HESSIANS AND THE MODULI SPACE OF CUBIC SURFACES

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Abstract. The Hessian of a general cubic surface is a nodal quartic surface, hence its desingularisation is a K3 surface. We determine the transcendental lattice of the Hessian K3 surface for various cubic surfaces (with nodes and/or Eckardt points for example). Classical invariant theory shows that the moduli space of cubic surfaces is a weighted projective space. We describe the singular locus and some other subvarieties of the moduli space.

Moduli spaces of cubic surfaces have been intensively studied recently. The main reason was the discovery by Allcock, Carlson and Toledo [ACT] that the moduli space of cubic surfaces is a ball quotient. Another proof of this fact, using K3 surfaces rather than cubic threefolds, is given in [DGK]. There is a classical, quite different, way to associate a K3 surface to a general cubic surface, it is simply by taking the Hessian of the cubic polynomial defining the surface. We refer to the desingularization of the corresponding nodal quartic surface as the Hessian K3 surface of the cubic surface.

In this paper we study these Hessian K3 surfaces. The Hessian, with its polarization given by the natural map to $\mathbf{P}^3$, determines the cubic surface. This gives a birational isomorphism between the moduli space of M-lattice polarized K3 surfaces, where M is the Néron-Severi group of the general Hessian, and the moduli space of cubic surfaces.

As a first step in the study of the moduli space of these K3 surfaces we determine the transcendental lattices of various types of Hessians. We find that certain naturally defined divisors in the moduli space (Hessians of cubics with a node, an Eckardt point, without a Sylvester form, Kummer surfaces) are Heegner divisors, that is, their transcendental lattices are sublattices of a lower rank of the transcendental lattice of a general Hessian. Various very special cubic surfaces (with many nodes and/or Eckardt points) have transcendental lattices of rank two, these are usually referred to as singular (or, nowadays, as attractive) K3 surfaces. As there is no general method to determine the transcendental lattice of a K3 surface, we have to use ad-hoc methods. To study the Hessians, we use the Sylvester form of a cubic surface which is also very useful in describing the moduli space.

In the final section of this paper we recall the classical description of the moduli space of cubic surfaces using 19th century invariant theory. The moduli space is a weighted projective space. We describe various divisors and subvarieties in the moduli space. We also describe its singular locus.

This paper is based on the PhD thesis of the first author under the direction of the second author.

1. The Sylvester form and the Hessian

1.1. **The Sylvester form.** Given a general homogeneous polynomial F (briefly, a form) of degree three in four variables, there are five linear forms x_i in four variables such that any four of the five x_i are linearly independent and such that

$$\sum_{i=0}^{4} x_i = 0, \qquad F = \sum_{i=0}^{4} \lambda_i x_i^3.$$

We refer to these equations as the Sylvester form of F.

The x_i are uniquely determined by F (up to permutation and multiplication by a common non-zero scalar) and the λ_i are uniquely determined by F and the x_i ([S], pp. 125-137, [Rod], pp. 72ff. and also [C] pp. 295 ff.).

1.2. **The pentahedron.** In case F has a Sylvester form, the union of the five planes in $\mathbf{P}^3$ defined by $x_i = 0$ is called the pentahedron of F. Each plane is called a face, the intersection of two distinct faces is called an edge and the intersection of three distinct faces is a vertex of the pentahedron. The ten edges and ten vertices are denoted by

$$L_{ij}: \quad x_i = x_j = 0, \qquad\qquad P_{ijk}: \quad x_i = x_j = x_k = 0.$$

1.3. **The discriminant divisor.** It is not hard to verify that the cubic surface defined by a Sylvester form is smooth iff, for all choices of signs, we have:

$$\sum_{i=0}^{4} \pm\sqrt{\lambda_0 \dots \hat{\lambda}_i \dots \lambda_4} \neq 0.$$

Assuming the first sign to be $+1$, this condition is equivalent to the product of the 2^4 such expressions, with all possible signs, to be non-zero. This product is homogeneous of degree 32 in in the λ_i (in fact the 'discriminant' divisor which parametrizes singular cubic surfaces in $\mathbf{P}^{19} = \mathbf{P}H^0(\mathbf{P}^3, \mathcal{O}(3))$ has degree 32, cf. [S], App. III p. 179, [GKZ], p. 38).

For a general cubic form F we can use the linear forms $x_0, \dots, x_3$ as coordinates on $\mathbf{P}^3$ and $x_4 = -(x_0 + \dots + x_3)$. The general cubic surface defined by $F = 0$ is now determined by $p = (\lambda_0 : \dots : \lambda_4) \in \mathbf{P}^4$, and p is unique up to permutation of the λ_i.

1.4. **The Hessian.** The Hessian H_G of a cubic form G in $n+1$ variables u_i is the homogeneous polynomial of degree $n+1$ defined by

$$H_G = \det\left(\frac{\partial^2 G}{\partial u_i \partial u_j}\right).$$

1.5. **The Hessian K3 surface.** The Hessian surface of a cubic surface S defined by $F = 0$ is the quartic surface Y defined by $H_F = 0$ (in case $H_F \equiv 0$ we do not define the Hessian surface). We will call $Y = Y_S$ the Hessian of S.

In case S is defined by a Sylvester form, its Hessian Y is defined by

$$Y: \quad H_F = \sum_{i=0}^{4} \lambda_0 \dots \hat{\lambda}_i \dots \lambda_4 x_0 \dots \hat{x}_i \dots x_4 = (\lambda_0 \dots \lambda_4)(x_0 \dots x_4) \sum_{i=0}^{4} \frac{1}{\lambda_i x_i} = 0, \qquad \sum_{i=0}^{4} x_i = 0.$$

The ten edges L_{ij} of the pentahedron lie on the Hessian and the ten vertices P_{ijk} are singular points of the Hessian. One verifies that S is smooth iff the singular locus of Y consists of these 10 points.

The desingularization X of Y is a K3 surface, which we refer to as the Hessian K3 surface of S (or simply Hessian if no confusion is likely). The natural map

$$\pi : X \longrightarrow Y, \qquad \pi(N_{ijk}) = P_{ijk}$$

contracts 10 smooth rational curves N_{ijk} to the singular points P_{ijk} of Y and is an isomorphism on the complement. The strict transforms in X of the lines L_{ij} in Y are denoted by N_{ij}.

1.6. **The Néron-Severi group of the general Hessian K3 surface.** The ten (-2)-curves N_{ijk} are disjoint, and so are the ten (-2)-curves N_{ij}. Moreover, $N_{ij}N_{klm} = 1$ iff $\sharp\{i,j,k,l,m\} = 3$ and is zero otherwise. A machine computation shows that the rank of the 20×20 matrix of intersection numbers of the N_α is equal to 16. A $\mathbf{Z}$-basis of the sublattice NS_{gen} of $NS(X)$ generated by these curves is given by all twenty curves except N_{234}, N_{14}, N_{23} and N_{24}. The discriminant of NS_{gen} is equal to $2^4 \cdot 3$. From [DK] it follows that NS_{gen} is the Néron-Severi group of a general Hessian K3 surface (i.e. one with Picard number 16). The perpendicular of NS_{gen} in $H^2(X, \mathbf{Z})$ is the transcendental lattice T_{gen} which was determined in [DK] (for an alternative method, see Proposition 2.5):

$$T_{gen} = U \oplus U(2) \oplus A_2(-2).$$

The following lemma is very useful for determining the transcendental lattices of 'singular' Hessian K3 surfaces, that is of those with Picard number 20.

1.7. **Lemma.** Let T be an even lattice of rank two:

$$T = \begin{pmatrix} 2n & a \\ a & 2m \end{pmatrix}.$$

Then there is a primitive embedding $T \hookrightarrow T_{gen}$ if and only if at least one among a, n and m is even. In this case T embeds into $U \oplus U(2)$.

Proof. An embedding $T \hookrightarrow T_{gen}$, is equivalent to giving vectors $x = (x_1, x_2, x_3, x_4, x_5, x_6)$, $y = (y_1, y_2, y_3, y_4, y_5, y_6) \in \mathbf{Z}^6 = T_{gen}$, such that

$$\begin{array}{ll} 2x_1x_2 + 4x_3x_4 - 4(x_5^2 + x_6^2 - x_5x_6) & = 2n, \\ 2y_1y_2 + 4y_3y_4 - 4(y_5^2 + y_6^2 - y_5y_6) & = 2m, \\ x_1y_2 + x_2y_1 + 2(x_3y_4 + x_4y_3) - 2(2x_5y_5 + 2x_6y_6 - x_5y_6 - x_6y_5) & = a. \end{array}$$

If $n \equiv m \equiv a \equiv 1(2)$, this system has no solution, since from the first two equations we get $x_1x_2 \equiv 1(2)$ and $y_1y_2 \equiv 1(2)$, so $x_1 \equiv x_2 \equiv y_1 \equiv y_2 \equiv 1(2)$ and hence $x_1y_2 + x_2y_1 \equiv 0(2)$. Thus in the third equation the first member is even, while the second is odd.

