K}3_{M,\rho}^m \setminus \Delta^m \longrightarrow \mathcal{M}_{\text{cub}}^m,$$

where  $\mathcal{M}_{\text{cub}}^m$  is the moduli space of marked smooth cubic surfaces, i.e., smooth cubic surfaces with an ordered set of six skew lines  $L_1, \dots, L_6$ .

Let  $[(X, \phi)] \in \mathcal{K}3_{M,\rho}^m \setminus \Delta^m$  be an ample  $(M, \rho)$ -polarized  $K3$  surface. We use the notation of Lemma 6.21 and its proof. For simplicity we consider the case where  $\Phi_\phi$  has two singular fibres of type  $II$  and five singular fibres of type  $IV$ . The construction for the other two cases is similar. It follows from the proof of lemma 6.21 that on each reducible fibre  $\sigma$  has one fixed point, the point of intersection of the three components. The bisection  $b$  intersects two components, and the section  $s$  intersects the third one. Let  $X'$  be the blow-up of the five isolated fixed points of  $\sigma$  as in the proof of the lemma. The quotient  $\bar{X}'$  of  $X'$  by the action of  $\sigma$  is a smooth rational surface and the images of the components of the fibers of type  $IV$  are  $(-1)$ -curves in  $\bar{X}'$ . The marking  $\phi$  gives an ordering of the 2 components in each fibre which meet the bisection  $b$ , and we blow down the first one in each of the 5 fibres as well as the component in the fibre which meets the section. The result is a smooth rational surface  $S$  which has  $(-1)$ -curves  $L_1, \dots, L_5$  the images of the remaining components in the type  $IV$  fibres (these are numbered by the marking  $\phi$ ) as well as the  $(-1)$ -curve  $m$  which is the image of the section  $s$ . These six curves do not intersect and thus can be blown down to get a smooth rational surface with  $b_2 = 1$ , hence this surface must be  $\mathbb{P}^2$ . Therefore  $S$  is a cubic surface and the six  $(-1)$ -curves define a marking on  $S$ . It is easy to see that this marked cubic surface  $S$  depends only on the isomorphism class of  $(X, \phi)$ . We may now define:

$$G : [(X, \phi)] \longmapsto (S, L_1, \dots, L_5, L_6 = m).$$

Note that the 2-section  $C$  maps to a line  $l$  in  $S$  which is skew with  $m$  and does meet  $L_1, \dots, L_5$ . By the uniqueness of the triple cover (Theorem 4.9) we have that  $X \cong X_{S,l,m}$  and, by construction (see 6.13)  $\sigma^* = \tilde{\phi} \circ \rho \circ \tilde{\phi}^{-1}$  for some extension  $\tilde{\phi} : L \rightarrow H^2(X, \mathbb{Z})$  of  $\phi$ .

**7.2. Theorem.** *The map  $G$  defines a  $W(E_6)$ -equivariant isomorphism*

$$G : \mathcal{K}3_{M,\rho}^m \setminus \Delta^m \xrightarrow{\cong} \mathcal{M}_{\text{cub}}^m.$$

*Proof.* We first construct the inverse map

$$G^{-1} : \mathcal{M}_{\text{cub}}^m \longrightarrow \mathcal{K}3_{M,\rho}^m \setminus \Delta^m.$$

Given  $(S, L_1, \dots, L_6) \in \mathcal{M}_{\text{cub}}^m$ , let  $m = L_6$  and let  $l$  be the (unique) line which meets  $L_1, \dots, L_5$  but not  $m$  (if we blow down the  $L_i$  to points  $x_i \in \mathbb{P}^2$ ,  $l$  maps to the conic on  $x_1, \dots, x_5$ ).

Let  $X_{l,m}$  be the  $K3$  surface associated to  $(S, l, m)$  and let  $f : X_{l,m} \rightarrow \mathbb{P}^1$  be the elliptic fibration from subsection 4.3. We define a marking  $\phi_{l,m} : M \rightarrow \text{Pic}(X_{l,m})$  as in the proof of Lemma 5.8 by fixing an order on the set of reducible fibres and the order on the set of components of fibres of type  $IV$  which do not intersect the section  $s$ . Thus  $\phi(f_1)$  is the class of a fibre of  $f$  and  $\phi(f_2)$  is the sum of the class of a fibre and the class of the section (see 6.20). The image of  $r_1$  in the  $i$ -th copy of  $A_2 \subset M$  is the first component of the  $i$ -th fibre if it is of type  $IV$ , and it is the divisor class  $E + E_1 + E_2 + E_3$  if the  $i$ -th fibre is of type  $I_0^*$  (see the notation in the proof of Lemma 6.21).

The  $K3$  surface  $X_{l,m}$  is a triple cyclic covering of  $S$  with an automorphism  $\sigma$ . We proved in Lemma 5.8 that  $\sigma^*$  acts identically on  $\phi(M)$  and has the trace  $-5$  on  $\phi(M)^\perp$ . This implies that  $\sigma^*$  has no eigenvectors in  $\phi(M)^\perp \otimes \mathbb{Q}$ , and hence  $\phi(M)^\perp$  is a free module of rank 5 over the ring of Eisenstein integers  $\mathbb{Z}[\zeta_3]$ . In particular, the maps  $\sigma^*$  glue to a locally constant map on the local system with fibers  $H^2(X_{l,m}, \mathbb{Z})$ . The construction of the map  $G$  is such that if  $(S', L'_1, \dots, L'_6) = G(X, \phi)$  for some  $(X, \phi)$ , then  $\rho = \tilde{\phi}^{-1} \circ \sigma_{S'}^* \circ \tilde{\phi}$  where  $\tilde{\phi} : L \rightarrow H^2(X_{l,m}, \mathbb{Z})$  is a cohomology marking of  $X$  such that  $\tilde{\phi}|_M = \phi$  and  $\tilde{\phi}(T) = \phi(M)^\perp$ . As  $\sigma^*$  is locally constant we conclude that there is an extension  $\tilde{\phi}_{l,m}$  of the marking  $\phi_{l,m}$  such that  $\rho = \tilde{\phi}_{l,m}^{-1} \circ \sigma^* \circ \tilde{\phi}_{l,m}$ . This shows that  $G^{-1}[(S, L_1, \dots, L_6)] := [(X_{l,m}, \phi)]$  belongs to  $\mathcal{K}3_{M,\rho}^m \setminus \Delta^m$ . It is obvious that  $G^{-1}$  is the inverse of  $G$ .

We show that  $G^{-1}$  is  $W(E_6)$ -equivariant, then  $G = (G^{-1})^{-1}$  is obviously equivariant as well. The group  $W(E_6)$  acts on  $\mathcal{M}_{\text{cub}}^m$  in the standard way via symmetries of the set of lines and  $W(E_6) = \text{Gal}(\mathcal{M}_{\text{cub}}^m / \mathcal{M}_{\text{cub}})$ . Let  $\mu : \text{Gal}(\mathcal{M}_{\text{cub}}^m / \mathcal{M}_{\text{cub}}) \rightarrow \text{Aut}(\mathcal{K}3_{M,\rho}^m \setminus \Delta^m)$  be the action defined via the isomorphism  $G^{-1}$ , obviously  $\mu$  is injective. Let  $S \in \mathcal{M}_{\text{cub}}$ , the main result of the section 3 (Theorem 4.9) was that  $X_{l,m}$  is independent of the choice of the lines  $l, m$  in  $S$ , hence  $\mu(g)$  is a covering transformation of  $\mathcal{K}3_{M,\rho}^m \setminus \Delta^m \rightarrow \mathcal{K}3_{M,\rho} \setminus \Delta$  for any  $g \in W(E_6)$ . Thus we have an injection:

$$\mu : W(E_6) \cong \text{Gal}(\mathcal{M}_{\text{cub}}^m / \mathcal{M}_{\text{cub}}) \longrightarrow \text{Gal}(\mathcal{K}3_{M,\rho}^m / \mathcal{K}3_{M,\rho}).$$

Since  $\text{Gal}(\mathcal{K}3_{M,\rho}^m / \mathcal{K}3_{M,\rho}) \cong W(E_6)$  (see 6.18),  $\mu$  is an isomorphism.  $\square$

7.3. The moduli space of cubic surfaces  $\mathcal{M}_{\text{cub}}$  is the quotient of  $\mathcal{M}_{\text{cub}}^m$  by  $W(E_6)$ . Let  $W(E_6)_l \subset W(E_6) \subset \text{Aut}(\text{Pic}(S))$  be the subgroup which fixes the class of a line  $l$  on  $S$ . It is well-known that  $W(E_6)_l \cong W(D_5)$ , which is the semi-direct product of  $(\mathbb{Z}/2)^4$  and  $S_5$ .

