# Real multiplication on K3 surfaces and Kuga Satake varieties 

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#### Abstract

The endomorphism algebra of a K3 type Hodge structure is a totally real field or a CM field. In this paper we give a low brow introduction to the case of a totally real field. We give existence results for the Hodge structures, for their polarizations and for certain K3 surfaces. We consider the Kuga Satake variety of these Hodge structures and we discuss some examples. Finally we indicate various open problems related to the Hodge conjecture.


A K3 type Hodge structure is a simple, rational, polarized weight two Hodge structure $V$ with $\operatorname{dim} V^{2,0}=1$. Zarhin $[Z]$ proved that the endomorphism algebra of a K3 type Hodge structure is either a totally real field or a CM field. Conversely, K3 type Hodge structures whose endomorphism algebra is a given such field exist under fairly obvious conditions. For the totally real case, see Lemma 3.2

Similar to the case of abelian varieties and their polarized weight one Hodge structures, given a polarization and a totally positive endomorphism, one can define a new polarization (see Lemma 4.2). For a polarized abelian variety, this follows from the well-known relation between the Rosati invariant endomorphisms and the Néron Severi group. In case the K3 type Hodge structure is a Hodge substructure of the $H^{2}$ of a smooth surface, it comes with a natural polarization induced by the cupproduct. It is then interesting to consider whether the polarization obtained by means of a totally real element $a$ is also realized as the natural polarization for some other surface $S_{a}$. Thus $H^{2}\left(S_{a}\right)$ has a Hodge substructure isomorphic to the original one, but the isomorphism does not preserve the natural polarizations.

It follows easily from general results on K3 surfaces that, under a condition on the dimension of the Hodge structure, such K3 surfaces do exist (see section 4.7). The isomorphism of Hodge substructures, in combination with the Hodge conjecture, then leads one to wonder whether there is an algebraic cycle realizing the isomorphism. We discuss some aspects of this question in

[^0]the last section. Mukai Muk proved that Hodge isometries between rational Hodge structures of K3 surfaces are realized by algebraic cycles, but this deep result does not apply to the general case. It does imply that if the endomorphism algebra of the (transcendental) Hodge structure of the K3 surface is a CM field, then any endomorphism is induced by an algebraic cycle on the selfproduct of the surface Ma.

We consider the Kuga Satake variety of a K3 type Hodge structure with real multiplication in sections 5 and 6. The CM case was already studied in vG2]. In particular, we consider the endomorphism algebra of the Kuga Satake variety in the presence of real multiplications on the Hodge structure and we discuss some examples. We show that the Kuga Satake construction is related to the corestriction of (Clifford) algebras. From this result we obtain a better understanding of previous work of Mumford Mum and Galluzzi Ga.

## 1 Hodge structures, polarizations and endomorphisms

1.1 Hodge structures. We recall the basic notions and we refer to [Z], section 0 and vG1], section 1 for further details.

For a $\mathbf{Q}$-vector space $V$ and a $\mathbf{Q}$-algebra $R$ we write $V_{R}:=V \otimes_{\mathbf{Q}} R$. The $R$-linear extension of a Q-linear map $f: V \rightarrow W$ is denoted by $f_{R}: V_{R} \rightarrow W_{R}$.

A rational Hodge structure of weight $k$ is a $\mathbf{Q}$-vector space $V$ with a decomposition of its complexification:

$$
V_{\mathbf{C}}=\oplus_{p+q=k} V^{p, q} \quad \text { such that } \overline{V^{p, q}}=V^{q, p}
$$

and $p, q \in \mathbf{Z}_{\geq 0}$. Equivalently, a rational Hodge structure of weight $k$ is a $\mathbf{Q}$-vector space $V$ with a representation of algebraic groups over $\mathbf{R}$ :

$$
h: U(1)=\{z \in \mathbf{C}:|z|=1\} \longrightarrow G L\left(V_{\mathbf{R}}\right) \quad \text { such that } \quad h(z) \sim_{\mathbf{C}} \operatorname{diag}\left(\ldots, z^{p} \bar{z}^{q}, \ldots\right)
$$

with $p+q=k, p, q \geq 0$, that is the set of eigenvalues of $h(z)$ on $V_{\mathbf{C}}$ is a subset of $\left\{z^{k}, \ldots, z^{p} \bar{z}^{q}, \ldots, \bar{z}^{k}\right\}$. The subspace of $V_{\mathbf{C}}$ on which $h(z)$ acts as $z^{p} \bar{z}^{q}$ is $V^{p, q}$. Note that $h$ is defined over $\mathbf{R}$ if and only if $\overline{V^{p, q}}=V^{q, p}$. We write $(V, h)$, or simply $V$, for the Hodge structure on $V$ defined by $h$.
1.2 Polarizations. (See [Z], section 0.3.1, 0.3 .2 and vG1], section 1.7.) Let ( $V, h$ ) be a rational Hodge structure of weight $k$. A polarization $\psi$ of $(V, h)$ is first of all a Q-bilinear map

$$
\psi: V \times V \longrightarrow \mathbf{Q} \quad \text { such that } \quad \psi_{\mathbf{R}}(h(z) v, h(z) w)=\psi(v, w)
$$

for all $z \in U(1), v, w \in V_{\mathbf{R}}$. That is, $\psi_{\mathbf{R}}$ must be $U(1)$-invariant.
Let $C:=h(i) \in \operatorname{End}\left(V_{\mathbf{R}}\right), C$ is called the Weil operator. Then $\psi$ must also satisfy:

$$
\psi_{\mathbf{R}}(v, C w)=\psi_{\mathbf{R}}(w, C v) \quad\left(\forall v, w \in V_{\mathbf{R}}\right)
$$

hence $\psi_{\mathbf{R}}(\cdot, C \cdot)$ is a symmetric $\mathbf{R}$-bilinear form. As $C^{2}=h(-1)$ and $h(-1)$ has eigenvalues $(-1)^{p+q}=(-1)^{k}$, we have $C^{2}=(-1)^{k}$ on $V_{\mathbf{R}}$. Using also the symmetry of $\psi_{\mathbf{R}}(\cdot, C \cdot)$, we get:

$$
\psi(v, w)=\psi_{\mathbf{R}}(v, w)=\psi_{\mathbf{R}}(C v, C w)=\psi_{\mathbf{R}}\left(w, C^{2} v\right)=(-1)^{k} \psi_{\mathbf{R}}(w, v)=(-1)^{k} \psi(w, v)
$$

for $v, w \in V$. Thus $\psi$ is symmetric if the weight $k$ of $V$ is even and alternating if the weight is odd.

Finally one requires that $\psi_{\mathbf{R}}(\cdot, C \cdot)$ is positive definite:

$$
\psi_{\mathbf{R}}(v, C v)>0 \quad\left(\forall v \in V_{\mathbf{R}}, v \neq 0\right)
$$

A polarized rational Hodge structure $(V, h, \psi)$ is a rational Hodge structure $(V, h)$ with a polarization $\psi$.
1.3 Endomorphisms. A homomorphism $f$ of Hodge structures $\left(V, h_{V}\right)$ and $\left(W, h_{W}\right)$ is a Q-linear map whose $\mathbf{R}$-linear extension intertwines the representations $h_{V}$ and $h_{W}$ of $U(1)$ :

$$
f: V \longrightarrow W, \quad \text { such that } \quad f_{\mathbf{R}}\left(h_{V}(z) v\right)=h_{W}(z) f_{\mathbf{R}}(v) \quad\left(\forall z \in U(1), \forall v, w \in V_{\mathbf{R}}\right)
$$

Equivalently, there is an integer $a$ such that the $\mathbf{C}$-linear extension of $f$ satisfies

$$
f_{\mathbf{C}}\left(V^{p, q}\right) \subset W^{p+a, q+a}
$$

(so we work up to Tate twists, cf. [Z, 0.3.0, vG1, 1.6). In particular kernels and images of homomorphisms of Hodge structures are Hodge substructures, i.e. they are rational Hodge structures with the decomposition induced by the one of $V$ ( $W$ respectively).

We write $H o m_{H o d}(V, W)$ for the $\mathbf{Q}$-vector space of homomorphisms of Hodge structures and $E n d_{H o d}(V)=\operatorname{Hom}_{H o d}(V, V)$, note that $E n d_{H o d}(V)$ is a Q-algebra with product given by the composition.

A Hodge substructure of $(V, h)$ is a subspace $W \subset V$ such that $h(z) W_{\mathbf{R}} \subset W_{\mathbf{R}}$ for all $z \in U(1)$. Thus $\left(W, h_{\mid W}\right)$ is a rational Hodge structure and the inclusion $W \hookrightarrow V$ is a homomorphism of Hodge structures.

A rational Hodge structure $(V, h)$ is said to be simple if it does not contain non-trivial rational Hodge structures. If $(V, h)$ is simple then any non-zero $f \in \operatorname{End}_{H o d}(V)$ must be an isomorphism, i.e. $E n d_{H o d}(V)$ is a division algebra.
1.4 Polarizations on weight two Hodge structures. Let $(V, h)$ be a rational Hodge structure of weight two. Define a decomposition of $V_{\mathbf{R}}$ (this is actually the eigenspace decomposition for the Weil operator $C$ ) by:

$$
V_{\mathbf{R}}=V_{2} \oplus V_{0}, \quad V_{2}:=V_{\mathbf{R}} \cap\left(V^{2,0} \oplus V^{0,2}\right), \quad V_{0}:=V_{\mathbf{R}} \cap V^{1,1}
$$

Then $V_{2}$ is a real vector space which is $h(U(1))$-invariant and the eigenvalues of $h(z)$ on $V_{2} \otimes \mathbf{C}$ are $z^{2}$ and $\bar{z}^{2}$. In particular, $C=-1$ on $V_{2}$. The subspace $V_{0}$ is a complementary $h(U(1))$ invariant subspace of $V_{2}$ on which $h(U(1))$ acts trivially. In particular $C=1$ on $V_{0}$.

Let $\psi: V \times V \rightarrow \mathbf{Q}$ be a morphism of Hodge structures. Then $\psi_{\mathbf{R}}$ is $U(1)$-invariant, hence $V_{0}$ and $V_{2}$ are perpendicular w.r.t. $\psi_{\mathbf{R}}$ :

$$
\psi_{\mathbf{R}}(v, w)=\psi_{\mathbf{R}}(C v, C w)=-\psi_{\mathbf{R}}(v, w) \quad\left(v \in V_{0}, w \in V_{2}\right)
$$

Now assume that $\psi$ is a polarization. Then $\psi_{\mathbf{R}}(\cdot, C \cdot)$ is positive definite on $V_{\mathbf{R}}$ and as $C=-1$ on $V_{2},+1$ on $V_{0}$, the symmetric form $\psi_{\mathbf{R}}$ is negative definite on $V_{2}$ and positive definite on $V_{0}$. In particular, the signature of $\psi_{\mathbf{R}}$ is $((d-2 e)+, 2 e-)$ where $d=\operatorname{dim}_{\mathbf{Q}} V, e=\operatorname{dim}_{\mathbf{C}} V^{2,0}$.
1.5 K3 type Hodge structures. A Hodge structure of K3 type is a simple, polarized, weight two Hodge structure $(V, h, \psi)$ with

$$
\operatorname{dim} V^{2,0}=1
$$

1.6 Periods of K3 type Hodge structures. Let $(V, \psi, h)$ be a K3 type Hodge structure. Any non-zero element $\omega \in V^{2,0}$ will be called a period of $(V, h, \psi)$. Note that $V^{0,2}=\mathbf{C} \bar{\omega}$ and $V^{1,1}=\left(V^{2,0} \oplus V^{0,2}\right)^{\perp}$, the perpendicular is taken w.r.t. the polarization $\psi$. Thus the polarized weight two Hodge structure $(V, h, \psi)$ is determined by the bilinear form $\psi$, of signature $((d-2)+, 2-)$ where $d=\operatorname{dim} V$ and the period $\omega$.

As $h(z) \omega=z^{2} \omega$ for all $z \in U(1)$, we get

$$
\psi_{\mathbf{C}}(\omega, \omega)=\psi_{\mathbf{C}}(h(z) \omega, h(z) \omega)=z^{4} \psi_{\mathbf{C}}(\omega, \omega), \quad(\forall z \in U(1))
$$

hence $\psi_{\mathbf{C}}(\omega, \omega)=0$. As $\omega+\bar{\omega} \in V_{2}$ we have $0>\psi_{\mathbf{R}}(\omega+\bar{\omega}, \omega+\bar{\omega})=2 \psi_{\mathbf{C}}(\omega, \bar{\omega})$. Thus the period satisfies:

$$
\psi_{\mathbf{C}}(\omega, \omega)=0, \quad \psi_{\mathbf{C}}(\omega, \bar{\omega})<0, \quad\left(V^{2,0}=\mathbf{C} \omega\right)
$$

Conversely, let $V$ be a $\mathbf{Q}$-vector space of dimension $d$, with a bilinear form $\psi$ of signature $((d-2)+, 2-)$. Any $\omega \in V_{\mathbf{C}}$ which satisfies $\psi_{\mathbf{C}}(\omega, \omega)=0$ and $\psi_{\mathbf{C}}(\omega, \bar{\omega})<0$ determines a polarized weight two Hodge structure $(V, h, \psi)$ by $V^{2,0}:=\mathbf{C} \omega$. The following lemma gives a criterion for this Hodge structure to be simple, equivalently, to be of K3 type.
1.7 Lemma. Let $(V, h, \psi)$ be a weight two Hodge structure with $\operatorname{dim} V^{2,0}=1$ and let $V^{2,0}=\mathbf{C} \omega$. The $(V, h, \psi)$ is of K3 type if and only if $\psi_{\mathbf{C}}(\omega, v)=0$, with $v \in V$, implies $v=0$.

