THE TANGENT SPACE TO THE MODULI SPACE OF VECTOR BUNDLES ON A CURVE AND THE SINGULAR LOCUS OF THE THETA DIVISOR OF THE JACOBIAN

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Abstract. We complete the proof of the fact that the moduli space of rank two bundles with trivial determinant embeds into the linear system of divisors on $Pic^{g-1}C$ which are linearly equivalent to 2Θ. The embedded tangent space at a semi-stable non-stable bundle $\xi \oplus \xi^{-1}$, where ξ is a degree zero line bundle, is shown to consist of those divisors in $|2\Theta|$ which contain $Sing(\Theta_\xi)$ where Θ_ξ is the translate of Θ by ξ. We also obtain geometrical results on the structure of this tangent space.

Introduction

Let C be a smooth complete irreducible algebraic curve of genus g, with $g \geq 2$ over the field $\mathbb{C}$ of complex numbers. For an integer d, let $Pic^d C$ be the variety parametrizing line bundles of degree d on C. Then $Pic^{g-1}C$ has a natural theta divisor which is the reduced divisor with underlying set

$$\Theta := \{L \in Pic^{g-1}C : \ h^0(L) > 0\}\,.$$

For a line bundle L on C, let $\mathcal{M}_L$ be the moduli space of semi-stable vector bundles of rank 2 and determinant L on C. Put $\mathcal{O} := \mathcal{O}_C$. The variety $\mathcal{M}_{\mathcal{O}}$ is a normal and Cohen-Macaulay projective variety of dimension $3g-3$ with only rational singularities. Its singular locus is the set of (semi-stable equivalence classes of) semi-stable non-stable bundles, i.e., bundles of the form $\xi \oplus \xi^{-1}$ with $\xi \in Pic^0C$.

The Picard group of $\mathcal{M}_{\mathcal{O}}$ is isomorphic to $\mathbb{Z}$ and we let $\mathcal{L}$ be its ample generator. The dimensions of the vector spaces $H^0(\mathcal{M}_{\mathcal{O}}, \mathcal{L}^{\otimes k})$ are given by the celebrated Verlinde formula. The natural map $\mathcal{M}_{\mathcal{O}} \to \mathbb{P}H^0(\mathcal{M}_{\mathcal{O}}, \mathcal{L})^*$ coincides with the morphism

$$\Delta : \mathcal{M}_{\mathcal{O}} \longrightarrow |2\Theta|$$

which to a semi-stable vector bundle E of rank 2 and trivial determinant associates the divisor D_E on $Pic^{g-1}C$ defined set-theoretically as

$$D_E := \{L \in Pic^{g-1}C : \ h^0(E \otimes L) > 0\}\,.$$

We complete the proof of the following theorem:

Theorem 1. *The morphism Δ is an embedding for any non-hyperelliptic curve of genus $g \geq 4$.*

Narasimhan and Ramanan showed that Δ is an isomorphism for $g = 2$ [NR1] and an embedding for $g = 3$ and C non-hyperelliptic [NR2]. Beauville proved [Bea1] that for C hyperelliptic Δ induces an embedding of the quotient of $\mathcal{M}_{\mathcal{O}}$ by the hyperelliptic involution of C (the hyperelliptic involution induces the identity on $\mathcal{M}_{\mathcal{O}}$ if $g = 2$). He also proved [Bea1] that if C is non-hyperelliptic, then Δ has generic degree 1. Subsequently, Laszlo proved [L2] that Δ is an embedding for C general. Brivio and Verra proved [BV] that Δ is injective and an immersion on the smooth locus of $\mathcal{M}_{\mathcal{O}}$ for any non-hyperelliptic C. The singular locus of $\mathcal{M}_{\mathcal{O}}$ can be identified with the Kummer variety $K^0(C) := Pic^0C/\pm 1$ via the morphism

$$Pic^0C \longrightarrow \mathcal{M}_{\mathcal{O}}, \qquad \xi \longmapsto \xi \oplus \xi^{-1}$$

which induces an embedding of $K^0(C)$. Using [BV], Theorem 1 follows from Theorem 2 which we prove in 2.9 and 6.9:

Theorem 2. *Suppose C is non-hyperelliptic of genus $g \geq 4$. Then, for any $\xi \in Pic^0C$, the morphism Δ is an immersion at $\xi \oplus \xi^{-1}$.*

Since Δ is an embedding, we can obtain geometric information on the tangent spaces to $\mathcal{M}_{\mathcal{O}}$ in terms of divisors in the linear system $|2\Theta|$. Let $\mathbb{T}_\xi$ denote the embedded tangent space to $\Delta(\mathcal{M}_{\mathcal{O}})$ at the image of $\xi \oplus \xi^{-1}$. Then $\mathbb{T}_\xi$ is a projective subspace of $|2\Theta|$ of the same dimension as $T_{\xi\oplus\xi^{-1}}\mathcal{M}_{\mathcal{O}}$. Each point of $\mathbb{T}_\xi$ is a divisor on $Pic^{g-1}C$. The following theorem identifies these divisors.