Conversely, let $x := (n, 1, 0, 0, 0, 0)$, then y must satisfy

$$\begin{array}{ll} y_1y_2 + 2y_3y_4 - 2(y_5^2 + y_6^2 - y_5y_6) & = m, \\ ny_2 + y_1 & = a. \end{array}$$

Substituting $y_1 = a - ny_2$ in the first equation, we get:

$$2y_3y_4 - 2(y_5^2 + y_6^2 - y_5y_6) = ny_2^2 - ay_2 + m.$$

Hence we can choose:

$$y = (a - ny_2, y_2, 1, (ny_2^2 - ay_2 + m)/2, 0, 0),$$

unless $n \equiv a(2)$ and $m \equiv 1(2)$, the only case in which $ny_2^2 - ay_2 + m$ is odd for all $y_2 \in \mathbf{Z}$. Since, by assumption, one of n, m and a is even, in this case n (and a) must be even. Now we begin with $y = (1, m, 0, 0, 0, 0)$ and the same argument provides an x. It is easy to see that with these choices of x and y the embedding of T is primitive. $\square$

2. Eckardt points

2.1. **Definition.** A smooth cubic surface contains 27 lines and has 45 plane sections which are unions of three lines. In case three coplanar lines meet in a single point, that point is called an Eckardt point.

2.2. **The Sylvester form and Eckardt points.** A smooth cubic surface S defined by a Sylvester form $\sum \lambda_i x_i^3$ as in 1.1 has an Eckardt point iff $\lambda_i = \lambda_j$ for some $i \neq j$ ([S], pp. 145 ff.)

If $\lambda_i = \lambda_j$, the corresponding Eckardt point P of S is the vertex P_{klm} of the pentahedron (with $\{i, \ldots, m\} = \{0, \ldots, 4\}$). The plane $x_i + x_j = 0$ (which is the tangent plane T_PS to S in P_{klm}) cuts out the three lines meeting in P_{klm} and S has an involution (induced by permuting x_i and x_j). In particular, S has an Eckardt point iff a vertex of the pentahedron lies on S.

A Sylvester form with $\lambda_0 = \lambda_1 = \lambda_2$ defines a cubic surface with three Eckardt points in general, there are 4 Eckardt points if also $\lambda_3 = \lambda_4$ and there are 10 Eckardt points if all λ_i are equal (this surface is known as the Clebsch diagonal surface). In case $\lambda_0 = \lambda_1$ and $\lambda_2 = \lambda_3$, but otherwise the λ_i's are distinct, the surface has 2 Eckardt points. In case $\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3$ but $\lambda_0 \neq \lambda_4$, the surface has six Eckardt points.

2.3. **New curves on the Hessian.** For an Eckardt point P the intersection of the tangent plane T_PS and the Hessian Y_S consists of the line L_{ij} (with multiplicity two) and of the pair of lines defined by

$$\lambda_k\lambda_l x_k x_l + \lambda_k\lambda_m x_k x_m + \lambda_l\lambda_m x_l x_m = 0, \qquad x_k + x_l + x_m = 0,$$

which meet in P_{klm}, we refer to these as new lines. The strict transform of these lines in the Hessian K3 surface are two new disjoint (-2)-curves which both meet N_{klm}, but none of the other N_{abc}'s.

2.4. **Clebsch diagonal cubic surface.** The diagonal cubic surface is the smooth cubic surface S_{10} defined by the Sylvester form:

$$S_{10}: \qquad \sum_{i=0}^{4} x_i^3 = 0, \qquad \sum_{i=0}^{4} x_i = 0,$$

so all $\lambda_i = 1$. This surface has 10 Eckardt points and thus its Hessian Y_{10} has 10 new pairs of lines. Let ω be a primitive cube root of unity and let

$$C_{234} = \{(s : -s : t : t\omega : t\omega^2) : \ (s : t) \in \mathbf{P}^1\} \qquad \subset Y_{10} \qquad (\omega^2 + \omega + 1 = 0)$$

be one of the new lines on the Eckardt point $P_{234} = (1 : -1 : 0 : 0 : 0)$. For $\sigma \in S_5$, the symmetric group on $\{0, \dots, 4\}$, we denote by $C_{\sigma(2),\sigma(3),\sigma(4)}$ the line obtained by permuting the coordinates. In particular,

$$C_{234} = C_{342} = C_{423}, \qquad C_{243} = C_{324} = C_{432} = \{(s : -s : t : t\omega^2 : t\omega) : \ (s : t) \in \mathbf{P}^1\}$$

is the other new line on P_{234}. We use the same names to denote the corresponding (-2)-curves on the Hessian K3 surface X_{10} of S_{10}.

The intersection numbers of N_{ab} and N_{abc} with C_{klm} are zero unless $\sharp\{a, b, k, l, m\} = 5$ resp. $\sharp\{a, b, c, k, l, m\} = 3$, in which case the intersection number is 1. Moreover, C_{abc} and C_{klm} intersect, with intersection number 1, iff there are two equal indices in the same order (up to cyclic permutations, e.g. C_{124} intersects C_{041}), C_{abc} and C_{abd} meet in a point on X_{10} whose image in Y_{10} has coordinates $x_a = \omega$, $x_b = \omega^2$, $x_c = x_d = 1$ and $x_e = -1$ where $\{a, b, c, d, e\} = \{0, 1, 2, 3, 4\}$. The following classes in the Néron-Severi group NS_{10} of X_{10}

$$c_{ij} = C_{abc} - C_{acb} \quad \in NS_{10} := NS(X_{10}), \qquad \{i, j, a, b, c\} = \{0, \dots, 4\},$$

are easily seen to be perpendicular to NS_{gen}. They satisfy:

$$c_{ij}^2 = -4, \qquad c_{ik} \cdot c_{il} = 2, \qquad c_{ij} \cdot c_{kl} = 0$$

for distinct indices i, j, k, l, so we have a copy of $A_4(-2)$ in T_{gen} (cf. Remark 2.6). The symmetric group S_5 is a subgroup of $Aut(X_{10})$ (acting via permutations of the coordinates) and acts via permutation of the indices on the c_{ij}, so it acts as $W(A_4) \cong S_5$ on the copy of $A_4(-2)$.

2.5. **Proposition.** Let X_k be the Hessian K3 surface of a smooth cubic surface with Sylvester form having k Eckardt points (in particular, $k \in \{0, 1, 2, 3, 4, 6, 10\}$). Let T_k be the transcendental lattice of X_k.

$$\begin{array}{ll}
T_0 = T_{gen} = U \oplus U(2) \oplus A_2(-2) & \mathrm{discr}(T_{gen}) = 48, \\
T_1 = U \oplus U(2) \oplus < -12 > & \mathrm{discr}(T_1) = -48, \\
T_2 = U \oplus < 4 > \oplus < -12 > & \mathrm{discr}(T_2) = 48, \\
T_3 = U \oplus U(6) & \mathrm{discr}(T_3) = 36, \\
T_4 = U(3) \oplus < 4 > & \mathrm{discr}(T_4) = -36, \\
T_6 = U \oplus < 24 > & \mathrm{discr}(T_6) = -24. \\
T_{10} = \left(\mathbf{Z}^2, \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix} \right) & \mathrm{discr}(T_{10}) = 15.
\end{array}$$

Proof. We start with the computation of NS_{10} and T_{10}. The 40×40 matrix of intersection products of the curves N_α and C_β has rank 20, hence the Néron-Severi group of X_{10} has maximal rank. A $\mathbf{Z}$-basis of the lattice generated by these 40 curves is given by the basis of NS_{gen} (given in 1.6) and the 4 curves C_{234}, C_{134}, C_{124} and C_{032}. A machine verified that the discriminant of this lattice is -15. Since 15 is square free, we can conclude that NS_{10} is generated by these 40 curves. It is easy to find two orthogonal $E_8(-1)$'s in this lattice, for example:

$$\begin{array}{ccccccccccccc}
N_{034} & -- & N_{04} & -- & N_{024} & -- & C_{024} & -- & C_{124} & -- & C_{041} & -- & N_{23} \\
 & & & & | & & & & & & & & \\
 & & & & N_{24} & & & & & & & &
\end{array}$$

$$\begin{array}{ccccccccccccc}
N_{12} & -- & N_{012} & -- & N_{01} & -- & C_{234} & -- & C_{142} & -- & C_{143} & -- & N_{134} \\
 & & & & | & & & & & & & & \\
 & & & & N_{013} & & & & & & & &
\end{array}$$