Proof. We will prove that $V$ has a non-trivial Hodge substructure iff there is a non-zero $v \in V$ perpendicular to $\omega$.

Let $W \subset V$ be a non-trivial Hodge substructure. As $W_{\mathbf{R}}$ must be invariant under $C=h(i)$, it is the direct sum of the perpendicular eigenspaces $W_{2}:=W_{\mathbf{R}} \cap V_{2}$ and $W_{0}=W_{\mathbf{R}} \cap V_{0}$.

In case $W_{2}=0, W \subset V_{0}$ and thus $\psi_{\mathbf{C}}(\omega, w)=0$ for all $w \in W$. In case $W_{2} \neq 0$, we get $W_{2}=V_{2}$ since $V_{2}$ is an irreducible (over $\mathbf{R}$ ) representation of $U(1)$. In particular, $\omega, \bar{\omega} \in W_{\mathbf{C}}$. We consider the subspace

$$
W^{\perp}:=\left\{v \in V: \psi_{\mathbf{C}}(v, w)=0 \quad \forall w \in W\right\} .
$$

Then $W^{\perp}$ is also a non-trivial Hodge substructure of $V$ (the $h(U(1))$-invariance of $W_{\mathbf{R}}^{\perp}$ follows from the $h(U(1))$-invariance of $W_{\mathbf{R}}$ and $\left.\psi_{\mathbf{R}}(h(z) v, h(z) w)=\psi_{\mathbf{R}}(v, w)\right)$. As $\psi_{\mathbf{R}}$ is positive definite on $V_{2}$, it follows that $W_{\mathbf{R}}^{\perp} \cap V_{2}=0$. As before, we find $\psi_{\mathbf{C}}(\omega, w)=0$ for all $w \in W^{\perp}$.

Conversely, given a non-zero $v \in V$ with $\psi_{\mathbf{C}}(\omega, v)=0$, the C-linearity of $\psi_{\mathbf{C}}$ and the fact that $\left(\psi_{\mathbf{C}}\right)_{\left.\right|_{\mathbf{R}}}=\psi_{\mathbf{R}}$ imply that also $\psi_{\mathbf{C}}(\bar{\omega}, v)=0$. Thus $v \in V_{2}^{\perp}=V_{0}$. As $h(z) w=w$ for all $z \in U(1)$ and $w \in V_{0}$, we conclude that $W=\langle v\rangle$ is a non-trivial Hodge substructure.
1.8 The transcendental lattice of a K3 surface. Let $S$ be a K3 surface, then $H^{2}(S, \mathbf{Z}) \cong \mathbf{Z}^{22}$ and the natural Hodge structure on $H^{2}(S, \mathbf{Q})$ has $\operatorname{dim} H^{2,0}(S)=1$ (cf. [BPV]). The orientation of $S$ gives a natural isomorphism $H^{4}(S, \mathbf{Z}) \cong \mathbf{Z}$, so we obtain a cupproduct $H^{2}(S, \mathbf{Z}) \times H^{2}(S, \mathbf{Z}) \rightarrow \mathbf{Z}$. This cupproduct is an even unimodular bilinear form of signature $(3+, 19-)$ on $H^{2}(S, \mathbf{Z})$ and (cf. [BPV] I.2.7, with $U=H$, VIII.3):

$$
H^{2}(S, \mathbf{Z}) \cong \Lambda_{K 3}, \quad \Lambda_{K 3}=U^{3} \oplus E_{8}(-1)^{2}
$$

We will assume that $S$ is algebraic, thus $S$ has an ample divisor with class $h \in H^{2}(S, \mathbf{Z}) \cap$ $H^{1,1}(S)$, in particular $h \cup h>0$. The primitive rational cohomology of $S$ (w.r.t. to $h$ ) is

$$
h^{\perp}=H^{2}(S, \mathbf{Q})_{\text {prim }}=\left\{v \in H^{2}(S, \mathbf{Q}): v \cup h=0\right\}
$$

The inclusion $H^{2}(S, \mathbf{Q})_{\text {prim }} \subset H^{2}(S, \mathbf{Q})$ defines a rational Hodge structure on the primitive cohomology. The map $\psi_{S}$, defined by $\psi_{S}(v, w):=-(v \cup w)$, gives a polarization on $H^{2}(S, \mathbf{Q})_{\text {prim }}$.

The Néron Severi group of $S$ is $N S(S)=H^{2}(S, \mathbf{Z}) \cap H^{1,1}(S)$ (BPV], IV, 2.13), note that $N S(S)_{\mathbf{Q}}=H^{2}(S, \mathbf{Q}) \cap V_{0}$ is the maximal Hodge structure of type $(1,1)$ contained in $H^{2}(S, \mathbf{Q})$. The transcendental lattice of $S$ is defined as

$$
T_{S}:=N S(S)^{\perp} \quad\left(\subset H^{2}(S, \mathbf{Z})_{\text {prim }}\right)
$$

As $N S(S) \subset V_{0}$, we get $H^{2,0}(S) \subset T_{S, \mathbf{C}}$. The Hodge substructure $T_{S, \mathbf{Q}}$ of $H^{2}(S, \mathbf{Q})_{\text {prim }}$, with the polarization induced by $\psi_{S}$, is of K3 type.

The 'surjectivity of the period map' ([BPV], VIII.14) implies that any $\omega \in \Lambda_{K 3} \otimes_{\mathbf{Z}} \mathbf{C}$ with $\omega \cdot \omega=0, \omega \cdot \bar{\omega}>0$ and such that there is a $h \in \Lambda_{K 3}$ with $h \cdot h>0$ and $h \cdot \omega=0$, defines an algebraic K3 surface $S$ with an isomorphism $H^{2}(S, \mathbf{Z}) \cong \Lambda_{K 3}$ whose C-linear extension induces an isomorphism $H^{2,0}(S) \cong \mathbf{C} \omega$.

## 2 Real multiplication for K3 type Hodge structures

2.1 Endomorphisms of K3 type Hodge structures. Zarhin showed that for a Hodge structure of K3 type $(V, h, \psi)$, the division algebra $E n d_{H o d}(V)$ is a (commutative) field which is either totally real, in which case we write $E n d_{H o d}(V)=F$ or it is a CM field $E$, that is $E$ is
an imaginary quadratic extension of a totally real field $F$ ( $[\bar{Z}$, Theorem 1.5.1). Moreover, for any polarization $\psi$ of $(V, h)$, one has ( $\mathbb{Z}], 1.5$, Thm 1.5.1)

$$
\psi(a v, w)=\psi(v, \bar{a} w), \quad\left(\forall a \in E_{n d}{ }_{H o d}(V), \forall v, w \in V\right)
$$

where $\bar{a}$ is the complex conjugate of $a$, in particular $\bar{a}=a$ for $a \in F$.
2.2 Notation. From now on $(V, h, \psi)$ will be a polarized Hodge structure of K3 type with $F=E n d_{H o d}(V)$ a totally real field and

$$
d=\operatorname{dim}_{\mathbf{Q}} V, \quad n=[F: \mathbf{Q}], \quad m=n / d=\operatorname{dim}_{F} V .
$$

2.3 Totally real fields. Recall that a finite extension $F$ of $\mathbf{Q}$ is said to be totally real if for any embedding $\sigma: F \hookrightarrow \mathbf{C}$ one has $\sigma(F) \subset \mathbf{R}$.

It is well-known that for any number field $F$ there is an irreducible polynomial $p \in \mathbf{Q}[X]$ such that $F \cong \mathbf{Q}[X] /(p)$. Then $[F: \mathbf{Q}]=n$, where $n$ is the degree of $p$. Let $\alpha_{1}, \ldots, \alpha_{n} \in \mathbf{C}$ be the roots of $p$ in $\mathbf{C}$. Then the maps

$$
\sigma_{j}: F \cong \mathbf{Q}[X] /(p) \hookrightarrow \mathbf{C}, \quad \sigma_{j}: \sum a_{i} X^{i}+(p) \mapsto \sum a_{i} \alpha_{j}^{i}
$$

$j=1, \ldots, n$, are the embeddings of $F$ into $\mathbf{C}$. In particular, $F$ is totally real iff all roots of $p$ are real.

Let $F$ be a totally real field. An element $a \in F$ is called totally positive if $\sigma(a)>0$ for all complex embeddings $\sigma$ of $F$. For example, $b^{2}$, for any non-zero $b \in F$, is totally positive. For any $b \in F$, the element $k+b \in F$, with $k \in \mathbf{Z}_{>0}$, is totally positive if $k>-\sigma(b)$ for any embedding $\sigma \in S$.
2.4 Splitting over extensions. To study the action of the field $F=\operatorname{End}_{H o d}(V)$ on the Q-vector space $V$ with the bilinear form $\psi$, it is convenient to have a decomposition into eigenspaces. So let $\tilde{F}$ be the Galois closure of $F$. Then $\tilde{F}$ is a Galois extension of $\mathbf{Q}$ which contains $F$ as a subfield. Let

$$
H:=\operatorname{Gal}(\tilde{F} / F) \hookrightarrow G:=\operatorname{Gal}(\tilde{F} / \mathbf{Q}), \quad[G: H]=[F: \mathbf{Q}]=n .
$$

Note that $h(a)=a$ for any $a \in F$ and $h \in H$, thus any coset $g H$ gives a well-defined embedding $F \hookrightarrow \tilde{F}, a \mapsto g(a)$.

Let again $F=\mathbf{Q}(\alpha)=\mathbf{Q}[X] /(p)$ with $\alpha=X+(p) \in F$ a root of $p$. Then $p=\prod_{g \in G / H}(X-$ $g(\alpha)) \in \tilde{F}[X]$. The Chinese remainder theorem gives an isomorphism of rings

$$
F_{\tilde{F}}:=F \otimes_{\mathbf{Q}} \tilde{F} \cong \prod_{g \in G / H} \tilde{F}_{g}, \quad a \otimes t \longmapsto(\ldots, g(a) t, \ldots,)_{g \in G}
$$

where the $F$-algebra $\tilde{F}_{g}$ is the field $\tilde{F}$ on which $F$ acts via the automorphism $g$ of $\tilde{F}: a \cdot t:=$ $(a \otimes 1) t=g(a) t$ for $t \in \tilde{F}_{g}$.

For $h \in G$, let $\pi_{h} \in \tilde{F}_{h} \subset F_{\tilde{F}}$ be the idempotent corresponding to the projection on $\tilde{F}_{h}$, so $\pi_{h}=\left(\ldots,\left(\pi_{h}\right)_{g}, \ldots\right) \in \prod \tilde{F}_{g}$ with $\left(\pi_{h}\right)_{h}=1$ and $\left(\pi_{h}\right)_{g}=0$ if $g \neq h$. Note that $a \cdot \pi_{g}=g(a) \pi_{g}$.

Then $V_{\tilde{F}}$ has the decomposition, with $V_{g}:=\pi_{g} V$ :

$$
V_{\tilde{F}}=\oplus_{g \in G / H} V_{g}, \quad v=\sum_{g \in G / H} v_{g} \quad \text { with } \quad v_{g}:=\pi_{g} v
$$

This is also the decomposition of $V_{\tilde{F}}$ into eigenspaces for the $F$-action because, with $a \cdot v=$ $(a \otimes 1) v$ for $a \in F$ and $v \in V_{\tilde{F}}$, we have

$$
a \cdot v_{g}=a \cdot \pi_{g} v_{g}=g(a) \pi_{g} v_{g}=g(a) v_{g} .
$$

Note that $\operatorname{dim}_{\mathbf{Q}} V=\operatorname{dim}_{\tilde{F}} V_{\tilde{F}}=d=n m$ and that $\operatorname{dim}_{\tilde{F}} V_{g}=m$ for any $g \in G / H$.
2.5 The Galois action on $V_{\tilde{F}}$. The Galois group $G$ acts on $V_{\tilde{F}}=V \otimes_{\mathbf{Q}} \tilde{F}$ via the second factor of the tensor product. This action commutes with the one of $F$ on the first factor. Under the isomorphism $V_{\tilde{F}} \cong \oplus V_{g}$, this action permutes the eigenspaces $V_{g}$ : if $a \cdot v=g(a) v$ then for $h \in G$ one has:

$$
a \cdot h(v)=h(a \cdot v)=h(g(a) v)=(h g)(a) h(v) \quad \text { hence } \quad h\left(V_{g}\right)=V_{h g}
$$

(use that if $v=\sum_{i} v_{i} \otimes t_{i}$, then $g(a) v=(1 \otimes g(a))\left(\sum_{i} v_{i} \otimes t_{i}\right)$ etc.).
Let $v \in V=V \otimes 1 \subset V_{\tilde{F}}$ then $v=h(v)$. Writing $v=\sum v_{g}$ we get

$$
v=\sum v_{g}=\sum h\left(v_{g}\right) \quad \text { hence } \quad h\left(v_{g}\right)=v_{h g}, \quad(v \in V) .
$$

