# A FAMILY OF MARKED CUBIC SURFACES AND THE ROOT SYSTEM $D_{5}$ 

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#### Abstract

We define and study a family of cubic surfaces in the projectivized tangent bundle over a four dimensional projective space associated to the root system $D_{5}$. The 27 lines are rational over the base and we determine the classifying map to the moduli space of marked cubic surfaces. This map has degree two and we use it to get short proofs for some results on the Chow group of the moduli space of marked cubic surfaces.


A marked cubic surface is a smooth cubic surface $S$ with an ordered set of six skew lines on $S$. The symmetry group of the configuration of the 27 lines on a smooth cubic surface is the Weyl group of the root system $E_{6}$. This group acts on the moduli space of marked cubic surfaces $\mathcal{M}$. In various constructions of $\mathcal{M}$ certain subgroups of $W\left(E_{6}\right)$ play a prominent role.

Starting with the description of a marked cubic surface as the blow-up of the projective plane in six ordered points the action of the symmetric group $S_{6} \subset W\left(E_{6}\right)$ on $\mathcal{M}$ is very evident. In this approach, $\mathcal{M}$ is constructed as a double cover of a projective space branched along a quartic threefold, cf. [DGK], 2.11. The universal family of marked cubic surfaces is given there by the Cremona hexaedral form of a cubic surface (cf. [Co1], section 4, [D], section 3), this family has an obvious $S_{6}$-action.

In his construction of $\mathcal{M}$ and its desingularisation $\mathcal{C}$, Naruki $[\mathrm{N}]$ used the subgroup $W\left(F_{4}\right)$, which is associated to a tritangent plane of a cubic surface. In his approach, $\mathcal{M}$ is a compactification of the maximal torus of a complex Lie group of type $D_{4}$. The universal family is given explicitly, but it is rather complicated.

In this paper we consider the subgroup $W\left(D_{5}\right)$ of $W\left(E_{6}\right)$. It corresponds to the stabiliser of a line on a cubic surface. The standard five dimensional represention of $W\left(D_{5}\right)$ gives an action of this Weyl group on projective four space. We define a family of marked cubic surfaces $\mathcal{X}$ in the projectivized tangent bundle of this projective space. The family is defined by a very simple determinantal equation and $W\left(D_{5}\right)$ acts on it. The 27 lines and the 45 tritangent planes are also given by simple expressions.

Naruki's construction of $\mathcal{M}$ and subsequent work of Allcock and Freitag on an explicit projective embedding of $\mathcal{M}$ into $\mathbf{P}^{9}$ (using modular forms on a four ball), allows us to find a vector space of quintic polynomials in five variables which gives the classifying map from an open subset of the projective four space to $\mathcal{M} \subset \mathbf{P}^{9}$. We find that the classifying map has degree two and that the covering involution is simply inversion of the (natural) coordinates. We extend the classifying map to an explicit blow-up of the projective space. This allows us to recover some results on the Chow group of the moduli space of marked cubic surfaces which we obtained earlier in [CvG] using Naruki's model.

## 1. The root system $D_{5}$

1.1. We recall some basic facts on root systems and cubic surfaces. For root systems we refer to $[\mathrm{Hu}]$, for cubic surfaces to $[\mathrm{DGK}]$ and the references given there.
1.2. The root lattice $Q\left(D_{5}\right)$. Let $Q\left(D_{5}\right) \cong \mathbf{Z}^{5}$ be the root lattice of type $D_{5}$. It can be realised in the complex vector space of linear forms on $\mathbf{C}^{5}$ as

$$
Q\left(D_{5}\right)=\left\{\sum_{i=1}^{5} a_{i} x_{i}: a_{i} \in \mathbf{Z}, a_{1}+\ldots+a_{5} \equiv 0 \bmod 2\right\}, \quad x_{i} \cdot x_{j}=\delta_{i j}
$$

where '.' indicates the bilinear form on $Q_{\mathbf{C}}:=Q\left(D_{5}\right) \otimes_{\mathbf{Z}} \mathbf{C}$. A Z-basis of $Q\left(D_{5}\right)$ of simple roots is given by:

$$
\alpha_{1}=h_{12}=x_{1}-x_{2}, \quad \alpha_{2}=h_{123}=x_{4}+x_{5}, \quad \alpha_{3}=h_{23}=x_{2}-x_{3}, \quad \ldots \quad \alpha_{5}=h_{45}=x_{4}-x_{5}
$$

The roots of $D_{5}$ are the $\pm x_{i} \pm x_{j} \in Q\left(D_{5}\right), i \neq j$, the positive roots are $h_{i j}=x_{i}-x_{j}$, $1 \leq i<j \leq 5$, and $h_{k l m}=x_{i}+x_{j}$ where $\{i, \ldots, m\}=\{1, \ldots, 5\}$.
1.3. The Weyl group $W\left(D_{5}\right)$. The Weyl group $W\left(D_{5}\right)$ is the subgroup of $G L\left(Q_{\mathbf{C}}\right)$ generated by the reflections in the hyperplanes perpendicular to the roots of $D_{5}$. The reflection defined by $h_{i j}$ is denoted by $s_{i j}$. It permutes $x_{i}$ and $x_{j}$ and fixes the other $x_{k}$, this gives a subgroup $S_{5} \subset W\left(D_{5}\right)$. The reflection $s_{k l m}$ defined by $h_{k l m}$ maps $x_{i}$ to $-x_{j}$ and $x_{j}$ to $-x_{i}$ and fixes $x_{k}, x_{l}, x_{m}$.
1.4. The root system $E_{6}$ and cubic surfaces. The Picard group of a smooth complex cubic surface $S$ is a free $\mathbf{Z}$-module of rank 7 which has a basis $a_{0}, a_{1}, \ldots a_{6}$ such that the intersection form is given by $x_{0}^{2}-\left(x_{1}^{2}+\ldots+x_{6}^{2}\right)$. The canonical class is $k=-3 a_{0}+\left(a_{1}+\ldots+a_{6}\right)$ and $k^{\perp}$ is isomorphic, up to sign, to the root lattice $Q\left(E_{6}\right)$. The classes in $\operatorname{Pic}(S)$ of the 27 lines on $S$ are the elements $l \in \operatorname{Pic}(S)$ with $l^{2}=-1$ and $k \cdot l=-1$. In particular, they are not in $Q\left(E_{6}\right)$, but they project onto a $W\left(E_{6}\right)$-orbit of 27 weights in $Q\left(E_{6}\right) \otimes \mathbf{Q}$. The (classes of the) lines are

$$
a_{i}, \quad b_{j}=2 a_{0}-\left(a_{1}+\ldots+\widehat{a_{j}}+\ldots+a_{6}\right), \quad c_{k l}=a_{0}-\left(a_{k}+a_{l}\right)
$$

with $i, j, k, l \in\{1, \ldots, 6\}, k \neq l$. The six lines $a_{1}, \ldots, a_{6}$ are skew and can be blown down to give a birational morphism $S \rightarrow \mathbf{P}^{2}$ which maps these lines to six points $p_{i} \in \mathbf{P}^{2}$. The line $b_{i}$ maps to the conic through the points $p_{1}, \ldots, \widehat{p_{i}}, \ldots, p_{6}, 1 \leq i \leq 6$, and the line $c_{i j}=c_{j i}$ maps to the line through $p_{i}$ and $p_{j}$. The Weyl group of $E_{6}$ is generated by reflections defined by the -2-classes in $Q\left(E_{6}\right) \subset \operatorname{Pic}(S)$, in particular, $W\left(E_{6}\right)$ acts by isometries on $\operatorname{Pic}(S)$.
1.5. Orbits of $W\left(D_{5}\right)$ on the 27 lines. We identify $W\left(D_{5}\right) \subset W\left(E_{6}\right)$ with the stabiliser of the line $a_{6}$. Equivalently, $D_{5} \subset E_{6}$ is generated by the simple roots $\alpha_{1}, \ldots, \alpha_{5}$ in the $E_{6}$-diagram below.


The subgroup $S_{5} \subset W\left(D_{5}\right)$ acts on the lines by permutation of the indices. The reflection $s_{123}$, whose action on the six points is given by the Cremona transformation in $p_{1}, p_{2}, p_{3}$, maps

$$
s_{123}: \quad a_{1} \leftrightarrow c_{23}, \quad a_{2} \leftrightarrow c_{13}, \quad a_{3} \leftrightarrow c_{12}, \quad b_{4} \leftrightarrow c_{56}, \quad b_{5} \leftrightarrow c_{46}, \quad b_{6} \leftrightarrow c_{45}
$$

and fixes the other lines. Thus the $W\left(D_{5}\right)$-orbits in the set of lines are:

$$
\left\{a_{6}\right\}, \quad\left\{a_{1}, \ldots, a_{5}, c_{12}, \ldots, c_{45}, b_{6}\right\}, \quad\left\{b_{1}, \ldots, b_{5}, c_{16}, \ldots, c_{56}\right\} .
$$

Note that the 5 pairs of lines $\left(b_{1}, c_{16}\right), \ldots,\left(b_{5}, c_{56}\right)$ meet $a_{6}$ and the other $27-1-10=16$ lines are disjoint from $a_{6}$.
1.6. Orbits of $W\left(D_{5}\right)$ on the $\mathbf{4 5}$ tritangent planes. A tritangent of a smooth cubic surface $S$ is a plane section of $S$ which consists of three lines. As the embedding $S \hookrightarrow \mathbf{P}^{3}$ is anticanonical, the tritangents correspond to sets of three lines whose sum, in $\operatorname{Pic}(S)$, is $-k$.

There are 45 tritangent planes, they correspond to the following sets of lines:

$$
\left\{a_{i}, b_{j}, c_{i j}\right\}, \quad\left\{c_{i j}, c_{k l}, c_{m n}\right\}
$$

where $i \neq j$ and $\{i, j, \ldots, n\}=\{1,2, \ldots, 6\}$. There are two $W\left(D_{5}\right)$-orbits on the set of tritangent divisors, of length 5 and 40 respectively, they are:

$$
\left\{\ldots,\left\{a_{6}, b_{i}, c_{i 6}\right\}, \ldots\right\}_{1 \leq i \leq 5}, \quad\left\{\ldots,\left\{a_{p}, b_{q}, c_{p q}\right\}, \ldots,\left\{c_{i j}, c_{k l}, c_{m n}\right\}, \ldots\right\}
$$

where in the last set we have $p \neq q, 1 \leq p \leq 5,1 \leq q \leq 6$ and $\{i, j, \ldots, n\}=\{1,2, \ldots, 6\}$.