Theorem 3. *For $\xi \in Pic^0C$, let Θ_ξ be the translate of Θ by ξ. Then the embedded tangent space to $\Delta(\mathcal{M}_{\mathcal{O}})$ at $\Delta(\xi \oplus \xi^{-1})$ is:*

$$\mathbb{T}_\xi = \{D \in |2\Theta| : \; Sing(\Theta_\xi) \subset D \; \}.$$

The equations for $\mathbb{T}_\xi$ are linear and thus are elements of $|2\Theta|^*$. The natural morphism defined by the global sections of 2Θ:

$$h : Pic^{g-1}C \longrightarrow |2\Theta|^*, \qquad L \longmapsto H_L := \{D \in |2\Theta| : \; L \in D\} \quad \subset |2\Theta|$$

gives a 'natural' supply of candidate equations H_L for $\mathbb{T}_\xi$. Let $\langle Sing(\Theta_\xi)\rangle \subset |2\Theta|^*$ be the span of the image of $Sing(\Theta_\xi)$ by h. An equivalent form of Theorem 3 is (cf. section 1.1):

Theorem 4. *For any $\xi \in Pic^0C$ the embedded tangent space $\mathbb{T}_\xi$ to $\Delta(\mathcal{M}_{\mathcal{O}})$ at $\Delta(\xi\oplus\xi^{-1})$ is the intersection of the hyperplanes H_L where L runs over $Sing(\Theta_\xi)$:*

$$\mathbb{T}_\xi = \cap_{L\in Sing(\Theta_\xi)} H_L \, .$$

The difficult part of the proof is the inclusion '$\supset$' which follows from a dimension computation in section 7.

The space $\mathbb{T}_{\mathcal{O}}$ contains much geometric information about $\mathcal{M}_{\mathcal{O}}$. Using geometrical invariant theory Laszlo [L3] proved

$$T_{\mathcal{O}^{\oplus 2}}\mathcal{M}_{\mathcal{O}} \cong S^2H^1(\mathcal{O}) \oplus \wedge^3 H^1(\mathcal{O}).$$

The subspace $S^2H^1(\mathcal{O})$ of $T_{\mathcal{O}^{\oplus 2}}\mathcal{M}_{\mathcal{O}}$ is the tangent space to the Kummer variety $K^0(C) \subset \mathcal{M}_{\mathcal{O}}$. We define $\mathbb{T}_0$ ($\subset \mathbb{T}$) to be the embedded tangent space to the image of the Kummer variety $K^0(C)$ in $|2\Theta|$. To investigate the $\wedge^3H^1(\mathcal{O})$ quotient of $T_{\mathcal{O}^{\oplus 2}}\mathcal{M}_{\mathcal{O}}$ we construct a family of hyperplanes in $|2\Theta|$ which contain $\mathbb{T}_0$ but do not contain $\mathbb{T}$. These hyperplanes thus correspond to elements in $\mathbb{P}((\wedge^3H^1(\mathcal{O})))^*$.

Let ω be the canonical bundle on C. There is a morphism, similar to Δ (see 1.3): $\Delta_\omega^* : \mathcal{M}_\omega \to |2\Theta|^*$ where $\mathcal{M}_\omega$ is the moduli space of rank two bundles with canonical determinant. In Section 4 we define a natural rational map $\beta : Gr(3, H^0(\omega)) \longrightarrow \mathcal{M}_\omega$, where $Gr(3, H^0(\omega))$ is the grassmannian of 3-planes in $H^0(\omega)$. Composing β with Δ_ω^* we obtain a rational map

$$Gr(3, H^0(\omega)) \xrightarrow{\beta} \mathcal{M}_\omega \xrightarrow{\Delta_\omega^*} |2\Theta|^*, \qquad W \longmapsto H_W.$$

Note that $H^0(\omega)$ and $H^1(\mathcal{O})$ are dual vector spaces so, with abuse of notation, $(\mathbb{T}/\mathbb{T}_0)^* \cong \mathbb{P}\wedge^3 H^0(\omega)$. We prove in 6.11:

Theorem 5. *Let $W \in Gr(3, H^0(\omega))$ be general. Then $\mathbb{T}_0 \subset H_W$. The rational restriction map:*

$$Gr(3, H^0(\omega)) \longrightarrow (\mathbb{T}/\mathbb{T}_0)^*, \qquad W \longmapsto \bar{H}_W := H_W/(H_W \cap \mathbb{T}_0)$$

extends to a morphism which is the composition of the Plücker map $W \mapsto \wedge^3 W$ with a linear isomorphism $\mathbb{P}(\wedge^3H^0(\omega)) \cong (\mathbb{T}/\mathbb{T}_0)^*$.

The proof of the theorem involves the construction of small subvarieties of $\mathcal{M}_{\mathcal{O}}$ which map to points in $\mathbb{T}/\mathbb{T}_0$. We construct such subvarieties, parametrized by p, q, $r \in C$, in section 6 and show that we obtain a rational map $C^{(3)} \to \mathbb{T}/\mathbb{T}_0 = \mathbb{P} \wedge^3 H^1(\mathcal{O})$, which is essentially the map which to $p+q+r$ associates the plane it spans in the canonical space. We then study the intersection of these subvarieties with the hyperplanes H_W to prove Theorem 5 and Theorem 2 in the case that $\xi^{\otimes 2} \cong \mathcal{O}$.

Recently Pauly and Previato constructed bundles in $\mathbb{T} \cap \Delta(\mathcal{M}_{\mathcal{O}})$ which allowed them to give another description of the above subvarieties. They obtain a second proof of the incidence relations in our Proposition 6.5, see [PP]. They also provide generalizations of the space Γ_{00} and relate these to the geometry of $\mathcal{M}_{\mathcal{O}}$.

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1. The geometry of the $|2\Theta|$ system.

Here we collect some facts on the geometry of the linear system $|2\Theta|$ on $Pic^{g-1}(C)$.