Thus for $v \in V$ we have the decomposition $v=\sum g\left(v_{e}\right)$. In particular, the composition of the inclusion $V \hookrightarrow V_{\tilde{F}}$ with the projection of $V_{\tilde{F}} \rightarrow V_{e}$ is an injective $F$-linear map:

$$
V \hookrightarrow V_{e}, \quad v \longmapsto v_{e}
$$

This inclusion induces an isomorphism of $\tilde{F}$-vector spaces $V \otimes_{F} \tilde{F} \cong V_{e}$.
2.6 Lemma. The $\tilde{F}$-bilinear extension of the polarization $\psi$ on $V$ will be denoted by $\psi_{\tilde{F}}$,

$$
\psi_{\tilde{F}}: V_{\tilde{F}} \times V_{\tilde{F}} \longrightarrow \tilde{F}, \quad \text { let } \quad \psi_{e}: V_{e} \times V_{e} \longrightarrow \tilde{F}
$$

be the restriction of $\psi_{\tilde{F}}$ to $V_{e} \times V_{e}$. Let $\Phi$ be the restriction of $\psi_{e}$ to $V \times V \subset V_{e} \times V_{e}$,

$$
\Phi: V \times V \longrightarrow F, \quad \Phi(v, w):=\psi_{e}\left(v_{e}, w_{e}\right) .
$$

Then $\Phi$ is an $F$-bilinear map and for $v=\sum v_{g}, w=\sum w_{g} \in V \subset V_{\tilde{F}}$ we have:

$$
\psi(v, w)=\sum_{g \in G / H} g\left(\Phi\left(v_{e}, w_{e}\right)\right)=\operatorname{tr}_{F / \mathbf{Q}}\left(\Phi\left(v_{e}, w_{e}\right)\right)
$$

where $\operatorname{tr}_{F / \mathbf{Q}}: F \rightarrow \mathbf{Q}, t \mapsto \sum_{g \in G / H} g(t)$ is the trace map.
Proof. The polarization $\psi$ on $V$ satisfies $\psi(a v, w)=\psi(v, a w)$ for all $a \in F, v, w \in V$. Using the idempotents $\pi_{g}$ we get:

$$
\psi_{\tilde{F}}\left(v_{g}, v_{h}\right)=\psi_{\tilde{F}}\left(\pi_{g} v_{g}, v_{h}\right)=\psi_{\tilde{F}}\left(v_{g}, \pi_{g} v_{h}\right)
$$

and $\pi_{g} v_{h}=0$ if $g \neq h$. Thus the eigenspaces $V_{g}$ are perpendicular w.r.t. $\psi_{\tilde{F}}$ and we get

$$
\psi_{\tilde{F}}(v, w)=\sum_{g \in G / H} \psi_{g}\left(v_{g}, w_{g}\right), \quad \text { with } \quad \psi_{g}=\left(\psi_{\tilde{F}}\right)_{\mid V_{g} \times V_{g}}: V_{g} \times V_{g} \longrightarrow \tilde{F}
$$

and $v=\sum v_{g}, w=\sum w_{g} \in V_{\tilde{F}}$. Note that $a \psi_{g}\left(v_{g}, w_{g}\right)=\psi_{g}\left(g(a) v_{g}, w_{g}\right)=\psi_{g}\left(v_{g}, g(a) w_{g}\right)$, thus $\psi_{e}$ is $\tilde{F}$-bilinear.

As $\psi$ is defined over $\mathbf{Q}$, for $h \in G$ and $v, w \in V_{\tilde{F}}$ we have:

$$
h\left(\psi_{\tilde{F}}(v, w)\right)=\psi_{\tilde{F}}(h(v), h(w))
$$

(alternatively, use $v=\sum_{i} v_{i} \otimes t_{i}$ etc.). In particular, for $v, w \in V \subset V_{e}$ and $h \in H$ we get $\psi_{e}\left(v_{e}, w_{e}\right) \in F$. For $v=\sum g\left(v_{e}\right), w=\sum g\left(w_{e}\right) \in V \subset V_{\tilde{F}}$ we then have:

$$
g\left(\psi_{e}\left(v_{e}, w_{e}\right)\right)=g\left(\psi_{\tilde{F}}\left(v_{e}, w_{e}\right)\right)=\psi_{\tilde{F}}\left(g\left(v_{e}\right), g\left(w_{e}\right)\right)=\psi_{g}\left(g\left(v_{e}\right), g\left(w_{e}\right)\right)
$$

hence $\psi(v, w)=\psi_{\tilde{F}}(v, w)=\sum \psi_{g}\left(g\left(v_{e}\right), g\left(w_{e}\right)\right)=\sum g\left(\psi_{e}\left(v_{e}, w_{e}\right)\right)$. The lemma now follows from the definition of $\Phi$.
2.7 The Hodge group. The Hodge group of a K3 type Hodge structure was determined by Zarhin. For the definition and properties of the Mumford Tate group $M T(V)$ and its subgroup, the Hodge group $H d g(V)=S M T(V)$, of a Hodge structure $V$ we refer to [Z], 0.3.1 and [Go. Both are algebraic subgroups, defined over $\mathbf{Q}$, of $G L(V)$ and $S L(V)$ respectively.

Let $S O(V, \Phi)$ be the special orthogonal group of the bilinear form $\Phi$ on the $F$-vector space $V$ (defined in Lemma [2.6), viewed as an algebraic group over $\mathbf{Q}$. Then $S O(V, \Phi) \cong S O\left(V_{e}, \psi_{e}\right)$. For a Q-algebra $R$, the group of $R$-valued points of $S O(V, \Phi)$ is:

$$
S O(V, \Phi)(R)=\left\{A \in S L\left(V_{R}\right): a A=A a, \quad \Phi_{R}(A v, A w)=\Phi_{R}(v, w), \quad \forall a \in F, \forall v, w \in V_{R}\right\}
$$

2.8 Theorem([Z], Thm. 2.2.1) Let $(V, h, \psi)$ be a K3 type Hodge structure with $F=$ $E n d_{H o d}(V)$ a totally real field, then

$$
H d g(V)=S O(V, \Phi), \quad S O(V, \Phi)(\mathbf{R}) \cong S O(2, m-2) \times S O(m, \mathbf{R})^{n-1}
$$

and $S O(V, \Phi)(\mathbf{C}) \cong S O(m, \mathbf{C})^{n}$. The representations of these Lie groups on the $d=n m$ dimensional vector spaces $V_{\mathbf{R}}, V_{\mathbf{C}}$ respectively, are the direct sum of the standard representations of the factors.

Proof. With our definition of $\Phi$, Lemma 2.6 shows that $\psi(v, w)=\operatorname{tr}_{F / \mathbf{Q}}(\Phi(v, w))$ for all $v, w \in$ $V$, and this is also Zarhin's definition of $\Phi([\mathbb{Z}, 2.1$, with $e=1)$. Thus $H d g(V)=S O(V, \Phi)$ by [Z], Thm 2.2.1.

Similar to the various decompositions in [2.4], one has $F_{\mathbf{R}} \cong \oplus_{\sigma \in S} \mathbf{R}_{\sigma}$ and

$$
V_{\mathbf{R}}=\oplus_{\sigma \in S} V_{\sigma}, \quad v=\left(\ldots, v_{\sigma}, \ldots\right) \quad \text { and } \quad a \cdot v=\left(\ldots, \sigma(a) v_{\sigma}, \ldots\right)_{\sigma \in S}
$$

for $a \in F$. Thus this is also the decomposition of $V_{\mathbf{R}}$ into eigenspaces for the $F$-action.
To obtain these from the decomposition of $F_{\tilde{F}}$, choose an embedding $\tilde{\epsilon}: \tilde{F} \hookrightarrow \mathbf{R}$, for example one which extends $\epsilon: F \hookrightarrow \mathbf{R}$, where $a \cdot \omega=\epsilon(a) \omega$ and $V^{2,0}=\mathbf{C} \omega$. (We are only interested in the action of $F$ and $\mathbf{R}$ on $F_{\mathbf{R}}$, so the choice of the extension of $\epsilon$ does not matter). Then $\mathbf{R}$ becomes an $\tilde{F}$-module which we denote by $\mathbf{R}_{\epsilon}$ and we get

$$
F_{\mathbf{R}}:=F \otimes_{\mathbf{Q}} \mathbf{R} \cong\left(F \otimes_{\mathbf{Q}} \tilde{F}\right) \otimes_{\tilde{F}} \mathbf{R}_{\epsilon} \cong\left(\oplus_{g \in G / H} \tilde{F}_{g}\right) \otimes_{\tilde{F}} \mathbf{R}_{\epsilon} \cong \oplus_{\sigma \in S} \mathbf{R}_{\sigma}
$$

where the bijection between the set of complex embeddings $S$ of $F$ and $G / H$ is given by $\sigma(a)=\tilde{\epsilon}(g(a))$, for all $a \in F$. The idempotent in $F_{\mathbf{R}}$ which corresponds to the projection on $\mathbf{R}_{\sigma}$ will be denoted by $\pi_{\sigma}$ and $V_{\sigma}=\pi_{\sigma} V$.

The fact that $\Phi$ is $F$-bilinear implies that its $\mathbf{R}$-bilinear extension $\Phi_{\mathbf{R}}$, with values in $F_{\mathbf{R}} \cong$ $\prod \mathbf{R}_{\sigma}$, is $F_{\mathbf{R}}$-bilinear. Thus $\Phi_{\mathbf{R}}=\left(\ldots, \Phi_{\sigma}, \ldots\right)_{\sigma \in S}$ with $\Phi_{\sigma}: V_{\sigma} \times V_{\sigma} \rightarrow \mathbf{R}_{\sigma}$.

Any $F$-linear endomorphism $A$ of $V$ extends $\mathbf{R}$-linearly to an $F_{\mathbf{R}}$-linear endomorphism $A_{\mathbf{R}}$ of $V_{\mathbf{R}}$. In particular, $A_{\mathbf{R}}$ commutes with the idempotents $\pi_{\sigma} \in F_{\mathbf{R}}$, hence for $v_{\sigma} \in V_{\sigma}$ we get $A_{\mathbf{R}} v_{\sigma}=A_{\mathbf{R}} \pi_{\sigma} v_{\sigma}=\pi_{\sigma} A_{\mathbf{R}} v_{\sigma}$ hence $A_{\mathbf{R}} v_{\sigma} \in \pi_{\sigma} V=V_{\sigma}$.

Since the elements of $S O(V, \Phi)$ are $F$-linear, we get: $S O(V, \Phi)_{\mathbf{R}} \subset \prod_{\sigma \in S} G L\left(V_{\sigma}\right)$. Here $S O(V, \Phi)_{\mathbf{R}}=S O(V, \Phi) \times_{\mathbf{Q}} \mathbf{R}$ is the algebraic group over $\mathbf{R}$ obtained by extension of scalars (that is, by base change) from the algebraic group $S O(V, \Phi)$ over $\mathbf{Q}$. As $(V, \Phi) \cong\left(V_{e}, \psi_{e}\right)$ and the Galois group $G$ permutes the $V_{g}$, we obtain the following isomorphism of algebraic groups:

$$
S O(V, \Phi)_{\mathbf{R}} \cong \prod_{\sigma \in S} S O\left(V_{\sigma}, \Phi_{\sigma}\right)
$$

and the representation is factorwise on $V_{\mathbf{R}}=\oplus V_{\sigma}$.
By definition of $\epsilon$ we have $V^{2,0} \subset V_{\epsilon, \mathbf{C}}:=V_{\epsilon} \otimes_{\mathbf{R}} \mathbf{C}$. Since $h$ commutes with $F$, each $V_{\sigma}$ is a real Hodge structure, hence also $V^{0,2} \subset V_{\epsilon, \mathbf{C}}$ :

$$
V^{2,0} \oplus V^{0,2} \subset V_{\epsilon, \mathbf{C}}:=V_{\epsilon} \otimes_{\mathbf{R}} \mathbf{C} \quad \text { hence } \quad V_{2} \subset V_{\epsilon}
$$

Recall from 1.4 that $V_{\mathbf{R}}=V_{2} \oplus V_{0}$ and that $\psi_{\mathbf{R}}$ is negative definite on $V_{2}$ and positive definite on $V_{0}$. As $V_{\sigma} \subset V_{0}$ if $\sigma \neq \epsilon$, we get $S O\left(V_{\sigma}, \Phi_{\sigma}\right)_{\mathbf{R}} \cong S O(m)_{\mathbf{R}}$ if $\sigma \neq \epsilon$ and $S O\left(V_{\epsilon}, \Phi_{\epsilon}\right)_{\mathbf{R}} \cong S O(m-$ $2,2)_{\mathbf{R}}$. Extending scalars to $\mathbf{C}$ and taking $\mathbf{C}$-points, we get $S O(V, \Phi)(\mathbf{C}) \cong S O(m, \mathbf{C})^{n}$.

## 3 The existence of Hodge structures of K3 type with real multiplication.

3.1 We prove an easy existence result for K3 type Hodge structures with real multiplication. Next we apply results of Nikulin on embeddings of lattices and the surjectivity of the period map to show the existence of K3 surfaces whose transcendental lattice has real multiplication.

It would be interesting to have geometrical (and not just 'Hodge theoretic') examples of such surfaces. A first step in this direction is done in 3.4. where real multiplication for a certain 'geometric' four dimensional family of K3 surfaces is studied.
3.2 Lemma. Let $F$ be a totally real field with $[F: \mathbf{Q}]=n$. Then for any $m \in \mathbf{Z}_{\geq 3}$ there exist K3 type Hodge structures $(V, h, \psi)$ with $E n d_{H o d}(V)=F$ and $\operatorname{dim}_{F} V=m$.

However, there are no such K3 type Hodge structures with $\operatorname{dim}_{F} V \leq 2$.
Proof. We use the notation from the proof of Theorem 2.8. In case $m=1$ we would have $1=\operatorname{dim} V_{\epsilon}$, which contradicts the fact that the two dimensional subspace $V_{2}$ is a subspace of $V_{\epsilon}$.