## 2. A family of marked cubic surfaces

2.1. We will construct a family of marked cubic surfaces embedded in the projectivised tangent bundle over a $\mathbf{P}^{4}$ naturally associated to the root system $D_{5}$.

Recall that $Q_{\mathbf{C}}:=Q\left(D_{5}\right) \otimes \mathbf{C} \cong \mathbf{C}^{5}$. Let $\mathcal{T}$ be the tangent bundle of the projective space $\mathbf{P}\left(Q_{\mathbf{C}}\right) \cong \mathbf{P}^{4}$. The fibre of $\mathcal{T} \rightarrow \mathbf{P}\left(Q_{\mathbf{C}}\right)$ over $<x>\in \mathbf{P}\left(Q_{\mathbf{C}}\right)$ is the vector space $Q_{\mathbf{C}} /\langle x\rangle$.

Let $\mathbf{P}(\mathcal{T}) \rightarrow \mathbf{P}\left(Q_{\mathbf{C}}\right)$ be the associated projective space bundle, its fibres are $\mathbf{P}^{3}$ 's. The Euler sequence, which maps the trivial bundle $\mathbf{P}\left(Q_{\mathbf{C}}\right) \times Q_{\mathbf{C}}$ over $\mathbf{P}\left(Q_{\mathbf{C}}\right)$ to $\mathcal{T}(-1)$, gives a natural $\operatorname{map} \pi$ :

$$
\begin{array}{ccc}
\mathbf{P}\left(Q_{\mathbf{C}}\right) \times\left(Q_{\mathbf{C}}-\{0\}\right) & \xrightarrow{\longrightarrow} & \mathbf{P}(\mathcal{T}) \\
\downarrow & & \downarrow \\
\mathbf{P}\left(Q_{\mathbf{C}}\right) & & = \\
\mathbf{P}\left(Q_{\mathbf{C}}\right) .
\end{array}
$$

We will denote the homogeneous coordinates on $\mathbf{P}\left(Q_{\mathbf{C}}\right)$ by $x_{i}$ and the $X_{i}$ are the coordinates on $Q_{\mathbf{C}}$. A family of surfaces $\mathcal{X} \subset \mathbf{P}(\mathcal{T})$ over $\mathbf{P}\left(Q_{\mathbf{C}}\right)$ is determined by the polynomial $F$ in the $x_{i}$ and $X_{i}$ which defines the inverse image of $\mathcal{X}$ in $\mathbf{P}\left(Q_{\mathbf{C}}\right) \times\left(Q_{\mathbf{C}}-\{0\}\right) \cong \mathbf{P}^{4} \times\left(\mathbf{C}^{5}-\{0\}\right)$.
2.2. The determinant. Consider the following polynomial in $\mathbf{C}\left[x_{1}, \ldots, x_{5}, X_{1}, \ldots, X_{5}\right]$, bihomogeneous of degree $(7,3)$ which is defined as the determinant of the following matrix:

$$
F(x, X)=\operatorname{det}\left(\begin{array}{rrrrr}
1 & 1 & 1 & 1 & 1 \\
x_{1}^{2} & x_{2}^{2} & x_{3}^{2} & x_{4}^{2} & x_{5}^{2} \\
x_{1}^{4} & x_{2}^{4} & x_{3}^{4} & x_{4}^{4} & x_{5}^{4} \\
x_{1} X_{1} & x_{2} X_{2} & x_{3} X_{3} & x_{4} X_{4} & x_{5} X_{5} \\
X_{1}^{2} & X_{2}^{2} & X_{3}^{2} & X_{4}^{2} & X_{5}^{2}
\end{array}\right) .
$$

Thus we have:

$$
F(x, X)=\sum_{i \neq j} f_{i j}(x) X_{i} X_{j}^{2}, \quad f_{12}=-x_{1}\left(x_{3}^{2}-x_{4}^{2}\right)\left(x_{3}^{2}-x_{5}^{2}\right)\left(x_{4}^{2}-x_{5}^{2}\right),
$$

and the other polynomials $f_{i j}$ are determined by:

$$
f_{\sigma^{-1}(i) \sigma^{-1}(j)}\left(x_{1}, \ldots, x_{5}\right)=\operatorname{sign}(\sigma) f_{i j}\left(x_{\sigma(1)}, \ldots, x_{\sigma(5)}\right),
$$

where $\sigma \in S_{5}$.
2.3. Descend to the tangent bundle. We define an open subset of $\mathbf{P}\left(Q_{\mathbf{C}}\right)$ by:

$$
U:=\left\{\left(x_{1}: \ldots: x_{5}\right) \in \mathbf{P}\left(Q_{\mathbf{C}}\right): x_{i} \neq \pm x_{j} \text { for } i \neq j\right\}
$$

The following lemma implies that for $x \in U$ the polynomial $F$ from section 2.2 defines the cone over a cubic surface in $\left(Q_{\mathbf{C}}-\{0\}\right) / \mathbf{C}^{*} \cong \mathbf{P}^{4}$ with vertex $\langle x\rangle$.
2.4. Lemma. There is a family of cubic surfaces $\mathcal{X} \hookrightarrow \mathbf{P}(\mathcal{T})_{\mid U}$ over $U$ whose inverse image in $U \times\left(Q_{\mathbf{C}}-\{0\}\right)$ under $\pi$ is defined by $F(x, X)=0$.

Proof. Note that for $x \in U$ the polynomial $F(x, X)$ in the $X_{i}$ is not identically zero. We verify that $F(x, X+t x)=F(x, X)$ for all $t$, so we get a well defined family of cubic surfaces $\mathcal{X}$ in $\mathbf{P}\left(\mathcal{T}_{Q}\right)$, i.e. for $x \in \mathbf{P}\left(Q_{\mathbf{C}}\right)$ the surface $\mathcal{X}_{x} \subset \mathbf{P}\left(Q_{\mathbf{C}} /<x>\right)$ is defined by $F(x, X)=0$.

Substituting $X_{i}:=X_{i}+t x_{i}, 1 \leq i \leq 5$, in the matrix defining $F$, one easily sees that the last two rows become linear combinations of the rows of the matrix and that the determinant doesn't change.
2.5. The $W\left(D_{5}\right)$-action on the family. The subset $U$ is invariant under the action of $W\left(D_{5}\right)$ and this action lifts to an action of $W\left(D_{5}\right)$ on $\mathcal{T}_{\mid U}$. In fact, $g \in W\left(D_{5}\right)$ acts on the $x_{i}$ via the standard representation. As this action is linear, the action on $\mathcal{T}$ is the one induced by the standard representation on the $X_{i}$. For $g \in W\left(D_{5}\right)$ and $(x, X) \in \mathbf{P}\left(Q_{\mathbf{C}}\right) \times\left(Q_{\mathbf{C}}-\{0\}\right)$ we thus have $g(x, X):=(g x, g X)$.

The group $W\left(D_{5}\right)$ is generated by $S_{5}$ and $s_{123}$, which maps $x_{4} \mapsto-x_{5}, x_{5} \mapsto-x_{4}$ and fixes the other $x_{i}$. It is then easy to check from the determinant defining $F$ that $F(g x, g X)=$ $\operatorname{det}(g) F(x, X)$. Thus the family $\mathcal{X}$ is invariant under the action of $W\left(D_{5}\right)$ on $\mathbf{P}(\mathcal{T})_{\mid U}$.
2.6. Determinants. To find lines in these cubic surfaces we need the following determinantal identity:

$$
\operatorname{det}\left(\begin{array}{ccccc}
1 & \cdots & 1 & \cdots & 1 \\
x_{1} & \cdots & x_{i} & \cdots & x_{n} \\
\vdots & \cdots & \vdots & \cdots & \vdots \\
x_{1}^{n-2} & \cdots & x_{i}^{n-2} & \cdots & x_{n}^{n-2} \\
x_{2} \cdots x_{n-1} & \cdots & x_{1} \cdots \widehat{x_{i}} \cdots x_{n} & \cdots & x_{1} \cdots x_{n-1}
\end{array}\right)=(-1)^{(n-1)(n-2) / 2} \prod_{i<j}\left(x_{i}-x_{j}\right) \text {. }
$$

This holds because the determinant is a homogeneous polynomial of degree $n(n-1) / 2$ which is divisible by $x_{i}-x_{j}$ for every $i \neq j$ and the coefficients of $x_{1} x_{2}^{2} \ldots x_{n-1}^{n-1}$ on both sides are equal. Note that the Van der Monde determinant is obtained by replacing the last row by $x_{1}^{n-1}, \ldots, x_{n}^{n-1}$ and that it equals $(-1)^{n(n-1)} \prod_{i<j}\left(x_{i}-x_{j}\right)$. In particular, the sum of the Van der Monde determinant and the one above is zero if $n=4$.
2.7. Lines in $\mathcal{X}$. A line in $\mathbf{P}\left(\mathcal{T}_{x}\right)$ is defined by a two dimensional subspace of $\mathcal{T}_{x}=Q_{\mathbf{C}} /\langle x\rangle$ and hence by a three dimensional subspace of $Q_{\mathbf{C}}$ which contains $x$. In the following theorem we give some lines on $\mathcal{X}_{x}$ by specifying these three dimensional subspaces as $\langle x, s(x), t(x)\rangle$ with $s(x), t(x) \in Q_{\mathbf{C}} \cong \mathbf{C}^{5}$.

The Weyl group $W\left(D_{5}\right)$ acts on $\mathcal{X}$. As $g \in W\left(D_{5}\right)$ maps a section $\sigma$ with $\sigma(x)=(x, s(x))$ to $(g x, g s(x))$, we get an action of $W\left(D_{5}\right)$ on the sections with $g(\sigma)(x)=\left(x, g s\left(g^{-1} x\right)\right)$.