Using a machine, it is not hard to find the rank 4 lattice L' such that $NS_{10} \cong E_8(-1)^2 \oplus L'$, we found a copy of U in L' (a standard basis is: $-2N_{134} - 5N_{124} - N_{123} + 4N_{034} + 5N_{024} + N_{023} + 7N_{014} + 6N_{013} + 8N_{012} + 13N_{01} + 2N_{02} + 8N_{04} + N_{12} - N_{13} + 5C_{234} - 3C_{134} - 4C_{124} + C_{032}$ and $-2N_{134} - 5N_{124} - N_{123} + 4N_{034} + 5N_{024} + N_{023} + 7N_{014} + 6N_{013} + 8N_{012} + 13N_{01} + 2N_{02} + 8N_{04} + N_{12} - N_{13} + 5C_{234} - 3C_{134} + 4C_{124}$). The orthogonal complement of U in L' is isomorphic to the lattice denoted by $T_{10}(-1)$ in the theorem (a basis for it is: $2N_{124} + 2N_{123} - 5N_{034} - 2N_{024} - 2N_{023} - N_{014} + 2N_{012} + N_{01} - 3N_{03} - 4N_{04} + 3N_{12} + 2N_{13} - 3N_{34} + C_{234} + C_{134} - C_{032}$ and $(-4N_{134} - 9N_{124} - 4N_{123} + 8N_{034} + 9N_{024} + 4N_{023} + 11N_{014} + 9N_{013} + 11N_{012} + 19N_{01} + 4N_{02} + 2N_{03} + 14N_{04} - N_{12} - 3N_{13} + 7C_{234} - 5C_{134} - 6C_{124} + 2C_{032}$). In general, the embedding of a rank 20 lattice like NS_{10} into the K3 lattice $E_8(-1)^2 \oplus U^3$ is not unique up to isomorphism. However, in this case we know that the orthogonal complement of NS_{10} in $H^2(X_{10}, \mathbf{Z})$ is an even rank two lattice with discriminant 15. The classification of positive definite binary quadratic forms shows that there are two such lattices, one is T_{10}, the other has matrix with rows $2, 1$ and $1, 8$. The discriminant form of the Néron-Severi lattice is the opposite of the one of the transcendental lattice (after identifying the discriminant groups). The discriminant form NS_{10} is equal to minus the one of T_{gen}, but the discriminant form of the other lattice is different, hence we found T_{10}.

A result of Nikulin implies that the embedding of T_{10} in the K3 lattice is unique up to isometry (cf. [M], cor. 2.10). Choosing a convenient embedding, it is then easy to find an explicit isomorphism of NS_{10} with a sublattice of the K3 lattice.

To find the other lattices, we use that the number of moduli d_k of K3 surfaces with a fixed number k of Eckardt points is easy to find using the Sylvester form. By the Torelli theorem the rank of the transcendental lattice of a general K3 in this family is at least $d_k + 2$, hence NS_k has rank at most $20 - d_k$. Computations show that in each case the lattice spanned by NS_{gen} and the C_α's on the general member of the family has rank $20 - d_k$. It is not hard to check that this lattice is perpendicular to certain elements in NS_{10}, choosing the Sylvester forms defining the X_k such that $\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3$ for X_6, $\lambda_0 = \lambda_1 = \lambda_2$ and $\lambda_3 = \lambda_4$ for X_4, $\lambda_0 = \lambda_1 = \lambda_2$ for X_3, $\lambda_0 = \lambda_1$ and $\lambda_2 = \lambda_3$ for X_2 and $\lambda_0 = \lambda_1$ for X_1 we have:

(1) $NS_6 = < c_{04} - c_{14} + c_{24} - c_{34} >^\perp \hookrightarrow NS_{10}$.
(2) $NS_4 = < c_{14} - c_{24} + c_{03} - c_{13} + c_{23} >^\perp \hookrightarrow NS_{10}$;
(3) $NS_3 = < c_{34} >^\perp \hookrightarrow NS_4$ and $NS_3 = < c_{23} - c_{13} + c_{03} >^\perp \hookrightarrow NS_6$;
Obviously, to embed NS_2 in NS_6 we need to make a permutation of the indices and consider X_2', i.e. the Hessian of the cubic with two Eckardt points and Sylvester

representation with coefficients $\lambda_0 = \lambda_1$ and $\lambda_2 = \lambda_3$. Thus the embedding is: $NS_2' =< c_{02} - c_{03} - c_{12} + c_{13} >^\perp \hookrightarrow NS_6$;

(4) $NS_2 =< c_{02} - c_{12} >^\perp \hookrightarrow NS_4$;

(5) $NS_1 =< c_{34} >^\perp \hookrightarrow NS_2$ and $NS_1 =< c_{02} - c_{12} >^\perp \hookrightarrow NS_3$;

(6) $NS_{Hess} =< c_{01} >^\perp \hookrightarrow NS_1$.

Next we determined the transcendental lattice T_i as $NS_i^\perp$ in the K3 lattice. In particular, in this way we determined NS_{gen} and its perpendicular T_{gen} in the K3 lattice. $\square$

2.6. **Remark.** We give an explicit primitive embedding of $A_4(-2)$ in T_{gen}. Recall that $T_{gen} = U \oplus U(2) \oplus A_2(-2)$, an element $x \in T_{gen}$ will be written as $x = (x_1, \dots, x_6) \in \mathbf{Z}^6$, the quadratic form on T_{gen} is $2x_1x_2 + 4x_3x_4 - 2(2x_5^2 - x_5x_6 + 2x_6^2)$.

A basis of $A_4(-2)$ is given by the c_{ij} (cf. section 2.4) and the embedding of $A_4(-2)$ in T_{gen} is:

$$c_{01} = (0, 0, 0, 0, 0, 1), \qquad c_{12} = (0, 0, 0, -2, 1, 0)$$

$$c_{23} = (0, 0, 1, 2, -2, -1), \qquad c_{34} = (4, -2, -3, -5, 4, 2).$$

The orthogonal complement of this sublattice in T_{gen} is isomorphic to T_{10} and is spanned by

$$t_1 = (2, 1, 0, 0, 0, 0), \qquad t_2 = (5, -2, -3, -6, 4, 2), \qquad \text{with} \quad (t_i \cdot t_j) = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}.$$

Moreover, the lattice

$$T_{10} \oplus A_4(-2) \hookrightarrow T_{gen} = U \oplus U(2) \oplus A_2(-2)$$

has index 5 and T_{gen} is generated by $T_{10} \oplus A_4(-2)$ and

$$\tfrac{1}{5}(2(t_1 + t_2) + c_{01} + 2c_{12} + 3c_{23} + 4c_{34})) = (6, -2, -3, -6, 4, 2).$$

3. Nodal cubic surfaces

3.1. **Cayley's four nodal cubic surface.** A cubic surface can have up to four nodes. There is a unique surface S_{4n} having four nodes, it is Cayley's cubic surface defined by

$$\frac{1}{z_0} + \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} = 0,$$

it has the Sylvester form:

$$S_{4n}: \qquad \sum x_i = 0, \qquad x_0^3 + x_1^3 + x_2^3 + x_3^3 + \frac{1}{4}x_4^3 = 0.$$

Its nodes (in the Sylvester form) are $p_0 = (-1 : 1 : 1 : 1 : -2)$ and the other three are obtained by permuting the first four coordinates.

These points are also singular on the Hessian Y_{4n}, and they give rise to (-2)-curves M_i on the Hessian K3 surface X_{4n}. The lines L_{ij} connecting two nodes p_i, p_j of the Hessian are on the Hessian. Thus we find another six (-2)-curves on X_{4n} which we also denote by L_{ij}.

In case the cubic surface has k nodes, we denote the (-2)-curves on the Hessian K3 by $M_1, \dots, M_k$ and L_{ij} $(1 \leq i < j \leq k)$.

3.2. **Proposition.** Let T_{kn} be the transcendental lattice of the Hessian K3 surface of a general cubic surface with k nodes. Then we have:

$$T_{4n} =< 2 > \oplus < 6 >$$

and:

$$T_{3n} = T_{4n} \oplus < -2 >, \qquad T_{2n} = T_{4n} \oplus < -2 >^2, \qquad T_{1n} = T_{4n} \oplus < -2 >^3 .$$

Proof. The proof is similar to the one of Proposition 2.5. The non-zero intersection numbers involving the ten new (-2)-curves on the Hessian K3 surface X_{4n} of the Cayley cubic are:

$$\begin{array}{lll} (M_k, L_{ij}) = 1 & \text{if} & \#\{i,j,k\} = 2, \\ (N_{ij4}, L_{kl}) = 1 & \text{if} & \#\{i,j,k,l\} = 4, \\ (N_{ij}, L_{kl}) = 1 & \text{if} & \#\{i,j,k,l\} = 2, \end{array}$$

and of course $M_i^2 = L_{ij}^2 = -2$. A machine computation confirms that the rank of the lattice spanned by the 30 (-2)-curves in X_{4n} is indeed 20 and that a basis of this lattice is given by the 16 curves in the basis of $NS(X_{gen})$ (cf. 1.6) and the 4 curves $L_{01}, L_{02}, L_{03}, L_{12}$. The discriminant of this lattice is -12. Thus the discriminant of the Néron-Severi lattice is either -12 or -3, but if it were -3 the transcendental lattice had to be A_2, which is impossible by Lemma 1.7. Thus these 20 curves span the Néron-Severi lattice.

There are two perpendicular $E_8(-1)$'s in the Néron-Severi lattice:

$$\begin{array}{ccccccccccccc} N_{012} & -- & N_{02} & -- & N_{024} & -- & N_{24} & -- & N_{234} & -- & N_{34} & -- & N_{134} \\ & & & & | & & & & & & & & \\ & & & & N_{04} & & & & & & & & \end{array}$$

$$\begin{array}{ccccccccccccc} N_{013} & -- & N_{03} & -- & L_{03} & -- & M_3 & -- & L_{23} & -- & M_2 & -- & L_{12} \\ & & & & | & & & & & & & & \\ & & & & M_0 & & & & & & & & \end{array}$$

A further computation then shows that

$$NS(X_{4n}) \cong E_8(-1)^2 \oplus U \oplus < -2 > \oplus < -6 > .$$

Thus T_{4n} has rank two, discriminant 12 and its discriminant form is the opposite of the one on $NS(X_{4n})$ and there is a unique such lattice, which is $< 2 > \oplus < 6 >$.