If $m=2$ and $\operatorname{End}_{H o d}(V)$ were equal to $F$, then by Zarhin's theorem we would have $H d g(V)(\mathbf{C}) \cong S O(2, \mathbf{C})^{n}$. A basic property of $H d g(V)$ is that $\operatorname{End}_{H d g}(V)=\operatorname{End}_{H o d}(V)$. Note that $S O(2, \mathbf{C}) \cong \mathbf{C}^{*}$ and its standard representation on $\mathbf{C}^{2}$ is equivalent to $t \mapsto \operatorname{diag}\left(t, t^{-1}\right)$. With this action, $E n d_{\mathbf{C}^{*}}\left(\mathbf{C}^{2}\right)$, the endomorphisms commuting with $\mathbf{C}^{*}$, consists of the diagonal matrices in $\operatorname{End}\left(\mathbf{C}^{2}\right)$. Hence

$$
\operatorname{End}_{H o d}(V)_{\mathbf{C}} \cong\left(\operatorname{End}_{\mathbf{C}^{*}}\left(\mathbf{C}^{2}\right)\right)^{n} \cong \mathbf{C}^{2 n},
$$

and thus $\operatorname{dim}_{\mathbf{Q}} \operatorname{End}_{H o d}(V)=2 n$ which contradicts that $E n d_{H o d}(V)=F$.
Fix an integer $m \geq 3$. It remains to show that there exist K3 type Hodge structures with $\operatorname{dim}_{F} V=m$. Let $V=F^{m}$ and choose an embedding $\epsilon: F \hookrightarrow \mathbf{R}$. Using the isomorphism $F_{\mathbf{R}} \cong \prod \mathbf{R}_{\sigma}$ it is easy to see that there are $a_{i} \in F$ such that:

$$
\epsilon\left(a_{1}\right)<0, \quad \epsilon\left(a_{2}\right)<0, \quad \epsilon\left(a_{j}\right)>0 \quad \text { and } \quad \sigma\left(a_{i}\right)>0 \quad \text { if } \quad \sigma \neq \epsilon,
$$

with $3 \leq j \leq n, 1 \leq i \leq n$. We define an $F$-bilinear form:

$$
\Phi: V \times V \longrightarrow F, \quad \Phi(x, y)=\sum_{k=1}^{m} a_{k} x_{k} y_{k}
$$

Then $\Phi$ induces the bilinear form defined by $\sum_{k=1}^{m} \sigma\left(a_{k}\right) x_{k} y_{k}$ on $V_{\sigma} \cong \mathbf{R}^{m}$. Note that $\Phi_{\epsilon}$ has signature $((m-2)+, 2-)$ and $\Phi_{\sigma}$ is positive definite if $\sigma \neq \epsilon$. Thus the signature of the Q-bilinear form $\psi:=\operatorname{tr}(\Phi)$ on $V$ is $((n m-2)+, 2-)$.

To define a K3 type Hodge structure, with polarization $\psi$, on $V$ it suffices to give a period $\omega \in V_{\mathbf{C}}$ (cf. 1.6) such that $\psi_{\mathbf{C}}(\omega, v) \neq 0$ for all non-zero $v \in V$ (Lemma 1.7). Since $\psi_{\mathbf{R}}<0$ on $V_{2}$, we must choose $\omega \in V_{\epsilon, \mathbf{C}}:=V_{\epsilon} \otimes_{\mathbf{R}} \mathbf{C}$. As $\omega$ and $\lambda \omega, \lambda \in \mathbf{C}-\{0\}$ define the same Hodge structure, we consider

$$
\mathcal{D}:=\left\{[\omega] \in \mathbf{P}\left(V_{\epsilon, \mathbf{C}}\right): \psi_{\mathbf{C}}(\omega, \omega)=0, \quad \psi_{\mathbf{C}}(\omega, \bar{\omega})<0\right\}
$$

which is a non-empty open subset in a quadric in a complex projective space of dimension $m-1 \geq 2$. Any non-zero $v \in V$ defines a hyperplane

$$
H_{v}:=\left\{[w] \in \mathbf{P}\left(V_{\epsilon, \mathbf{C}}\right): \psi_{\mathbf{C}}(v, w)=0\right\} \quad \subset \mathbf{P}\left(V_{\epsilon, \mathbf{C}}\right)
$$

As an open subset of a quadric is not contained in a hyperplane, $H_{v} \cap \mathcal{D}$ is an analytic subset of codimension at least one in $\mathcal{D}$. As $V$ is a countable set, we get $\mathcal{D} \neq \cup_{v}\left(H_{v} \cap \mathcal{D}\right)$, where the union is over the non-zero $v \in V$. Hence there is an $\omega \in \mathcal{D}$ which defines a simple Hodge structure with $V^{2,0}=\mathbf{C} \omega$. This construction shows that such (integral) Hodge structures have $m-2$ moduli.

As $\omega \in V_{\epsilon, \mathbf{C}}$ and $a \in F$ acts by scalar multiplication by $\epsilon(a) \in \mathbf{R}$ on $V_{\epsilon, \mathbf{C}}$, it follows that $a V^{2,0} \subset V^{2,0}$ and, taking the complex conjugates, $a V^{0,2} \subset V^{0,2}$. As $V^{1,1}=\left(V^{2,0} \oplus V^{0,2}\right)^{\perp}$ and $\psi_{\mathbf{C}}(a v, w)=\psi_{\mathbf{C}}(v, a w)$ we get $a H^{1,1} \subset H^{1,1}$, so $F \subset \operatorname{End}_{H o d}(V)$.

In case $F \neq \operatorname{End}_{H o d}(V)$, the K3 type Hodge structure $V$ has $E n d_{H o d}(V)=F^{\prime}$, with $F^{\prime}$ an extension of the field $F$. Multiplication by $b \in F^{\prime}$ defines an $F$-linear map $V \rightarrow V$, and thus each eigenspace $V_{\sigma}$ for the $F$-action is mapped into itself. The splitting of $V_{\mathbf{C}}$ into $F^{\prime}$ eigenspaces thus splits each

$$
V_{\sigma, \mathbf{C}}=\oplus_{\rho} V_{\rho}
$$

where the $\rho: F^{\prime} \rightarrow \mathbf{C}$ are the embeddings of $F^{\prime}$ which restrict to $\sigma$ on $F \subset F^{\prime}$. In particular, each $V_{\rho}$ has dimension at most $\left(\operatorname{dim}_{\mathbf{C}} V_{\sigma, \mathbf{C}}\right) / 2$. As $F^{\prime} \in \operatorname{End}_{H o d}(V)$ we must have $\omega \in V_{\rho}$ for some $\rho$ which extends $\epsilon$. Since the $V_{\rho}$ are eigenspaces of elements $b \in F^{\prime} \subset \operatorname{End}_{F}(V)$ and the set $\operatorname{End}_{F}(V)$ is countable, we conclude that the general $\omega \in \mathcal{D}$ defines a K3 type Hodge structure $V$ with $E n d_{H o d}(V)=F$.
3.3 Proposition. Given a totally real number field $F$ and an integer $m \geq 3$ such that $m[F: \mathbf{Q}] \leq 10$, there exist $(m-2)$-dimensional families of K 3 surfaces such that $F=\operatorname{End}\left(T_{S}\right)$ for the general surface $S$ in the family.

Proof. Let $(V, h, \psi)$ be a K3 type Hodge structure with $E n d_{H o d}(V)=F$ and period $\omega$. Choose a free Z-module $T \subset V$ of rank $d=\operatorname{dim}_{\mathbf{Q}} V$ such that $\psi$ is integer valued on $T \times T$. Theorem 1.10.1 of [N] shows that there is a primitive embedding of lattices $T \hookrightarrow \Lambda_{K 3}$. The surjectivity of the period map implies that $\omega \in T \otimes_{\mathbf{Z}} \mathbf{C}=V_{\mathbf{C}} \subset \Lambda_{K 3, \mathbf{C}}$ defines a K3 surface $S$ with $T_{S} \cong T$ as integral polarized Hodge structures. The proof of Lemma 3.2 shows that there are $m-2$ moduli.
3.4 Example. The minimal model of a double cover of $\mathbf{P}^{2}$ branched over six lines is a K3 surface. The general surface $S$ in this four dimensional family has (cf. MSY 0.3):

$$
T_{S} \cong U(2)^{2} \oplus<-2>^{2}, \quad \text { hence } \quad T_{S, \mathbf{Q}} \cong<1>^{2} \oplus<-1>^{4}
$$

where $U(2)$ is the lattice $\mathbf{Z}^{2}$ with quadratic form $4 x_{1} x_{2}$, which is isomorphic over $\mathbf{Q}$ to $y_{1}^{2}-y_{2}^{2}$ (put $x_{1}=y_{1}+y_{2}, x_{2}=\left(y_{1}-y_{2}\right) / 2$ ), and $<-2>^{2}$ is $\mathbf{Z}^{2}$ with quadratic form $-2 x_{1}^{2}-2 x_{2}^{2}$ which is isomorphic to $<-1>^{2}$ (put $\left.x_{1}=\left(y_{1}+y_{2}\right) / 2, x_{2}=\left(y_{1}-y_{2}\right) / 2\right)$.

We will show that there are one parameter families of K3 surfaces such that $T_{S} \cong T$ and $E n d_{H o d}\left(T_{S}\right)$ is a real quadratic field for the general $S$ in the family.

We consider the vector space $V=\mathbf{Q}^{6}=\left(\mathbf{Q}^{2}\right)^{3}$ with the following bilinear form $\psi$ :

$$
(V, \psi)=\left(\mathbf{Q}^{2}, Q_{1}\right) \oplus\left(\mathbf{Q}^{2}, Q_{2}\right) \oplus\left(\mathbf{Q}^{2}, Q_{3}\right), \quad Q_{i}:=\left(\begin{array}{cc}
1 & 0 \\
0 & r_{i}
\end{array}\right)
$$

so we identify the bilinear form with the symmetric matrix which defines it and $r_{i} \in \mathbf{Q}$ are to be chosen later. Next we want to consider the $a \in \operatorname{End}(V)$ such that $\psi(a x, y)=\psi(x, a y)$ for all $x, y \in V$. We will restrict ourselves to those which preserve the direct sum decomposition. Thus $a=\operatorname{block}\left(A_{1}, A_{2}, A_{3}\right)$ with $A_{i} \in \operatorname{End}\left(\mathbf{Q}^{2}\right)$ and ${ }^{t} A_{i} Q_{i}=Q_{i} A_{i}$, so we have to consider the matrix equation

$$
\left(\begin{array}{cc}
a & c \\
b & d
\end{array}\right)\left(\begin{array}{ll}
1 & 0 \\
0 & r
\end{array}\right)=\left(\begin{array}{cc}
1 & 0 \\
0 & r
\end{array}\right)\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right), \quad \text { let } \quad A_{e, c}:=\left(\begin{array}{cc}
e & c r \\
c & -e
\end{array}\right),
$$

then the solutions to the matrix equation are $A=A_{e, c}+\lambda I$, note that $A_{e, c}$ is just the 'traceless part' of $A$. As $A_{e, c}^{2}=\left(e^{2}+r c^{2}\right) I, \mathbf{Q}\left(A_{e, c}\right) \cong \mathbf{Q}\left(\sqrt{e^{2}+r c^{2}}\right)$ is a quadratic extension $\mathbf{Q}$ if $e^{2}+r c^{2}$ is not a square in $\mathbf{Q}$.

Now we consider the case $r_{1}=r_{2}=-1$ and $r_{3}=+1$, so $(V, \psi) \cong T_{S, \mathbf{Q}}$ for a general $S$ as above.

Assume that $d \in \mathbf{Z}_{>0}$ is odd, square free and that $d=e^{2}+c^{2}$ for some integers $c, e$. Write $d=2 d^{\prime}+1$, then $d=\left(d^{\prime}+1\right)^{2}-\left(d^{\prime}\right)^{2}$. Hence if we define

$$
a=\left(A_{1}, A_{2}, A_{3}\right), \quad \text { with } \quad A_{1}=A_{2}=A_{d^{\prime}+1, d^{\prime}}, \quad A_{3}=A_{e, c}, \quad\left(c^{2}+e^{2}=d=2 d^{\prime}+1\right)
$$

then $a^{2}=d$ so we have an action of the real field $F=\mathbf{Q}(a) \cong \mathbf{Q}(\sqrt{d})$ on $V$ and the elements of $F$ are self adjoint for the bilinear form $\psi$.

It is easy to see that one eigenspace for the $F$-action on $V_{\mathbf{R}}$ is positive definite for $\psi$ and the other, call it $V_{\sigma}$, has signature $(1+, 2-)$. Next we choose $T \cong \mathbf{Z}^{6} \hookrightarrow \mathbf{Q}^{6}$ such that $(T, \psi) \cong T_{S}=U(2)^{2} \oplus<-2>^{2}$ so the lattice $\left(T, \psi_{T}\right)$ is isometric to the lattice of double cover of $\mathbf{P}^{2}$ branched over 6 lines. Choosing a general $\omega \in V_{\sigma, \mathbf{C}}$ with $\psi_{T, \mathbf{C}}(\omega, \omega)=0, \psi_{T, \mathbf{C}}(\omega, \bar{\omega})>0$ defines a polarized integral Hodge structure on $\left(T, \psi_{T}\right)$ with $\operatorname{End}_{H o d}\left(T_{\mathbf{Q}}\right)=F$ (cf. the proof of Lemma (3.2). Thus we obtain a one parameter family of Hodge structures on $\left(T, \psi_{T}\right)$ with $E n d_{\text {Hod }}\left(T_{\mathbf{Q}}\right)=F$ for the general member of the family.