Recall that a double six on a cubic surface is a set of twelve lines $\left\{a_{1}, \ldots, a_{6}, b_{1}, \ldots, b_{6}\right\}$ such that the intersection products are (cf. $[\mathrm{H}]$ ):

$$
a_{i} \cdot a_{j}=0, \quad b_{i} \cdot b_{j}=0, \quad a_{i} \cdot b_{j}=1, \quad a_{i} \cdot b_{i}=0, \quad \text { for } i \neq j
$$

2.8. Theorem. For $x \in U$, the cubic surface $\mathcal{X}_{x}$ has a double six formed by the following twelve lines:

$$
a_{1}=\left\langle x,(-1,1,1,1,1),\left(-x_{1}^{2}, x_{2}^{2}, x_{3}^{2}, x_{4}^{2}, x_{5}^{2}\right)\right\rangle
$$

the lines $a_{2}, \ldots, a_{5}$ defined as $a_{i}=s_{1 i}\left(a_{1}\right)$ with $s_{1 i} \in S_{5} \subset W\left(D_{5}\right)$,

$$
\begin{gathered}
a_{6}=\left\langle x,\left(x_{1}^{-1}, x_{2}^{-1}, x_{3}^{-1}, x_{4}^{-1}, x_{5}^{-1}\right),\left(x_{1}^{3}, x_{2}^{3}, x_{3}^{3}, x_{4}^{3}, x_{5}^{3}\right)\right\rangle \\
b_{1}=\left\langle x,(1,0,0,0,0),\left(0, x_{2}^{3}-x_{3} x_{4} x_{5}, x_{3}^{3}-x_{2} x_{4} x_{5}, x_{4}^{3}-x_{2} x_{3} x_{5}, x_{5}^{3}-x_{2} x_{3} x_{4}\right)\right\rangle,
\end{gathered}
$$

the lines $b_{2}, \ldots, b_{5}$ defined as $b_{i}=s_{1 i}\left(b_{1}\right)$ with $s_{1 i} \in S_{5} \subset W\left(D_{5}\right)$, and

$$
b_{6}=\left\langle x,(1,1,1,1,1),\left(x_{1}^{2}, x_{2}^{2}, x_{3}^{2}, x_{4}^{2}, x_{5}^{2}\right)\right\rangle .
$$

Proof. In view of Lemma 2.4, to check that $\langle x, s(x), t(x)\rangle$ defines a line in $F(x, X)=0$ it suffices to check that $F(x, s(x)+\lambda t(x))=0$ for all $\lambda$.

Thus $b_{6}$ lies in $F(x, X)=0$ since upon substituting $X_{i}:=1+\lambda x_{i}^{2}$ in the matrix defining $F$, the last row becomes a linear combination of the first three hence its determinant is zero.

Recall that $g \in W\left(D_{5}\right)$ acts on sections by $g:(x, s(x)) \mapsto\left(x, g s\left(g^{-1} x\right)\right)$ and maps lines to lines. Thus $b_{6}$ is fixed under the $s_{i j}$ (which permute $x_{i}$ and $x_{j}$ and fix the other coordinates). Its image under $s_{123}$ (which maps $x_{4} \mapsto-x_{5}, x_{5} \mapsto-x_{4}$ and fixes the other $x_{i}$ ) is:

$$
c_{45}=s_{123}\left(b_{6}\right)=\left\langle x,(1,1,1,-1,-1),\left(x_{1}^{2}, x_{2}^{2}, x_{3}^{2},-x_{4}^{2},-x_{5}^{2}\right)\right\rangle .
$$

Applying $s_{145}$ (which maps $x_{2} \mapsto-x_{3}, x_{3} \mapsto-x_{2}$ and fixes the other $x_{i}$ ) we get:

$$
a_{1}=s_{145}\left(c_{45}\right)=\left\langle x,(-1,1,1,1,1),\left(-x_{1}^{2}, x_{2}^{2}, x_{3}^{2}, x_{4}^{2}, x_{5}^{2}\right)\right\rangle .
$$

As the $a_{i}, 1 \leq i \leq 5$, are in the $W\left(D_{5}\right)$-orbit of $b_{6}$, these lines are also on the cubic surface $\mathcal{X}_{x}$.
An argument similar to the one above shows that $a_{6}$ is a line on $\mathcal{X}_{x}$. Under the $W\left(D_{5}\right)$-action this line is invariant. In fact, all coordinate functions are of odd degree so sign changes are merely reparametrisations of the same line.

The linearity of the determinant in each row of a matrix and the identity given in 2.6 easily show that $b_{1}$ lies on $\mathcal{X}_{x}$, and so the $b_{i}, 1 \leq i \leq 5$ are on $\mathcal{X}_{x}$. Thus the twelve lines in the theorem are on $\mathcal{X}_{x}$.

Two lines $l=\left\langle x, v_{1}, v_{2}\right\rangle, m=\left\langle x, w_{1}, w_{2}\right\rangle$ in $\mathbf{P}\left(\mathcal{T}_{x}\right)$ are skew iff the $5 \times 5$ matrix whose rows are $x, v_{1}, v_{2}, w_{1}, w_{2}$ has maximal rank. In particular, the lines $b_{6}$ and $a_{6}$ are skew because the $5 \times 5$ matrix they define is, up to multiplication of the $i$-th column by $x_{i}$, a Van der Monde matrix. Using the $W\left(D_{5}\right)$-action on the lines (this action fixes $a_{6}$ ) we find that each of the 16 lines in the $W\left(D_{5}\right)$-orbit of $b_{6}$ is skew with $a_{6}$. In particular, $a_{6}$ is skew with the lines denoted by $a_{1}, \ldots, a_{5}$ above. It is also easy to see that $a_{i}$ and $a_{j}$ are skew if $1 \leq i<j \leq 5$. Hence the six $a_{i}$ are skew.

The lines $b_{i}$ and $b_{j}$ are skew if $1 \leq i<j \leq 5$, in fact in case $i=1, j=2$ the $5 \times 5$ matrix now has maximal rank since:

$$
\operatorname{det}\left(\begin{array}{ccc}
x_{3} & x_{4} & x_{5} \\
x_{3}^{3}-x_{2} x_{4} x_{5} & x_{4}^{3}-x_{2} x_{3} x_{5} & x_{5}^{3}-x_{2} x_{3} x_{4} \\
x_{3}^{3}-x_{1} x_{4} x_{5} & x_{4}^{3}-x_{1} x_{3} x_{5} & x_{5}^{3}-x_{1} x_{3} x_{4}
\end{array}\right)=\left(x_{1}-x_{2}\right)\left(x_{3}^{2}-x_{4}^{2}\right)\left(x_{3}^{2}-x_{5}^{2}\right)\left(x_{4}^{2}-x_{5}^{2}\right),
$$

and the other cases are similar. The lines $b_{i}, b_{6}, 1 \leq i \leq 5$, are skew since the $5 \times 5$ matrix (essentially a $4 \times 4$ matrix) can be seen to have maximal rank using the remarks in section 2.6. Hence the six $b_{i}$ are skew.

We already observed that $b_{6}$ and $a_{6}$ are skew. It is not hard to verify that $a_{i}$ and $b_{i}$ are skew. We show that $a_{i}$ and $b_{j}$ intersect if $i \neq j$. The line $b_{6}$ meets $a_{1}$ in the point:

$$
\begin{aligned}
\left(0, x_{1}^{2}-x_{2}^{2}, \ldots, x_{1}^{2}-x_{5}^{2}\right) & =x_{1}^{2}(1,1,1,1,1)-\left(x_{1}^{2}, x_{2}^{2}, x_{3}^{2}, x_{4}^{2}, x_{5}^{2}\right) \\
& =x_{1}^{2}(-1,1,1,1,1)-\left(-x_{1}^{2}, x_{2}^{2}, x_{3}^{2}, x_{4}^{2}, x_{5}^{2}\right) .
\end{aligned}
$$

Similarly, $b_{6}$ meets $a_{i}$ for $i=2, \ldots, 5$. The lines $a_{1}$ and $b_{5}$ meet, since the $5 \times 5$ matrix reduces to a $4 \times 4$ matrix which, upon substituting $x_{1}:=-x_{1}$ and multiplying the first column by -1 , is of the type discussed in section 2.6. Applying suitable elements $W\left(D_{5}\right)$ we get that $a_{i}$ and $b_{j}$ meet if $i \neq j$ and $1 \leq i, j \leq 5$. Finally one verifies with similar arguments that $a_{6}$ meets $b_{i}$ if $1 \leq i \leq 5$.
2.9. Using the information on the lines we found in the general $\mathcal{X}_{x}$, it is not hard to show that this surface must be smooth. All we will need is that for $x \in U$, the cubic surface $\mathcal{X}_{x}$ has a double six.
2.10. Lemma. The cubic surface $\mathcal{X}_{x}$ is smooth if $x \in U$, that is, $x=\left(x_{1}: \ldots: x_{5}\right)$ and $x_{i} \neq \pm x_{j}$ for $i \neq j$.

Proof. Let $x \in U$, then $\mathcal{X}_{x}$ has six skew lines, the $a_{i}$, and for each $j \in\{1, \ldots, 6\}$ there is a line, $b_{j}$, which meets all $a_{i}$ except $a_{j}$. This easily implies that $\mathcal{X}_{x}$ is irreducible. Suppose that $p \in \mathcal{X}_{x}$ is a singular point. We choose coordinates such that $p=(0: 0: 0: 1)$. Then the equation defining $\mathcal{X}_{x}$ is $G(x, y, z) t+H(x, y, z)=0$ for some $G, H \in \mathbf{C}[x, y, z]$ homogeneous of degree two and three respectively.

If $G=0$, the surface $\mathcal{X}_{x}$ is a cone over the plane curve defined by $H=0$, which must be irreducible, thus all lines on $\mathcal{X}_{x}$ pass through $p$, a contradiction. If $H=0$ the surface would be reducible, so we conclude that that both $G$ and $H$ are non-zero and have no common factor. Projection from $p$ to the plane $t=0$ gives a birational isomorphism $\mathcal{X}_{x} \rightarrow \mathbf{P}^{2}$. The inverse is given by $q=(x: y: z) \mapsto(G(q) x: G(q) y: G(q) z:-H(q))$, which is an isomorphism on the complement of the finite set $B=(G=0) \cap(H=0)$. The lines containing $p$ project to the points in $B$, hence there are at most six such lines. Let $l$ be a line on $\mathcal{X}_{x}$ not containing $p$. The plane spanned by $p$ and $l$ intersects $\mathcal{X}_{x}$ in $l$ and a conic containing the singular point $p$. Hence this conic is reducible and consists of two lines (possibly a double line) passing through $p$. If there are no double lines, the lines on $\mathcal{X}_{x}$ project to the points of $B$ or to the lines spanned by two points in $B$. It is now easy to see that $\mathcal{X}_{x}$ does not contain six skew lines.