The action of  $S_5 \subset W(D_5)$  on a marking  $(L_1, \dots, L_6 = l)$  of a cubic surface is by permuting the first 5 lines. The group  $W(D_5)$  is generated by these permutations and an element  $c_{123}$  of order two which acts as the standard Cremona transformation on  $\mathbb{P}^2$  defined by the points  $p_1, p_2$  and  $p_3$  where  $\pi : S \rightarrow \mathbb{P}^2$  is the blow down of the  $L_i$  and  $p_i = \pi(L_i)$ . Thus  $c_{123}$  maps  $L_1$  to  $L'_1$ , the strict transform of the line on  $p_2$  and  $p_3$ , and it fixes  $L_4, L_5$  and  $L_6$ . It also permutes the  $2 \cdot 5$  lines on  $S$  which meet  $l$ . Let  $l_i$  be the line which maps to the line through  $p_i$  and  $p_6$  and let  $m_i$  be the conic through all 6 points except  $p_i$ . Then  $c_{123}$  fixes the  $l_i$  and  $m_i$  except for permuting  $l_4 \leftrightarrow m_5$  and  $l_5 \leftrightarrow m_4$ . This implies that an element in  $W(D_5)$  permutes the indices and exchanges an even number of  $l_i$  with an even number of  $m_i$ .

7.4. Recall from Proposition 6.11 that

$$\Gamma_\rho / \Gamma_{M,\rho} \cong O(D) \cong W(E_6) \times \{\pm 1\}$$

acts on the discriminant lattice  $D = D(T) \cong \mathbb{F}_3^5$ . The subgroup of  $O(D)$  which consists of isometries preserving an unordered basis (up to signs) of  $D(T)$  is isomorphic to  $W(D_5) \times \{\pm 1\}$ . This provides us with a natural copy of  $W(D_5)$  in  $\Gamma_\rho/\Gamma_{M,\rho}$ . Let  $\Gamma'_{M,\rho}$  be the inverse image in  $\Gamma_\rho$  of this subgroup. The group  $\Gamma'_{M,\rho}$  acts on  $\mathcal{K}3_{M,\rho}^m$  by changing the markings without changing the standard elliptic fibration defined by the marking. Since  $W(D_5)$  is a maximal subgroup of  $W(E_6)$  we see that any  $w \in W(E_6) \setminus W(D_5)$  does not preserve the isomorphism class of the standard elliptic fibration. This implies the following corollaries:

**7.5. Corollary.** *Let  $\mathcal{M}_{cub}$  be the moduli space of cubic surfaces. There are isomorphisms*

$$(\mathcal{B} \setminus \mathcal{H})/\Gamma_{M,\rho} \cong \mathcal{K}3_{M,\rho} \setminus \Delta \cong \mathcal{M}_{cub}.$$

*Let  $\mathcal{M}_{cub}^1$  be the moduli space of cubic surfaces with a line. There are isomorphisms*

$$(\mathcal{B} \setminus \mathcal{H})/\Gamma'_{M,\rho} \cong (\mathcal{K}3_{M,\rho}^m \setminus \Delta^m)/W(D_5) \cong \mathcal{M}_{cub}^1.$$

*as well as a birational isomorphism*

$$\mathcal{B}/\Gamma'_{M,\rho} \simeq \mathcal{M}_{cub}^1$$

*where  $\Gamma'_{M,\rho}$  is the inverse image of  $W(E_6)_l \times \{\pm 1\} \subset W(E_6) \times \{\pm 1\} \cong \Gamma_\rho/\Gamma_{M,\rho}$  in  $\Gamma_\rho$ .*

**7.6. Corollary.** *Assume that  $S$  is a generic cubic surface. Then  $X_S$  has exactly 27 (= the index of  $W(D_5)$  in  $W(E_6)$ ) non-isomorphic standard elliptic fibrations.*

## 8. THE GEOMETRY OF THE DISCRIMINANT LOCUS.

**8.1.** Here we will give a geometric interpretation of the points in  $\mathcal{K}3_{M,\rho}^m$  belonging to the discriminant locus  $\Delta^m$ . We know that each such point represents the isomorphism class of a non-amply  $M$ -polarized  $K3$  surface  $(X, \phi)$ . For such a surface there is a  $(-2)$ -vector  $r$  in  $\phi(M)^\perp \cap \text{Pic}(X)$ . This implies that  $\rho$  (cf. 6.6) can not be represented by an automorphism of  $X$ . Let  $R$  be the sublattice of  $\text{Pic}(X)$  generated by all  $(-2)$ -vectors in  $\phi(M)^\perp \cap \text{Pic}(X)$ . Then  $R$  is a negative definite lattice generated by  $(-2)$ -vectors, i.e., a root lattice. Hence  $R$  is an orthogonal direct sum

$$R = R_1 \oplus \cdots \oplus R_r,$$

where  $R_i$  is an indecomposable root lattices of type  $A_m, D_n, E_k$ . Obviously  $\rho$  preserves  $R$ . Since  $\rho$  has no non-zero fixed vectors in  $R$ ,  $\rho$  preserves each  $R_i$ . Thus  $R_i$  is an indecomposable root lattice with an isometry of order 3 without non-zero fixed vectors. In the following we shall show that  $R_i \cong A_2$  and  $r \leq 4$  (see 8.7).

**8.2. Lemma.**  $R_i \cong A_2$  for any  $i$ .

*Proof.* First of all, note that the rank of  $R_i$  is even because it has an isometry of order 3 without non-zero fixed vectors. Since the rank of  $\text{Pic}(X) \leq 20$ ,  $R_i$  is isometric to  $A_{2n}, D_{2n}, E_6$  or  $E_8$  ( $n \leq 4$ ). Let  $K$  be a primitive sublattice of  $H^2(X, \mathbb{Z})$  generated by  $M$  and  $R$ . Let  $l(K)$  be the number of minimal generator of the 3-elementary subgroup of  $K^*/K$ . Then  $K^*/K \cong (K^\perp)^*/K^\perp$  and  $l(K) = l(K^\perp) \leq \text{rank}(K^\perp)$ . Using this observation and the fact  $l(M) = 5$ , we can easily see that  $R$  is isometric to  $D_4, A_2^{\oplus n}$  ( $1 \leq n \leq 4$ ) or  $E_6$  (for example if  $R = E_8$ , then  $K = M \oplus E_8$  and  $l(K) = 5$ . This contradicts to the fact  $l(K^\perp) \leq \text{rank}(K^\perp) = 2$ ). Next we shall show that  $R$  is not isometric to  $D_4$ . In this case  $K = M \oplus D_4$  and the elliptic fibration defined by  $M$ -marking has five singular fibres of type  $IV$  and one of type  $I_0^*$ . This contradicts the fact that the Euler number of  $K3$  surface is 24. By the same argument, the case  $R = E_6$  does not occur.  $\square$

8.3. We remark that all  $R_i$  are 3-elementary, i.e.,  $R_i^*/R_i \cong (\mathbb{Z}/3\mathbb{Z})^l$  for some non-negative integer  $l$  and  $\rho$  acts trivially on  $R_i^*/R_i$ .

Let

$$T' = (\phi(M) \oplus R)^\perp, \quad S = (T')^\perp \quad (\subset H^2(X, \mathbb{Z})).$$

Thus  $S$  is the smallest primitive sublattice of  $H^2(X, \mathbb{Z})$  containing  $\phi(M) \oplus R$ . By definition, the lattice  $T' \cap \text{Pic}(X)$  contains no  $(-2)$ -vectors.

8.4. **Lemma.** *Let  $(X, \phi)$  be an  $(M, \rho)$ -polarized K3 surface. Let  $S, R, T'$  be as above. Then  $S, T'$  are 3-elementary lattices, and  $\rho$  acts trivially on  $(T')^*/T'$ . Moreover  $X$  has an automorphism  $\sigma'$  of order three such that  $S = H^2(X, \mathbb{Z})^{(\sigma')^*}$ .*

*Proof.* We have a chain of lattices:

$$\phi(M) \oplus R \subset S \subset S^* \subset (\phi(M) \oplus R)^*$$

and  $S^*/S \cong (S^*/(\phi(M) \oplus R))/(S/(\phi(M) \oplus R))$ . Since  $M$  and  $R$  are 3-elementary,  $S$  is a 3-elementary lattice, i.e.,  $S^*/S \cong (\mathbb{Z}/3\mathbb{Z})^l$ . Since  $\rho$  acts trivially on  $(\phi(M) \oplus R)^*/(\phi(M) \oplus R) \cong \phi(M)^*/\phi(M) \oplus R^*/R$ ,  $\rho$  acts trivially on  $S^*/S$ . Since  $T'$  is the orthogonal complement of  $S$  in unimodular lattice  $H^2(X, \mathbb{Z})$ ,  $T'$  is 3-elementary and  $\rho$  acts trivially on  $(T')^*/T'$  (see Nikulin [N1], Proposition 1.6.1). Hence the isometry  $(1_S, \rho | T')$  can be extended to an isometry  $\rho'$  of  $H^2(X, \mathbb{Z})$  (Nikulin [N1], Corollary 1.5.2). Then  $\rho'$  is represented by an automorphism  $\sigma'$  of  $X$  (see [Na], Theorem 3.1).  $\square$

The following fact was first observed by Vorontsov [Vor].