## 4 Twisting the polarization

4.1 Real multiplication and polarizations. Let $(V, h, \psi)$ be a K3 type Hodge structure. Let $B_{1}(V) \subset V^{*} \otimes V^{*}$ be the subspace of $\mathbf{Q}$-bilinear maps $\phi: V \times V \rightarrow \mathbf{Q}$ such that $\phi_{\mathbf{R}}$ is $h(U(1))$-invariant.

The isomorphism $\operatorname{End}(V)=V^{*} \otimes V \cong V^{*} \otimes V^{*}$, given by the isomorphism $V \rightarrow V^{*}$ defined by $\psi$, defines an isomorphism

$$
\operatorname{End}_{H o d}(V) \longrightarrow B_{1}, \quad a \longmapsto \psi_{a}, \quad \text { with } \quad \psi_{a}(v, w)=\psi(a v, w) .
$$

The bilinear form $\psi_{a}$ is symmetric iff $\mathbf{Q}(a)$ is a totally real field (use $\psi(a v, w)=\psi(v, \bar{a} w)$ ).
We now consider when the bilinear form $\psi_{a}$ gives a polarization on $V$.
4.2 Lemma. Let $(V, h, \psi)$ be a Hodge structure of K3 type such that $F=\operatorname{End}_{H o d}(V)$ is a totally real field. Then $\psi_{a}$ is a polarization of the Hodge structure ( $V, h$ ) if and only if $a \in F$ is totally positive.

Proof. As $\psi_{\mathbf{R}}$ is $U(1)$-invariant and $a$ commutes with $h(U(1))$ it follows that $\psi_{a, \mathbf{R}}$ is also $U(1)-$ invariant. As $C=h(i)$ we get $a C=C a$ and as $\psi(a v, w)=\psi(v, a w)$ it follows that $\psi_{a, \mathbf{R}}(\cdot, C \cdot)$ is symmetric. Thus $\psi_{a}$ is a polarization iff $\psi_{a, \mathbf{R}}(v, C v)>0$ for all non-zero $v \in V_{\mathbf{R}}$.

Using the decomposition $V_{\mathbf{R}}=\oplus V_{\sigma}$ (cf. the proof of 2.8 and 2.6), for $v=\left(\ldots, v_{\sigma}, \ldots\right) \in V_{\mathbf{R}}$ we have:

$$
\psi_{a, \mathbf{R}}(v, C v)=\sum_{\sigma \in S} \psi_{\sigma}\left(a \cdot v_{\sigma}, C v_{\sigma}\right)=\sum_{\sigma \in S} \sigma(a) \psi_{\sigma}\left(v_{\sigma}, C v_{\sigma}\right) .
$$

As $\psi$ is a polarization, $\psi_{\sigma}(\cdot, C \cdot)$ is positive definite for any $\sigma \in S$. Thus $\psi_{a}$ is a polarization iff $\sigma(a)>0$ for all $\sigma \in S$ iff $a$ is totally positive.
4.3 Example. Let $(V, h, \psi)$ and $F$ be as in the Lemma (or equivalently, as in (2.2). In case $a=b^{2}$ for some $b \in F$ we have $\psi(a v, w)=\psi(b v, b w)$. Thus the map $B: V \rightarrow V$ given by multiplication by $b, B v:=b v$ is an isometry between $\left.V, \psi_{a}\right)$ and $(V, \psi)$. As $b$ is a map of Hodge structures, $B$ is a Hodge isometry from $\left(V, h, \psi_{b}\right)$ to $(V, h, \psi)$.

To find examples where there is no Hodge isometry between the K3 type Hodge structure and its twist, the determinant of a bilinear form is useful.
4.4 Determinants and discriminants. Let $\psi, \psi_{a}$ be two bilinear forms on a $\mathbf{Q}$ vector space $V$. Choose a basis of $V$. If $Q, Q_{a}$ are the symmetric matrices defining the symmetric bilinear forms $\psi, \psi_{a}$ respectively, and $B$ is (the matrix of) an isometry between $\left(V, \psi_{a}\right)$ and $(V, \psi)$, we must have ${ }^{t} B Q B=Q_{a}$, in particular, $\operatorname{det}\left(Q_{a}\right)=\operatorname{det}(B)^{2} \operatorname{det}(Q)$. We will write $\operatorname{det}(\psi)$ for the class of) $\operatorname{det}(Q)$ in $\mathbf{Q}^{*}$ (modulo the subgroup of squares). This gives a well-known invariant (often called disciriminant) of a quadratic space.

If $F$ is a finite extension of $\mathbf{Q}$, the discriminant of $F$ is the rational number $D_{F}$, well defined up to squares in $\mathbf{Q}$, defined as $D_{F}:=\operatorname{det}\left(\operatorname{tr}\left(e_{i} e_{j}\right)\right)$ where the $e_{i} \in F$ are a $\mathbf{Q}$-basis of $F$.

### 4.5 Lemma.

1. Let $\psi=\operatorname{tr}(\Phi)$ as in Lemma 2.6 and $m=\operatorname{dim}_{F} V$. Then

$$
\operatorname{det}(\psi)=D_{F}^{m} N(\operatorname{det}(\Phi))
$$

where $m=\operatorname{dim}_{F} V$.
2. For $a \in F$, let $\psi_{a}(\cdot, \cdot):=\psi(a \cdot, \cdot)$. Then

$$
\operatorname{det}\left(\psi_{a}\right)=N(a)^{m} \operatorname{det}(\psi)
$$

where $N(a):=\prod_{\sigma \in S} \sigma(a)$ is the norm of $a$.

Proof. Choose an $F$-basis of $V$ for which $\Phi$ is diagonal: $\Phi(x, y)=\sum a_{k} x_{k} y_{k}$. Then

$$
\operatorname{det}(\psi)=\prod \operatorname{det}\left(\psi^{(k)}\right) \quad \text { with } \quad \psi^{(k)}: F \times F \longrightarrow \mathbf{Q}, \quad \psi^{(k)}(x, y)=\operatorname{tr}\left(a_{k} x y\right)
$$

Let $e_{1}, \ldots, e_{n}$ be a Q-basis of $F$ and let $S=\left\{\sigma_{1}, \ldots, \sigma_{n}\right\}$. As $D_{F}=\operatorname{det}\left(\sigma_{i}\left(e_{j}\right)\right)^{2}$ (cf. Sa, Proposition II.3), one finds

$$
\begin{aligned}
\operatorname{det}\left(\operatorname{tr}\left(a e_{i} e_{j}\right)\right) & =\operatorname{det}\left(\sum_{k} \sigma_{k}(a) \sigma_{k}\left(e_{i}\right) \sigma_{k}\left(e_{j}\right)\right) \\
& =\operatorname{det}\left(\sigma_{k}(a) \sigma_{k}\left(e_{i}\right)\right) \operatorname{det}\left(\sigma_{k}\left(e_{j}\right)\right) \\
& =\left(\prod_{k} \sigma_{k}(a)\right) \operatorname{det}\left(\sigma_{k}\left(e_{i}\right)\right) \operatorname{det}\left(\sigma_{k}\left(e_{j}\right)\right) \\
& =N(a) D_{F} .
\end{aligned}
$$

Thus $\operatorname{det}(\psi)=D_{F}^{m} \prod_{k} N\left(a_{k}\right)=D_{F}^{m} N(\operatorname{det}(\Phi))$. To compute $\operatorname{det}(\psi(a)$ use that

$$
\psi_{a}(v, w)=\psi(a v, w)=\psi_{\mathbf{R}}(a v, w)=\oplus_{\sigma \in S} \sigma(a) \Phi_{\sigma}\left(v_{\sigma}, w_{\sigma}\right)
$$

thus, with $m=\operatorname{dim}_{F} V=\operatorname{dim}_{\mathbf{R}} V_{\sigma}$, we get:

$$
\operatorname{det}\left(\psi_{a}\right)=\prod_{\sigma \in S} \sigma(a)^{m} \operatorname{det}\left(\Phi_{\sigma}\right)=N(a)^{m} \operatorname{det}(\psi)
$$

4.6 Examples. In case $F \cong \mathbf{Q}(\sqrt{d})$, with $d$ square free, and $m=\operatorname{dim}_{F} V$ is odd, it is easy to produce examples of totally positive $a \in F$ such that $(V, \psi)$ and ( $V, \psi_{a}$ ) are not isometric. In fact, $d \pm \sqrt{d}>0$, so $a=d+\sqrt{d}$ is totally positive. As $N(a)=d^{2}-d=d(d-1)$ and $d$ is square free, $N(a)$ is not a square in $\mathbf{Z}$, hence $\operatorname{det}\left(\psi_{a}\right) / \operatorname{det}(\psi)$ is not a square in $\mathbf{Q}$.
4.7 Twisting K3 surfaces with real multiplication. Let $(V, h, \psi)$ be a Hodge structure of K3 type, with $\operatorname{dim} V \leq 11$. Then for any totally positive $a \in F=E n d_{H o d}(V)$ we obtain the polarized Hodge structure of K3 type $\left(V, h, \psi_{a}\right)$.

Results of Nikulin and the surjectivity of the period map (cf. the proof of Proposition 3.3) imply that there exist K3 surfaces $S, S_{a}$ such that

$$
(V, h, \psi) \cong T_{S, \mathbf{Q}}, \quad\left(V, h, \psi_{a}\right) \cong T_{S_{a}, \mathbf{Q}}
$$

where the polarizations on the right hand sides are induced by (minus) the cup product in the corresponding surface. We will call $S_{a}$ a real twist of $S$. Note that there are isomorphisms of rational Hodge structures $(V, h) \cong T_{S, \mathbf{Q}} \cong T_{S_{a}, \mathbf{Q}}$, but in general there is no isomorphism of polarized Hodge structures between $T_{S, \mathbf{Q}}$ and $T_{S_{a}, \mathbf{Q}}$.

## 5 The Kuga-Satake variety

5.1 The Kuga Satake construction. We briefly recall the construction of 'the' Kuga Satake variety, actually an isogeny class of abelian varieties, of a rational, polarized, Hodge structure $(V, h, \psi)$ of weight two with $\operatorname{dim} V^{2,0}=1$ (cf. vG1], section 5).

The Clifford algebra $C(\psi)$ is the quotient of the tensor algebra $T(V)=\oplus_{n} V^{\otimes n}$ by the two sided ideal generated by $v \otimes v-\psi(v, v)$, where $v$ runs over $V$. The dimension of $C(\psi)$ is $2^{d}$, where $d=\operatorname{dim} V$. The Clifford algebra has a subalgebra $C^{+}(\psi)$ of dimension $2^{d-1}$, the quotient of $\oplus_{m} V^{\otimes 2 m}$, called the even Clifford algebra.

The Hodge decomposition of $V$ defines the subspace $V_{2} \subset V_{\mathbf{R}}$. Choosing a basis $f_{1}, f_{2}$ of $V_{2}$ such that $\left\langle f_{1}+i f_{2}\right\rangle=V^{2,0}$ and $\psi_{\mathbf{R}}\left(f_{1}, f_{1}\right)=-1$ defines an element $f_{1} f_{2} \in C^{+}(\psi)_{\mathbf{R}}$. Multiplication by $f_{1} f_{2}$ defines a map $J: c \mapsto f_{1} f_{2} c$, which is a complex structure on $C^{+}(\psi)_{\mathbf{R}}$, $J^{2}=-I$ (cf. vG1, Lemma 5.5). This defines a weight one Hodge structure $h_{s}$ on $C^{+}(\psi)$ by

$$
h_{s}: U(1) \longrightarrow G L\left(C^{+}(\psi)_{\mathbf{R}}\right), \quad a+b i \longmapsto a+b J
$$

for $a, b \in \mathbf{R}$. The choice of $e_{1}, e_{2} \in V$ with $\psi\left(e_{1}, e_{1}\right)>0, \psi\left(e_{2}, e_{2}\right)>0$ and $\psi\left(e_{1}, e_{2}\right)=0$ determines a Riemann form, i.e. an alternating form

$$
E: C^{+}(\psi) \times C^{+}(\psi) \longrightarrow \mathbf{Q} \quad \text { such that } \quad E(J x, J y)=E(x, y), \quad E(x, J x)>0
$$

for all $x, y \in C^{+}(\psi)_{\mathbf{R}}$ (cf. vG1, Proposition 5.9). The complex structure $J$ is uniquely determined by $(V, h, \psi)$, but the polarization, as constructed in vG1, 5.7, depends on the choice of a positive 2-plane in $V$ and is not unique in general.

Thus $(V, h, \psi)$ defines a polarized rational Hodge structure of weight one $\left(C^{+}(\psi), h_{s}, E\right)$. Each abelian variety $A$ in the isogeny class of abelian varieties of dimension $2^{d-2}$ defined by $\left(C^{+}(\psi), h_{s}, E\right)$ will be called a Kuga Satake variety for $(V, h, \psi)$, so $A$ is characterized by an isomorphism of Hodge structures $H^{1}(A, \mathbf{Q}) \cong\left(C^{+}(\psi), h_{s}\right)$.
5.2 The endomorphism algebra of the Kuga Satake variety. Let $(V, h, \psi)$ be a general polarized Hodge structure of K3 type, more precisely, assume that $M T(V)=G O(\psi)$ where $G O(\psi)=\left\{g \in G L(V): \psi(g v, g w)=\lambda_{g} \psi(v, w)\right\}$. Then

$$
M T(A)=C \operatorname{Spin}(\psi), \quad \operatorname{End}(A)_{\mathbf{Q}} \cong C^{+}(\psi)
$$

(VG1, Proposition 6.3.1, Lemma 6.5), where $C \operatorname{Spin}(\psi)$ is the Spin group of $(V, \psi)$. There is an isomorphism of complex Lie algebras $H d g(A)_{\mathbf{C}} \cong s o(d)_{\mathbf{C}}, d=\operatorname{dim} V$, and $H^{1}(A, \mathbf{C})$ is a direct sum of copies of the Spin representation $S(d)$ of $s o(d)_{\mathbf{C}}$. These results are useful for decomposing the Kuga Satake variety into simple abelian subvarieties.
5.3 K3 type Hodge structures with real multiplication. Assume now that $V$ is of K3 type with $F=E n d_{H o d}(V)$ is a real field. Then

$$
H d g(V)(\mathbf{C}) \cong S O(m, \mathbf{C})^{n}
$$

One would again like to know $\operatorname{End}(A)_{\mathbf{Q}}=\operatorname{End}_{H d g(A)}\left(H^{1}(A, \mathbf{Q})\right)$, but this seems rather hard. As a first step, we consider the decomposition of the Spin representation of the Lie algebra $s o(d)$ upon restriction to $s o(m)^{n}$, where $d=m n$.