Another way to obtain this result is by realizing the Hessian as a double cover of the plane by projecting from the node M_0. This double cover branches over the union of three lines and a cubic with a node. The pencil of lines on the node of the cubic curve defines an elliptic fibration on X_{4n} which turns out to have six singular fibers, three of type I_6 and three of type I_2. According to the table in [SZ] there is only one such K3 surface (case 4) and its transcendental lattice is indeed $< 2 > \oplus < 6 >$. This surface also appears as a double cover of the plane branched over three conics in [P], Example 3, p.304.

There is a unique, up to isomorphism, embedding of T_{4n} into the K3 lattice. It is not hard to find an explicit isomorphism of the Néron-Severi lattice with the orthogonal complement of the image of T_{4n}. The Néron Severi lattice NS_{kn} of X_{kn} is now easily seen to be:

$$NS_{3n} = (T_{4n} \oplus < M_3 >)^{\perp}, \quad \ldots \quad , NS_{1n} = (T_{4n} \oplus < M_1, M_2, M_3 >)^{\perp},$$

next one finds T_{kn} as the orthogonal complement of NS_{kn}. $\square$

4. Cubics with Eckardt points and nodes

4.1. In this section we consider the Hessians X_{anb} of three cubic surfaces with a nodes and b Eckardt points with $(a,b)=(1,6),(1,4)$ and $(3,4)$. These K3 surfaces are 'singular', i.e. have Néron-Severi groups of rank 20. In two of the three cases we succeeded in determining the transcendental lattice completely.

4.2. **Proposition.**

(1) The Hessian K3 surface X_{1n6} of the cubic surface S_{1n6} with one node and 6 Eckardt points with Sylvester form
$$\sum x_i=0,\qquad x_0^3+x_1^3+x_2^3+x_3^3+\frac{1}{16}x_4^3=0$$
has transcendental lattice
$$T_{1n6}=<2>\oplus<24>.$$

(2) The Hessian K3 surface X_{1n4} of the cubic surface S_{1n4} with one node and 4 Eckardt points with Sylvester form
$$\sum x_i=0,\qquad x_0^3+x_1^3+x_2^3+\frac{4}{9}x_3^3+\frac{4}{9}x_4^3=0$$
has transcendental lattice which is either
$$<6>\oplus<12>\qquad\text{or}\qquad<2>\oplus<4>.$$

(3) The Hessian K3 surface X_{3n4} of the cubic surface S_{3n4} with three nodes and 4 Eckardt points with Sylvester form
$$\sum x_i=0,\qquad x_0^3+x_1^3+x_2^3+4x_3^3+4x_4^3=0$$
has transcendental lattice
$$T_{3n4}=<4>\oplus<6>.$$

Proof. The only singular point of S_{1n6} is the point $p_0=(1:\ldots:1:-4)$ which is also singular on the Hessian, which has eleven singular points. This point does not lie on any of the 10 lines of the Hessian, so the Néron-Severi group of X_{1n6} contains the direct sum of the Néron-Severi group of the general Hessian K3 with 6 Eckardt points and the lattice $<-2>$, spanned by the (-2)- curve over p_0:
$$NS_6\oplus<-2>=E_8(-1)^2\oplus U\oplus<-24>\oplus<-2>\hookrightarrow NS_{1n6}.$$
As the discriminant of this sublattice is -48, the discriminant of NS_{1n6} is either -48 or there is a vector v, primitive in $NS_6\oplus<-2>$, such that $v/2\in NS_{1n6}$. As $E_8(-1)^2\oplus U$ is a direct summand, we can write $v=ae+bf$ with $a,b\in\{0,1\}$ and $e^2=-24, ef=0, f^2=-2$. As $(v/2)^2=-6a^2+b^2/2\in 2\mathbf{Z}$ (since NS_{1n6} is even) we get $b=0$, as $<-24>$ is primitive in NS_6 and hence, after specialization, in NS_{1n6}, we get $a=0$. Therefore $NS_6\oplus<-2>\cong NS_{1n6}$.

The transcendental lattice T_{1n6} is a positive definite even rank two lattice whose discriminant form is the opposite of the one of NS_{1n6}, and there is only one such lattice.

The unique singular point of S_{1n4} is $p_0 = (-2 : -2 : -2 : 3 : 3)$ which does not lie on any of the lines in the Hessian. Thus one finds as above an embedding of lattices:

$$NS_4 \oplus < -2 > = E_8(-1)^2 \oplus U(3) \oplus < -4 > \oplus < -2 > \hookrightarrow NS_{1n4}.$$

In this case we cannot decide whether the sublattice has index 1 or 3. In case the index is one, one has to use the discriminant form to find T_{1n4}, if the index is three then the discriminant of T_{1n4} is 8, and there is a unique such even positive definite lattice.

The three singular points of S_{3n4} are:

$$p_0 = (-2 : 2 : 2 : -1 : -1), \qquad p_1 = (2 : -2 : 2 : -1 : -1), \qquad p_2 = (2 : 2 : -2 : -1 : -1).$$

The lines L_{ij} spanned by p_i and p_j lie on the Hessian. The p_i's do not lie on any of the 10 lines N_{ij} but the lines L_{ij} and N_{ij} intersect, finally L_{ij} and N_{klm} intersect if $\sharp\{i,j,k,l,m\} = 5$. Of the lines C_{abc} on the Hessian of a cubic with Eckardt points, only the lines C_{012} and C_{021} remain, the others collapse onto the L_{ij}. A machine computation shows that the lattice spanned by these (-2)-curves has rank 20 and discriminant -24. A basis is given by the 16 curves in the basis of $NS(X_{gen})$ (cf. 1.6) and $C_{012}, L_{01}, L_{02}, L_{12}$. Since the determinant of an even rank two lattice cannot be 6, we conclude that this lattice is NS_{gen}. To find its discriminant form we used the following two copies of $E_8(-1)$:

$$\begin{array}{ccccccccccccc}
N_{23} & -- & N_{123} & -- & N_{13} & -- & N_{013} & -- & N_{01} & -- & L_{01} & -- & M_0 \\
 & & & & | & & & & & & & & \\
 & & & & N_{134} & & & & & & & & \\
 & & & & & & & & & & & & \\
N_{124} & -- & N_{24} & -- & N_{024} & -- & N_{04} & -- & N_{034} & -- & L_{12} & -- & M_2 \\
 & & & & | & & & & & & & & \\
 & & & & N_{02} & & & & & & & &
\end{array}$$

and we found an isomorphism

$$NS_{3n4} \cong E_8(-1)^2 \oplus U \oplus < -4 > \oplus < -6 > .$$

There is a unique even, rank 2, positive definite lattice with opposite discriminant group. $\square$

5. Cubic forms without a Sylvester form

5.1. **No Sylvester form.** We defined the Sylvester form of a cubic form F in 4 variables using 5 linear forms $x_0, \ldots, x_4$ in 4 variables such that any four of the five are linearly independent, cf. 1.1, normalized such that $\sum x_i = 0$ and satisfying $F = \sum \lambda_i x_i^3$ for some $\lambda_i \in \mathbf{C}$.

There are basically two types of cubic forms defining smooth cubic surfaces which do not admit a Sylvester form (we do not know precise results on forms defining singular surfaces). In the first case one still has $F = \sum \lambda_i x_i^3$, but at least four of the five x_i are linearly dependent, say $x_4 = a_1x_1 + a_2x_2 + a_3x_3$. Then one has $F = \lambda_0 x_0^3 + G(x_1, x_2, x_3)$ with a homogeneous polynomial G in three variables. The corresponding cubic surfaces are called cyclic surfaces. Note that the Hessian of a cyclic cubic surface is reducible, which proves that these surfaces do not have a Sylvester form.

The second type of cubic forms F which do not admit a Sylvester forms are those which are not a sum of five cubes of linear forms. These are obtained as a limit of Sylvester forms where at least two of the five x_i coincide, we will refer to these as non-Sylvester forms. The corresponding surfaces were described in [S] and [Rod]. They are of two types, denoted here by $ns1$ and $ns2$, the dimensions of these families are three and two respectively. We give explicit one parameter families of Sylvester forms specializing to these surfaces in the proof of Theorem 6.6.

5.2. **The first case.** A cubic surface of type $ns1$ is defined by a form F of the following type:

$$S_{ns1}: \qquad F = x_1^3 + x_2^3 + x_3^3 - x_0^2(a_0x_0 + 3a_1x_1 + 3a_2x_2 + 3a_3x_3) = 0.$$

This form defines a smooth cubic surface iff, for all choices of sign,

$$a_0 + 2(\pm a_1^{3/2} \pm a_2^{3/2} \pm a_3^{3/2}) \neq 0.$$

In case one of the $a_i = 0$ for $1 \leq i \leq 3$, the form F defines a cyclic surface. We will assume that S_{ns1} is smooth and that $a_1a_2a_3 \neq 0$ from now on.