8.5. **Lemma.** *We keep the same assumption as in Lemma 8.4. Define a non-negative integer  $l(T')$  by:  $(T')^*/T' \cong (\mathbb{Z}/3\mathbb{Z})^{l(T')}$ . Then*

$$\text{rank}(T') \geq 2l(T').$$

*Proof.* Let  $x \in T'$ . Since

$$(x, \rho'(x)) = (\rho'(x), (\rho')^2(x)) = (\rho'(x), -x - \rho'(x)),$$

we get  $2(x, \rho(x)) = -(x, x)$ . Hence  $x$  and  $\rho'(x)$  generate a sublattice  $A_2(m)$ , where  $m = (x, x)$ . From this we can find a sublattice  $K = A_2(m_1) \oplus \cdots \oplus A_2(m_k)$  of  $T'$  of finite index. Moreover we have  $(T')^*/T' \cong ((T')^*/K)/(T'/K)$ . If  $m_i$  is not divisible by 3, the contribution from  $A_2(m_i)$  to  $l(T')$  is at most 1. In case  $m_i$  is divisible by 3, the fixed part under  $\rho'$  in  $A_2(m_i)^*/A_2(m_i)$  is  $\mathbb{Z}/3\mathbb{Z}$ . Since  $\rho$  acts trivially on  $(T')^*/T'$ , the contribution from  $A_2(m_i)$  is at most 1. This implies the assertion.  $\square$

8.6. **Lemma.** *We keep the same notation as in Lemma 8.4. Then  $R \cong A_2^{\oplus r}$  and  $l(S) = 5 - r$ .*

*Proof.* Let

$$R = R_1 \oplus \cdots \oplus R_r$$

be the orthogonal decomposition of  $R$  into indecomposable root lattices  $R_i$ . We know that  $R_i$  is isomorphic to  $A_2$  (Lemma 8.2). Obviously  $R_i^*/R_i$  is  $\mathbb{Z}/3\mathbb{Z}$ . Since  $S^*/S \cong (S^*/(\phi(M) \oplus R))/(S/(\phi(M) \oplus R))$ , we have  $l(T') = l(S) \geq (l(M) + r) - 2r = 5 - r$ . On the other hand, it follows from Lemma 8.5 that  $10 - 2r \geq \text{rank}(T') \geq 2l(T')$ . Hence  $l(S) = 5 - r$ .  $\square$

Let us summarize the previous lemmas by stating the following:

**8.7. Theorem.** *Let  $(X, \phi) \in \mathcal{K}3_{M,\rho}^m$ . Then  $X$  admits an automorphism  $\sigma'$  of order 3 such that  $H^2(X, \mathbb{Z})^{(\sigma')^*} = S$ , the smallest primitive sublattice of  $\text{Pic}(X)$  which contains  $\phi(M)$  and the sublattice  $R$  generated by all  $(-2)$ -vectors in  $\phi(M)^\perp \cap \text{Pic}(X)$ . The sublattices  $\phi(M)$  and  $R$  are orthogonal to each other and the lattice  $R$  is isomorphic to  $r$  ( $\leq 4$ ) copies of the lattice  $A_2$ . The number  $r$  will be called the degeneracy rank of  $(X, \phi)$ .*

The degeneracy rank of  $(X, \phi)$  is equal to the number of nodes of the associated nodal cubic surface (see 2.15). This is easy to see from Table 2 by computing the quotient of  $M(\mathbf{t})$  by  $M = U \oplus A_2^{\oplus 5}$  and comparing the result with the value of  $r$  in Table 1. The next theorem generalizes Lemma 6.21.

**8.8. Theorem.** *Let  $[(X, \phi)] \in \mathcal{K}3_{M,\rho}^m$ . Then the  $M$ -marking  $\phi$  of  $X$  defines an elliptic fibration. Its singular fibres are given in the column Kodaira fibres of Table 1 from above. The Picard lattice  $S_X$  and its lattice of transcendental cycles  $T_X$  can be found in the corresponding rows of Table 2 (under the assumption in Proposition 5.3). The degeneracy level is given in the column  $r$  in Table 1.*

*Proof.* By the same arguments as in 6.19, 6.20, the  $M$ -marking on  $X$  defines an elliptic fibration with a section. The proof of the assertion about possible combinations of singular fibres is very similar to the proof of Lemma 6.21 and is omitted. The description of the transcendental lattice follows from the following easy facts:

$$q_{E_6} = -q_{A_2}, \quad q_{A_2(-1)} = -q_{A_2}, \quad q_{A_2} \oplus q_{A_2} = q_{A_2(-1)} \oplus q_{A_2(-1)}, \quad q_{A_2(-2)} = q_{D_4} \oplus q_{A_2}$$

and Theorem 1.14.2 from [N1]. □

**8.9. The Eckardt locus.** Let  $[(X, \phi)] \in \mathcal{K}3_{M,\rho}^m \setminus \Delta^m$ . We know that the corresponding marked cubic surface  $(S, L_1, \dots, L_6)$  has an Eckardt point on the unique line  $l$  intersecting  $L_1, \dots, L_5$  if and only if the standard elliptic fibration  $\Phi_\phi$  on  $(X, \phi)$  has a fibre of type  $I_0^*$ . In that case  $\phi(M) \neq \text{Pic}(X)$ , but for general  $S$  with such property, the orthogonal complement  $\phi(M)_{\text{Pic}(X)}^\perp$  of  $\phi(M)$  in  $\text{Pic}(X)$  is isomorphic to  $A_2(2)$ . In fact if  $F = 2E_0 + E_1 + \dots + E_4$  is the fibre of type  $I_0^*$  and  $E_4$  meets the section, then  $\phi(M)_{\text{Pic}(X)}^\perp$  is spanned by  $E_1 - E_2$  and  $E_2 - E_3$ .

The involution  $\tau$  (cf. 5.6) defined by the elliptic fibration also acts on  $\phi(M)$ , via  $\iota = \tau^*$ , in a different way. If all fibres are of type  $IV$ , then the action of  $\iota$  on  $\phi(M) \cong U \oplus A_2^5$  permutes the simple root basis in each copy of  $A_2$ . Let  $N = \phi(M)^\iota$  be the sublattice of the invariant elements, then

$$N \cong U \oplus A_1^5.$$

However, if one of the fibres is of type  $I_0^*$ , then  $\phi(M)^\iota \cong U \oplus A_2 \oplus A_1^4$ . The orthogonal complement of  $\phi(N)$  in  $\phi(M)^\iota$  is spanned by the class of the divisor  $E_1 + E_2 + E_3$ . Also  $r = [E_1] \in \phi(N)_{\mathbb{Z}}^\perp$  but not in  $\phi(M)$ .