Finally assume there is a plane $P$ such that there are only two lines $l, m$ with $P \cap \mathcal{X}_{x}=l \cup m$, possibly $l=m$. As each line in $\mathcal{X}_{x}$ intersects $P$ (or lies in $P$ ), there are at least three of the six $a_{i}$ 's which properly intersect one of the two lines $l, m$. Call that line $l$ and let $a_{1}, a_{2}, a_{3}$ be three
$a_{i}$ 's meeting $l$ in points. The lines $b_{4}, b_{5}, b_{6}$ all meet $a_{1}, a_{2}, a_{3}$. In case $b_{j}=l$ for one such $b_{j}$, say $j=6$, the lines $a_{6}, b_{1}, \ldots, b_{5}$ must intersect $m$ properly. The plane spanned by $a_{6}$ and $m$ then meets at least four of the five lines $b_{1}, \ldots, b_{5}$ each in at least two distinct points, so these lines must lie in that plane and hence they meet, a contradiction, so $b_{j} \neq l$ for $j=4,5,6$. Three skew lines in $\mathbf{P}^{3}$ are contained in a unique smooth quadric, let $Q$ be the quadric containing $a_{1}, a_{2}, a_{3}$. Then $b_{4}, b_{5}, b_{6}$ also lie in $Q$ since each meets any of these three $a_{i}$ and similarly $l \subset Q$. Thus $Q \cap \mathcal{X}_{x}$ contains at least 7 distinct lines, but an irreducible cubic surface can meet a quadric in at most 6 lines. We conclude that a cubic surface having a double six must be smooth.
2.11. The 27 lines. A smooth cubic surface contains 27 lines. For $x \in U$, we already found a double six $a_{1}, \ldots, b_{6}$ in $\mathcal{X}_{x}$. The remaining 15 lines are the $c_{i j}=c_{j i}, 1 \leq i<j \leq 6$. The line $c_{i j}$ is the third line in the intersection of the plane spanned by $a_{i}$ and $b_{j}$ with $\mathcal{X}_{x}$. Alternatively, one can find the $c_{i j}$ using the $W\left(D_{5}\right)$ action on the lines in $\mathcal{X}_{x}$. For example, $c_{45}$ was given in the proof of Theorem 2.8 and

$$
c_{16}=\left\langle x,(1,0,0,0,0),\left(0, x_{2}^{3}+x_{3} x_{4} x_{5}, x_{3}^{3}+x_{2} x_{4} x_{5}, x_{4}^{3}+x_{2} x_{3} x_{5}, x_{5}^{3}+x_{2} x_{3} x_{4}\right)\right\rangle .
$$

## 3. Tritangent planes and cross ratios

3.1. Tritangent planes. The lines $a_{6}, b_{1}, c_{16}$ lie on the tritangent plane:

$$
t_{61}: \quad \sum_{j=1}^{5} b_{1 j} X_{j}:=\operatorname{det}\left(\begin{array}{rrrr}
1 & 1 & 1 & 1 \\
x_{2}^{2} & x_{3}^{2} & x_{4}^{2} & x_{5}^{2} \\
x_{2}^{4} & x_{3}^{4} & x_{4}^{4} & x_{5}^{4} \\
x_{2} X_{2} & x_{3} X_{3} & x_{4} X_{4} & x_{5} X_{5}
\end{array}\right)=0,
$$

Explicitly, we get

$$
b_{11}=0, \quad b_{12}=x_{2}\left(x_{3}^{2}-x_{4}^{2}\right)\left(x_{3}^{2}-x_{5}^{2}\right)\left(x_{4}^{2}-x_{5}^{2}\right), \ldots
$$

The lines $a_{1}, b_{6}, c_{16}$ lie on the tritangent plane:

$$
t_{16}: \quad \sum_{j=1}^{5} a_{1 j} X_{j}:=\operatorname{det}\left(\begin{array}{rrrr}
1 & 1 & 1 & 1 \\
x_{2} & x_{3} & x_{4} & x_{5} \\
x_{2}^{2} & x_{3}^{2} & x_{4}^{2} & x_{5}^{2} \\
X_{2} & X_{3} & X_{4} & X_{5}
\end{array}\right)=0
$$

Explicitly, we get

$$
a_{11}=0, \quad a_{12}=\left(x_{3}-x_{4}\right)\left(x_{3}-x_{5}\right)\left(x_{4}-x_{5}\right), \ldots .
$$

Since the determinant is linear in each row, a line $l=\langle x, v, w\rangle$ lies in $t_{i j}$ if upon substituting $X:=\left(X_{1}, \ldots, X_{5}\right)=x, v, w$ the determinant defining $t_{i j}$ becomes zero. To check that $c_{16}$ lies on $t_{16}$ one can use the linearity of the matrix in the last row and the identity 2.6 to see that the two determinants cancel.

The set of tritangent planes is the union of two $W\left(D_{5}\right)$-orbits. The tritangent planes $t_{16}, t_{61}$ are in distinct orbits, so we essentially determined all tritangent planes.
3.2. Cross ratios. Given a tritangent plane $t$ of a smooth cubic surface $X$ and one of the three lines $l \subset t \cap X$, there are 4 other tritangent planes $t_{1}, \ldots, t_{4}$ of $X$ which contain $l$. These four planes give four points in the $\mathbf{P}^{1}$ of planes containing $l$. Any one of the six cross ratios of these 4 points is called a cross ratio associated to $X$ and $t$. In fact, Cayley already observed that these six cross ratios do not depend on the choice of the line $l \subset t \cap X$, cf. [ N$]$.

The pencil of lines on $l$ is spanned by $t_{1}$ and $t_{2}$ and if we write

$$
t_{3}=a t_{1}+b t_{2}, \quad t_{4}=c t_{1}+d t_{2}
$$

then these four tritangents define the points $(1: 0),(0: 1),(a: b),(c: d)$ on $\mathbf{P}^{1}$. One of the six cross ratios of these points is

$$
\gamma_{t}:=\frac{\operatorname{det}\left(\begin{array}{ll}
1 & a \\
0 & b
\end{array}\right) \operatorname{det}\left(\begin{array}{ll}
0 & c \\
1 & d
\end{array}\right)}{\operatorname{det}\left(\begin{array}{ll}
1 & c \\
0 & d
\end{array}\right) \operatorname{det}\left(\begin{array}{ll}
0 & a \\
1 & b
\end{array}\right)}=\frac{b c}{a d}
$$

3.3. Lemma. The cross ratio $\gamma_{56}$ defined by $t=t_{56}, l=b_{6}, t_{i}=t_{i 6}$ and the cross ratio $\gamma_{65}$ defined by $t=t_{65}, l=a_{6}, t_{i}=t_{6 i}$, where $i=1, \ldots, 4$, are given by:

$$
\gamma_{56}=\frac{\left(x_{1}-x_{3}\right)\left(x_{2}-x_{4}\right)}{\left(x_{1}-x_{4}\right)\left(x_{2}-x_{3}\right)}, \quad \gamma_{65}=\frac{\left(x_{1}^{2}-x_{3}^{2}\right)\left(x_{2}^{2}-x_{4}^{2}\right)}{\left(x_{1}^{2}-x_{4}^{2}\right)\left(x_{2}^{2}-x_{3}^{2}\right)}
$$

Proof. Let $t_{i 6}=\sum_{i} a_{i j} X_{j}$ be defined by a determinant as in section 3.1, with the obvious change of indices. Then $a_{i i}=0$. Thus we must have:

$$
a_{12} a_{21} t_{36}=a_{21} a_{32} t_{16}+a_{12} a_{31} t_{26}, \quad a_{12} a_{21} t_{46}=a_{21} a_{42} t_{16}+a_{12} a_{41} t_{26}, \quad \text { hence } \quad \gamma_{56}=\frac{a_{31} a_{42}}{a_{32} a_{41}}
$$

For $\gamma_{65}$ a similar formula, with $a_{i j}$ replaced by $b_{i j}$, applies.

## 4. The classifying map

4.1. The moduli space of marked cubics. A point in the moduli space of smooth marked cubic surfaces $\mathcal{M}^{m}$ can be identified with an isomorphism class $\left[S, a_{1}, \ldots, a_{6}\right]$ of a smooth cubic surface with 6 skew lines, the other 21 lines are then naturally labelled by $b_{i}, c_{i j}=c_{j i}$, $i, j \in\{1, \ldots, 6\}, i \neq j$, cf. section 1.4.

The moduli space of smooth marked cubic surfaces $\mathcal{M}^{m}$ has a natural compactification $\mathcal{M}$ (denoted by $\overline{\mathcal{M}}_{\text {cub }}^{m}$ in [DGK], 2.8). The Weyl group $W\left(E_{6}\right)$ acts biregularly on $\mathcal{M}$ and the quotient $\mathcal{M} / W\left(E_{6}\right)$ is the GIT quotient of the space of cubic surfaces in $\mathbf{P}^{3}$.

The boundary $\mathcal{M}-\mathcal{M}^{m}$ consists of 36 (irreducible) boundary divisors which are parametrised by the positive roots of $E_{6}$. The divisor corresponding to a root $\alpha$ will be denoted by $D_{\alpha}$. The projective variety $\mathcal{M}$ is smooth except for 40 singular points, called the cusps of $\mathcal{M}$, which form one $W\left(E_{6}\right)$-orbit and which map to the unique non-stable point in $\mathcal{M} / W\left(E_{6}\right)$.

A tritangent plane $t=\{l, m, n\}$ with $l, m, n \in\left\{a_{i}, b_{j}, c_{r s}\right\}$ defines a divisor $D_{t}$ in $\mathcal{M}$, the tritangent divisor associated to $t$, which is the closure of the locus of marked surfaces such that the lines $l, m, n$ meet in a point. Such a point is called an Eckardt point.
4.2. The line bundle $\mathcal{L}$ on $\mathcal{M}$. The cross ratios associated to a tritangent plane extend to rational functions on $\mathcal{M}$. Naruki [ N ] showed that the rational map $\mathcal{M} \rightarrow\left(\mathbf{P}^{1}\right)^{270}$ defined by the $270=6 \cdot 45$ cross ratios is an embedding on the complement of the 40 cusps and blows up each cusp to a copy of $\left(\mathbf{P}^{1}\right)^{3}$.