The Hessian of F is, up to a scalar multiple,

$$H_F = x_1x_2x_3(a_0x_0 + \ldots + a_3x_3) + x_0^2(a_1^2x_2x_3 + a_2^2x_1x_3 + a_3^2x_1x_2).$$

Obviously, $p_0 = (1:0:0:0)$ is singular on the Hessian surface Y. Projecting from p_0 realizes the Hessian surface as a double cover of the plane (with coordinates x_1, x_2, x_3) branched over the curve defined by:

$$x_1x_2x_3[a_0^2x_1x_2x_3 - 4(a_1x_1 + a_2x_2 + a_3x_3)(a_1^2x_2x_3 + a_2^2x_1x_3 + a_3^2x_1x_2)] = 0,$$

that is, the union of three lines and a nonsingular cubic curve. As a singular point of the discriminant curve is the image of a singular point or the image of a line on the Hessian surface it is now easy to determine the singular locus of Y. One finds that Y has exactly 7 singular points, p_0 and the six points:

$$p_1 = (0:1:0:0), \quad p_2 = (0:0:1:0), \quad p_3 = (0:0:0:1),$$

(these points lie on lines $x_i = x_j = 0$ $(1 \leq i < j \leq 3)$ in Y which project to points),

$$q_1 = (0:0:a_3:-a_2), \quad q_2 = (0:-a_3:0:a_1), \quad q_3 = (0:a_2:-a_1:0).$$

In particular, the cubic surface S_0 defined by F cannot have a Sylvester form (in that case the Hessian has exactly 10 singular points) nor is it cyclic (in that case the Hessian surface is reducible).

5.3. **The second case.** A cubic surface of type $ns2$ is defined as follows.

$$S_{ns2}: \qquad G = x_1^3 + x_2^3 + 2\lambda x_3^3 - 3x_3(\mu x_1x_3 + x_2x_3 + x_0^2) = 0.$$

If $\mu = 0$ this is a cyclic cubic surface and the surface S_{ns2} is smooth iff for all choices of sign:

$$\lambda \pm \mu^{3/2} \pm 1 \neq 0.$$

The Hessian of G is (up to scalar multiple):

$$H_G = x_1x_2x_3(-2\lambda x_3 + \mu x_1 + x_2) + x_3^3(x_1 + \mu^2x_2) - x_0^2x_1x_2.$$

Again $p_0 = (1:0:0:0)$ is a singular point. Projecting from p_0 gives the branch curve defined by:

$$x_1x_2x_3[x_1x_2(-2\lambda x_3 + \mu x_1 + x_2) + x_3^2(x_1 + \mu^2 x_2)] = 0,$$

which is the union of three lines and a smooth cubic curve. The Hessian surface has only four singular points, p_0, and

$$p_1 = (0:1:0:0), \quad p_2 = (0:0:1:0), \quad q = (0:1:-\mu:0).$$

As before, we conclude that these surfaces do not admit a Sylvester form nor are they cyclic.

5.4. **Proposition.** Let X_{nsk}, $k = 1,2$, be the Hessian K3 surface of a general cubic surface S_{nsk} which does not admit a Sylvester form. Let T_{nsk} be the transcendental lattice of X_{nsk}. Then we have:

$$T_{ns1} = U \oplus U(2) \oplus < -4 >, \qquad T_{ns2} = U \oplus U(2).$$

Proof. First we determine the Néron-Severi group of X_{ns2} using the elliptic fibration defined by the pencil of lines on the image $\bar{q}$ of q in $\mathbf{P}^2$. From a study of the branch curve one finds that there are 6 bad fibers, of type $I_0^*, I_8^*, I_1, I_1, I_1, I_1$. The lines $x_i = x_3 = 0$ $(i = 1,2)$ define sections of the fibration. The rank of the Néron-Severi group of the general X_{ns2} is 18 (the surfaces X_{ns2} have two moduli), which is two plus the number of components of the bad fibers not meeting one of the sections. Hence the Mordell-Weil group of the fibration is finite. The torsion subgroup of a fiber of type I_0^* is $(\mathbf{Z}/2\mathbf{Z})^2$ and the two-torsion of an I_1-fiber is $\mathbf{Z}/2\mathbf{Z}$. As the torsion of the Mordell-Weil group injects in the torsion subgroup of the fibers it must be either trivial or be isomorphic to $\mathbf{Z}/2\mathbf{Z}$. As there are two sections, we conclude that the Mordell-Weil group of the elliptic fibration is $\mathbf{Z}/2\mathbf{Z}$. The Shioda-Tate formula ([SI], Lemma 1.3) now shows that the discriminant of $NS(X_{ns2})$ is 4.

The 13 (-2)-curves in the I_8^* fiber (which map to the line $x_3 = 0$ in $\mathbf{P}^2$, note the configuration of curves is two copies of D_6, mapping to $x_1 = x_3 = 0$ and $x_2 = x_3 = 0$, linked with a vertex which is a (-2)-curve mapping to the line $x_3 = 0$), the (-2)-curve over $\bar{q}$, the two sections and the four of the five (-2)-curves in the I_0^* fiber (these 4 curves map to the point $x_1 = x_2 = 0$ in $\mathbf{P}^2$) give a diagram with 20 vertices. It is a square with 3 points on each edge and each of the four vertices of the square is linked to a further point, which is not linked to anything else. In particular, it is very easy to find two perpendicular E_8's in the lattice generated by these 20 curves and to compute that the orthogonal complement to this sublattice is $U(2)$. Hence $NS(X_{ns2}) \cong E_8(-1)^2 \oplus U(2)$. By [Ni], Thm. 1.14.4, there is, up to isometry, a unique embedding of $U(2)$ into U^3. From this one finds that $T_{ns2} = NS(X_{ns2})^\perp \cong U(2) \oplus U$.

Similar to the case of X_{ns2}, the family of lines on the image of q_3 in $\mathbf{P}^2$ defines an elliptic fibration X_{ns1} with two sections, $x_i = x_3 = 0$, $i = 1,2$. There are 7 singular fibers of type $I_4^*, I_4, I_0^*, I_1, I_1, I_1, I_1$. There are 15 components of bad fibers which do not meet one of the sections, so the Mordell-Weil group of this fibration finite. As before we conclude that the Mordell-Weil group of the elliptic fibration is $\mathbf{Z}/2\mathbf{Z}$. The Shioda-Tate formula shows that the discriminant of $NS(X_{ns1})$ is 2^4.

It is amusing to observe that the following 20 (-2)-curves have an intersection diagram which is a cube with a point on the middle of each edge: the 9 components of the I_4^* fiber; the following 3 components of the I_4-fiber: the line $l:\ x_0 = a_1x_1 + a_2x_2 + a_3x_3 = 0$ in Y_{ns1} and the (-2)-curves over q_1 and q_2 in X_{ns2}; the (-2)-curves over p_0, q_3; the two sections; and 4 of the 5 components of the I_0^* fiber.

It is clear that X_{ns2} is a limit of X_{ns1} (two of the lines in the branch curve become tangent to the cubic component, equivalently, the fibers of type I_4 and I_4^* coalesce to a fiber of type I_8^*). Thus we get an inclusion $NS(X_{ns1}) \subset NS(X_{ns2})$ and it is not hard to see that $NS(X_{ns1}) = (n_1 - n_2)^{\perp}$ where n_1, n_2 are two disjoint (-2)-curves in X_{ns2}, they are the inverse image of the strict transform of the exceptional divisor in the first blow up of $\mathbf{P}^2$ in the image of p_1 and p_2. Thus the lattice $T(X_{ns2}) \oplus < -4 >$ is a sublattice of $T(X_{ns1})$, but since these lattices have the same discriminant, they are equal. $\square$

5.5. **Remark.** Specializing a general Hessian to one of type $ns1$ induces an inclusion $NS_{gen} \hookrightarrow NS_{ns1}$. One can verify that $NS_{gen}^{\perp}$ in NS_{ns1} is generated by a class with selfintersection -12. Note that $t = (1,2) \in A_2(-2)$ has $t^2 = -12$ and that its orthogonal complement is $< (1,0) >$ with $(1,0)^2 = -4$, so

$$(0,0,(1,2))^{\perp} \cong U \oplus U(2) \oplus < -4 > \cong T_{ns1} \qquad (\subset U \oplus U(2) \oplus A_2(-2) = T_{gen}).$$

5.6. **Eckardt points.** As the Eckardt points of a cubic surface S are the singular points of its Hessian Y which are on S ([S], p.146), it is easy to find the possible configurations of Eckardt points on a surface of type $ns1$. We determine the transcendental lattices of these surfaces below.