1.1. For any subset Y of a projective space $\mathbb{P}$, denote by $Y^{\perp}$ its polar in the dual projective space $\mathbb{P}^*$. Note that the linear subspace in $|2\Theta|$ defined by the H_L's with $L \in Sing(\Theta_\xi)$ is:

$$\langle Sing(\Theta_\xi)\rangle^{\perp} := \cap_{L\in Sing(\Theta_\xi)} H_L = \{D \in |2\Theta| :\ Sing(\Theta_\xi) \subset D\ \},$$

this shows that Theorem 4 is indeed equivalent to Theorem 3.

1.2. **Wirtinger's duality.** The dual of the projective space $|2\Theta|$ has an intrinsic description as a linear system on Pic^0C. This is Wirtinger's duality d_w (cf. [Mu] page 335): a canonical isomorphism

$$d_w : |2\Theta_0| \xrightarrow{\cong} |2\Theta|^*,$$

where Θ_0 is any symmetric theta divisor on Pic^0C. It is characterized by the property that for $\alpha \in Pic^{g-1}C$ one has $d_w(\Theta_L + \Theta_{\omega\otimes L^{-1}}) = \{D \in |2\Theta| :\ L \in D\}$ with $\Theta_L = \{\xi \in Pic^0C :\ h^0(\xi \otimes L) > 0\}$ the translate of Θ by L.

1.3. **Moduli of bundles with canonical determinant.** Recall that $\mathcal{M}_\omega$ is the moduli space of semistable rank 2 bundles with determinant ω. Similar to the morphism Δ, we have a morphism:

$$\Delta_\omega : \mathcal{M}_\omega \longrightarrow |2\Theta_0|, \qquad E \longmapsto D_E := \{L \in Pic^0C :\ h^0(E \otimes L) \neq 0\ \}.$$

Define

$$\Delta^*_\omega : \mathcal{M}_\omega \longrightarrow |2\Theta|^*, \qquad \Delta^*_\omega = d_w\Delta_\omega.$$

Let $e_\omega : Pic^{g-1}C \to \mathcal{M}_\omega$ be the morphism which to a line bundle of degree $g-1$ associates the vector bundle $L \oplus (\omega \otimes L^{-1})$. Then the composition $d_w\Delta_\omega e_\omega : Pic^{g-1}C \to |2\Theta|^*$ is equal to the natural map $h : Pic^{g-1}C \to |2\Theta|^*$.

1.4. Recall that we defined $\mathbb{T}_0$ ($\subset \mathbb{T}$) to be the embedded tangent space to the image of the Kummer variety $K^0(C)$ in $|2\Theta|$. To recall a nice description of the linear equations for $\mathbb{T}_0$ we define Γ_{00} to be the space of global sections of $\mathcal{O}_{Pic^0C}(2\Theta_0)$ which have multiplicity at least 4 at the origin:

$$\mathbb{P}\Gamma_{00} := \{D \in |2\Theta_0| : mult_{\mathcal{O}}(D) \geq 4\} \qquad \subset |2\Theta_0| = |2\Theta|^*.$$

The vector space Γ_{00} has dimension $2^g - g(g+1)/2 - 1$. Also define $C - C$ to be the surface

$$C - C := \{\mathcal{O}(p-q) : \; p, q \in C\} \quad \subset Pic^0C$$

and let $h_0 : Pic^0C \to |2\Theta| = |2\Theta_0|^*$ be the natural map associated to the linear system $|2\Theta_0|$.

1.5. **Lemma.** The span of the image of $C - C \;\subset Pic^0C$ in $|2\Theta|$ is $\mathbb{P}\Gamma_{00}^{\perp}$ which is equal to $\mathbb{T}_0$.

Proof. Let $\delta : C \times C \longrightarrow C - C \subset Pic^0C$ be the difference map and put $\Gamma_0 = \{s \in H^0(Pic^0C, \mathcal{O}(2\Theta_0)) : s(0) = 0\}$. By [W] page 18, the restriction $\delta^* : \Gamma_0 \longrightarrow H^0(C \times C, 2\Theta)$ induces a surjection $\Gamma_0 \longrightarrow S^2H^0(\omega)$ which coincides with the map which to an element of Γ_0 associates the quadratic term of its Taylor expansion at 0. It follows from this that the span of $C - C$ in $|2\Theta|$ is contained in $\mathbb{P}\Gamma_{00}^{\perp}$ and has dimension $g(g+1)/2$. Therefore the span of $C - C$ is equal to $\mathbb{P}\Gamma_{00}^{\perp}$.

To see that $\mathbb{P}\Gamma_{00}^{\perp} = \mathbb{T}_0$ or equivalently $\mathbb{P}\Gamma_{00} = \mathbb{T}_0^{\perp}$, we consider the cotangent space of the Kummer variety at the origin. The maximal ideal of the local ring of the Kummer variety at the origin consists of the (-1)-invariant elements in the maximal ideal at the origin of Pic^0C. The degree d part of the Taylor series of a regular function at $\mathcal{O}$ is canonically identified with an element of $S^dH^0(\omega)$. Therefore the cotangent space at the origin of the Kummer variety is canonically identified with $S^2H^0(\omega)$ and thus has dimension $g(g+1)/2$. Since the map $r : \Gamma_0 \to S^2H^0(\omega)$ is surjective, the differential of the map $K^0(C) \to |2\Theta|$ is injective and thus $\dim \mathbb{T}_0 = g(g+1)/2$. Moreover, $\mathbb{T}_0$ is defined by the kernel of r which is $\mathbb{P}\Gamma_{00}^{\perp}$. $\square$