Allcock and Freitag [AF] showed that there is a very ample line bundle $\mathcal{L}$ over $\mathcal{M}$ with the property that any cross ratio is the quotient of two global sections of $\mathcal{L}$. The vector space $H^{0}(\mathcal{M}, \mathcal{L})$ is 10 -dimensional and the group $W\left(E_{6}\right)$ acts on $H^{0}(\mathcal{M}, \mathcal{L})$. The corresponding representation is the unique irreducible 10-dimensional representation of $W\left(E_{6}\right)$, cf. [vG], section 5.
4.3. Crosses. The three lines in a tritangent plane $t$ correspond to three weights in $Q\left(E_{6}\right) \otimes \mathbf{Q}$, the orthogonal complement in $Q\left(E_{6}\right)$ of these three weights is a root lattice $t^{\perp}$ of type $D_{4}$. A cross is a divisor

$$
D_{c}:=D_{t}+D_{\alpha}+D_{\beta}+D_{\gamma}+D_{\delta}
$$

where $\alpha, \ldots, \delta$ are positive, mutually perpendicular, roots of $E_{6}$ in $t^{\perp}$ (cf. [AF], Definition 3.2). Each tritangent divisor determines three crosses, so there are $3 \cdot 45=135$ crosses. These crosses are linearly equivalent and $\mathcal{L} \cong \mathcal{O}_{\mathcal{M}}\left(D_{c}\right)$ for any cross $D_{c}$. The space $H^{0}(\mathcal{M}, \mathcal{L})$ is spanned by sections whose divisors are crosses. Three sections whose zero divisors are the three crosses associated to a given tritangent $t$ span a two dimensional subspace of $H^{0}(\mathcal{M}, \mathcal{L})$ and the quotient of any two of these three sections is, up to a scalar multiple, a cross ratio associated to $t$.
4.4. Divisors and involutions. The boundary divisor $D_{\alpha}$ is the fixed point locus of the reflection $s_{\alpha} \in W\left(E_{6}\right)$ in $\mathcal{M}$. The tritangent divisor $D_{t}$ is the fixed point locus of the involution $\gamma(t) \in W\left(E_{6}\right)$ which is the the product of the reflections in (any) four perpendicular roots in $t^{\perp}$ (cf. [N], Section 8). In particular, $\gamma(t)=-I$ on $t^{\perp}$ and $+I$ on the span of the weights in $t$.
4.5. The vector space $V$. Given the marked family of cubic surfaces $\mathcal{X} \rightarrow U$ (with marking given by the lines $a_{1}, \ldots, a_{6}$, we obtain a classifying map

$$
\Phi_{\mathcal{X}}: U \longrightarrow \mathcal{M}
$$

This map is equivariant for the action of $W\left(D_{5}\right)$. Let

$$
V=\Phi_{\mathcal{X}}^{*} H^{0}(\mathcal{M}, \mathcal{L})
$$

The ten dimensional vector space $V$ can be described as follows.
4.6. Lemma. The vector space $V$ is the 10 dimensional vector space spanned by the following functions:

$$
\begin{array}{ll}
x_{1}\left(x_{2}^{2}-x_{3}^{2}\right)\left(x_{4}^{2}-x_{5}^{2}\right), & x_{1}\left(x_{2}^{2}-x_{5}^{2}\right)\left(x_{3}^{2}-x_{4}^{2}\right), \\
x_{2}\left(x_{1}^{2}-x_{3}^{2}\right)\left(x_{4}^{2}-x_{5}^{2}\right), & x_{2}\left(x_{1}^{2}-x_{4}^{2}\right)\left(x_{3}^{2}-x_{5}^{2}\right), \\
x_{3}\left(x_{1}^{2}-x_{2}^{2}\right)\left(x_{4}^{2}-x_{5}^{2}\right), & x_{3}\left(x_{1}^{2}-x_{4}^{2}\right)\left(x_{2}^{2}-x_{5}^{2}\right), \\
x_{4}\left(x_{1}^{2}-x_{2}^{2}\right)\left(x_{3}^{2}-x_{5}^{2}\right), & x_{4}\left(x_{1}^{2}-x_{3}^{2}\right)\left(x_{2}^{2}-x_{5}^{2}\right), \\
x_{5}\left(x_{1}^{2}-x_{2}^{2}\right)\left(x_{3}^{2}-x_{4}^{2}\right), & x_{5}\left(x_{1}^{2}-x_{4}^{2}\right)\left(x_{2}^{2}-x_{3}^{2}\right) .
\end{array}
$$

Proof. The morphism $\Phi_{\mathcal{X}}$ extends to a rational map $\mathbf{P}\left(Q_{\mathbf{C}}\right) \rightarrow \mathcal{M}$ which, since $\mathbf{P}\left(Q_{\mathbf{C}}\right)$ is smooth, is a morphism on an open set $U^{\prime}$ whose complement has codimension at least two in
$\mathbf{P}\left(Q_{\mathbf{C}}\right)$. Then $\operatorname{Pic}\left(U^{\prime}\right) \cong \operatorname{Pic}\left(\mathbf{P}\left(Q_{\mathbf{C}}\right)\right) \cong \mathbf{Z}$, so the global sections of $\mathcal{L}$ pull-back to homogeneous polynomials of a fixed degree.

The map $\Phi_{\mathcal{X}}$ extends to the general points in the hyperplanes $x_{i}= \pm x_{j}, i \neq j$. This extension is obviously still $W\left(D_{5}\right)$-equivariant. The positive roots $x_{i}-x_{j}, x_{i}+x_{j}, 1 \leq i<j \leq 5$, are zero on the fixed point locus of the reflections $s_{i j}, s_{k l m}$ (with $\{i, j, k, l, m\}=\{1, \ldots, 5\}$ ) respectively, so these 20 hyperplanes are mapped to the corresponding boundary divisors. As points of $U$ correspond to smooth cubic surfaces, we conclude that $\Phi_{\mathcal{X}}^{-1}\left(D_{\alpha}\right)=\left(h_{\alpha}=0\right) \cap U^{\prime}$, where $\alpha \in D_{5}\left(\subset E_{6}\right)$ is a positive root.

The root system $D_{4}\left(\subset D_{5} \subset E_{6}\right)$ with roots $\pm x_{i} \pm x_{j}, 2 \leq i<j \leq 6$, is $t^{\perp}$ for $t=\left\{a_{6}, b_{1}, c_{16}\right\}$. The involution $\gamma_{t}$ is thus $\left(x_{1}, x_{2}, \ldots, x_{5}\right) \mapsto\left(x_{1},-x_{2},-x_{3},-x_{4},-x_{5}\right)$ which fixes the divisor $x_{1}=0$ (and the point (1:0:0:0:0) which is not in $U$ ). Thus $\Phi_{\mathcal{X}}$ maps $\left(x_{1}=0\right) \cap U$ to $D_{t}$. We now show that $\Phi_{\mathcal{X}}^{-1}\left(D_{t}\right)=\left(x_{1}=0\right) \cap U$. The lines $b_{1}$ and $c_{16}$ (cf. 2.11) meet in the point in $\mathbf{P}\left(\mathcal{T}_{x}\right)$ defined by $\langle x,(1,0,0,0,0)\rangle \subset Q_{\mathbf{C}}$, we write simply $b_{1} \cap c_{16}=(1,0,0,0,0)$. Similarly:

$$
a_{6} \cap b_{1}=\left(x_{2} x_{3} x_{4} x_{5}, x_{1}\left(x_{2}^{3}-x_{3} x_{4} x_{5}\right), x_{1}\left(x_{3}^{3}-x_{2} x_{4} x_{5}\right), x_{1}\left(x_{4}^{3}-x_{2} x_{3} x_{5}\right), x_{1}\left(x_{5}^{3}-x_{2} x_{3} x_{4}\right)\right) .
$$

It is now easy to verify that $x \in \Phi_{\mathcal{X}}^{-1}\left(D_{t}\right)$ iff $x_{1}=0$.
Thus there is a section of $\mathcal{L}$, which defines a cross associated to the tritangent divisor $D_{t}$, which pulls back to $x_{1}^{a_{0}}\left(x_{2}-x_{3}\right)^{a_{1}}\left(x_{2}+x_{3}\right)^{a_{2}}\left(x_{4}-x_{5}\right)^{a_{4}}\left(x_{4}+x_{5}\right)^{a_{4}}$ for certain $a_{n} \in \mathbf{Z}_{>0}$. Applying $s_{34} \in W\left(D_{4}\right)$, which fixes the tritangent $t$, we get the pull-back of another section, and the quotient of these two should be a cross ratio associated to $t$, like $\gamma_{61}$, cf. Lemma 3.3. So we must have $a_{1}=a_{2}=a_{3}=a_{4}=1$.

To show that $a_{0}=1$ if suffices to show that the classifying map is not ramified at a general point of $x_{1}=0$. For this it suffices to exhibit just 4 cross ratios $\gamma_{i}, i=1, \ldots, 4$, such that the differential of the map $x \mapsto\left(\gamma_{1}(x), \ldots, \gamma_{4}(x)\right) \in\left(\mathbf{P}^{1}\right)^{4}$ has maximal rank in some point $x \in\left(x_{1}=0\right) \cap U$. We took $\gamma_{1}=t_{56}, \gamma_{2}=t_{65}$ and $\gamma_{3}, \gamma_{4}$ obtained from these two by permuting $x_{4} \leftrightarrow x_{5}$, for the point we took $x=(0: 2: 3: 4: 1)$. We found that the differential is indeed injective at this point.
4.7. The $W\left(D_{5}\right)$-action on $V$ and the cross ratios. Note that the sum of the first two basis functions is:

$$
x_{1}\left(x_{2}^{2}-x_{3}^{2}\right)\left(x_{4}^{2}-x_{5}^{2}\right)+x_{1}\left(x_{2}^{2}-x_{5}^{2}\right)\left(x_{3}^{2}-x_{4}^{2}\right)=x_{1}\left(x_{2}^{2}-x_{4}^{2}\right)\left(x_{3}^{2}-x_{5}^{2}\right),
$$

thus all polynomials of the form $x_{i}\left(x_{j}^{2}-x_{k}^{2}\right)\left(x_{l}^{2}-x_{m}^{2}\right)$, with $\{i, \ldots, m\}=\{1, \ldots, 5\}$, are contained in $V$. This verifies that $W\left(D_{5}\right)$ acts on $V$ via its action on the variables.