5.7. **Proposition.** Let S_{ns1} be a cubic surface without a Sylvester form as in section 5.2, from which we also adopt the notation and conventions:

$$S_{ns1}: \qquad a_0x_0^3 + x_1^3 + x_2^3 + x_3^3 - 3x_0^2(a_1x_1 + a_2x_2 + a_3x_3) = 0.$$

(1) In case $a_0 = 0$ and all a_i^3's are distinct, the point p_0 is the unique Eckardt point on S_{ns1}. The general Hessian K3 surface has Picard number 18 and transcendental lattice

$$U(2)^2.$$

(2) In case $a_0 \neq 0$ but $a_i^3 = a_j^3 \neq a_k^3$ the point q_k is the unique Eckardt point on S_{ns1}. The general Hessian K3 surface has Picard number 18 and transcendental lattice

$$U \oplus < -4 > \oplus < 4 > .$$

(3) In case $a_0 = 0$ and $a_i^3 = a_j^3 \neq a_k^3$, the points p_0, q_k are the only Eckardt points on S_{ns1}. The general Hessian K3 surface has Picard number 19 and transcendental lattice

$$U(2) \oplus < 4 > .$$

(4) In case $a_0 \neq 0$ and $a_i^3 = a_j^3 = a_k^3$, the points q_1, q_2, q_3 are the only Eckardt points on S_{ns1}. The general Hessian K3 surface has Picard number 19 and transcendental lattice

$$U \oplus < 12 > .$$

(5) In case $a_0 = 0$ and $a_i^3 = a_j^3 = a_k^3$, the points p_0, q_1, q_2, q_3 are the only Eckardt points on S_{ns1}. The general Hessian K3 surface has Picard number 20 and transcendental lattice

$$A_2(2) = \begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix}.$$

Proof. We start with the case $a_0 = 0$ and $a_i^3 = a_j^3 = a_k^3$, we denote the cubic surface by S. (Note that the transformation $x_j \mapsto \omega^{n_j} x_j$, with $j = 1, 2, 3$ and with a cube root of unity ω, also gives an equation of type $ns1$ but with $a_j \mapsto \omega^{-n_j} a_j$). For every Eckardt point p the tangent plane to the cubic surface in p intersects the Hessian in a double line and two new lines. In particular, p_0 gives the pair of lines lines $d^{\pm}$ on S defined by $\sum_{i=1}^3 x_i = 0, \sum_{1\leq i<j\leq 3} x_i x_j = 0$, and similarly the q_i give the pairs of lines $x_k + x_l = x_0^2 + x_i^2 = 0$. The lattice L generated by a basis of $NS(X_{ns1})$, d^+ and the lines $c_1^+ : x_2 + x_3 = x_0 + ix_1 = 0$, $c_2^- : x_1 + x_3 = x_0 - ix_2 = 0$ has rank 20 and discriminant 12. The only rank two, even, positive definite lattice with discriminant 3 is A_2, but this cannot be the transcendental lattice of a Hessian by Lemma 1.7. Hence L must be the Néron-Severi lattice of the Hessian K3 surface X of S and a computation of the discriminant group shows that the transcendental lattice is $A_2(2)$. It is also not hard to find two orthogonal E_8's in the Néron-Severi group of X and thus to find an embedding of $NS(X)$ into the K3 lattice.

For the other cases it is easy to find the rank r of the transcendental lattice from a count of the moduli and to find curves on the surface which span a lattice of rank $22 - r$. Then one can verify that these curves all lie in the following sublattices of $L = NS(X)$, and using the embedding of $NS(X)$ into the K3 lattice, one finds the transcendental lattices.

(1) The Néron-Severi group is $< (c_1^+ - c_1^-), (c_2^+ - c_2^-) - (c_3^+ - c_3^-) >^{\perp}$ in $NS(X)$.
(2) The Néron-Severi group is $< (c_i^+ - c_i^-) - (c_j^+ - c_j^-), d^+ - d^- >^{\perp}$ in $NS(X)$.
(3) The Néron-Severi group is $< (c_i^+ - c_i^-) - (c_j^+ - c_j^-) >^{\perp}$ in $NS(X)$.
(4) The Néron-Severi group is $< d^+ - d^- >^{\perp}$ in $NS(X)$.

□

5.8. **Remark.** The one dimensional family of Hessians in Proposition 5.7.4 with transcendental lattice $U \oplus < 12 >$ was studied by Peters and Stienstra in [PS]. They also observed the 'cube' formed by 20 of the curves in the Néron-Severi lattice (see the proof of Proposition 5.4). The total space of this family is a Calabi-Yau threefold (the Fermi threefold) and was studied by Verrill [V], section 4.

6. The moduli space of cubic surfaces

6.1. Classical invariant theory shows that the moduli space of cubic surfaces $\mathcal{M}$ is isomorphic to the weighted projective space $\mathbf{P}(1, 2, 3, 4, 5)$. Each point in $\mathcal{M}$ corresponds to the isomorphism class of a cubic surface with at most 4 nodes, except for one point. That point corresponds to the semi stable, non-stable, cubic surfaces, the unique closed orbit in this set is the orbit of $t^3 = xyz$. In Theorem 6.6 we show that it maps to $(8 : 1 : 0 : 0 : 0) \in \mathcal{M}$ where we use Salmon's generating invariants (cf. section 6.3).

We recall the classical description of this moduli space using the Sylvester forms. We discuss some divisors, and their classes in the Chow group of the moduli space. We determine the divisor parametrizing cubic surfaces without a Sylvester form in Theorem 6.6. Finally we make some comments on the singular locus of the moduli space.

6.2. **Invariants.** The ring of invariants of the action of $SL(4,\mathbf{C})$ on the space of cubic forms in 4 variables is generated by the invariant polynomials I_n of degree n for $n = 8, 16, 24, 32, 40, 100$. Since 100 is not divisible by 8 and I_{100}^2 is polynomial in the other generators, the moduli space of cubic surfaces (the *Proj* of the ring of invariants) is the weighted projective space $\mathbf{P}(1,2,3,4,5)$. Note that I_8 and I_{100} are, up to a scalar multiple, unique whereas $I_{16},\ldots,I_{40}$ are only unique up to the addition of homogeneous weighted polynomials of lower degree (for example, for any $a,b,c \in \mathbf{C}$ $a \neq 0$, the invariant $aI_{24} + bI_8^3 + cI_8I_{16}$ can also be used as a generator of degree 24). We will use Salmon's convention to choose the generators.

6.3. **Generating invariants and the Sylvester form.** The generating invariants are easily computed for a Sylvester form

$$\sum_{i=0}^{4} x_i = 0, \qquad \sum_{i=0}^{4} \lambda_i x_i^3 = 0.$$

Let σ_i be the i-th symmetric function in $\lambda_0,\ldots,\lambda_4$. Then ([Sal] p. 197):

$$I_8 = \sigma_4^2 - 4\sigma_3\sigma_5, \quad I_{16} = \sigma_5^3\sigma_1, \quad I_{24} = \sigma_5^4\sigma_4, \quad I_{32} = \sigma_5^6\sigma_2, \quad I_{40} = \sigma_5^8.$$

The quotient of $\mathbf{P}^4_\lambda$, the parameter space of the Sylvester forms, by the group S_5 is also a weighted projective space $\mathbf{P}(1,2,3,4,5)_\sigma$ with coordinates the elementary symmetric functions. The formulas above define a birational isomorphism:

$$\phi : \mathbf{P}(1,2,3,4,5)_\sigma \dashrightarrow \mathbf{P}(1,2,3,4,5)_I, \qquad (\sigma_1 : \ldots : \sigma_5) \longmapsto (I_8 : \ldots : I_{40}).$$

The base locus of ϕ is given by $\sigma_4 = \sigma_5 = 0$. The birational inverse of ϕ is:

$$\psi : \mathbf{P}(1,2,3,4,5)_I \dashrightarrow \mathbf{P}(1,2,3,4,5)_\sigma,$$

$$(I_8 : \ldots : I_{40}) \longmapsto (I_{16} : I_{32} : (I_{24}^2 - I_8I_{40})/4 : I_{24}I_{40} : I_{40}^2),$$

the base locus of ψ is the point $(1:0:0:0:0)$ which corresponds to the Fermat cubic surface, see Theorem 6.6.

6.4. **The boundary divisor.** The locus of singular cubic surfaces, the boundary of the moduli space, is the divisor in $\mathbf{P}(1,2,3,4,5)$ defined by:

$$(I_8^2 - 2^6 I_{16})^2 = 2^{14}(I_{32} + 2^{-3} I_8 I_{24}).$$

The degree of this invariant is 32 (cf. [S], App. III p. 179, [GKZ], p. 38), the formula in [Sal], p. 198, omits the exponent -3, but we verified that upon substituting the σ_i in the I_n in our formula one finds the expression given in section 1.3. As I_{40} does not appear in this formula, the boundary divisor is a cone over a $\mathbf{P}(1,2,3,4)$.

6.5. **The tritangent divisor.** The tritangent divisor parametrizes cubic surfaces with an Eckardt point. The general such surface has a Sylvester form in which two of the λ_i coincide. Thus this divisor is defined by the discriminant of the polynomial $\prod_{i=0}^4(x-\lambda_i)$, which is the invariant I_{100} ([Sal], p. 197). The discriminant is a polynomial of weight 20 in the symmetric functions in the λ_i and gives, upon pull-back along the map ψ from section 6.3, a polynomial of weight $320 = 20\cdot 16$ in the basic invariants $I_8,\ldots,I_{40}$. This polynomial has a factor I_{40}^3, the other factor is $I_{100}^2 = f(I_8,\ldots,I_{40})$ which defines the tritangent divisor in $\mathbf{P}(1,2,3,4,5)$.