1.6. **Spaces and annihilators.** We have the following diagram of projective subspaces and their annihilators in the dual:

$$\begin{array}{ccccccccc} & & \Delta(\mathcal{O}^{\oplus 2}) & \subset & \mathbb{T}_0 & \subset & \mathbb{T} & \subset & |2\Theta| \\ & & | & & | & & | & & \\ |2\Theta|^* & \supset & \langle\Theta\rangle & \supset & \mathbb{P}\Gamma_{00} & \supset & \langle Sing(\Theta)\rangle & & \end{array}$$

where $\langle\Theta\rangle$ and $\langle Sing(\Theta)\rangle$ are the spans of the images of Θ and $Sing(\Theta)$ respectively.

2. The general unstable bundle

2.1. In this section we prove the injectivity of the differential of Δ at the non-stable bundles $\xi \oplus \xi^{-1}$ with $\xi^{\otimes 2} \not\cong \mathcal{O}$. The proof is in Corollary 2.9, the proof of Theorem 2 is completed in 6.9. We will use Bertram's results on extensions of line bundles. These also play an important role in the case $\xi^{\otimes 2} \cong \mathcal{O}$ which will be considered in section 6.

2.2. **Bertram's maps.** For a divisor class $D \in Pic^dC$ with $d > 0$ we let

$$C_D := \mathrm{im}\left(C \longrightarrow |\omega(2D)|^* \cong \mathbb{P}H^1(\mathcal{O}(-2D))\right)$$

be the image of C under the natural map. Each $\epsilon \in \mathbb{P}H^1(\mathcal{O}(-2D))$ defines, up to isomorphism, an extension

$$0 \longrightarrow \mathcal{O}(-D) \longrightarrow F_\epsilon \longrightarrow \mathcal{O}(D) \longrightarrow 0.$$

The rational classifying map

$$\phi_D : \mathbb{P}H^1(\mathcal{O}(-2D)) \longrightarrow \mathcal{M}_{\mathcal{O}}, \qquad \epsilon \longmapsto F_\epsilon$$

was studied in [Ber]. We denote the composition $\Delta\phi_D$ by

$$\psi_D : \mathbb{P}H^1(\mathcal{O}(-2D)) \cong |\omega(2D)|^* \xrightarrow{\phi_D} \mathcal{M}_{\mathcal{O}} \xrightarrow{\Delta} |2\Theta|, \qquad \epsilon \longmapsto \Delta(F_\epsilon).$$

In this section we will need the case $deg(D) = 1$. The case $deg(D) = 2$ is studied in section 5.

2.3. **The case** $deg(D) = 1$**.** In this case ϕ_D is an injective morphism ([Ber], Cor. 4.4). Moreover, from Theorems 1 and 2 of [Ber] one deduces that ψ_D is given by global sections of $H^0(\mathbb{P}H^1(\mathcal{O}(-2D)), \mathcal{O}(1))$, hence ψ_D is a linear embedding given by the complete linear system $|\mathcal{O}(1)|$. Put

$$\mathbb{P}^g_D := \psi_D(\mathbb{P}H^1(\mathcal{O}(-2D))) \hookrightarrow \Delta(\mathcal{M}_{\mathcal{O}}) \hookrightarrow |2\Theta|.$$

The non-stable bundles correspond to the points of the curve C_D ([Ber], page 451). For $p \in C_D$, we have $\phi_D(p) = \mathcal{O}(D-p) \oplus \mathcal{O}(p-D)$. This implies that $\psi_D(C_D)$ is the image of the abel-jacobi embedded curve $C \to Pic^0 C$, $p \mapsto \mathcal{O}(p-D)$ under the composition of the maps

$$Pic^0 C \longrightarrow K^0(C) \hookrightarrow \mathcal{M}_{\mathcal{O}} \xrightarrow{\Delta} |2\Theta|.$$

From this one concludes that the g-dimensional linear space $\mathbb{P}^g_D$ is the span of the image of the abel-jacobi image of C. See [OPP] for applications of these facts to the study of $\mathcal{M}_{\mathcal{O}}$.

2.4. **The tangent space** $T_{\xi\oplus\xi^{-1}}\mathcal{M}_{\mathcal{O}}$**.** We consider a line bundle ξ of degree zero with $\xi^{\otimes 2} \ncong \mathcal{O}$. Let $\xi \oplus \xi^{-1}$ ($\in \mathcal{M}_{\mathcal{O}}$) be the corresponding S-equivalence class. According to Laszlo [L3] we have:

$$T_{\xi\oplus\xi^{-1}}\mathcal{M}_{\mathcal{O}} \cong H^1(\mathcal{O}) \oplus \left(H^1(\xi^{\otimes 2}) \otimes H^1(\xi^{\otimes -2})\right), \qquad \dim T_{\xi\oplus\xi^{-1}}\mathcal{M}_{\mathcal{O}} = g + (g-1)^2 = g^2 - g + 1.$$

(in fact $H^1(\mathcal{O}) \cong (Ext^1(\xi,\xi) \oplus Ext^1(\xi^{-1},\xi^{-1}))_0 = (H^1(\mathcal{O}) \oplus H^1(\mathcal{O}))_0$ and $H^1(\xi^{\otimes 2}) = Ext^1(\xi^{-1},\xi)$). The quotient map $Pic^0 C \to K^0(C)$ induces an isomorphism from $T_\xi Pic^0 C \cong H^1(\mathcal{O})$ to the tangent space to the Kummer variety at the image of ξ. The subspace $H^1(\mathcal{O})$ of $T_{\xi\oplus\xi^{-1}}\mathcal{M}_{\mathcal{O}}$ is the image of $T_\xi Pic^0 C$.