Another useful function in $V$ is:

$$
\begin{aligned}
g_{126} & :=-\left(x_{3}\left(x_{1}^{2}-x_{2}^{2}\right)\left(x_{4}^{2}-x_{5}^{2}\right)-x_{4}\left(x_{1}^{2}-x_{2}^{2}\right)\left(x_{3}^{2}-x_{5}^{2}\right)+x_{5}\left(x_{1}^{2}-x_{2}^{2}\right)\left(x_{3}^{2}-x_{4}^{2}\right)\right) \\
& =\left(x_{1}^{2}-x_{2}^{2}\right)\left(x_{3}-x_{4}\right)\left(x_{3}-x_{5}\right)\left(x_{4}-x_{5}\right) .
\end{aligned}
$$

As $V$ is $W\left(D_{5}\right)$-invariant, it contains all polynomials of the form

$$
\left(x_{i}-\epsilon_{1} x_{j}\right)\left(x_{i}-\epsilon_{2} x_{k}\right)\left(x_{j}-\epsilon_{3} x_{l}\right)\left(x_{l}^{2}-x_{m}^{2}\right) ; \quad \epsilon_{a} \in\{1,-1\}, \quad \epsilon_{1} \epsilon_{2} \epsilon_{3}=1,
$$

and $\{i, j, \ldots, m\}=\{1,2, \ldots, 5\}$.
To write a cross like $\gamma_{56}$ as quotient of functions in $V$ we observe first of all that

$$
\left(x_{1}^{2}-x_{2}^{2}\right)\left(x_{3}-x_{4}\right)\left(x_{3} x_{4}+x_{5}^{2}\right)=-x_{3}\left(x_{1}^{2}-x_{2}^{2}\right)\left(x_{4}^{2}-x_{5}^{2}\right)+x_{4}\left(x_{1}^{2}-x_{2}^{2}\right)\left(x_{3}^{2}-x_{5}^{2}\right) \quad \in V .
$$

Using the $W\left(D_{5}\right)$ action, we then also have $g_{12,34} \in V$ with:

$$
\begin{aligned}
g_{12,34} & :=\left(x_{1}-x_{2}\right)\left(x_{3}^{2}-x_{4}^{2}\right)\left(x_{1} x_{2}+x_{5}^{2}\right)+\left(x_{1}^{2}-x_{2}^{2}\right)\left(x_{3}-x_{4}\right)\left(x_{3} x_{4}+x_{5}^{2}\right) \\
& =\left(x_{1}-x_{2}\right)\left(x_{3}-x_{4}\right)\left(x_{1} x_{2} x_{3}+x_{1} x_{2} x_{4}+x_{1} x_{3} x_{4}+x_{2} x_{3} x_{4}+\left(x_{1}+x_{2}+x_{3}+x_{4}\right) x_{5}^{2}\right) .
\end{aligned}
$$

Obviously, crosses like $\gamma_{56}$ are quotients of suitable polynomials in the $W\left(D_{5}\right)$-orbit of $g_{12,34}$.

## 5. Extending the classifying map

5.1. The rational map. The composition of the classifying map $\Phi_{\mathcal{X}}: U \rightarrow \mathcal{M}$ and the embedding $\phi_{\mathcal{L}}: \mathcal{M} \longrightarrow \mathbf{P}^{9}$, defined by the global section of $\mathcal{L}$, extends to a rational map

$$
\Phi: \mathbf{P}^{4}=\mathbf{P}\left(Q_{\mathbf{C}}\right) \longrightarrow \mathbf{P}^{9} .
$$

As coordinate functions of this map one can take the basis of $V$ given in 4.6. We will study the extension of this map to a suitable blow-up $\tilde{\mathbf{P}}^{4}$ of $\mathbf{P}^{4}$. The smooth projective variety $\tilde{\mathbf{P}}^{4}$ is a compactification of the open set $U$ from section 2.3. In particular, this gives a morphism $\tilde{\mathbf{P}}^{4} \rightarrow \mathcal{M}$ which extends ('compactifies') the classifying map $\Phi_{\mathcal{X}}: U \rightarrow \mathcal{M}$.
5.2. The involution $\iota$. It is obvious that the map $\Phi$ factors over the birational involution:

$$
\iota: \mathbf{P}^{4} \longrightarrow \mathbf{P}^{4}, \quad\left(x_{1}: \ldots: x_{5}\right) \longmapsto\left(x_{1}^{-1}: \ldots: x_{5}^{-1}\right)=\left(x_{2} x_{3} x_{4} x_{5}: \ldots: x_{1} x_{2} x_{3} x_{4}\right)
$$

in fact, the coordinate functions of $\Phi$ are $x_{i}\left(x_{j}^{2}-x_{k}^{2}\right)\left(x_{l}^{2}-x_{m}^{2}\right)$ and after putting $x_{a}:=x_{a}^{-1}$ and multiplying all coordinates by $\left(x_{1} x_{2} \ldots x_{5}\right)^{2}$ one obtains again $x_{i}\left(x_{j}^{2}-x_{k}^{2}\right)\left(x_{l}^{2}-x_{m}^{2}\right)$.
5.3. Lemma. The map $\Phi$ has degree two and $\Phi$ is $\iota$-invariant: $\Phi \circ \iota=\Phi$.

Proof. We already observed that $\Phi \circ \iota=\Phi$ in 5.2. We study the general fiber of $\Phi$ :

$$
\Phi^{-1}(p), \quad p=\left(1: a_{2}, \ldots: a_{10}\right) \in \operatorname{im}(\Phi)
$$

A point $\left(x_{1}: \ldots: x_{5}\right) \in \Phi^{-1}(p)$ satisfies the following two equations:
$a_{5} x_{1}\left(x_{2}^{2}-x_{3}^{2}\right)\left(x_{4}^{2}-x_{5}^{2}\right)=x_{3}\left(x_{1}^{2}-x_{2}^{2}\right)\left(x_{4}^{2}-x_{5}^{2}\right), \quad a_{5} x_{2}\left(x_{1}^{2}-x_{3}^{2}\right)\left(x_{4}^{2}-x_{5}^{2}\right)=a_{3} x_{3}\left(x_{1}^{2}-x_{2}^{2}\right)\left(x_{4}^{2}-x_{5}^{2}\right)$.
Removing the common factor $\left(x_{4}^{2}-x_{5}^{2}\right)$ we obtain that $\left(x_{1}: x_{2}: x_{3}\right) \in \mathbf{P}^{2}$ must lie in the intersection of the two cubics:

$$
a_{5} x_{1}\left(x_{2}^{2}-x_{3}^{2}\right)-x_{3}\left(x_{1}^{2}-x_{2}^{2}\right)=0, \quad a_{5} x_{2}\left(x_{1}^{2}-x_{3}^{2}\right)-a_{3} x_{3}\left(x_{1}^{2}-x_{2}^{2}\right)=0 .
$$

There are 7 obvious points of intersection:

$$
(0: 0: 1), \quad(0: 1: 0), \quad(1: 0: 0), \quad(1: 1: 1), \quad(1: 1:-1), \quad(1:-1: 1), \quad(-1: 1: 1)
$$

The other two points can be found as follows. Put $x_{3}=1$. From the first equation we have:

$$
x_{2}^{2}=x_{1}\left(a_{5}+x_{1}\right) /\left(a_{5} x_{1}+1\right)
$$

substituting this in the second equation gives:

$$
x_{2}=a_{3}\left(x_{1}^{2}-x_{2}^{2}\right) / a_{5}\left(x_{1}^{2}-1\right)=a_{3} x_{1} /\left(a_{5} x_{1}+1\right)
$$

To have compatibility we need:

$$
x_{1}\left(a_{5}+x_{1}\right) /\left(a_{5} x_{1}+1\right)=\left(a_{3} x_{1} /\left(a_{5} x_{1}+1\right)\right)^{2} \quad \text { hence } \quad a_{5}+\left(1-a_{3}^{2}+a_{5}^{2}\right) x_{1}+a_{5} x_{1}^{2}=0
$$

is, essentially, the only remaining equation. Note that if $x_{1}$ is a root, then so is $x_{1}^{-1}$. We already found $x_{2}=a_{3} x_{1} /\left(a_{5} x_{1}+1\right)$. Proceeding in this way with the other equations one finds the result.
5.4. Lemma. Let $B$ be the base locus of $\Phi$ :

$$
B:=\left\{x \in \mathbf{P}^{4}: f(x)=0 \quad \forall f \in V\right\} .
$$

Then $B$ is the union of 50 lines, they are the $W\left(D_{5}\right)$-orbits of the lines

$$
l=\left\{(s: t: t: t: t):(s: t) \in \mathbf{P}^{1}\right\}, \quad m=\left\{(s: t: 0: 0: 0):(s: t) \in \mathbf{P}^{1}\right\}
$$

These orbits consist of 40 and 10 lines respectively.
Proof. It is easy to check that $l$ and $m$ lie in $B$. As $B$ is invariant under $W\left(D_{5}\right)$, it suffices to show that each base point can be mapped into $l \cup m$ using an element of $W\left(D_{5}\right)$. Let $p=\left(x_{1}: \ldots: x_{5}\right)$ be a base point. Assume first that none of the coordinates of $p$ is zero. Then we must have $x_{i}= \pm x_{j}$ for some $i \neq j$ and using a suitable element of $W\left(D_{5}\right)$ we may assume that $x_{4}=x_{5}$. Next we must still have $x_{l}= \pm x_{m}$ with $1 \leq l<m \leq 3$, so we may assume that $x_{2}=x_{3}$. As $x_{1}\left(x_{2}^{2}-x_{4}^{2}\right)\left(x_{3}^{2}-x_{5}^{2}\right) \in V$ it follows that $x_{2}= \pm x_{4}$ or $x_{3}= \pm x_{5}$ so using the $W\left(D_{5}\right)$ action again we get $p=(s: t: t: t: t)$ for some $(s: t) \in \mathbf{P}^{1}$.

Assume now that at least one of the coordinates of $p$ is zero. Using a suitable element of $W\left(D_{5}\right)$ we may then assume that $x_{5}=0$ and, using the bases of $V$ given in 4.6 we then have $x_{i} x_{j}^{2}\left(x_{k}^{2}-x_{l}^{2}\right)=0$ whenever $\{i, j, k, l\}=\{1,2,3,4\}$. If all other coordinates are non-zero, then, as above, we find that $p$ is in the $W\left(D_{5}\right)$-orbit of $(0: t: t: t: t) \in l$. If one other coordinate is zero, we may assume that $x_{4}=x_{5}=0$ and we must have $x_{i} x_{j}^{2} x_{k}^{2}=0$ whenever $\{i, j, k\}=\{1,2,3\}$. Thus one more coordinate is zero and we may assume that $p=(s: t: 0: 0: 0) \in m$.
5.5. The singular points of $B$. The singular points of the base locus $B$ are the points of intersection of the lines in $B$. There are 21 such points, they are the $W\left(D_{5}\right)$-orbits of the points

$$
p_{1}:=(1: 1: 1: 1: 1), \quad q_{1}:=(1: 0: 0: 0: 0) .
$$

These orbits consist of 16 and 5 points respectively. The points in these orbits are denoted by $p_{i}(1 \leq i \leq 16)$ and $q_{i}(1 \leq i \leq 5)$, only when necessary will we define these other points precisely.
5.6. The blow-up $\tilde{\mathbf{P}}^{4}$. In order to extend $\Phi: \mathbf{P}^{4}-B \rightarrow \mathbf{P}^{9}$ to a morphism, we first blow-up $\mathbf{P}^{4}$ in the 21 singular points of $B$, and next we blow-up the strict transforms of the 50 lines in $B$. The variety which we obtain in this way is denoted by $\tilde{\mathbf{P}}^{4}$.