For any  $(-2)$ -vector  $r \in N^\perp \setminus T \subset L$  consider the hyperplane  $r^\perp$  in  $\mathbb{P}(V_+)$  of lines orthogonal to  $r$ . Let  $H(r)_\iota$  be the intersection of this hyperplane with the ball  $\mathcal{B} \subset \mathbb{P}(V_+)$ . Let  $\mathcal{H}_\iota$  be the union of the hyperplanes  $H(r)_\iota$ . If an ample  $(M, \rho)$ -marked surface  $(V, \phi)$  has a fibre of type  $I_0^*$  in its standard elliptic fibration  $\Phi_\phi$ , then its period belongs to  $\mathcal{H}_\iota$ . Let  $\Delta_\iota^m$  (resp. in  $\Delta_\iota$ ) be the image of  $\mathcal{H}_\iota$  in  $\mathcal{K}3_{M,\rho}^m$  (resp. in  $\mathcal{K}3_{M,\rho}$ ). In this notation we have

**8.10. Theorem.** *Under the isomorphism  $\mathcal{M}_{\text{cub}} \cong \mathcal{K}3_{M,\rho} \setminus \Delta$ , the image of the locus of smooth cubic surfaces with Eckardt points (the Eckardt locus) is mapped to  $\Delta_\iota \setminus (\Delta \cap \Delta_\iota)$ .*

8.11. It is well-known that any nonsingular cubic surface contains 45 tritangent planes, i.e. plane sections which split into the union of three lines. A marking of a cubic surface defines an order on the set of tritangent planes. Let  $\mathcal{E}_i$  be the locus of points in  $\mathcal{M}_{\text{cub}}^m$  corresponding to marked cubic surfaces which contain an Eckardt point in the  $i$ -th tritangent plane. The Weyl group  $W(E_6)$  acts on  $\mathcal{M}_{\text{cub}}^m$  and permutes the loci  $\mathcal{E}_i$ 's transitively. Let  $(S, L_1, \dots, L_6)$  be a marked cubic surface and let  $M_i$  be the line on  $S$  which meets  $L_i$  and  $L_{i+3}$  for  $i = 1, 2, 3$  but none of the other  $L_j$ . The  $M_i$  lie in a tritangent plane and they meet in a point if and only if the points  $p_1, \dots, p_6 \in \mathbb{P}^2$  obtained by blowing down the  $L_i$  are such that the three lines  $\langle p_i, p_{i+3} \rangle$  (the images of the  $M_i$ ), intersect at some point  $q$ . Let  $\mathcal{E}_j$  be the corresponding component of the Eckardt locus in  $\mathcal{M}_{\text{cub}}^m$ . Its pre-image  $Z$  in  $(\mathbb{P}^2)^6$  consists of 6-tuples of points  $(p_1, \dots, p_6)$  such that the lines  $\langle p_i, p_{i+3} \rangle, i = 1, 2, 3$  intersect. The assigning the intersection point  $q$  defines a surjective map from  $Z$  to  $\mathbb{P}^2$  whose fibres, as is easy to see, are irreducible. This shows that  $Z$ , and hence  $\mathcal{E}_j$  is irreducible. The image of each  $\mathcal{E}_i$  in  $\mathcal{M}_{\text{cub}}$  is then an irreducible hypersurface.

The irreducibility of the Eckardt locus in 8.16 follows also from our ball uniformization of  $\mathcal{M}_{\text{cub}}$ . We follow the proof given in [AF].

8.12. **Lemma.** *Let  $D = T^*/T$  be the discriminant group of  $T$  as in 5.1 and let  $N = M^\vee$ . The group  $W(E_6) = O(D)/\{\pm 1\}$  acts transitively on the subsets of  $(D - \{0\})/\{\pm 1\}$  of vectors of the same norm. There are three such subsets.*

- (i) *The set of vectors of norm 0 has 40 elements. Each non-zero isotropic vector is represented by  $(e + 2\rho(e))/3$ , where  $e \in T$  is a primitive isotropic vector.*
- (ii) *The set of vectors of norm  $-2/3$  has 36 elements. Each  $(-2/3)$ -vector is represented by a vector  $(r + 2\rho(r))/3$  in  $T^*$  with  $r \in T$ ,  $r^2 = -2$  and  $(r, \rho(r)) = 1$ .*
- (iii) *The set of vectors of norm  $-4/3$  has 45 elements. Each  $(-4/3)$ -vector in  $D(T)$  is represented by  $r''$  where  $r = r' + r'' \in N^\perp \setminus T$  is a  $(-2)$ -vector and  $r', r''$  is the projection of  $r$  into  $(N^\perp \cap M)^*, T^*$  respectively.*

*Proof.* If we consider  $T$  as a free Hermitian module  $\Lambda$  over  $\mathbb{Z}[\zeta_3]$  (see 6.9), then [ACT, AF] define an isotropic vector, a short vector and a long vector as a vector with Hermitian square equal to 0,  $-1$ ,  $-2$ , respectively. The images of these vectors in  $T^*$  with respect to the isomorphism  $h : \Lambda \rightarrow T^*$  (6.1) are vectors with square 0,  $-\frac{2}{3}$ ,  $-\frac{4}{3}$ , respectively. It is proven in [AF], Proposition 2.1 that there are exactly three  $\Gamma_\rho$ -orbits of the images of these vectors in  $D(T)$ . Their cardinality is 40, 36 and 45, respectively. This gives three orbits of  $O(D(T))$  in  $D(T)$  of the same cardinality. The assertions (i) and (ii) follow from the explicit formula for the isomorphism  $h$  (6.1). To prove (iii), we consider an ample  $(M, \rho)$ -polarized K3 surface  $X$  whose standard elliptic fibration acquires a fibre of type  $I_0^*$ . Let  $\tilde{\phi} : L \rightarrow H^2(X, \mathbb{Z})$  be a cohomology marking with  $\tilde{\phi}|_M = \phi$ . In the notation of 8.9, we may assume that the image of the first copy of  $A_2$  of  $M$  in  $\text{Pic}(X)$  is spanned by  $E_0$  and  $E_0 + E_1 + E_2 + E_3$ . Let  $r = \tilde{\phi}^{-1}([E_1])$ . Then  $r \in N^\perp \setminus T$  and  $r' = \frac{1}{3}(r + \rho(r) + \rho^2(r)) = \frac{1}{3}\tilde{\phi}^{-1}(E_1 + E_2 + E_3) \in (M \cap N^\perp)^*$ . We easily check that  $r'^2 = -\frac{2}{3}$ . Then  $r'' = r - r' \in T^*$  and  $(r'')^2 = -\frac{4}{3}$ .  $\square$

8.13. **Moduli interpretation.** Consider the three  $\Gamma_\rho$ -orbits of vectors from  $T^*$ :

- (1)  $\frac{1}{3}(e + 2\rho(e))$ , where  $e$  is a primitive isotropic vector in  $T$ ;
- (2)  $\frac{1}{3}(r + 2\rho(r))$ , where  $r$  is a  $(-2)$ -vector in  $T$  (this corresponds to a short root in  $\Lambda$ );
- (3)  $r''$  equal to the projection of a  $(-2)$ -vector  $r \in N^\perp \setminus T$  (this corresponds to a long root in  $\Lambda$ ).

Each vector  $v \in T^*$  defines a hyperplane  $v^\perp$  in  $\mathbb{P}(V_+)$  of lines orthogonal to  $v$ . So, we have three  $\Gamma_\rho$ -orbits of such hyperplanes corresponding to vectors from the above list. It is shown in [AF] that there is a bijective correspondence between the  $\Gamma_{M,\rho}$ -orbits of these vectors and their images in  $D(T)$ . Thus each  $\Gamma_\rho$ -orbit consists of 40, 36, 45  $\Gamma_{M,\rho}$ -orbits, respectively.

**8.14. The boundary divisors.** We know that the discriminant  $\mathcal{H}$  is equal to the union of hyperplanes  $H(r) = r^\perp \cap \mathcal{B}$ , where  $r$  is a  $(-2)$ -vector from  $T$ . For any  $x \in V_+$ , we can easily see that  $(r, x) = 0$  if and only if  $(r + 2\rho(r), x) = 0$ . This shows that the hyperplane corresponding to a vector of type (2) in 8.13 is one of the hyperplanes  $H(r)$ . Thus the discriminant locus  $\Delta^m$  in  $\mathcal{K}3_{M,\rho}^m$  consists of 36 hypersurfaces  $\Delta_\alpha^m$  ( $\alpha \in D/\{\pm 1\}$  with norm  $-2/3$ ) which are permuted transitively by  $W(E_6)$ . The discriminant locus  $\Delta$  in  $\mathcal{K}3_{M,\rho}$  is irreducible.

It is well-known that the stabilizer of each  $\Delta_\alpha^m$  in  $W(E_6)$  is  $S_6 \times \mathbb{Z}/2\mathbb{Z}$  (see 2.10). Let  $\alpha_1, \dots, \alpha_r$  be mutually orthogonal  $r$   $(-2/3)$ -vectors in  $D(T)$  ( $1 \leq r \leq 4$ ). These vectors correspond to a sublattice  $R = A_2^r$  in  $T$ . Let

$$\Delta_{\alpha_1, \dots, \alpha_r}^m = \Delta_{\alpha_1}^m \cap \dots \cap \Delta_{\alpha_r}^m.$$

It parametrizes the marked  $K3$  surfaces whose periods are orthogonal to  $R$ .

We fix an orthogonal basis  $\{\alpha_i\}$  of  $D$  such that  $q_T(\alpha_i) = -4/3$ . This defines an isomorphism of quadratic forms

$$D \simeq \mathbb{F}_3^5$$

where the quadratic form  $q$  on  $\mathbb{F}_3^5$  is given by

$$q(0, \dots, 0, 1, 0, \dots, 0) = -4/3.$$

Recall that the stabilizer of a basis of  $D$  in  $W(E_6)$  is  $W(D_5) \simeq (\mathbb{Z}/2\mathbb{Z})^4 \cdot S_5$ .