2.5. We define maps α_ξ, Φ_ξ and Ψ_ξ on $C^2 = C \times C$ as follows:

$$\alpha_\xi : C \times C \longrightarrow Pic^0 C, \qquad (p,q) \longmapsto \xi(p-q),$$

$$\Phi_\xi : C \times C \xrightarrow{\alpha_\xi} Pic^0 C \longrightarrow \mathcal{M}_{\mathcal{O}}, \qquad (p,q) \longmapsto \xi(p-q) \oplus \xi^{-1}(q-p),$$

$$\Psi_\xi : C \times C \xrightarrow{\Phi_\xi} \mathcal{M}_{\mathcal{O}} \xrightarrow{\Delta} |2\Theta|, \qquad (p,q) \longmapsto \Delta(\xi(p-q) \oplus \xi^{-1}(q-p)).$$

Obviously, the diagonal $\Delta_C \subset C \times C$ is contracted to a point by all these maps. Under Φ_ξ each 'ruling' $C \times \{q\}$ is mapped to the curve $\phi_D(C_D)$ where $D := \xi^{-1}(q)$. The span of $\Psi_\xi(C \times \{q\})$ is thus the projective space $\mathbb{P}^g_D \subset \Delta(\mathcal{M}_{\mathcal{O}})$ (see 2.3).

The following lemma relates the image of Ψ_ξ to the embedded tangent space $\mathbb{T}_\xi$ to $\Delta(\mathcal{M}_{\mathcal{O}})$ and to $Sing(\Theta_\xi)$. Next we compute the dimension of the span of $\Psi_\xi(C^2)$ by first determining the line bundle on C^2 which defines Ψ_ξ and its global sections (see Lemma 2.7) and then by showing that the pull-back map is surjective on global sections (the details of that proof are given in section 3). Since, fortunately, the dimension of this span is equal to $\dim T_{\xi\oplus\xi^{-1}}\mathcal{M}_{\mathcal{O}}$, we can deduce the injectivity of the differential in Corollary 2.9.

2.6. **Lemma.** Let $\xi\in Pic^0C$ with $\xi^{\otimes 2}\not\cong\mathcal{O}$. Then we have

$$\langle\Psi_\xi(C^2)\rangle\subset\mathbb{T}_\xi\qquad\text{and}\qquad\langle\Psi_\xi(C^2)\rangle\subset\langle Sing(\Theta_\xi)\rangle^\perp.$$

Proof. A point $\Psi_\xi(p,q)$ lies on the curve $\Psi_\xi(C\times\{q\})$ which lies in $\mathbb{P}^g_D$ as above. Since $\Delta(\xi\oplus\xi^{-1})=\Psi_\xi(q,q)\in\mathbb{P}^g_D$ and $\mathbb{P}^g_D$ is a linear space in $\Delta(\mathcal{M}_\mathcal{O})$ we have $\mathbb{P}^g_D\subset\mathbb{T}_\xi$. Thus also $\Psi_\xi(p,q)\in\mathbb{T}_\xi$.

The point $\Psi_\xi(p,q)\in|2\Theta|$ is the divisor $\Theta_{\xi(p-q)}+\Theta_{\xi^{-1}(q-p)}$ on $Pic^{g-1}C$. We must show that this divisor contains $Sing(\Theta_\xi)$. For $L\in Sing(\Theta)$, $h^0(L)\geq 2$ hence $h^0(L(q-p))\geq 1$, so $L\in\Theta_{\mathcal{O}_C(p-q)}$. Thus $Sing(\Theta)\subset\Theta_{\mathcal{O}_C(p-q)}$ and so $Sing(\Theta_\xi)\subset\Theta_{\xi(p-q)}\subset\Theta_{\xi(p-q)}+\Theta_{\xi^{-1}(q-p)}=\Psi_\xi(p,q)$.

This completes the proof in the case where $g\geq 5$ or $g=4$ and C has two distinct pencils of degree 3. In the case where $g=4$ and C has only one pencil of degree 3, we need to be a little more careful since $Sing\Theta$ is not reduced. In fact, the scheme we really need is not $Sing\Theta$ but the scheme W^1_{g-1} parametrizing line bundles of degree $g-1$ with at least two sections. When $g=4$ and C has only one pencil of degree 3, we have $Sing\Theta\neq W^1_{g-1}$ whereas in all other cases these two schemes are equal. Suppose now that $g=4$ and C has only one pencil g^1_3 of degree 3. The scheme W^1_3 has length 2 and a one-dimensional Zariski tangent space. The projectivization of this Zariski tangent space is the vertex of the singular quadric containing the canonical curve. The tangent space to $\Theta_{g^1_3(p-q)}$ at g^1_3 is equal to the tangent space to Θ at $\mathcal{O}(g^1_3-p+q)$. Since C is non-hyperelliptic, we have $h^0(g^1_3+q)=2$ for every $q\in C$ and, for $p\neq q$, we have $h^0(g^1_3+q-p)=1$. By Riemann's theorem (see for instance [ACGH] page 229), the projectivization of the tangent space to Θ at $\mathcal{O}(g^1_3-p+q)$ is the span of D_p+q where D_p is the unique divisor of g^1_3 containing p. This span contains the vertex of the singular quadric containing C because the quadric is ruled by the spans of the divisors of g^1_3. Therefore Θ contains the translate of W^1_3 by $\mathcal{O}(p-q)$ as a scheme and $\Theta_{\xi(p-q)}\subset\Theta_{\xi(p-q)}+\Theta_{\xi^{-1}(q-p)}$ contains the translate of W^1_3 by $g^1_3(p-q)$ as a scheme. □