6.6. **Theorem.** The divisor in the moduli space of cubic surfaces $\mathcal{M}\cong\mathbf{P}(1,2,3,4,5)_I$, which parametrizes cubic surfaces which do not admit a Sylvester form is defined by $I_{40}=0$.
The subvariety parametrizing the surfaces of type $ns2$ is defined by $I_{24}=I_{40}=0$.
The subvariety parametrizing the cyclic surfaces is defined by $I_{24}=I_{32}=I_{40}=0$.
The point $I_{16}=I_{24}=I_{32}=I_{40}=0$ parametrizes the Fermat cubic surface.
The non-stable, semi-stable cubic surface defined by $t^3=xyz$ maps to the point $(8:1:0:0:0)\in\mathcal{M}$.

Proof. A family of five linear forms where $x_0,\ldots,x_3$ are coordinates on a $\mathbf{P}^3$ and where two of the linear forms coincide in the limit $t\to 0$ is:

$$x_0(t)=x_0,\quad x_i(t)=a_itx_i\quad(i=2,3,4),\qquad x_4(t)=-x_0-t(a_1x_1+a_2x_2+a_3x_3).$$

Note that any four of the five linear forms are independent when the $a_i\neq 0$ and $t\neq 0$ and that $\sum x_i(t)=0$. The family of cubic surfaces, defined by Sylvester forms:

$$S_t:\quad\left\{\begin{array}{l}(a_0+t^{-1})x_0(t)^3+(a_1t)^{-3}x_1(t)^3+(a_2t)^{-3}x_2(t)^3+(a_3t)^{-3}x_3(t)^3+t^{-1}x_4(t)^3=0,\\ \sum_{i=0}^4 x_i(t)=0\end{array}\right.$$

has as limit the cubic surface S_{ns1} defined by:

$$S_{ns1}:=S_0:\qquad a_0x_0^3+x_1^3+x_2^3+x_3^3-3x_0^2(a_1x_1+a_2x_2+a_3x_3)=0.$$

Replacing x_0 by $-x_0$ we get the cubic form:

$$F=x_1^3+x_2^3+x_3^3-x_0^2(a_0x_0+3a_1x_1+3a_2x_2+3a_3x_3)$$

considered in 5.2.

To find the invariants of the general surface of type $ns1$ we compute the invariants of the surface S_t in the one parameter family S_t and take the limit $t\to 0$. The result is:

$$[S_{ns1}]=[\lim_{t\to 0}S_t]=(-4\rho_1+a_0^2:\rho_2:2\rho_3:\rho_1\rho_3:0)\qquad(\in\mathbf{P}(1,2,3,4,5)_I),$$

where ρ_i is the i-th elementary symmetric function in a_1^3,a_2^3,a_3^3. From the equation of S_{ns1} it is clear that its isomorphism class depends only on the symmetric functions of a_1^3,a_2^3,a_3^3. Thus the three dimensional family $ns1$ maps to the divisor $I_{40}=0$ which shows that this divisor is the closure of the locus of surfaces of type $ns1$. Note that for a general point of $I_{40}=0$, in particular one for which the other coordinates are non-zero, one can recover ρ_2 and ρ_3 from the second and third coordinate, the fourth coordinate then determines ρ_1 and finally a_0 is recovered from the first coordinate.

A one parameter family where three of the linear forms coincide in the limit $t \to 0$ is:

$$x_0(t) = -x_3 - t^2x_0, \quad x_1(t) = \mu t^2 x_1, \quad x_2 = t^2x_2, \quad x_3 = 2x_3 + tx_0$$

and $x_4(t) = -(x_0(t) + \ldots + x_3(t))$. Again any four of the 5 $x_i(t)$ are independent for $t \neq 0$. The family of cubic surfaces, defined by Sylvester forms:

$$S_t: \quad \begin{cases} t^{-2}x_0(t)^3 + (\mu t^2)^{-3}x_1(t)^3 + t^{-6}x_2(t)^3 + (\frac{\lambda}{4} + \frac{1}{4t^2})x_3(t)^3 + t^{-2}x_4(t)^3 = 0, \\ \sum_{i=0}^4 x_i(t) = 0 \end{cases}$$

has as limit the cubic surface S_{ns2} defined by (after substituting $\frac{1}{\sqrt{2}}x_0$ for x_0):

$$S_{ns2} := S_0: \qquad G = x_1^3 + x_2^3 + 2\lambda x_3^3 - 3x_3(\mu x_1x_3 + x_2x_3 + x_0^2) = 0.$$

The same procedure as above gives:

$$[S_{ns2}] = (-8\lambda : 1 + \mu^3 : 0 : \mu^3 : 0) \qquad (\in \mathbf{P}(1,2,3,4,5)_I),$$

where λ, μ are the coefficients of the equation of S_{ns2}.

Finally we consider the cyclic surfaces. These can be obtained from S_{ns1} with one of the $a_i = 0$, but the following family has a particularly nice limit, a sum of cubes of linear forms. Consider the one parameter family of Sylvester forms:

$$x_i(t) = (1+t)x_i, \quad (i = 0,1,2), \qquad x_4(t) = tx_4, \qquad x_3(t) = -(x_0(t) + x_1(t) + x_2(t) + x_4(t)).$$

Any four of the five $x_i(t)$ are independent for $t \neq 0$. The family of cubic surfaces, defined by Sylvester forms:

$$S_t: \quad \lambda_0x_0(t)^3 + \ldots + \lambda_3x_3(t)^3 + (\tfrac{\lambda_4}{t^3})x_4(t)^3 = 0, \quad \sum_{i=0}^4 x_i(t) = 0,$$

has as limit the cyclic cubic surface S_{cyc} defined by

$$S_{cyc} := S_0: \qquad G = \lambda_4x_4^3 - \lambda_3(x_0 + x_1 + x_2)^3 + \lambda_0x_0^3 + \lambda_1x_1^3 + \lambda_2x_2^3 = 0.$$

Computing the invariants as before gives:

$$[S_{cyc}] = (\tau_3^2 - 4\tau_2\tau_4 : \tau_4^3 : 0 : 0 : 0) \qquad (\in \mathbf{P}(1,2,3,4,5)_I),$$

where τ_i is the i-th elementary symmetric function in $\lambda_0, \ldots, \lambda_3$ (note that λ_4 does not appear as is obvious from the equation of S_{cyc}). The Fermat cubic is the surface S_{cyc} with $\lambda_3 = 0$, hence $\tau_4 = \prod_{i=0}^3 \lambda_i = 0$.

Putting $\lambda_i = 1$, $0 \leq i \leq 3$, we obtain the cubic surface $x_4^3 = 3(x_0 + x_1)(x_0 + x_2)(x_1 + x_2)$, which is isomorphic to the strictly semi-stable surface $t^3 = xyz$, and it defines the point $(-8:1:0:0:0) = (8:1:0:0:0) \in \mathcal{M} = \mathbf{P}(1,2,3,4,5)$. Note that this is the unique point in the intersection of the boundary divisor, defined by the equation from 6.4, with the curve $I_{24} = I_{32} = I_{40} = 0$ parametrizing cyclic surfaces. $\square$

6.7. **The Kummer divisor.** The Kummer K3 surface of a principally polarized abelian surface is the Hessian K3 surface of a cubic surface. The divisor in $\mathcal{M}$ which we obtain in this way will be called the Kummer divisor. Rosenberg ([Ros], Cor. 1.2) proved that the Kummer divisor in $\mathcal{M}$ is defined by:

$$I_8 I_{24} + 8 I_{32}.$$

The Néron-Severi group of the Kummer surface has rank 17 and has transcendental lattice isomorphic to $T_{kum} = U(2) \oplus U(2) \oplus < -4 >$. The lattice T_{kum} is the orthogonal complement of an element t with $t^2 = -12$ in T_{gen}.

6.8. **The locus** $I_{40} = I_{100}^2 = 0$. The intersection of the tritangent divisor $I_{100}^2 = 0$ and the non-Sylvester divisor $I_{40} = 0$ consists of three irreducible components (of dimension two). In fact, putting $I_{40} = 0$ in $I_{100}^2 = f(I_8, \dots, I_{40})$ one finds (up to scalar multiple):

$$f(I_8, \dots, I_{32}, 0) = I_{24}^3 (I_8 I_{24} + 8 I_{32}) g(I_8, \dots, I_{32})$$

where g is given by:

$$g = 16 I_{16}^3 I_{24}^2 + 27 I_{24}^4 - 72 I_{16} I_{24}^2 I_{32} - 16 I_{16}^2 I_{32}^2 + 64 I_{32}^3.$$

The first factor, $I_{24} = 0$, corresponds to the non-Sylvester surfaces of type $ns2$, cf. Theorem 6.6. These surfaces do indeed have an Eckardt point: $p_0 = (1 : 0 : 0 : 0)$ (notation as in section 5.3) is singular on the Hessian and lies on S_{ns2}, hence it is an Eckardt point.

The second factor is the intersection of $I_{40} = 0$ with the Kummer divisor. Parametrizing the non-Sylvester divisor as in the proof of Theorem 6.6, this component is given by $a_0 = 0$ and these surfaces in fact have an Eckardt point (cf. Proposition 5.7.1).