On the other hand, recall that the stabilizer  $G_k$  in  $W(E_6)$  of  $k$  mutually orthogonal  $(-2/3)$ -vectors in  $D$  is  $S_6 \times \mathbb{Z}/2\mathbb{Z}$ ,  $(\mathbb{Z}/2\mathbb{Z})^2 \times S_4 \times \mathbb{Z}/2\mathbb{Z}$ ,  $(\mathbb{Z}/2\mathbb{Z})^3 \times S_3 \times \mathbb{Z}/2\mathbb{Z}$ ,  $(\mathbb{Z}/2\mathbb{Z})^4 \cdot S_4$  for  $k = 1, 2, 3, 4$  respectively (see 2.10).

Now we list the  $W(D_5)$ -orbits of mutually orthogonal  $k$   $(-2/3)$ -vectors in  $D$ :

(i) There are 36  $(-2/3)$ -vectors in  $D$  which are divided into two orbits. One consists of 16 vectors containing  $(1, 1, 1, 1, 1)$  and another consists of 20 vectors containing  $(1, 1, 0, 0, 0)$ .

The stabilizer in  $W(D_5)$  of  $(1, 1, 1, 1, 1)$  is  $S_5$ , and that of  $(1, 1, 0, 0, 0)$  is  $(\mathbb{Z}/2\mathbb{Z})^3 \cdot (S_2 \times S_3)$ . Note that the sum of indices of these groups in  $G_1$  is  $12 + 15 = 27$ . The orbit of cardinality 20 corresponds to markings such that the marked line does not contain the node. For example, if the line corresponds to  $e_6$  under a geometric marking defined by  $(e_1, \dots, e_6)$ , then the effective class corresponding to the node could be either of type  $e_i - e_j$ ,  $1 \leq i < j < 6$  or  $e_0 - e_i - e_j - e_k$ ,  $1 \leq i < j < k < 6$ .

(ii) There are four types of mutually orthogonal pairs of  $(-2/3)$ -vectors in  $D$ :

$$\begin{aligned} & \{(1, 1, 1, 1, 1), (1, -1, 0, 0, 0)\}, \quad \{(1, 1, 1, 1, 1), (-1, 1, 1, 1, 1)\}, \\ & \{(1, 1, 0, 0, 0), (0, 0, 1, 1, 0)\}, \quad \{(1, 1, 0, 0, 0), (-1, 1, 0, 0, 0)\}. \end{aligned}$$

The stabilizer in  $W(D_5)$  of the first one is  $S_2 \times S_3$ , the second is  $(\mathbb{Z}/2\mathbb{Z}) \times S_4$ , the third is  $(\mathbb{Z}/2\mathbb{Z})^2 \cdot (S_2 \times S_2) \cdot S_2$ , and the fourth is  $(\mathbb{Z}/2\mathbb{Z})^4 \cdot (S_2 \times S_3)$ . The sum of indices of these groups in  $G_2$  is  $16 + 4 + 6 + 1 = 27$ .

(iii) There are three types of mutually orthogonal triples of  $(-2/3)$ -vectors in  $D$ :

$$\{(1, 1, 1, 1, 1), (1, -1, 0, 0, 0), (0, 0, 1, -1, 0)\}, \quad \{(1, 1, 1, 1, 1), (-1, 1, 1, 1, 1), (0, -1, 1, 0, 0)\},$$

$$\{(1, 1, 0, 0, 0), (-1, 1, 0, 0, 0), (0, 0, 1, 1, 0)\}.$$

The stabilizer in  $W(D_5)$  of the first one is  $(S_2 \times S_2) \cdot S_2$ , the second is  $(\mathbb{Z}/2\mathbb{Z}) \cdot (S_2 \times S_2)$ , and the third is  $(\mathbb{Z}/2\mathbb{Z})^3 \cdot (S_2 \times S_2)$ . The sum of indices of these groups in  $G_3$  is  $12 + 12 + 3 = 27$ .

(iv) There are two types of mutually orthogonal 4-tuples of  $(-2/3)$ -vectors in  $D$ :

$$\begin{aligned} &\{(1, 1, 1, 1, 1), (-1, 1, 1, 1, 1), (0, -1, 1, 0, 0), (0, 0, 0, -1, 1)\}, \\ &\{(1, 1, 0, 0, 0), (-1, 1, 0, 0, 0), (0, 0, 1, 1, 0), (0, 0, -1, 1, 0)\}. \end{aligned}$$

The stabilizer in  $W(D_5)$  of the first one is  $(\mathbb{Z}/2\mathbb{Z}) \cdot (S_2 \times S_2) \cdot S_2$ , and the second is  $(\mathbb{Z}/2\mathbb{Z})^4 \cdot (S_2 \times S_2) \cdot S_2$ . The sum of indices of these groups in  $G_4$  is  $24 + 3 = 27$ .

Let  $\Delta_k$  be the image in  $\Delta$  of all  $\Delta_{\alpha_1, \dots, \alpha_k}^m$  where  $\{\alpha_1, \dots, \alpha_k\}$  is a set of mutually orthogonal  $k$   $(-2/3)$ -vectors in  $D$ . Then the discriminant locus  $\Delta$  in  $\mathcal{K}3_{M, \rho}$  has the following stratification:

$$\Delta = \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4.$$

Similarly  $\Delta' = \Delta^m/W(D_5)$  in  $\mathcal{K}3_{M, \rho}^m/W(D_5)$  has the following stratification

$$\begin{aligned} \Delta' &= \Delta'_1 \cup \Delta'_2 \cup \Delta'_3 \cup \Delta'_4; \\ \Delta'_1 &= \Delta_1^{(1)} \cup \Delta_1^{(2)}; \\ \Delta'_2 &= \Delta_2^{(1)} \cup \Delta_2^{(2)} \cup \Delta_2^{(3)} \cup \Delta_2^{(4)}; \\ \Delta'_3 &= \Delta_3^{(1)} \cup \Delta_3^{(2)} \cup \Delta_3^{(3)}; \\ \Delta'_4 &= \Delta_4^{(1)} \cup \Delta_4^{(2)}. \end{aligned}$$

Here  $\Delta_k^{(r)}$  is the image of  $\Delta_{\alpha_1, \dots, \alpha_k}^m$  where  $\{\alpha_1, \dots, \alpha_k\}$  is as follows:

In case  $\Delta_1^{(1)}$ ,  $\{\alpha_1\} = \{(1, 1, 1, 1, 1)\}$ ;

In case  $\Delta_1^{(2)}$ ,  $\{\alpha_1\} = \{(1, 1, 0, 0, 0)\}$ ;

In case  $\Delta_2^{(1)}$ ,  $\{\alpha_1, \alpha_2\} = \{(1, 1, 1, 1, 1), (1, -1, 0, 0, 0)\}$ ;

In case  $\Delta_2^{(2)}$ ,  $\{\alpha_1, \alpha_2\} = \{(1, 1, 1, 1, 1), (-1, 1, 1, 1, 1)\}$ ;

In case  $\Delta_2^{(3)}$ ,  $\{\alpha_1, \alpha_2\} = \{(1, 1, 0, 0, 0), (0, 0, 1, 1, 0)\}$ ;

In case  $\Delta_2^{(4)}$ ,  $\{\alpha_1, \alpha_2\} = \{(1, 1, 0, 0, 0), (-1, 1, 0, 0, 0)\}$ ;

In case  $\Delta_3^{(1)}$ ,  $\{\alpha_1, \alpha_2, \alpha_3\} = \{(1, 1, 1, 1, 1), (1, -1, 0, 0, 0), (0, 0, 1, -1, 0)\}$ ;

In case  $\Delta_3^{(2)}$ ,  $\{\alpha_1, \alpha_2, \alpha_3\} = \{(1, 1, 1, 1, 1), (-1, 1, 1, 1, 1), (0, -1, 1, 0, 0)\}$ ;

In case  $\Delta_3^{(3)}$ ,  $\{\alpha_1, \alpha_2, \alpha_3\} = \{(1, 1, 0, 0, 0), (-1, 1, 0, 0, 0), (0, 0, 1, 1, 0)\}$ ;

In case  $\Delta_4^{(1)}$ ,  $\{\alpha_1, \dots, \alpha_4\} = \{(1, 1, 1, 1, 1), (-1, 1, 1, 1, 1), (0, -1, 1, 0, 0), (0, 0, 0, -1, 1)\}$ ;

In case  $\Delta_4^{(2)}$ ,  $\{\alpha_1, \dots, \alpha_4\} = \{(1, 1, 0, 0, 0), (-1, 1, 0, 0, 0), (0, 0, 1, 1, 0), (0, 0, -1, 1, 0)\}$ .