2.7. **Lemma.** Let $M:=\Psi_\xi^*\mathcal{O}_{|2\Theta|}(1)$, then

$$\dim H^0(C^2,M)=g^2-g+2.$$

Proof. The pull-back of $\mathcal{O}_{|2\Theta|}(1)$ to C^2 under Ψ_ξ is the pull-back of $\mathcal{O}_{Pic^0C}(2\Theta_0)$ by α_ξ which is

$$M:=\left(\pi_1^*(\omega\otimes\xi^{-2})\otimes\pi_2^*(\omega\otimes\xi^2)\right)(2\Delta)$$

(where now Δ is the diagonal in C^2). For the sake of completeness we sketch a proof. For $L\in Pic^{g-1}C$ let $\Theta_L:=\{x\in Pic^0C:\ L\otimes x\in\Theta\}$ be the translate of Θ by L. The intersection number between an abel-jacobi embedded curve and a theta divisor is g, hence $\alpha_\xi(C\times\{q\})\cdot\Theta_L=g$. Since $\alpha_\xi^{-1}(\Theta_L)=\{p\in C:\ 0<h^0(L\otimes\xi(p-q))=h^0(\omega\otimes L^{-1}\otimes\xi^{-1}(q-p))\}$, we have

$$\alpha_\xi^*(\Theta_L)_{|C\times\{q\}}\cong\omega\otimes L^{-1}\otimes\xi^{-1}(q)$$

for L such that $h^0(\omega\otimes L^{-1}\otimes\xi^{-1}(q))=1$. Since the pull-back is a morphism this must hold for all L. Thus, by the seesaw theorem, $\alpha_\xi^*(\Theta_L)\cong(\pi_1^*(\omega\otimes L^{-1}\otimes\xi^{-1})\otimes\pi_2^*N)(\Delta)$ for some line bundle N_ξ on C. Using the restriction to $\{p\}\times C$ one finds similarly that $N_\xi=L\otimes\xi$. Finally, take the inverse image of the divisor $\Theta_L+\Theta_{\omega\otimes L^{-1}}\in|2\Theta|$ by α_ξ.

In particular, we have $M(-2\Delta)=\pi_1^*(\omega\otimes\xi^{-2})\otimes\pi_2^*(\omega\otimes\xi^2)$ and the Künneth formula gives:

$$H^0(M(-2\Delta))\cong H^0(\omega\otimes\xi^{-2})\otimes H^0(\omega\otimes\xi^2)\cong\mathbb{C}^{(g-1)^2},\quad H^1(M(-2\Delta))=H^2(M(-2\Delta))=0,$$

because $H^1(\omega\otimes\xi^{\pm 2})=0$. The exact sequence of sheaves on C^2:

$$0\longrightarrow M(-2\Delta)\longrightarrow M(-\Delta)\longrightarrow M(-\Delta)_{|\Delta}\longrightarrow 0$$

and the isomorphisms $\Delta \cong C$, $M(-\Delta)_{|\Delta} \cong M(-2\Delta)_{|\Delta} \otimes \mathcal{O}_{C^2}(\Delta)_{|\Delta} \cong \omega^2 \otimes \omega^{-1} \cong \omega$ give:

$$H^0(M(-\Delta)) \cong H^0(\omega \otimes \xi^{-2}) \otimes H^0(\omega \otimes \xi^2) \oplus H^0(\omega), \qquad H^1(M(-\Delta)) \cong H^1(\omega), \quad H^2(M(-\Delta)) = 0.$$

Repeating this method we obtain:

$$0 \longrightarrow M(-\Delta) \longrightarrow M \longrightarrow M_{|\Delta} \longrightarrow 0.$$

Since $M_{|\Delta} \cong M(-\Delta)_{|\Delta} \otimes \mathcal{O}_{C^2}(\Delta)_{|\Delta} \cong \omega \otimes \omega^{-1} \cong \mathcal{O}$, we have $h^0(M_{|\Delta}) = 1$. Since there is a section of $\mathcal{O}_{|2\Theta|}(1)$ which is nonzero at the point $\Psi(\Delta)$, the inclusion $H^0(M(-\Delta)) \subset H^0(M)$ is strict and we conclude: $h^0(M) = 1 + h^0(M(-\Delta)) = 1 + (g-1)^2 + g = g^2 - g + 2$. $\square$

2.8. **Proposition.** Let $\xi \in Pic^0 C$ with $\xi^{\otimes 2} \not\cong \mathcal{O}$ then the pull-back map $\Psi_\xi^* : H^0(\mathcal{O}_{|2\Theta|}(1)) \to H^0(C^2, M)$ is surjective. In particular

$$\dim \langle \Psi_\xi(C^2) \rangle = g^2 - g + 1.$$

Proof. Restricting $\mathcal{O}(1)$ to the Kummer variety and then pulling back to $Pic^0(C)$ gives an isomorphism $H^0(\mathcal{O}_{|2\Theta|}(1)) \to H^0(Pic^0 C, 2\Theta_0)$ where Θ_0 is a symmetric theta divisor. Hence we must show that $\alpha_\xi^* : H^0(Pic^0 C, 2\Theta_0) \to H^0(M)$ is surjective. We will show that $\mathbb{P}H^0(C^2, M)$ is spanned by the pull-backs of the divisors D_E with $E \in \mathcal{M}_\omega$ (see 1.3).