The strict transform in $\tilde{\mathbf{P}}^{4}$ of the exceptional divisor over $p_{i}, q_{i}$ is denoted by $E_{p_{i}}$ and $E_{q_{i}}$ respectively, they are $\mathbf{P}^{3}$ 's blown up in a finite number of points, the points where the strict transforms of the lines in $B$ meet the exceptional divisors in the first blow-up.

The lines in the $W\left(D_{5}\right)$-orbit of $m, l$ will be denoted by $m_{\alpha}, l_{\beta}, 1 \leq \alpha \leq 40,1 \leq \beta \leq 10$. The exceptional divisors over the lines $m_{\alpha}$ and $l_{\beta}$ are denoted by $E_{m_{\alpha}}$ and $E_{l_{\beta}}$.

Explicit calculations, which are easy and which we omit (but see the next section), verify that $\Phi$ extends to a morphism, which we denote by the same name,

$$
\Phi: \tilde{\mathbf{P}}^{4} \longrightarrow \mathbf{P}^{9}
$$

## 6. Divisors

6.1. Contractions. The divisors $E_{l_{\alpha}}$ and $E_{m_{\beta}}$, which are birational to $\mathbf{P}^{2}$-bundles over the (strict transforms of the) lines $l_{\alpha}$ and $m_{\beta}$ respectively, are contracted to planes in $\mathbf{P}^{9}$. In fact, the image of a point $(x: y: z)$ in the fiber over $(1: a: a: a: a) \in l$, for $a$ general, is given by:
$\lim _{t \rightarrow 0} \Phi(1: a: a+t x: a+t y: a+t z)=(0: 0: y-z: x-z: y-z:-z: x-z:-z: x-y:-x)$,
which does not depend on $a$, so $\Phi$ collapses all fibers of the blow-up $E_{l} \rightarrow l$ to a fixed $\mathbf{P}^{2} \subset \mathbf{P}^{9}$. Similarly, the image of a point $(x: y: z)$ in the fiber over $(1: a: 0: 0: 0) \in m$ is given by:

$$
\lim _{t \rightarrow 0} \Phi(1: a: t x: t y: t z)=(0: 0: 0: 0: 0: x: 0: y: 0: z)
$$

6.2. Tritangent divisors. In the proof of Lemma 4.6 we observed that the classifying map $\Phi_{\mathcal{X}}$ maps the $\mathbf{P}^{3} \subset \mathbf{P}^{4}$ defined by $x_{i}=0(1 \leq i \leq 5)$ onto a tritangent divisor. The image under $\Phi$ of such a $\mathbf{P}^{3}$ spans a $\mathbf{P}^{7} \subset \mathbf{P}^{9}$. The covering involution maps this $\mathbf{P}^{3}$ to the point $q_{i}$ which has all coordinates equal to zero except $x_{i}=1$. Let $i=5$, then the map $\Phi$ induces the following map on the the exceptional divisor $E_{q_{5}}$, which is birational to $\mathbf{P}^{3}$, over $q_{5}$ :

$$
\Phi: E_{q_{5}} \sim \mathbf{P}^{3} \longrightarrow \mathbf{P}^{7} \subset \mathbf{P}^{9}, \quad\left(y_{1}: y_{2}: y_{3}: y_{4}\right) \longmapsto \lim _{t \rightarrow 0} \Phi\left(t y_{1}: t y_{2}: t y_{3}: t y_{4}: 1\right) .
$$

It is easy to verify that the image of $\left(y_{1}: \ldots: y_{4}\right)$ is given by:

$$
\left(y_{1}\left(y_{2}^{2}-y_{3}^{2}\right): y_{1}\left(y_{3}^{2}-y_{4}^{2}\right): \ldots: y_{4}\left(y_{1}^{2}-y_{2}^{2}\right): y_{4}\left(y_{1}^{2}-y_{3}^{2}\right): 0: 0\right)
$$

Thus on this $\mathbf{P}^{3}$ the map $\Phi$ coincides with the map on $\mathbf{P}_{\mathrm{w}}^{3}$ defined in terms of the root system $F_{4}$ considered in [vG], Thm. 6.5.

In particular, the inverse image of the tritangent divisor labelled by $\left\{a_{6}, b_{i}, c_{i 6}\right\}$, with $1 \leq$ $i \leq 5$, has two irreducible components in $\tilde{\mathbf{P}}^{4}$, the strict transforms of the hyperplane $x_{i}=0$ and the divisor $E_{q_{i}}$, which are exchanged by $\iota$.

The remaining 40 tritangent divisors are components of the divisors of the other crosses, so they are the images of the 40 cubics in the $W\left(D_{5}\right)$-orbit of (cf. 4.7):

$$
\begin{aligned}
X: & x_{1} x_{2} x_{3}+x_{1} x_{2} x_{4}+x_{1} x_{3} x_{4}+x_{2} x_{3} x_{4}+\left(x_{1}+x_{2}+x_{3}+x_{4}\right) x_{5}^{2}= \\
& =x_{1} x_{2} x_{3} x_{4}\left(x_{1}^{-1}+\ldots+x_{4}^{-1}\right)+\left(x_{1}+x_{2}+x_{3}+x_{4}\right) x_{5}^{2}=0 .
\end{aligned}
$$

From the equation it is obvious that $X$ is invariant under $\iota$.
Note that $X$ (and its $W\left(D_{5}\right)$-conjugates) are birationally covers of a $\mathbf{P}^{3}$ (with coordinates $\left.\left(x_{1}: x_{2}: x_{3}: x_{4}\right)\right)$ branched over the union of a plane $x_{1}+x_{2}+x_{3}+x_{4}=0$ and the 4 -nodal Cayley cubic surface $x_{1} x_{2} x_{3}+x_{1} x_{2} x_{4}+x_{1} x_{3} x_{4}+x_{2} x_{3} x_{4}=0$. In particular, $X$ is singular and projection from a singular point of $X$ gives a birational isomorphism between $X$ and $\mathbf{P}^{3}$.
6.3. Boundary divisors. There is a $W\left(E_{6}\right)$-equivariant bijection between the 36 boundary divisors of $\mathcal{M}$ and the 36 positive roots in $E_{6}$. There are two $W\left(D_{5}\right)$-orbits. One consists of the 20 roots of $h_{i j}, h_{i j k}, i, j, k \leq 5$, which are perpendicular to $a_{6}$ and these are the positive roots of $D_{5}$. Note that the involution $\iota$ fixes each of the divisors $h_{i j}=0, h_{i j k}=0$, hence $\Phi$ is a degree two map on the strict transform of such a divisor.

The other orbit consists of the 16 roots $h, h_{i 6}, h_{i j 6}, i, j \leq 5$. We already determined the image of each of the divisors in $\tilde{\mathbf{P}}^{4}-U$ except for the divisors $E_{p_{i}}, 1 \leq i \leq 16$. Thus these
must map to the remaining 16 boundary divisors. On the divisor $E_{p_{1}}$, which is birational to the exceptional divisor over $p_{1}$ in the blow-up of $\mathbf{P}^{4}$ in $p_{1}$, the map induced by $\Phi$ is
$\Phi: E_{p_{1}} \cong \mathbf{P}^{3} \longrightarrow \mathbf{P}^{4} \subset \mathbf{P}^{9}, \quad\left(y_{1}: y_{2}: y_{3}: y_{4}\right) \longmapsto \lim _{t \rightarrow 0} \Phi\left(t y_{1}+1: t y_{2}+1: t y_{3}+1: t y_{4}+1: 1\right)$
and it is easy to see that the image of $\left(y_{1}: \ldots: y_{4}\right)$ is given by:

$$
\left(y_{4}\left(y_{2}-y_{3}\right): y_{2}\left(y_{3}-y_{4}\right): \ldots:\left(y_{1}-y_{2}\right)\left(y_{3}-y_{4}\right):\left(y_{2}-y_{3}\right)\left(y_{1}-y_{4}\right)\right) .
$$

The 10 coordinate functions span the space of quadrics which vanish in the 5 points $(1: 0: 0$ : $0), \ldots,(0: 0: 0: 1),(1: 1: 1: 1)$. It is well-known that this map gives a birational isomorphism of $\mathbf{P}^{3}$ with the Segre threefold in $\mathbf{P}^{4}$, cf. [H], Theorem 3.2.1. It is well-known that a boundary divisor of $\mathcal{M}$ is isomorphic to a Segre cubic threefold. The nodes of the Segre threefold are cusps of $\mathcal{M}$. The line parametrized by $(0: 0: s: t)$ maps to a node of the cubic and explicit computation shows that this cusp has coordinates $1: 0: 1: 1: 0: 0: 0: 0: 0:-1) \in \mathcal{M} \subset \mathbf{P}^{9}$.

The inverse image of any of these 16 boundary divisors must be an $E_{p_{i}}$ since we accounted for all other divisors in $\tilde{\mathbf{P}}^{4}$. As $\Phi$ is a birational morphism on a $E_{p_{i}}$, these divisors lie in the ramification locus of $\Phi$.
6.4. Cusps. The inverse image of a cusp of $\mathcal{M}$ under $\Phi$ is a plane in $\mathbf{P}^{4}$. For example, the inverse image of the cusp $(1: 0: 1: 1: 0: 0: 0: 0: 0:-1)$ is easily seen to be the plane $\gamma$ parametrized by

$$
\gamma:=\left\{(s: s: t: u: s):(u: s: t) \in \mathbf{P}^{2}\right\}, \quad \Phi(\gamma)=(1: 0: 1: 1: 0: 0: 0: 0: 0:-1) .
$$

There are 40 planes in the $W\left(D_{5}\right)$-orbit of $\gamma$, these map to the 40 cusps of $\mathcal{M} \subset \mathbf{P}^{9}$.
6.5. Labelling of divisors. Naruki's cross ratio variety $\mathcal{C}$ is a natural desingularisation of the moduli space $\mathcal{M}$. The $\operatorname{map} \mathcal{C} \rightarrow \mathcal{M}$ contracts 40 disjoint divisors in $\mathcal{C}$, each isomorphic to $\left(\mathbf{P}^{1}\right)^{3}$, to the 40 cusps of $\mathcal{M}$ and is an isomorphism on the complements. We refer to these 40 divisors on $\mathcal{C}$ as cusp divisors.

The three types of divisors on $\mathcal{C}$ are conveniently labelled by points in a 4 -dimensional projective space over $\mathbf{F}_{3}$, canonically, it is $\mathbf{P}\left(P\left(E_{6}\right) / Q\left(E_{6}\right)\right)$, cf. [CvG], 1.7, 7.4-7.6. It is remarkable that these labels allow one to identify the corresponding loci in $\mathbf{P}^{4}$ easily. We will content ourselves with some examples.