Let $V_{i}, 1 \leq i \leq n$, be representations of $s o(m)$, then we write $V_{1} \boxtimes V_{2} \boxtimes \ldots \boxtimes V_{n}$ for the representation of $s o(m)^{n}$ where the $i$-th component of $s o(m)^{n}$ acts on $V_{i}$.
5.4 The Spin representation. Let $S(m)$ be the Spin representation of the complex Lie algebra $s o(m):=s o(m)_{\mathbf{C}}\left(c f\right.$. [FH], Chapter 20). In case $m=2 m^{\prime}+1$ is odd, the Spin representation is an irreducible representation of $s o(m)_{\mathbf{C}}$ of dimension $2^{m^{\prime}}$. In case $m=2 m^{\prime}$ is even, the Spin representation is the direct sum of two irreducible components $S^{ \pm}\left(2 m^{\prime}\right)$ :

$$
S\left(2 m^{\prime}\right)=S^{+}\left(2 m^{\prime}\right) \oplus S^{-}\left(2 m^{\prime}\right), \quad \operatorname{dim} S\left(2 m^{\prime}\right)=2^{m^{\prime}}, \quad \operatorname{dim} S^{ \pm}\left(2 m^{\prime}\right)=2^{m^{\prime}-1}
$$

5.5 Lemma. The restriction of the Spin representation $S(n m)$ of $s o(n m)$ to $s o(m)^{n}$ is given as follows:

$$
S(n m)_{\mid s o(m)^{n}}=S(m) \boxtimes \ldots \boxtimes S(m) \quad \text { if } \quad m \equiv 0(2)
$$

In case $m$ is odd, write $n=2 n^{\prime}$ if $n$ is even and $n=2 n^{\prime}+1$ if $n$ is odd. Then we have:

$$
S(n m)_{\mid s o(m)^{n}}=(S(m) \boxtimes \ldots \boxtimes S(m))^{2^{n^{\prime}}} \quad \text { if } \quad m \equiv 1(2)
$$

Proof. We use the conventions from [FH], Chapter 20. In case $m$ is even, so is $n m$ and the Lie algebras $s o(n m)$ and $s o(m)^{n}$ both have rank $n m / 2=n(m / 2)$. Thus we can assume that they have the same Cartan algebra $\mathfrak{h} \cong \mathbf{C}^{n m / 2}$ and the same dual $\mathfrak{h}^{*}$ generated by $L_{1}, \ldots, L_{n m / 2}$ The weights of $S(n m)$ are then ( FH , Propostion 20.5) the $\left( \pm L_{1} \pm L_{2} \ldots \pm L_{n m / 2}\right) / 2$, each with
multiplicity one. Any such weight is the sum, in a unique way, of the $n$ weights $\left( \pm L_{a m / 2+1} \pm\right.$ $\left.\ldots \pm L_{(a+1) m / 2}\right) / 2$ where $a=0,1, \ldots, n-1$. As these are the weights of $S(m) \boxtimes \ldots \boxtimes S(m)$, with the same multiplicity, one, we get the result.

In case $m$ is odd, the Lie algebra $s o(m)$ has rank $(m-1) / 2$. If $n$ is odd, then one has:

$$
r k(s o(n m))=(n m-1) / 2=n(m-1) / 2+(n-1) / 2=r k\left(s o(m)^{n}\right)+n^{\prime} .
$$

We can now assume that $L_{n(m-1) / 2+1}, \ldots, L_{n(m-1) / 2+n^{\prime}}$ are zero on the Cartan algebra of $s o(m)^{n}$. The weights of $S(\mathrm{~nm})$ are as in the previous case ( $[\mathrm{FH}]$, proof of Proposition 20.20), but in the restriction to $s o(m)^{n}$ there are $2^{n^{\prime}}$ weights which map to the same sum of weights. The case $n$ is even is similar.
5.6 In the remainder of this section we discuss two examples of Kuga Satake varieties of K3 type Hodge structures with real multiplication. In the first example we have $n=[F: \mathbf{Q}]=$ $2, m=\operatorname{dim}_{F} V=3$ and we assume moreover that $C^{+}(\psi) \cong M_{4}(E)$, the algebra of $4 \times 4$ matrices with coefficients in an imaginary quadratic field $E$ (cf. [ L], vG1], 9.2). In the other case we take $n=[F: \mathbf{Q}]=3, m=\operatorname{dim}_{F} V=3$.
5.7 Proposition. Let $(V, h, \psi)$ be a K3 type Hodge structure with $\operatorname{End}_{H o d}(V) \cong F$, a real quadratic field and with $d=\operatorname{dim} V=6$. Assume that $C^{+}(\psi) \cong M_{4}(E)$ for an imaginary quadratic field $E$.

Then the Kuga Satake variety $A$ of $V$ is an abelian variety of dimension 16,

$$
A \approx B^{4}, \quad D:=\operatorname{End}(B)_{\mathbf{Q}}, \quad N S(B) \cong \mathbf{Z}
$$

where $B$ is a simple abelian fourfold and $D$ is a definite quaternion algebra over $\mathbf{Q}$ which contains $E$. There are three copies of the Hodge structure $V$ in $H^{2}(B, \mathbf{Q})$ :

$$
H^{2}(B, \mathbf{Q}) \cong V^{3} \oplus W, \quad H o m_{H o d}(V, W)=0
$$

Proof. As $C^{+}(\psi) \subset \operatorname{End}(A)_{\mathbf{Q}}$, we get

$$
\operatorname{End}(A)_{\mathbf{Q}} \cong M_{4}(E), \quad \text { hence } \quad A \cong B^{4}, \quad \operatorname{End}(B)_{\mathbf{Q}}=E, \quad H^{1}(B, \mathbf{C}) \cong S(6)
$$

where $B$ is an abelian variety of dimension 4 and we use the isomorphism of so(6)representations $C^{+}(Q)_{\mathbf{C}} \cong S(6)^{4}$ (VG1], Example 6.6).

As $V$ has real multiplication by the real quadratic field $F$, the Lie algebra of $H d g(V)_{\mathbf{C}}$ is the subalgebra $s o(3)^{2}$ of $s o(6)$. Proposition 5.5 shows that the restriction of the Spin representation $S(6)$ of $s o(6)$ to $s o(3)^{2}$ is $(S(3) \boxtimes S(3))^{2}$. The Spin representation $S(3)$ of $s o(3) \cong s l(2)$ is wellknown to be the standard two dimensional representation $V_{1}$ of $s l(2)$. We write $V_{n}$ for the irreducible representation of highest weight $n$ of $s l(2)$, note that $\operatorname{dim} V_{n}=n+1$. Thus we have an isomorphism of $s l(2)^{2}$-representations:

$$
H^{1}(B, \mathbf{C}) \cong\left(V_{1} \boxtimes V_{1}\right)^{2}
$$

Next we compute $N S(B)_{\mathbf{C}}=\left(\wedge^{2} H^{1}(B, \mathbf{C})\right)^{H d g(B)}$, using that $s l(2)^{2}$ is the Lie algebra of $\operatorname{Hod}(B)_{\mathbf{C}}$. Note the following isomorphisms of $s l(2) \times s l(2)$-representations

$$
\begin{aligned}
\wedge^{2} H^{1}(B, \mathbf{C}) & \cong \wedge^{2}\left(\left(V_{1} \boxtimes V_{1}\right)^{2}\right) \\
& \cong\left(\wedge^{2}\left(V_{1} \boxtimes V_{1}\right)\right)^{2} \oplus\left(V_{1} \boxtimes V_{1}\right) \otimes\left(V_{1} \boxtimes V_{1}\right) \\
& \cong\left(\wedge^{2}\left(V_{1} \boxtimes V_{1}\right)\right)^{3} \oplus \operatorname{Sym}^{2}\left(V_{1} \boxtimes V_{1}\right) .
\end{aligned}
$$

It is not hard to check (using weights for example) that

$$
\operatorname{Sym}^{2}\left(V_{1} \boxtimes V_{1}\right) \cong V_{0} \boxtimes V_{0} \oplus V_{2} \boxtimes V_{2}, \quad \wedge^{2}\left(V_{1} \boxtimes V_{1}\right) \cong V_{2} \boxtimes V_{0} \oplus V_{0} \boxtimes V_{2},
$$

note that $V_{0}$ is the trivial representation of $s l(2)$. In particular, there is a unique invariant in $H^{2}(B)$, so $N S(B) \cong \mathbf{Z}$ and hence $B$ is simple.

As $H^{1}(B, \mathbf{C})$ is the direct sum of two copies of an irreducible $H d g(B)_{\mathbf{C}}$-representation, it follows that

$$
\operatorname{End}(B)_{\mathbf{C}}=\operatorname{End}_{H d g(B)}\left(H^{1}(B, \mathbf{C})\right) \cong M_{2}(\mathbf{C})
$$

thus $D=\operatorname{End}(B)_{\mathbf{Q}}$ is a (non-commutative) division algebra of degree four over $\mathbf{Q}$, which contains the imaginary quadratic field $E$. To see that $D$ is definite, we must to show that $D_{\mathbf{R}}$ is not isomorphic to $M_{2}(\mathbf{R})$. As elements of $D_{\mathbf{R}}$ are endomorphisms of $H^{1}(B, \mathbf{R})$ which commute with $H d g(B)(\mathbf{R})$, it suffices to show that $H^{1}(B, \mathbf{R})$ is an irreducible representation of $H d g(B)(\mathbf{R})$. From [2.8] we know that $H d g(V)(\mathbf{R}) \cong S O(2,1) \times S O(3, \mathbf{R})$. The Spin group is then $S L(2, \mathbf{R}) \times S U(2)$ and $H^{1}(B, \mathbf{R})$ is the $\boxtimes$-product of the standard two dimensional representation of $S L(2, \mathbf{R})$ and the standard two dimensional complex representation of $S U(2)$, which is an eight dimensional representation, irreducible over $\mathbf{R}$.

The Hodge structure $V$ corresponds to the $H d g(B)$-representation with complexification:

$$
V_{\mathbf{C}}=V_{2} \boxtimes V_{0} \oplus V_{0} \boxtimes V_{2},
$$

thus $V^{3}$ is a Hodge substructure of $H^{2}(B, \mathbf{Q})$.
5.8 The case $m=n=[F: \mathbf{Q}]=3$. For a general $h\left(\right.$ so $\left.\operatorname{End}_{H o d}(V)=\mathbf{Q}\right)$ we have the following decomposition, up to isogeny, of the Kuga Satake variety $A$ of $V$ :

$$
A \approx B^{8}, \quad \operatorname{dim} B=16, \quad \operatorname{End}(B)_{\mathbf{Q}} \cong D, \quad H^{1}(B, \mathbf{C}) \cong S(9)^{2}
$$

where $D$ is a quaternion algebra (cf. vG1, Proposition 7.7) and we used the isomorphism of so(9)-representations $C^{+}(\psi)_{\mathbf{C}} \cong S(9)^{16}$. In case $D \cong M_{2}(\mathbf{Q}), B$ is isogenous to $B_{1}^{2}$ for an abelian eightfold $B_{1}$.

Assume now that $\operatorname{End}_{H o d}(V)=F$, a totally real cubic extension of $\mathbf{Q}$, so $H d g(V)(\mathbf{C}) \cong$ $S O(3, \mathbf{C})^{3}$. Then the Lie algebra of the complex Hodge group of the Kuga Satake variety reduces from $s o(9)$ to $s o(3)^{3} \cong s l(2)^{3}$ and the Spin representation $S(9)$ of $s o(9)$ restricts to

2 copies of an 8 dimensional irreducible representation (cf. Proposition 5.5. Ga, Prop. 4.9, notation as in the proof of Proposition 5.7)

$$
S(9)_{\mid s l(2)^{3}} \cong W^{2}, \quad W:=V_{1} \boxtimes V_{1} \boxtimes V_{1}
$$

In particular, $E n d_{H o d}\left(H^{1}(B, \mathbf{Q})\right)$ is a Q-algebra of rank $4 \cdot 4=16$.
Assume that $E n d_{H o d}\left(H^{1}(B, \mathbf{Q})\right) \cong M_{4}(\mathbf{Q})$. Then we have an isogeny

$$
B \approx B_{2}^{4}, \quad \operatorname{dim} B_{2}=4, \quad \operatorname{End}\left(B_{2}\right)_{\mathbf{Q}}=\mathbf{Q}
$$

and $H^{1}\left(B_{2}, \mathbf{C}\right) \cong W$. This case is discussed in 6.4. The K3 type Hodge structure $V$ is not a Hodge substructure of $H^{2}\left(B_{2}, \mathbf{Q}\right)$, but $H^{2}\left(B_{2}^{2}, \mathbf{Q}\right) \cong V \oplus V^{\prime}$ with $H o m_{H o d}\left(V, V^{\prime}\right)=0$ ( Ga , proof of Thm 3.4).