Now we conclude

**8.15. Remark.** For each  $k$ , the degree of the natural map  $\Delta'_k \rightarrow \Delta_k$  is 27.

**8.16. Eckardt loci.** If  $v$  is of type (3) in 8.13 the hyperplane  $v^\perp \cap \mathcal{B}$  is equal to the hyperplane  $\mathcal{H}(r)_l$  defined in 8.9. Thus we obtain that the image of the Eckardt locus  $\Delta_l^m$  in  $\mathcal{K}3_{M,\rho}^m$  consists of 45 irreducible hypersurfaces. The Eckardt locus  $\Delta_l$  in  $\mathcal{K}3_{M,\rho}$  is irreducible. This shows that the Eckardt locus in  $\mathcal{M}_{\text{cub}}$  is irreducible (as promised).

**8.17. Cusps.** For a non-zero isotropic vector  $e$  in  $T$  we define a totally isotropic sublattice

$$I(e) := \langle e, \rho(e) \rangle \quad (\subset T).$$

Then  $\bar{\mathcal{B}} \cap (\mathbb{P}(I(e)) \otimes \mathbb{C})$  is a cusp of  $\mathcal{B}$  (i.e. a rational boundary component), and any cusp of  $\mathcal{B}$  corresponding to a parabolic subgroup of  $\Gamma_\rho$  is obtained in this manner. Thus we obtain that the Satake-Baily-Borel compactification of  $\mathcal{K}3_{M,\rho}^m = \mathcal{B}/\Gamma_{M,\rho}$  (resp.  $\mathcal{K}3_{M,\rho} = \mathcal{B}/\Gamma_\rho$ ) is obtained by adding 40 cusps (resp. one cusp).

## 9. EXTENSION OF THE ISOMORPHISM TO THE BOUNDARIES

The purpose of this section is to extend the isomorphisms  $\mathcal{K}3_{M,\rho}^m \setminus \Delta^m \rightarrow \mathcal{M}_{\text{cub}}^m$  to a  $W(E_6)$ -equivariant isomorphism

$$\mathcal{K}3_{M,\rho}^m \longrightarrow \mathcal{M}_{\text{ncub}}^m.$$

First we will prove the following.

**9.1. Theorem.** *The isomorphism  $f : (\mathcal{K}3_{M,\rho}^m \setminus \Delta^m)/W(D_5) \longrightarrow \mathcal{M}_{\text{cub}}^1$  extends to an isomorphism:*

$$\mathcal{K}3_{M,\rho}^m/W(D_5) \xrightarrow{\cong} \mathcal{M}_{\text{ncub}}^1.$$

*Proof.* It follows easily from Theorem 8.8 that the standard elliptic fibration defined by  $(X, \phi) \in \mathcal{K}3_{M,\rho}^m$  has Weierstrass form (4.5). Let  $\mathcal{E}$  be the moduli space of such elliptic fibrations. This defines a natural map from  $\mathcal{K}3_{M,\rho}^m$  to  $\mathcal{E}$  which obviously factors through a map  $\mathcal{K}3_{M,\rho}^m/W(D_5)$  to  $\mathcal{E}$ . By Corollary 4.11, we have a natural isomorphism  $\mathcal{E} \cong \mathcal{M}_{\text{ncub}}^1$ . It follows from the construction that the composition of these maps restricted to the complement of  $\Delta^m$  coincides with the isomorphism from Corollary 7.5. Thus it suffices to show that the map  $\mathcal{K}3_{M,\rho}^m/W(D_5) \rightarrow \mathcal{E}$  is an isomorphism. To construct the inverse we have to define a marking on an elliptic  $K3$  surface  $X$  from  $\mathcal{E}$ . We may assume that  $X = X_{S,l}$  for some nodal cubic surface  $S$ , and the elliptic fibration is defined by the pencil of planes through  $l$ . We have a line  $m$  skew to  $l$  such that  $W(D_5)$  corresponds to its stabilizer in  $W(E_6)$ .

Recall that the elliptic  $K3$  surface  $X_{S,l}$  is isomorphic to a minimal resolution of the double cover of  $\mathbb{P}^2$  branched along the curve

$$x_2(F_2(x_0, x_1)x_2^3 + F_5(x_0, x_1)) = 0$$

and the elliptic pencil  $f : X_{S,l} \rightarrow \mathbb{P}^1$  is defined by the pencil of lines through the point  $p = (0, 0, 1)$ . We shall define a  $M$ -marking of the elliptic  $K3$  surface  $X_{S,l} \simeq X_{S,l,m}$ .

First note that  $f$  has a section corresponding to the line  $x_2 = 0$  or  $m$ . This section and the class of a fibre of  $f$  define the hyperbolic plane  $U$ . Thus it suffices to give an embedding of  $A_2^5$  into  $\text{Pic}(X_{S,l})$ . If  $f$  has a singular fibre of type  $IV$  or of type  $I_0^*$ , the triple cover construction of  $X_{S,l,m}$  determines an embedding of  $A_2$  in  $\text{Pic}(X_{S,l})$  as in the smooth case. If  $f$  has a singular fibre of type  $IV^*$  or of type  $II^*$ , then we take and fix embeddings

$$\phi_1 : A_2 \rightarrow \text{Pic}(X_{S,l}), \quad \phi_2 : A_2^2 \rightarrow \text{Pic}(X_{S,l}), \quad \phi_3 : A_2^2 \rightarrow \text{Pic}(X_{S,l})$$

as follows:  $\phi_1, \phi_2$  sends simple roots to effective  $(-2)$ -curves on the singular fibre of type  $IV^*$  which do not meet the section, and  $\phi_3$  sends simple roots to effective  $(-2)$ -curves on the singular fibre of type  $II^*$  which do not meet the section. Now we define a  $M$ -marking of  $X_{S,l}$  according to types of lines on  $S$  listed in 2.16 (they determine the  $SL(2)$ -orbit of the stable pair  $(F_5, F_2)$  and the degenerate fibres of  $f$ ).

Cases 1, 2, 3: In these cases  $S$  is smooth and the marking has already been defined in the proof of Theorem 7.2. Recall that in the case 1), the subgroup of  $W(E_6)$  which preserves the elliptic fibration is the group  $W(D_5) \simeq (\mathbb{Z}/2\mathbb{Z})^4 \cdot S_5$ , cf. 7.3.

Case 4: in this case, five roots of  $F_5$  gives five singular fibres of type  $IV$ . Thus we have a  $M$ -marking

$$\phi : M \rightarrow \text{Pic}(X)$$

determined by the triple cover construction by sending  $A_2^5$  to the sublattice generated by ten components of these five singular fibres not meeting the section. The multi-section splits into two disjoint section and hence the stabilizer of  $\phi$  in  $W(D_5)$  is  $S_5$ .

Case 5: in this case, three simple roots of  $F_5$  give three singular fibres of type  $IV$  and one multiple root of  $F_5$  gives a singular fibre of type  $IV^*$ . Using  $\phi_2$ , we have a  $M$ -marking  $\phi$ . The stabilizer of  $\phi$  in  $W(D_5)$  is  $(\mathbb{Z}/2\mathbb{Z})^3 \cdot (S_2 \times S_3)$ .

Cases 6, 7: these cases are mixed one of case 2) and case 5).

Case 8: this case is the mixed one of case 4) and case 5). The stabilizer of  $\phi$  in  $W(D_5)$  is  $S_2 \times S_3$ .

Case 8\*: in this case, the common root of  $F_2$  and  $F_5$  gives a singular fibre of type  $IV^*$  and the remaining simple roots of  $F_5$  give four singular fibres of type  $IV$ . Using  $\phi_1$ , we have a  $M$ -marking  $\phi$ . The stabilizer of  $\phi$  in  $W(D_5)$  is  $\mathbb{Z}/2\mathbb{Z} \times S_4$ .

Case 9: this case is the mixed one of case 5). The stabilizer of  $W(D_5)$  is  $(\mathbb{Z}/2\mathbb{Z})^2 \cdot (S_2 \times S_2) \cdot S_2$ .

Case 10: the common root of  $F_2$  and  $F_5$  gives a singular fibre of type  $II^*$  and the remaining three roots of  $F_5$  give three singular fibres of type  $IV$ . Using  $\phi_3$ , we have a  $M$ -marking  $\phi$ . The stabilizer of  $\phi$  in  $W(D_5)$  is  $(\mathbb{Z}/2\mathbb{Z})^4 \cdot (S_2 \times S_3)$ .