Since the Kummer variety embeds into $|2\Theta|$, its embedded tangent space has dimension g at the point $p_\xi := \Psi_\xi(\Delta)$ if $\xi^{\otimes 2} \not\cong \mathcal{O}$. The global sections of $2\Theta_0$ which are singular at $\alpha_\xi(\Delta)$ thus have codimension $g+1$ in $H^0(2\Theta_0)$ and the inverse images of these by α_ξ lie in $H^0(M(-2\Delta))$. Since $H^0(M(-2\Delta))$ also has codimension $g+1$ in $H^0(M)$, the map α_ξ^* identifies the hyperplane sections of the embedded tangent space with $\mathbb{P}H^0(M)/H^0(M(-2\Delta)) \cong \mathbb{P}(H^0(\omega) \oplus \mathbb{C}) \cong \mathbb{P}^g$, hence the composition $H^0(Pic^0 C, 2\Theta_0) \to H^0(M) \to H^0(M)/H^0(M(-2\Delta))$ is surjective.

Recall that $H^0(M(-2\Delta)) \cong H^0(\omega \otimes \xi^2) \otimes H^0(\omega \otimes \xi^{-2})$, and note that the projective space $\mathbb{P}(H^0(\omega \otimes \xi^2) \otimes H^0(\omega \otimes \xi^{-2}))$ is spanned by the Segre image of $\mathbb{P}(H^0(\omega \otimes \xi^2)) \times \mathbb{P}(H^0(\omega \otimes \xi^{-2})) = |\omega \otimes \xi^2| \times |\omega \otimes \xi^{-2}|$. Therefore it is sufficient to prove that, for a general element (Z, Z') of $|\omega \otimes \xi^2| \times |\omega \otimes \xi^{-2}|$, there is a vector bundle E with $\alpha_\xi^{-1}(D_E) - 2\Delta_C = Z \times C + C \times Z'$. This is precisely Proposition 3.7 below. $\square$

2.9. **Corollary.** Let $\xi \in Pic^0 C$ with $\xi^{\otimes 2} \not\cong \mathcal{O}$. Then the differential

$$(\mathrm{d}\Delta)_{\xi \oplus \xi^{-1}} : T_{\xi \oplus \xi^{-1}} \mathcal{M}_{\mathcal{O}} \longrightarrow T_{\Delta(\xi \oplus \xi^{-1})} \Delta(\mathcal{M}_{\mathcal{O}})$$

is an isomorphism. Thus $\dim \mathbb{T}_\xi = g^2 - g + 1$. Moreover,

$$\mathbb{T}_\xi \subset \langle Sing(\Theta_\xi) \rangle^\perp.$$

Proof. The projective space $\langle \Psi_\xi(C^2) \rangle$ is spanned by the linear spaces $\mathbb{P}^g_D$'s (with $D = \xi^{-1}(q)$ and $q \in C$). Since $\mathbb{P}^g_D = \Delta \phi_D \mathbb{P}H^1(\mathcal{O}(-2D))$ it lies in the image of Δ, and thus

$$\dim \operatorname{im}(\mathrm{d}\Delta)_{\xi \oplus \xi^{-1}}) \geq \dim \langle \Psi_\xi(C^2) \rangle = g^2 - g + 1 = \dim T_{\xi \oplus \xi^{-1}} \mathcal{M}_{\mathcal{O}},$$

the last equalities are Proposition 2.8 and Laszlo's result quoted in 2.4. Hence $(\mathrm{d}\Delta)_{\xi \oplus \xi^{-1}}$ is an isomorphism.

As a consequence we find the dimension of the embedded tangent space (which is equal to the dimension of the tangent space itself): $\dim \mathbb{T}_\xi = g^2 - g + 1$. Since $\langle \Psi_\xi(C^2) \rangle \subset \mathbb{T}_\xi$ it follows for dimension reasons that $\langle \Psi_\xi(C^2) \rangle = \mathbb{T}_\xi$. Then Lemma 2.6 implies $\mathbb{T}_\xi \subset \langle Sing(\Theta_\xi) \rangle^\perp$. $\square$

3. Bundles with two sections

3.1. In this section we finish the proof of Proposition 2.8, which follows from Proposition 3.7. We say that a bundle E is generically generated by its global sections if for a general point t of C, there is no global section of E vanishing at t. Recall the map $\Psi_\xi : C^2 \to |2\Theta|$, $(p,q) \mapsto \Delta(\xi(p-q) \oplus \xi^{-1}(q-p))$. Our first result is:

3.2. **Lemma.** Suppose E is a semi-stable rank 2 vector bundle with determinant ω such that $h^0(E\otimes\xi) = 2$ and $\alpha_\xi(C \times C) \not\subset D_E$.