## 6 The Kuga Satake variety and corestriction of algebras

6.1 In this section we use the corestriction of algebras to describe the first cohomology group of the Kuga Satake variety as $H d g(V)$-representation where $V$ is a K3 type Hodge structure with real multiplication. In contrast to the previous section, where we restricted the complex Spin representation to $H d g(V)$, we now obtain a direct construction over the rational numbers. This construction generalizes the results of Galluzzi, cf. 6.45 .8 and Ga, which show that certain abelian varieties constructed by Mumford are Kuga Satake varieties.
6.2 The corestriction. We use the notation from 2.4 $\tilde{F}$ is the Galois closure of a finite extension $F$ of $\mathbf{Q}, H=\operatorname{Gal}(\tilde{F} / F) \subset G=\operatorname{Gal}(\tilde{F} / \mathbf{Q})$ and $n=[F: \mathbf{Q}]=[G: H]$. A coset $g H \in G / H$ gives a well-defined embedding $F \hookrightarrow \tilde{F}, a \mapsto g(a)$ and thus defines an $F$-algebra structure on $\tilde{F}$, this $F$-algebra is denoted by $\tilde{F}_{g}$.

For an $F$-algebra $R$ and a coset $g H \in G / H$ the twisted algebra $R_{g}=R_{g H}$ is defined to be (cf. Mum, cf. [Sc], 8.8, [R], 4.4)

$$
R_{g}:=R \otimes_{F} \tilde{F}_{g} \quad \text { so } \quad \text { ar } \otimes 1=r \otimes g(a) \quad(a \in F, r \in R)
$$

For $g \in G$ we have natural $\mathbf{Q}$-linear maps:

$$
g: R_{g^{\prime}} \longrightarrow R_{g g^{\prime}}, \quad r \otimes a \longmapsto r \otimes g(a) .
$$

To see that the map is well-defined, write $a=g^{\prime}(b)$, then

$$
r \otimes a=b r \otimes 1 \longmapsto r \otimes g(a)=r \otimes\left(g g^{\prime}\right)(b)=b r \otimes 1
$$

Thus we get an action of $G$ on

$$
Z_{R}:=R_{g_{1}} \otimes_{\tilde{F}} \ldots \otimes_{\tilde{F}} R_{g_{n}}, \quad \text { with } \quad G / H=\left\{g_{1} H, \ldots, g_{n} H\right\} .
$$

The corestriction of the $F$-algebra $R$ is the $\mathbf{Q}$-algebra ( Mum , cf. [Sc, 8.9, R , Thm 11, 5.5) of $G$-invariants in $Z_{R}$ :

$$
\operatorname{cores}_{F / \mathbf{Q}}(R)=Z_{R}^{G} .
$$

One has $\operatorname{cores}_{F / \mathbf{Q}}(R) \otimes_{\mathbf{Q}} \tilde{F} \cong Z_{R}$ and $\operatorname{dim}_{\mathbf{Q}} \operatorname{cores}_{F / \mathbf{Q}}(R)=\left(\operatorname{dim}_{F} R\right)^{n}$.
Let $R^{*}$ be the multiplicative group of invertible elements of $R$. Then there is a natural diagonal homomorphism:

$$
R^{*} \longrightarrow\left(\operatorname{cores}_{F / \mathbf{Q}}(R)\right)^{*}, \quad u \longmapsto u \otimes u \ldots \otimes u
$$

where $\left(\operatorname{cores}_{F / \mathbf{Q}}(R)\right)^{*}$ is the multiplicative group of units in the $\mathbf{Q}$-algebra $\operatorname{cores}_{F / \mathbf{Q}}(R)$.
6.3 Proposition. Let $(V, h, \psi=\operatorname{tr}(\Phi))$ be a K3 type Hodge structure with $F=\operatorname{End}_{H o d}(V)$ a totally real field. Let $C^{+}(\psi)$ be the Kuga Satake Hodge structure associated to $V$ and let $C_{F}^{+}(\Phi)$ be the even Clifford algebra, over $F$, of the $F$-bilinear form $\Phi: V \times V \rightarrow F$.

Then $H d g\left(C^{+}(\psi)\right)$ is a subgroup of $\left(\operatorname{cores}_{F / \mathbf{Q}} C_{F}^{+}(\Phi)\right)^{*}$ and there is an injective map of $H d g\left(C^{+}(\psi)\right)\left(\subset C \operatorname{Spin}_{F}^{+}(\Phi)\right)$ representations:

$$
\operatorname{cores}_{F / \mathbf{Q}} C_{F}^{+}(\Phi) \hookrightarrow C^{+}(\psi)
$$

Proof. We first extend scalars from $\mathbf{Q}$ to $\tilde{F}$. The Clifford algebra $C(\psi)$ of $\psi$ is the quotient of the tensor algebra of $V$ by the two-sided ideal generated by the $v \otimes v-\psi(v, v)$ for $v \in$ $V$. Extending scalars, we get an isomorphism $C(\psi)_{\tilde{F}} \cong C_{\tilde{F}}\left(\psi_{\tilde{F}}\right)$ where $\psi_{\tilde{F}}$ is the $\tilde{F}$-bilinear extension of $\psi$ to $V_{\tilde{F}} \times V_{\tilde{F}}$.

There is a direct sum decomposition of spaces with bilinear forms over $\tilde{F}$ (cf. the proof of Lemma (2.6):

$$
\left(V_{\tilde{F}}, \psi_{\tilde{F}}\right)=\oplus_{g \in G / H}\left(V_{g}, \psi_{g}\right) .
$$

This direct sum decomposition gives an isomorphism (cf. [Sc] 9.2.5):

$$
C_{\tilde{F}}\left(\psi_{\tilde{F}}\right)=C_{\tilde{F}}\left(\psi_{g_{1}}\right) \hat{\otimes}_{\tilde{F}} \ldots \hat{\otimes}_{\tilde{F}} C_{\tilde{F}}\left(\psi_{g_{n}}\right) \quad\left(G / H=\left\{H=g_{1} H, \ldots, g_{n} H\right\}\right)
$$

where $\hat{\otimes}$ is a graded tensor product. It is easy to see that $C_{\tilde{F}}^{+}\left(\psi_{g}\right) \cong C_{\tilde{F}}^{+}\left(\psi_{e}\right)_{g}$, as $F$-algebras. The weighted tensor product $\hat{\otimes}$ is the usual tensor product on the 'all even' part. So we have

$$
C_{\tilde{F}}^{+}\left(\psi_{g_{1}}\right) \otimes_{\tilde{F}} \ldots \otimes_{\tilde{F}} C_{\tilde{F}}^{+}\left(\psi_{g_{n}}\right) \cong C_{\tilde{F}}^{+}\left(\psi_{e}\right) \otimes_{\tilde{F}} \ldots \otimes_{\tilde{F}} C_{\tilde{F}}^{+}\left(\psi_{e}\right)_{g_{n}} \hookrightarrow C_{\tilde{F}}^{+}\left(\psi_{\tilde{F}}\right)
$$

As $V_{e}=V \otimes_{F} \tilde{F}_{e}$ and $\psi_{e}$ is the $\tilde{F}$-linear extension of $\Phi$ to $V_{e}$, we get

$$
C_{\tilde{F}}^{+}\left(\psi_{e}\right) \cong C_{F}^{+}(\Phi) \otimes_{F} \tilde{F}_{e}, \quad \text { hence } \quad Z_{C_{F}^{+}(\Phi)} \hookrightarrow C_{\tilde{F}}^{+}\left(\psi_{\tilde{F}}\right)=C^{+}(\psi)_{\tilde{F}}
$$

Taking $G$-invariants one finds that $\operatorname{cores}_{F / \mathbf{Q}}\left(C_{F}^{+}(\Phi)\right) \hookrightarrow C^{+}(\psi)$.

The Hodge group of the weight one Hodge structure $C^{+}(\psi)$ is an algebraic subgroup of $C \operatorname{Spin}(\psi) \subset C^{+}(\psi)^{*}$ and acts, by multiplication, on $C^{+}(\psi)$ and $C^{+}(\psi)_{\tilde{F}}: u \cdot c:=u c$ for $u \in C^{+}(\psi)^{*}, c \in C^{+}(\psi)$ or $C^{+}(\psi)_{\tilde{F}}$. Under the natural homomorphism $C \operatorname{Spin}(\psi) \rightarrow G L(V)$, $H d g\left(C^{+}(\psi)\right.$ maps onto $H d g(V)=S O(V, \Phi) \subset S O(V)$. In particular, $H d g\left(C^{+}(\psi)\right) \hookrightarrow$ $\operatorname{CSpin}_{F}(\Phi)$. The group $C \operatorname{Spin}_{F}(\Phi)$ is a subgroup of $C_{F}^{+}(\Phi)^{*}$ which again acts by multiplication on $C_{F}^{+}(\Phi)$. Upon extending scalars to $\tilde{F}$, it acts on $C_{\tilde{F}}^{+}\left(Q_{e}\right) \cong C_{F}^{+}(\Phi) \otimes_{F} \tilde{F}_{e}$. In particular, $H d g\left(C^{+}(\psi)\right)$ acts diagonally on $Z_{R}$, with $R=C_{F}^{+}(\Phi)$ and this gives the inclusion $H d g\left(C^{+}(\psi)\right) \subset\left(\operatorname{cores}_{F / \mathbf{Q}} C_{F}^{+}(\Phi)\right)^{*}$. This action is the restriction of the action of $H d g\left(C^{+}(\psi)\right)$ on $C^{+}(\psi)_{\tilde{F}}$.
6.4 Example. Let $(V, h, \psi)$ be a $K 3$ type Hodge structure with $E n d_{H o d}(V)=F$ a totally real field with $[F: \mathbf{Q}]=3$ and assume that $\operatorname{dim}_{F} V=3$. Then $\psi=\operatorname{tr}(\Phi)$ and as $\Phi$ is defined on a three dimensional $F$-vector space, $C_{F}^{+}(\Phi)$ is a quaternion algebra $D$ over $F$ (cf. vG1, 7.5).

As $\Phi_{\sigma}$ is indefinite for one embedding and positive definite for the other two embeddings $\sigma: F \hookrightarrow \mathbf{R}$ (cf. the proof of (2.8), the algebra $D$ splits for one embedding of $F \hookrightarrow \mathbf{R}$ and is isomorphic to the quaternions for the other two embeddings, so $D_{\mathbf{R}} \cong M_{2}(\mathbf{R}) \times \mathbf{H} \times \mathbf{H}$.

Conversely, a quaternion algebra defines a quadratic space $\left(D_{0}, N\right)$ over $F$, with $D_{0}$ the subspace of $D$ of elements with trace zero and $N: D_{0} \rightarrow F$ the restriction of the norm on $D$ to $D_{0}$. If $D_{\mathbf{R}}$ is as above, then one can define K3 type Hodge structures on $V=D_{0}$ with endomorphism algebra $F$ (cf. the proof of Lemma (3.2).

Assume that $\operatorname{cores}_{F / \mathbf{Q}}(D) \cong M_{8}(\mathbf{Q})$. Then Proposition 6.3 shows that $H d g\left(C^{+}(\psi)\right)$ is a subgroup of $G L(8, \mathbf{Q})$. Thus the Kuga Satake variety of $V$ has a four dimensional abelian subvariety. Actually, cf. Mum, Ga,

$$
H d g(A)=\operatorname{ker}\left(N: D^{*} \rightarrow F^{*}\right), \quad A \approx B_{2}^{32}
$$

with $B_{2}$ a four dimensional abelian variety.
Mumford Mum discovered these abelian varieties and he showed that $\operatorname{End}\left(B_{2}\right)_{\mathbf{Q}} \cong \mathbf{Q}$ but that nevertheless $H d g\left(B_{2}\right) \neq S p(8, \mathbf{Q})$. The relation with Kuga Satake varieties was established in Ga.

## 7 Predictions from the Hodge conjecture

7.1 The Hodge conjecture. The rational cohomology groups $H^{k}(X, \mathbf{Q})$ of a smooth projective variety $X$ have a (polarized) rational Hodge structure of weight $k$. The Hodge conjecture asserts that the space of codimension $p$ Hodge classes

$$
B^{p}(X):=H^{2 p}(X, \mathbf{Q}) \cap H^{p, p}(X)
$$

is spanned by classes of algebraic cycles. The conjecture is still very much open for $p \neq$ $0,1, \operatorname{dim} X-1, \operatorname{dim} X$.
7.2 Hodge classes on a product. Let $X, Y$ be smooth projective varieties. The Künneth formula and Poincarè duality $H^{k}(X, \mathbf{Q}) \cong H^{2 d_{X}-k}(X, \mathbf{Q})^{d u a l}$ imply that:

$$
H^{k}(X \times Y, \mathbf{Q}) \cong \oplus_{l+m=k} H^{l}(X, \mathbf{Q}) \otimes H^{m}(Y, \mathbf{Q}) \cong \oplus_{l+m=k} \operatorname{Hom}\left(H^{2 d_{X}-l}(X, \mathbf{Q}), H^{m}(Y, \mathbf{Q})\right)
$$

The summands $H^{l}(X, \mathbf{Q}) \otimes H^{m}(Y, \mathbf{Q})$ are Hodge substructures of $H^{k}(X \times Y, \mathbf{Q})$. The Hodge cycles in this summand are exactly the homomorphisms of Hodge structures:

$$
B^{p}(X \times Y) \cap \operatorname{Hom}\left(H^{2 d_{X}-l}(X, \mathbf{Q}), H^{m}(Y, \mathbf{Q})\right)=\operatorname{Hom}_{H o d}\left(H^{2 d_{X}-l}(X, \mathbf{Q}), H^{m}(Y, \mathbf{Q})\right)
$$

where $2 p=l+m$.
7.3 Products of K3 surfaces. Let $X=S_{1}, Y=S_{2}$ be (algebraic) K3 surfaces. We consider the Hodge classes in $H^{4}\left(S_{1} \times S_{2}\right)$. Note that $H^{1}(S)=H^{3}(S)=0$ for a K3 surface $S$. The summands $\operatorname{Hom}_{H o d}\left(H^{0}\left(S_{1}\right), H^{0}\left(S_{2}\right)\right), \operatorname{Hom}_{H o d}\left(H^{4}\left(S_{1}\right), H^{4}\left(S_{2}\right)\right)$ are obviously spanned by the classes of $\{p t\} \times S_{2}$ and $S_{1} \times\{p t\}$ respectively.