Case 11: this case is the mixed one of case 2) and case 5).

Case 12: this case is the mixed one of case 2) and case 10).

Case 13: this case is the mixed one of case 4) and case 5). The stabilizer of  $\phi$  in  $W(D_5)$  is  $(S_2 \times S_2) \cdot S_2$ .

Case 13\*: this case is the mixed one of case 5) and case 8\*). The stabilizer of  $\phi$  in  $W(D_5)$  is  $\mathbb{Z}/2\mathbb{Z} \cdot (S_2 \times S_2)$ .

Case 14: this case is the mixed one of case 5) and case 10). The stabilizer of  $\phi$  in  $W(D_5)$  is  $(\mathbb{Z}/2\mathbb{Z})^3 \cdot (S_2 \times S_2)$ .

Case 15: this case is the mixed one of case 2), case 5) and case 10).

Case 16: this case is the mixed one of case 5) and case 8\*). The stabilizer of  $\phi$  in  $W(D_5)$  is  $\mathbb{Z}/2\mathbb{Z} \cdot (S_2 \times S_2) \cdot S_2$ .

Case 17: this case is the mixed one of case 10). The stabilizer of  $\phi$  in  $W(D_5)$  is  $(\mathbb{Z}/2\mathbb{Z})^4 \cdot (S_2 \times S_2) \cdot S_2$ .

Thus we have defined a  $M$ -marking  $\phi$  of  $X_{S,l}$  modulo the action of  $W(D_5)$ . It gives the period of  $(X_{S,l}, \phi)$  in  $\mathcal{B}$  as in the case of smooth cubic surfaces. Recall that  $O(E_6)/\{\pm 1\} \simeq W(E_6)$  and  $O(E_8) \simeq W(E_8)$ . Since a reflection in  $O(M(\mathfrak{t}))$  acts trivially on the discriminant group of  $M(\mathfrak{t})$  (for  $M(\mathfrak{t})$  see 5.2), it can be extended to an isometry of  $L$  acting trivially on  $T(\mathfrak{t})$ . Now we can easily see that the image of the period of  $(X_{S,l}, \phi)$  in  $\mathcal{K}3_{M,\rho}^m/W(D_5)$  is independent of the choice of  $\phi_1, \phi_2, \phi_3$ . This defines the inverse map  $\mathcal{E} \rightarrow \mathcal{K}3_{M,\rho}^m/W(D_5)$  and proves the theorem.  $\square$

**9.2. Remark.** If  $S$  has a node, the period defines a point  $[X_{S,l}]$  in  $\Delta'$ . More precisely:

In the case 4, two singular fibres of type  $II$  degenerate to a singular fibre of type  $IV$  and the multi-section splits into two disjoint sections  $s_1, s_2$ . The projection of the class of  $s_1$  in  $D$  is of type  $(1, 1, 1, 1, 1)$ . Hence  $[X_{S,l}] \in \Delta_1^{(1)}$ .

In the cases 5, 6, 7, two singular fibres of type  $IV$  degenerate to a singular fibre of type  $IV^*$ . The projection of some component of the singular fibre of type  $IV^*$  is of type  $(1, 1, 0, 0, 0)$ . Hence  $[X_{S,l}] \in \Delta_1^{(2)}$ .

The case 8 is the mixed one of cases 4 and 5. Hence  $[X_{S,l}] \in \Delta_2^{(1)}$ .

In the case 8\*, two singular fibres of type  $II$  and a fibre of type  $IV$  degenerate to a singular fibre of type  $IV^*$ , and the multi-section splits into two disjoint sections  $s_1, s_2$ . The projections of  $s_1$  is of type  $(1, 1, 1, 1, 1)$  and that of a component of the fibre of type  $IV^*$  is of type  $(1, 0, 0, 0, 0)$ . Combining these two  $(-2/3)$ - and  $(-4/3)$ -vectors, we have a vector of type  $(-1, 1, 1, 1, 1)$ . Hence  $[X_{S,l}] \in \Delta_2^{(2)}$ .

The cases 9, 11 are the mixed ones of case 5. Hence  $[X_{S,l}] \in \Delta_2^{(3)}$ .

In the cases 10, 12, two singular fibres of type  $IV$  and one fibre of type  $II$  degenerate to a singular fibre of type  $II^*$ . The projections of some two components of the fibre of type  $II^*$  are of type  $(1, 1, 0, 0, 0), (1, -1, 0, 0, 0)$ . Hence  $[X_{S,l}] \in \Delta_2^{(4)}$ .

The case 13 is the mixed one of cases 5 and 8. Hence  $[X_{S,l}] \in \Delta_3^{(1)}$ .

The case 13\* is the mixed one of cases 5 and 8\*. Hence  $[X_{S,l}] \in \Delta_3^{(2)}$ .

The cases 14, 15 are the mixed one of cases 5 and 10. Hence  $[X_{S,l}] \in \Delta_3^{(3)}$ .

The case 16 is the mixed one of cases 5 and 13\*. Hence  $[X_{S,l}] \in \Delta_4^{(1)}$ .

The case 17 is the mixed one of cases 10 and 14. Hence  $[X_{S,l}] \in \Delta_4^{(2)}$ .

**9.3. Theorem.** *The isomorphism  $\mathcal{K}3_{M,\rho}^m \setminus \Delta^m \cong \mathcal{M}_{cub}^m$  extends to a  $W(E_6)$ -equivariant isomorphism*

$$\mathcal{K}3_{M,\rho}^m \cong \mathcal{M}_{ncub}^m.$$

*Passing to the quotients it defines an isomorphism*

$$\mathcal{K}3_{M,\rho} \cong \mathcal{M}_{ncub}.$$

*Proof.* The isomorphism  $\mathcal{K}3_{M,\rho}^m/W(D_5) \cong \mathcal{M}_{\text{ncub}}^m = \mathcal{M}_{\text{ncub}}^m/W(D_5)$  constructed in Theorem 9.1 lifts to a  $W(E_6)$ -equivariant isomorphism  $\mathcal{K}3_{M,\rho}^m \cong \mathcal{M}_{\text{ncub}}^m$ . In fact, this is true for open Zariski subsets defined by nonsingular cubic surfaces, hence each of the varieties is the normalization of the quotient in the field of rational functions  $\mathbb{C}(\mathcal{K}3_{M,\rho}^m) = \mathbb{C}(\mathcal{M}_{\text{ncub}}^m)$ . Now we have an isomorphism  $\alpha$  of varieties which defines a birational isomorphism of  $W(E_6)$ -varieties. Obviously, it is an isomorphism of  $W(E_6)$ -varieties (for each  $g \in W(E_6)$  the maps  $g \circ \alpha$  and  $\alpha \circ g$  coincide on an open Zariski subset, hence coincide everywhere).  $\square$

**9.4. Corollary.** *The isomorphism*

$$(\mathcal{B} \setminus \mathcal{H})/\Gamma_{M,\rho} \cong \mathcal{M}_{\text{cub}}$$

*from Corollary 7.5 extends to an isomorphism*

$$\mathcal{B}/\Gamma_{M,\rho} \cong \mathcal{M}_{\text{ncub}}.$$

**9.5. Remark.** The isomorphism  $\mathcal{M}_{\text{ncub}} \cong \mathcal{K}3_{M,\rho}$  can be extended to the compactification obtained by adding one strictly-semistable point. The image of this point goes to the cusp of the ball quotient. The corresponding  $K3$  surface is isomorphic to the double cover of  $\mathbb{P}^2$  ramified along the sextic curve

$$t_2(L_1(t_0, t_1)^3 L_2(t_0, t_1)^2 + t_2^3 L_2(t_0, t_1)^2) = 0,$$

where  $L_1, L_2$  are independent linear forms. This sextic appears as a semistable sextic in Shah [Sha], Theorem 2.4, Group II, (2). Its double cover is a Type II degeneration of  $K3$  surfaces, i.e. corresponding to a point on an 1-dimensional rational boundary component of the period domain of polarized  $K3$  surfaces of degree 2 (= a bounded symmetric domain of type IV and of dimension 19). The 1-dimensional rational boundary components of a bounded symmetric domain of type IV bijectively correspond to the set of totally isotropic primitive sublattices of rank 2 of its underlying lattice of signature  $(2, r)$ . In our situation,  $\rho$ -invariant totally isotropic primitive sublattices of rank 2 of  $T$  correspond to the set of cusps of  $\mathcal{B}$ . Thus the semistable points of type  $(6, 6)$  correspond to the boundary of the Satake's compactification of  $\mathcal{B}/\Gamma_{M,\rho}$ .