The last factor pulls-back to the discriminant of the polynomial $(x - a_1^3)(x - a_2^3)(x - a_3^3)$, hence gives the surfaces with Eckardt point considered in Proposition 5.7.2.

6.9. **The singular points of the moduli space.** The singular locus of a weighted projective space was determined in [DD], Prop. 7. For $\mathcal{M} = \mathbf{P}(1,2,3,4,5)$ we get:

$$\mathbf{P}(1,2,3,4,5)_{sing} = \{(0:0:1:0:0)\} \cup \{0:0:0:0:1)\} \cup \{(0:a:0:b:0)\},$$

with $a, b \in \mathbf{C}$ not both zero.

Let $\mathbf{P}(q_1, \dots, q_n)$ be a weighted projective space where we (may) assume that any $n-1$ of the q_i are relatively prime. This algebraic variety is covered by affine open subsets U_i which are cyclic quotients of vector spaces ([Do], Proposition 1.3.3 and its proof):

$$U_i = \{(x_1 : \dots : x_n) \in \mathbf{P}(q_1, \dots, q_n) : x_i \neq 0\} \;\cong\; \mathbf{C}^{n-1}/\mu_{q_i},$$

where the action of a q_i-th root of unity ζ on $\mathbf{C}^{n-1}$ is given by

$$(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \longmapsto (\zeta^{q_1} x_1, \dots, \zeta^{q_{i-1}} x_{i-1}, \zeta^{q_{i+1}} x_{i+1}, \dots, \zeta^{q_n} x_n).$$

In particular, $U_1 \subset \mathbf{P}(1,2,3,4,5)$ is isomorphic to $\mathbf{C}^4$, U_3, U_5 are the quotient of $\mathbf{C}^4$ by an automorphisms of order three and five respectively and both have an isolated singularity.

6.10. **The cubic surfaces in the singular locus.** The singular locus of $\mathcal{M}$ parametrizes, as expected, certain cubic surfaces with non-trivial automorphism groups.

The cubic surface corresponding to $(0:0:1:0:0)$ has $I_{40}=0, I_{24}\neq 0$, hence it is of non-Sylvester type $ns1$ (Proposition 6.6), moreover, it is a singular point (section 6.4). The parametrization given in the proof of Theorem 6.6 shows that $\rho_1=\rho_2=a_0=0$, from which one finds that $a_i^3=\omega^i a_3^3$ for a primitive cube of unity ω and the equation of the surface is: $x_1^3+x_2^3+x_3^3-3x_0^2(x_1+\epsilon x_2+\epsilon^2 x_3)=0$ where ϵ is a primitive 9th root of unity. A change of variables gives the equation ($\omega=\epsilon^6$):

$$x_1^3+\omega x_2^3+\omega^2 x_3^3-3x_0^2(x_1+x_2+x_3)=0.$$

This surface has two singular points, $p_\pm=(\pm 1:1:\omega:\omega^2)$. It has an automorphism of order three given by

$$(x_0:x_1:x_2:x_3)\longmapsto(\omega x_0:x_2:x_3:x_1).$$

The cubic surface corresponding to $(0:0:0:0:1)$ is singular and has the Sylvester form:

$$\sum_{i=0}^{4}\eta^i x_i^3=0,\qquad \sum_{i=0}^{4}x_i=0,$$

where η is a primitive 5-th root of unity. This surface has the unique singular point $(1:\eta^2:\eta^4:\eta:\eta^3)$, it is an ordinary double point. The map:

$$(x_0:\ldots:x_3:x_4)\longmapsto(x_1:\ldots:x_4:x_0)$$

is an automorphism of order five of this surface.

The rational curve defined by $I_8=I_{24}=I_{40}=0$ parametrizes cubic surfaces of non-Sylvester type $ns2$ (Theorem 6.6):

$$S_\mu:\quad x_1^3+x_2^3-3x_3(\mu x_1x_3+x_2x_3+x_0^2)=0$$

and $\mu^3/(1+\mu^3)^2=b/a^2$ (or, if $a=0$, $\mu^3=-1$). The point $(1:0:0:0)$ is an Eckardt point. Each of these surfaces has an automorphism of order 4 given by:

$$(x_0:x_1:x_2:x_3)\longmapsto(ix_0:x_1:x_2:-x_3).$$

This family was studied for example in [S], p. 151, [Na], Thm. 2, p. 3, [Ho], Thm. 5.3. Any surface in the family has the property that its automorphism group modulo the subgroup generated by the involutions defined by the Eckardt points is non-trivial. This quotient group has two elements except if $\mu^6=1$. When $\mu=\omega$, a primitive cube root of unity, the surface has two singular points $(1:0:\omega^2:\pm 1)$ and it has another Eckardt point: $(0:1:-\omega:0)$. In case $\mu^3=-1$ one obtains a surface with quotient group isomorphic to $\mathbf{Z}/4\mathbf{Z}$. This surface defines the point $(0:0:0:1:0)\in\mathcal{M}$.

The surface with $\mu=0$, which corresponds to $(0:1:0:0:0)\in\mathcal{M}$, has affine equation (putting $x_0=y, x_1=t, x_2=x, x_3=1$ in the equation above):

$$S_0:\quad t^3+3y^2+x^3-3x=0$$

which shows that it is a cyclic surface (as expected, see Theorem 6.6) which branches over the smooth elliptic curve $E:\ 3y^2+x^3-3x=0$ in $\mathbf{P}^2$, it has $j(E)=1728$, in fact the automorphism of S_0 of order 4 above induces one on E with a fixed point.

References

[ACT] D. Allcock, J. A. Carlson and D. Toledo, *The Complex Hyperbolic Geometry of the Moduli Space of Cubic Surfaces*, J. Algebraic Geom. **11** (2002) 659–724.

[C] O. Chisini, *La superficie cubica II*, Period. Mat. **35** (1957) 286–300.

[DD] A. Dimca, S. Dimiev, *On analytic coverings of weighted projective spaces*, Bull. London Math. Soc. **17** (1985) 234–238.

[DGK] I. Dolgachev, B. van Geemen, S. Kondo, *A Complex Ball Uniformization of the moduli space of cubic surfaces via periods of K3 surfaces*, eprint math.AG/0310342.

[DK] I. V. Dolgachev and J. Keum, *Birational Automorphisms of Quartic Hessian Surfaces*, Trans. Amer. Math. Soc. **354** (2002) 3031–3057.

[Do] I. Dolgachev, *Weighted projective varieties*, in: Group actions and vector fields (Vancouver, B.C., 1981), 34–71, LNM **956**, Springer, Berlin, 1982.

[GKZ] I. M. Gelfand, M. M. Kapranov and A. V. Zelevinsky, *Discriminants, Resultants and Multidimensional Determinants*, Birkhäuser 1994.

[H] J. I. Hutchinson, *The Hessian of the Cubic Surface*, Bull. Amer. Math. Soc. **5** (1899) 282–292.

[Ho] T. Hosoh, *Automorphism groups of Cubic surfaces*, J. Algebra **192** (1997) 651–677.

[M] D. R. Morrison, *On K3 surfaces with large Picard number*, Invent. Math. **75** (1984) 105–121.

[Na] I. Naruki, *Cross Ratio Variety as a Moduli Space of Cubic Surfaces*, Proc. London Math. Soc. **42** (1982) 1–30.

[Ni] V. Nikulin, *Integral Quadratic Bilinear Forms and some of their Applications*, Math. USSR Izvestija **14** (1980) 103–167.

[P] U. Persson, *Double Sextics and Singular K3 Surfaces*, in: *Algebraic geometry* (Sitges 1983),262–328, LNM **1124**, Springer, Berlin 1985.

[PS] C. Peters and J. Stienstra, *A pencil of K3-surfaces related to Apéry's recurrence for $\zeta(3)$ and Fermi surfaces for potential zero*, Arithmetic of complex manifolds (Erlangen, 1988), 110–127, LNM **1399**, Springer, Berlin, 1989.

[Rod] C. Rodenberg, *Zur Classification der Flaechen dritter Ordnung*, Math. Ann. **14** (1879) 46–110.

[Ros] J. E. Rosenberg, *Hessian Quartic Surfaces that are Kummer Surfaces*, eprint math.AG/9903037.

[S] B. Segre, *The non-singular cubic surfaces*, Oxford 1942.

[Sal] G. Salmon, *A Treatise on the Analytic Geometry of Three Dimensions*, Chelsea Publishing Company, New York 1965.

[SI] T. Shioda and H. Inose, *On Singular K3 Surfaces*, Complex Analysis and Algebraic Geometry, A collection of Papers Dedicated to K. Kodaira, Cambridge University Press 1977.

[SZ] I. Shimada, D.-Q. Zhang, *Classification of extremal elliptic K3 surfaces and fundamental groups of open K3 surfaces*, Nagoya Math. J. **161** (2001) 23–54.

[V] H. A. Verrill, *The L-series of certain Rigid Calabi-Yau Threefolds*, J. Number Theory **81** (2000) 310–334.

[Vi] È. B. Vinberg, *The Two Most Algebraic K3 Surfaces*, Math. Ann. **265** (1983) 1–21.

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