Recall that the Hodge structure on $H^{2}$ splits (cf. (1.8)

$$
H^{2}\left(S_{i}, \mathbf{Q}\right)=N S\left(S_{i}\right)_{\mathbf{Q}} \oplus T_{S_{i}, \mathbf{Q}}
$$

and, as $N S\left(S_{i}\right)_{\mathbf{Q}}^{2,0}=0$ and $T_{S_{i}, \mathbf{Q}}$ is simple,

$$
\operatorname{Hom}_{H o d}\left(N S\left(S_{1}\right)_{\mathbf{Q}}, T_{S_{2}, \mathbf{Q}}\right)=0, \quad \operatorname{Hom}_{H o d}\left(T_{S_{1}, \mathbf{Q}}, N S\left(S_{2}\right)_{\mathbf{Q}}\right)=0
$$

The vector space $\operatorname{Hom}_{H o d}\left(N S\left(S_{1}\right)_{\mathbf{Q}}, N S\left(S_{2}\right)_{\mathbf{Q}}\right)$ is spanned by classes of products of curves $C_{1} \times C_{2} \subset S_{1} \times S_{2}$. Thus there remains the summand

$$
\operatorname{Hom}_{H o d}\left(T_{S_{1}, \mathbf{Q}}, T_{S_{2}, \mathbf{Q}}\right) .
$$

7.4 Hodge isometries. Let $S$ be a K3 surface. Then the Hodge structure $T_{S, \mathbf{Q}}$ comes with a polarization $\psi_{S}$, induced by the cup product on $H^{2}(S)$. A homomorphism of Hodge structures

$$
f \in \operatorname{Hom}_{H o d}\left(T_{S_{1}, \mathbf{Q}}, T_{S_{2}, \mathbf{Q}}\right), \quad \text { such that } \quad \psi_{S_{2}}(f(v), f(w))=\psi_{S_{1}}(v, w)
$$

for all $v, w \in T_{S_{1}, \mathbf{Q}}$ is called a Hodge isometry.
Mukai has announced that if $f \in \operatorname{Hom}_{H o d}\left(T_{S_{1}, \mathbf{Q}}, T_{S_{2}, \mathbf{Q}}\right)$ is a Hodge isometry, then $f$ is the class of an algebraic cycle on $S_{1} \times S_{2}$ (Muk, Theorem 2). Under certain conditions on the dimension of $T_{S_{1}, \mathbf{Q}}$ proofs were given earlier by Mukai and Nikulin (cf. [Muk], section 4). This solves the Hodge conjecture for Hodge isometries, but below we recall that there is still a lot to do.
7.5 Complex multiplication. In case $S_{1}=S_{2}$ and $E n d_{H o d}\left(T_{S_{1}, \mathbf{Q}}\right)$ is a CM field, Ramon Mari Ma] showed that $E n d_{H o d}\left(T_{S_{1}, \mathbf{Q}}\right)$ is spanned by Hodge isometries. Thus, by Mukai's results, any $f \in \operatorname{End}_{H o d}\left(T_{S_{1}, \mathbf{Q}}\right)$ is the class of an algebraic cycle.
7.6 Real multiplication. In case $S_{1}=S_{2}$, the rational multiples of the identity can be obtained from the projection to $E n d_{H o d}\left(T_{S_{1}, \mathbf{Q}}\right)$ of rational multiples of the class of the diagonal in $S_{1} \times S_{1}$.

However, if $E n d_{H o d}\left(T_{S_{1}, \mathbf{Q}}\right)$ is a totally real field, distinct from $\mathbf{Q}$, I do not know of any example where a non-trivial endomorphism is represented by an algebraic cycle.
7.7 Scaling the polarization. Let $\left(T_{S, \mathbf{Q}}, \psi_{S}\right)$ be the polarized Hodge structure defined by a K3 surface $S$. For any positive integer $n$, there is the polarized Hodge structure of K3 type $\left(T_{S, \mathbf{Q}}, n \psi_{S}\right)$.

In general, these Hodge structures are not Hodge isometric. For example, if $d=\operatorname{dim} T_{S, \mathbf{Q}}$ is odd, and $n$ is not a square, then as $\operatorname{det}\left(n \psi_{S}\right)=n^{d} \operatorname{det}\left(\psi_{S}\right)$, we get an obstruction to the existence of isometries (cf. 4.4).

This construction is a special case of the real twist of 4.7, $n \psi=\psi_{n}$ for $a=n \in \mathbf{Z}$. Thus for $n \leq 11$ there exists a K3 surface $S_{n}$ such that its transcendental lattice $\left(T_{S_{n}}, \psi_{S_{n}}\right)$ is isometric to $\left(T_{S}, n \psi_{S}\right)$. In particular, the identity on $T_{S}$ gives a non-trivial element in $\operatorname{Hom}_{H o d}\left(T_{S, \mathbf{Q}}, T_{S_{n}, \mathbf{Q}}\right)$.

In case $n=2$, some of these homomorphisms of Hodge structures can be shown to be the classes of algebraic cycles by using Nikulin involutions of K3 surfaces, cf. GL, vGS. In general, it seems to be an interesting open problem to find such algebraic cycles.
7.8 Twisting the polarization. Let $S$ be a K3 surface with $E n d_{H o d}\left(T_{S, \mathbf{Q}}\right)=F$ a totally real field. Any totally positive $a \in F$ defines a polarization $\left(\psi_{S}\right)_{a}$ on $T_{S, \mathbf{Q}}$ (cf. 4.2). In case $F=\mathbf{Q}$ we recover the scaling operation described above. In general, the polarized Hodge structures $\left(T_{S, \mathbf{Q}}, \psi_{S}\right)$ and $\left(T_{S, \mathbf{Q}},\left(\psi_{S}\right)_{a}\right)$ are not isometric.

If $d=\operatorname{dim} T_{S, \mathbf{Q}} \leq 11$, there exists a K3 surface $S_{a}$ such that $\left(T_{S_{a}}, \psi_{S_{a}}\right)$ is Hodge isometric to $\left(T_{S},\left(\psi_{S}\right)_{a}\right)$, cf. 4.7. The identity map $T_{S, \mathbf{Q}} \rightarrow T_{S, \mathbf{Q}}=T_{S_{a}, \mathbf{Q}}$ is a homomorphism of Hodge structures and, again, it seems to be an interesting open problem to show that is induced by an algebraic cycle.
7.9 Remark. Let $S_{1}, S_{2}$ be K3 surfaces. Let $Z$ be a smooth surface, with maps

$$
S_{1} \stackrel{\pi_{1}}{\longleftrightarrow} Z \xrightarrow{\pi_{1}} S_{2} \quad \text { let } \quad f=\pi_{2 *} \pi_{1}^{*} \in \operatorname{Hom}_{H o d}\left(T_{S_{1}, \mathbf{Q}}, T_{S_{2}, \mathbf{Q}}\right)
$$

be the homomorphism of Hodge structures defined by the class of $\left(\pi_{1} \times \pi_{2}\right)(Z) \subset S_{1} \times S_{2}$. Assume that $f \neq 0$. Let $V_{i}:=T_{S_{i}, \mathbf{Q}}$, as the $V_{i}$ are simple and $f \neq 0$ we get an isomorphism of rational Hodge structures $f: V_{1} \xrightarrow{\cong} V_{2}$.

The map $f$ is not necessarily an isometry $\left(V_{1}, \psi_{1}\right) \rightarrow\left(V_{2}, \psi_{2}\right)$, where $\psi_{i}:=\psi_{S_{i}}$, the polarization induced by cupproduct on the surface $S_{i}$. In fact, $f=\pi_{2 *} \pi_{1}^{*}$ and as $\pi_{1}^{*}: V_{1} \hookrightarrow H^{2}(Z, \mathbf{Q})$ is compatible with the cupproduct:

$$
\psi_{Z}\left(\pi_{1}^{*} x_{1}, \pi_{1}^{*} y_{1}\right):=\pi_{1}^{*} x_{1} \cup \pi_{1}^{*} y_{1}=\pi_{1}^{*}\left(x_{1} \cup y_{1}\right)=d_{1}\left(x_{1} \cup y_{1}\right)=d_{1} \psi_{1}\left(x_{1}, y_{1}\right)
$$

where $d_{1}$ is the degree of $\pi_{1}$ (that is, the cardinality of a general fiber). So we have Hodge
isometries

$$
\left(V_{1}, \psi_{1}\right) \longrightarrow\left(V_{1}, d_{1} \psi_{1}\right) \hookrightarrow\left(H^{2}(Z, \mathbf{Q}), \psi_{Z}\right)
$$

The map $\pi_{2 *}$ is not compatible with cupproducts, but the projection formula gives:

$$
\psi_{2}\left(x_{2}, \pi_{2 *} y_{z}\right)=\psi_{Z}\left(\pi_{2}^{*} x_{2}, y_{z}\right) \quad\left(x_{2} \in H^{2}\left(S_{2}, \mathbf{Q}\right), y_{z} \in H^{2}(Z, \mathbf{Q})\right)
$$

Assume now that $H^{2}(Z, \mathbf{Q})=\pi_{1}^{*}\left(V_{1}\right) \oplus W$ with $\operatorname{Hom}_{H o d}\left(V_{1}, W\right)=0$, so there is a unique copy of the Hodge structure $V_{1} \cong V_{2}$ in $H^{2}(Z, \mathbf{Q})$. The composition $\pi_{2 *} \pi_{2}^{*}$ is multiplication by $d_{2}$, the degree of the map $\pi_{2}$, on $H^{2}\left(S_{2}, \mathbf{Q}\right)$. Thus the map $\pi_{2}^{*}: V_{2} \cong \pi_{1}^{*}\left(V_{1}\right)$ is an isomorphism and given $x_{z}, y_{z} \in f\left(V_{1}\right) \subset H^{2}(Z, \mathbf{Q})$, there are $x_{2}, y_{2} \in V_{2}$ with $x_{z}=\pi_{2}^{*} x_{2}, y_{z}=\pi_{2}^{*} y_{2}$. Therefore:

$$
\psi_{2}\left(\pi_{2 *} x_{z}, \pi_{2 *} y_{z}\right)=\psi_{2}\left(\pi_{2 *} \pi_{2}^{*} x_{2}, \pi_{2 *} \pi_{2}^{*} y_{2}\right)=d_{2}^{2} \psi_{2}\left(x_{2}, y_{2}\right)=d_{2} \psi_{Z}\left(\pi_{2}^{*} x_{2}, \pi_{2}^{*} y_{2}\right)=d_{2} \psi_{Z}\left(x_{z}, y_{z}\right)
$$

In particular, the isomorphism $f: V_{1} \rightarrow V_{2}$ induces a scaling on the polarizations:

$$
\psi_{2}\left(f x_{1}, f x_{2}\right)=d_{1} d_{2} \psi_{1}\left(x_{1}, x_{2}\right)
$$

In general, given a codimension two cycle $\sum n_{i} Z_{i}$ on $S_{1} \times S_{2}$, after replacing each surface $Z_{i}$ by its desingularization (which maps to $S_{1} \times S_{2}$ with image $Z_{i}$ ), the homomorphism of Hodge structures induced by $\sum n_{i} Z_{i}$ is a linear combination of maps as above. Thus to get an 'interesting' map $f: V_{1} \rightarrow V_{2}$, induced by an algebraic cycle, one needs surfaces $Z_{i}$ whose $H^{2}$ contains more than one copy of $V$.
7.10 The Kuga Satake Hodge conjecture. The Kuga Satake variety $A$ of a K3 type Hodge structure $(V, h, \psi)$ has the property that there is a homomorphism of Hodge structures $V \hookrightarrow H^{2}\left(A^{2}, \mathbf{Q}\right)\left(\mathrm{cf}\right.$. vG1], 6.3.3). In particular, if $V \cong T_{S, \mathbf{Q}} \subset H^{2}(S, \mathbf{Q})$ for a K3 surface $S$, then the Hodge conjecture predicts the existence of a cycle $Z$ on $S \times A \times A$ which induces an isomorphism from the copy of $V$ in $H^{2}(S)$ to the one in $H^{2}\left(A^{2}\right)$.

If there is such a cycle $Z$ and if $E n d_{H o d}\left(T_{S, \mathbf{Q}}\right)$ is also generated by algebraic cycles then any homomorphism of Hodge structures $T_{S, \mathbf{Q}} \rightarrow V \subset H^{2}\left(A^{2}, \mathbf{Q}\right)$ is represented by an algebraic cycle on $S \times A^{2}$.

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