The strictly semistable cubic surface defined by

$$(9.1) \quad X_3^3 - X_0 X_1 X_2 = 0$$

(cf. [ACT] (4.6)) has three double rational points of type  $A_2$  and has only three lines which lie in one  $\text{Aut}(S)$ -orbit. This defines three planes in the cubic fourfold  $X$  defined by  $X_5^3 + X_4^3 + X_3^3 - X_0 X_1 X_2 = 0$  (one such plane is  $\Pi : X_2 = X_3 = X_4 + X_5 = 0$ ) and projection away from such a plane defines a quadric bundle structure on  $X$ . The discriminant curve is easily computed and is a sextic as above.

It follows from Proposition 3.2 that the pair  $(F_5, F_2) = (L_1^3 L_2^2, L_2^2)$  represents a semi-stable but not stable point in  $\mathbb{P}(V(5)) \times \mathbb{P}(V(2))$  whose orbit is closed in the set of semi-stable points. The corresponding point in  $(\mathbb{P}(V(5)) \times \mathbb{P}(V(2)))^{\text{ss}}//\text{SL}(2)$  compactifies  $(\mathbb{P}(V(5)) \times \mathbb{P}(V(2)))^s/\text{SL}(2)$ . Thus we see that  $\mathcal{M}_{\text{ncub}}^1$  admits a one-point compactification corresponding to the surface (9.1) together with its unique (up to automorphism) line.

**9.6. Configurations of 7 points in  $\mathbb{P}^1$ .** Recall from Theorem 3.4 that we have a natural isomorphism

$$\mathcal{M}_{\text{ncub}}^1 \cong (\mathbb{P}(V(5)) \times \mathbb{P}(V(2)))^s / \text{SL}(2),$$

where  $(\mathbb{P}(V(5)) \times \mathbb{P}(V(2)))'$  is the open subset corresponding to stable pairs of binary forms  $(F_5, F_2)$ . Consider the product  $(\mathbb{P}^1)^7$  as the product  $(\mathbb{P}^1)^5 \times (\mathbb{P}^1)^2$ . We have an isomorphism

$$\psi : (\mathbb{P}^1)^7 / S_5 \times S_2 \rightarrow \mathbb{P}(V(5)) \times \mathbb{P}(V(2)).$$

Let  $p : (\mathbb{P}^1)^7 \rightarrow \mathbb{P}(V(5)) \times \mathbb{P}(V(2))$  be the composition of the quotient map and  $\psi$  and

$$\mathcal{L} = p^*(\mathcal{O}_{\mathbb{P}(V(5))}(2) \boxtimes \mathcal{O}_{\mathbb{P}(V(2))}(1)) \cong \boxtimes_{i=1}^5 \mathcal{O}_{\mathbb{P}^1}(2) \otimes (\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)).$$

Since the stability is preserved under the action of finite groups, we see that semi-stable (stable) points in  $\mathbb{P}(V(5)) \times \mathbb{P}(V(2))$  with respect to the action of  $\text{SL}(2)$  and the linearization defined by the invertible sheaf  $\mathcal{O}_{\mathbb{P}(V(5))}(2) \boxtimes \mathcal{O}_{\mathbb{P}(V(2))}(1)$  correspond to semi-stable (stable) points in  $(\mathbb{P}^1)^7$  with respect to the diagonal action of  $\text{SL}(2)$  and the linearization defined by the line bundle  $\mathcal{L}$ . Let

$$P_1(2^5, 1, 1) = ((\mathbb{P}^1)^7)^s / \text{SL}(2).$$

We have

$$(\mathbb{P}(V(5)) \times \mathbb{P}(V(2)))^s / \text{SL}(2) \cong P_1(2^5, 1, 1) / S_5 \times S_2.$$

We know that  $\mathcal{M}_{\text{ncub}}^1 = \mathcal{M}_{\text{ncub}}^m / W(D_5)$ . The group  $W(D_5)$  is equal to the semi-direct product  $(\mathbb{Z}/2\mathbb{Z})^4 \ltimes S_5$ . Here  $S_5$  is the subgroup of  $W(D_5)$  which acts on markings on nonsingular surfaces by permuting the divisor classes  $e_1, \dots, e_5$ . It stabilizes the divisor class  $2e_0 - e_1 - \dots - e_5$  of a line  $l$ . The subgroup  $H = (\mathbb{Z}/2\mathbb{Z})^4$  is generated by the conjugates of the product of two commuting reflections  $s_{e_0 - e_1 - e_2 - e_6} \circ s_{e_1 - e_2}$ . Let  $l'_i$  be the lines representing the classes  $e_0 - e_i - e_6$ . Then  $H$  acts by switching even number of  $l'_i$ 's with  $l'_i$ 's. The proof of Theorem 3.4 shows that the map  $\mathcal{M}_{\text{ncub}}^1 \rightarrow (\mathbb{P}(V_5) \times \mathbb{P}(V_2))' / \text{SL}(2)$  induces a  $S_5$ -equivariant isomorphism

$$\mathcal{M}_{\text{ncub}}^m / H \cong P_1(2^5, 1, 1) / S_2.$$

**9.7. Monodromy groups.** According to Deligne and Mostow [DM], the variety  $P_1(2^5, 1, 1)$  is isomorphic to the quotient of a complex 4 ball by a reflection subgroup  $\Pi'$  corresponding to hypergeometric function defined by the multi-valued form

$$\omega = z^{-1/6} [(z-1)(z-a_1)(z-a_2)(z-a_3)(z-a_4)]^{-1/3} dz.$$

They also show that  $\Pi'$  and  $S_2$  generate a reflection subgroup  $\Pi$  such that the ball quotient is isomorphic to  $P_1(2^5, 1, 1) / S_2$ . As shown in 4.15,  $X$  is the minimal model of a quotient  $(C \times E) / (\mathbb{Z}/6\mathbb{Z})$ . This correspondence gives us an isogeny between our group  $\Gamma_\rho$  and  $\Pi$ .

## 10. HALF TWISTS.

**10.1.** To a smooth cubic surface  $S$  one can associate a principally polarized Hodge structure of rank 10 and weight 1, it is  $H^1(P, \mathbb{Z})$  where  $P$  is the intermediate Jacobian of the cubic threefold  $V$  (cf. 4.13) associated to  $S$ . In [ACT], see also [MT], it is shown that this Hodge structure, with its automorphism of order three, determines  $S$ .

The automorphism of order three defines the structure of a free  $\mathbb{Z}[\zeta]$ -module on  $H^1(P, \mathbb{Z})$ . It defines eigenspaces  $H^{1,0}(P)_\chi$  and  $H^{1,0}(P)_{\bar{\chi}}$  of dimension 4 and 1 respectively. This allows one to define a

weight two Hodge structure  $W$ , with Hodge numbers  $(1, 8, 1)$ , and with the same underlying lattice  $W = H^1(P, \mathbb{Z})$  as follows:

$$W^{2,0} = H^{1,0}(P)_{\bar{\chi}}, \quad W^{1,1} = H^{1,0}(P)_{\chi} \oplus H^{0,1}(P)_{\bar{\chi}}, \quad W^{0,2} = H^{0,1}(P)_{\chi},$$

in fact it is easy to check that  $W^{p,q} = \overline{W^{q,p}}$ . The automorphism of order three of  $H^1(P, \mathbb{Z})$  preserves this decomposition, hence also  $W$  has an automorphism of order three. The polarization  $E$  on  $H^1(P, \mathbb{Z})$  defines a  $\mathbb{Q}[\zeta_3]$ -valued Hermitian form  $H$  on  $H^1(P, \mathbb{Z}) \cong \mathbb{Z}[\zeta_3]^5$  (cf. [ACT]) with imaginary part  $E$ . The real part  $Q$  of  $H$  is a polarization of  $W$ . The lattice  $(W, Q)$  is of type  $A_2^4 \oplus A_2(-1)$ . The polarized Hodge structure  $(W, Q)$  is the (negative) half twist of  $(H^1(P, \mathbb{Z}), E)$  ([vG1]).

10.2. The lattice  $(W, Q) \cong A_2^{\oplus 4} \oplus A_2(-1)$  has a unique (up to an isometry) embedding in the  $K3$  lattice  $L$  and the automorphism of order three on  $W$  extends to an automorphism of order three on the  $K3$  lattice. The polarized Hodge structure  $(W, Q)$  is invariant under this automorphism and defines a  $K3$  surface with an automorphism of order three. So the half twist of  $H^1(P, \mathbb{Z})$  provides a purely Hodge theoretic approach to the  $K3$  surfaces which were constructed as triple covers of cubic surfaces in this paper.

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