The boundary divisors are parametrized by the 36 points obtained from ( $0: 0: 0: 1: 1$ ) and (1:1:1:1:1) by permutation and sign changes of the coordinates. The boundary divisor labelled by $(0: 0: 0: 1:-1)$ is the image of the $\mathbf{P}^{3} \subset \mathbf{P}\left(Q_{\mathbf{C}}\right)$ defined by $h_{45}=x_{4}-x_{5}=0$. The boundary divisor labelled by $(1:-1: 1:-1: 1)$ is the image of the exceptional divisor over the singular point $(1:-1: 1:-1: 1)$ in the base locus of $\Phi$.

The tritangent divisors are parametrized by the 45 points obtained from ( $0: 0: 0: 0$ : $1)$ and $(0: 1: 1: 1: 1)$ by permutation and sign changes of the coordinates. The point ( $0: 0: 0: 0: 1$ ) labels the boundary divisor which is the image of the exceptional divisor over $(0: 0: 0: 0: 1) \in \mathbf{P}\left(Q_{\mathbf{C}}\right)$, equivalently, it is the image of the hyperplane $x_{5}=0$. The point $(0: 1:-1: 1: 1)$ labels the tritangent divisor which is the image of the cubic $x_{2} x_{3} x_{4} x_{5}\left(x_{2}^{-1}-x_{3}^{-1}+x_{4}^{-1}+x_{5}^{-1}\right)+\left(x_{2}-x_{3}+x_{4}+x_{5}\right) x_{1}^{2}=0$.

The cusp divisors are parametrized by the 40 points in $\mathbf{P}^{4}\left(\mathbf{F}_{3}\right)$ obtained from $(0: 0: 1: 1: 1)$ by permutation and sign changes of the coordinates. The plane in $\mathbf{P}\left(Q_{\mathbf{C}}\right)$ which maps to the
cusp labelled by $(0: 0: 1:-1: 1)$ is parametrized by $(s: t: u:-u: u)$. The plane $\gamma \subset \mathbf{P}\left(Q_{\mathbf{C}}\right)$ in 6.4 above maps to the cusp labelled ( $1: 1: 0: 0: 1$ ).

## 7. The Chow Ring

7.1. In the paper $[\mathrm{CvG}]$ we determined the Chow ring of Naruki's cross ratio variety $\mathcal{C}$ which is a desingularisation of the moduli space $\mathcal{M}$ of marked cubic surfaces. We will now see that the extension of the classifying map, which we denote again by $\Phi_{\mathcal{X}}: \tilde{\mathbf{P}}^{4} \longrightarrow \mathcal{M}$ (cf. sections 5.1, 5.6) allows us to recover quickly some of these results.

The singular locus of $\mathcal{M}$ consists of the 40 cusps which are in one $W\left(D_{5}\right)$-orbit. The inverse image of one cusp is the surface $\tilde{\gamma} \subset \tilde{\mathbf{P}}^{4}$ which is the strict transform of the plane $\gamma \subset \mathbf{P}^{4}$, cf. 6.4. We will thus be interested in the open subset

$$
\left(\tilde{\mathbf{P}}^{4}\right)_{0}:=\tilde{\mathbf{P}}^{4}-\cup \tilde{\delta}, \quad \text { and } \quad \Phi_{0}:=\left(\Phi_{\mathcal{X}}\right)_{\mid\left(\tilde{\mathbf{P}}^{4}\right)_{0}}:\left(\tilde{\mathbf{P}}^{4}\right)_{0} \longrightarrow \mathcal{M}^{s m}
$$

where the union is over the 40 surfaces in the $W\left(D_{5}\right)$-orbit of $\tilde{\gamma}$. The variety $\left(\tilde{\mathbf{P}}^{4}\right)_{0}$ maps, generically $2: 1$, onto the smooth locus $\mathcal{M}^{s m}$ of $\mathcal{M}$. The smooth, quasi-projective variety $\mathcal{M}^{s m}$ is the moduli space of marked nodal cubic surfaces ([DGK], section 2.8).
7.2. Proposition. The Picard group, tensored by $\mathbf{Q}$, of the moduli space of marked nodal cubic surfaces $\mathcal{M}^{s m}$ is:

$$
A^{1}\left(\mathcal{M}^{s m}\right)_{\mathbf{Q}} \cong\left(\oplus_{i=1}^{16} \mathbf{Q} B_{\alpha_{i}}\right) \oplus\left(\oplus_{j=1}^{5} \mathbf{Q} T_{j}\right)
$$

where

$$
B_{\alpha_{i}}:=\left(\Phi_{0}\right)_{*} E_{p_{i}}, \quad T_{j}:=\left(\Phi_{0}\right)_{*} E_{q_{j}} .
$$

Here $B_{\alpha_{i}}$ is the boundary divisor parametrised by $\alpha_{i}$ which runs over the 16 positive roots of $E_{6}$ which are not perpendicular to the weight $a_{6}$ (cf. section 6.3) and $T_{j}$ is the tritangent divisor labelled by $\left\{a_{6}, b_{j}, c_{j 6}\right\}$.

Proof. Basic results on the Chow group of a blow-up imply:

$$
A^{1}\left(\left(\tilde{\mathbf{P}}^{4}\right)_{0}\right)=\mathbf{Z} H \oplus\left(\oplus^{16} E_{p_{i}}\right) \oplus\left(\oplus^{5} E_{q_{i}}\right) \oplus\left(\oplus^{10} E_{m_{\alpha}}\right) \oplus\left(\oplus^{40} E_{l_{\beta}}\right)
$$

where $H$ is the class of the strict transform of a hyperplane in $\mathbf{P}^{4}$. The divisors $E_{l_{\beta}}, E_{m_{\alpha}}$ are contracted by $\Phi_{0}$ (cf. section 6.1), hence $\left(\Phi_{0}\right)_{*} E_{l_{\beta}}=\left(\Phi_{0}\right)_{*} E_{m_{\alpha}}=0$.

The strict transform $D_{i} \subset\left(\tilde{\mathbf{P}}^{4}\right)_{0}$ of the hyperplane $x_{i}=0$ in $\mathbf{P}^{4}$ is mapped birationally onto a tritangent divisor $T_{i}$, so $\left(\Phi_{0}\right)_{*} D_{i}=T_{i}$. The same tritangent divisor is also the birational image of $E_{q_{i}},\left(\Phi_{0}\right)_{*} E_{q_{i}}=T_{i}$. As there are 4 singular points, $q_{j}$ with $j \neq i$, of the base locus in $x_{i}=0$, we have:

$$
D_{i}=H-\sum_{j \neq i} E_{q_{j}}, \quad \text { thus } \quad\left(\Phi_{0}\right)_{*} H=\left(\Phi_{0}\right)_{*}\left(D_{i}+\sum_{j \neq i} E_{q_{j}}\right)=\sum_{j=1}^{5} T_{j} .
$$

Hence $\left(\Phi_{0}\right)_{*} A^{1}\left(\left(\tilde{\mathbf{P}}^{4}\right)_{0}\right) \cong\left(\oplus^{16} B_{\alpha_{i}}\right) \oplus\left(\oplus^{5} T_{j}\right)$. As $\left(\Phi_{0}\right)_{*}\left(\Phi_{0}\right)^{*}$ is multiplication by 2 on $A^{1}\left(\mathcal{M}^{s m}\right)$, the proposition follows.
7.3. The group $W\left(E_{6}\right)$ acts biregularly on $\mathcal{M}^{s m}$ and $\mathcal{C}$. It permutes the 45 tritangent and the 36 boundary divisors transitively. The sum of the tritangent divisors and of the boundary divisors are denoted by $\hat{T}$ and $\hat{B}$ respectively. These $W\left(E_{6}\right)$ invariant classes on $\mathcal{M}^{s m}$ are related as follows.
7.4. Proposition. In the Picard group of $\mathcal{M}^{s m}$ we have:

$$
4 \hat{T}=25 \hat{B}
$$

Proof. The strict transform $\tilde{H}_{12}$ in $\tilde{\mathbf{P}}^{4}$ of the hyperplane $x_{1}-x_{2}=0$ in $\mathbf{P}^{4}$ is mapped 2:1 onto a boundary divisor $B_{h_{12}}$ in $\mathcal{M}^{s m}$ by $\Phi_{0}$. As this hyperplane contains exactly 8 points of the base locus $B$ whose exceptional divisor maps birationally onto a boundary divisor and exactly 3 points, $q_{3}, q_{4}, q_{5}$ whose exceptional divisor maps to a tritangent divisor, we get $\tilde{H}_{12}=$ $H-\sum^{8} E_{p_{i}}-\sum^{3} E_{q_{i}}+\ldots$, where we omit the precise indices and divisors which are contracted by $\Phi_{0}$. In the proof of Prop. 7.2 we saw that $\left(\Phi_{0}\right)_{*} H=\sum_{j=1}^{5} T_{j}$. As $\left(\Phi_{0}\right)_{*} \tilde{H}_{12}=2 B_{h_{12}}$, we find that a sum of ten boundary divisors is linearly equivalent to a sum of two tritangent divisors in $\mathcal{M}^{s m}$. If $B$ is any boundary divisor, then $\sum_{\sigma} \sigma^{*} B=\left(\left|W\left(E_{6}\right)\right| / 36\right) \hat{B}$, since $W\left(E_{6}\right)$ is transitive on the set of 36 boundary divisors. Similarly $\sum_{\sigma} \sigma^{*} T=\left(\left|W\left(E_{6}\right)\right| / 45\right) \hat{T}$ for any tritangent divisor $T$. In this way the linear equivalence of the sum of ten boundary divisors with the sum of two tritangent divisors leads to $\frac{10}{36} \hat{B}=\frac{2}{45} \hat{T}$.
7.5. Remark. In $[\mathrm{CvG}]$, Theorem 2, we proved that Naruki's smooth compactification $\mathcal{C}$ of $\mathcal{M}^{s m}$ has $A^{1}(\mathcal{C}) \cong \mathbf{Z}^{61}$. As $\mathcal{C}-\mathcal{M}^{\text {sm }}$ is the disjoint union of 40 cusp divisors, whose classes in $\mathcal{C}$ generate a direct summand isomorphic to $\mathbf{Z}^{40}$, we see that Proposition 7.2 is consistent with the results of [CvG].

Proposition 7.4 follows from the relation ([CvG], Theorem 2.4(2)):

$$
\hat{T}=(25 \hat{B}+27 \hat{C}) / 4 \quad\left(\in A^{1}(\mathcal{C})\right)
$$

because $\hat{\mathcal{C}}$, the sum of the 40 cusp divisors, is zero on $\mathcal{M}^{\text {sm }}$.

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