Then $E \otimes \xi$ and $E \otimes \xi^{-1}$ are generically generated by their global sections and

$$\alpha_\xi^{-1}(D_E) - 2\Delta = C \times Z + Z' \times C$$

where Z and Z' are the divisors on C such that we have the exact sequences

$$0 \longrightarrow H^0(E \otimes \xi) \otimes \mathcal{O} \longrightarrow E \otimes \xi \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

$$0 \longrightarrow H^0(E \otimes \xi^{-1}) \otimes \mathcal{O} \longrightarrow E \otimes \xi^{-1} \longrightarrow \mathcal{O}_{Z'} \longrightarrow 0\ .$$

Proof. Set-theoretically $\alpha_\xi^{-1}(D_E)$ is the set of pairs (p,q) such that $h^0(E \otimes \xi(p-q)) > 0$. Therefore, if D_E does not contain the image of α_ξ, then $E \otimes \xi$ is generically generated by its global sections. Since D_E is symmetric, $C - C + \xi \subset D_E \Leftrightarrow C - C + \xi^{-1} \subset D_E$. Therefore $E \otimes \xi^{-1}$ is also generically generated by its global sections.

It follows from the first sequence in the statement of the lemma that for any point t of Z there is a section of $E \otimes \xi$ which vanishes at t. In other words $h^0(E \otimes \xi(-t)) > 0$. Hence, for any $p \in C$, $h^0(E \otimes \xi(p-t)) > 0$ and $\alpha_\xi^*(D_E)$ contains $C \times Z$. Similarly, for any point t of Z', $h^0(E \otimes \xi^{-1}(-t)) > 0$. By Riemann-Roch and because the determinant of E is ω, this is equivalent to $h^0(E \otimes \xi(t)) > 2$. Hence $\alpha_\xi^*(D_E)$ contains $Z' \times C$. Since $\alpha_\xi^*(D_E) - 2\Delta$ and $C \times Z + Z' \times C$ are linearly equivalent, it follows that they are equal. $\square$

3.3. **Lemma.** There exists an element (Z, Z') of $|\omega \otimes \xi^2| \times |\omega \otimes \xi^{-2}|$ and a vector bundle E as in Lemma 3.2.

Proof. Since $\xi^{\otimes 2} \not\cong \mathcal{O}$, there are distinct effective nonzero divisors D and D' on C such that $\xi^{\otimes 2} \cong \mathcal{O}(D' - D)$ and $h^0(D) = h^0(D') = 1$. Since $g \geq 4$, the difference map

$$C^{(g-2)} \times C^{(g-2)} \longrightarrow Pic^0 C, \qquad (D, D') \longmapsto \mathcal{O}(D - D')$$

is surjective and we can find D and D' of degree at most $g-2$. If D and D' have degree less than $g-2$, we can replace them by $D + G$ and $D' + G$ where G is a general effective divisor to obtain $\deg(D) = \deg(D') = g-2$, keeping the equality $h^0(D) = h^0(D') = 1$. Now consider the Bertram map:

$$\phi_{\xi^{-1}(-D)} : \mathbb{P}H^1((\omega^{-1} \otimes \xi^{\otimes 2})(2D)) \longrightarrow \mathcal{M}_\omega, \qquad \epsilon \longmapsto E_\epsilon,$$

where E_ϵ is the (S-equivalence class of) the bundle given by the extension defined by ϵ:

$$0 \longrightarrow \xi(D) \longrightarrow E_\epsilon \longrightarrow (\omega \otimes \xi^{-1})(-D) \longrightarrow 0.$$

Since $deg((\omega \otimes \xi^{-1}(-D))^{\otimes 2}) = 2(2g - 2 - (g-2)) = 2g$, the Bertram map is a morphism, in particular E_ϵ is indeed semistable.

Next we tensor the defining sequence for E_ϵ by ξ, the boundary map in the cohomology will be denoted by:

$$\delta_\epsilon : H^0(\omega(-D)) \longrightarrow H^1(\xi^{\otimes 2}(D)) = H^1(D') = H^0(\omega(-D'))^*.$$

Choose ϵ such that δ_ϵ has rank 1. Then $h^0(E_\epsilon \otimes \xi) = 2 (= h^0(E_\epsilon \otimes \xi^{-1}))$.

3.3.1. *Claim.* The image of α_ξ is not contained in D_{E_ϵ} for such ϵ.

Proof. Set-theoretically $\alpha_\xi^{-1}(D_{E_\epsilon})$ is the set of pairs (p,q) such that $h^0(E_\epsilon \otimes \xi(p-q)) > 0$. Let (p,q) be a general element of $C \times C$. We will show that (p,q) is not contained in $\alpha_\xi^{-1}(D_E)$. Since $deg(D) = g-2$ and $h^0(D) = 1$ we have $h^0(\omega(-D)) = 2$ and, since q is general, $h^0(\omega(-D-q)) = 1$. Since $deg(\omega(-D-q)) = 2g-2-(g-2)-1 = g-1$ and p is general, we then have $h^0(\omega(-D-q+p)) = 1$ so that p is a base point of the linear system (of dimension zero) $|\omega(-D-q+p)|$. We have a similar result for D'.

Tensoring the defining sequence of E_ϵ by $\xi(p-q)$ and using $\xi^{\otimes 2}(D) \cong \mathcal{O}(D')$ we obtain

$$0 \longrightarrow \mathcal{O}(D'+p-q) \longrightarrow E_\epsilon \otimes \xi(p-q) \longrightarrow \omega(-D+p-q) \longrightarrow 0\ .$$

Let $\delta_{p,q} : H^0(\omega(-D+p-q)) \to H^0(\omega(-D'-p+q)^*$ be the connecting homomorphism for the associated sequence of cohomology. We have a diagram:

$$\begin{array}{ccc} H^0(\omega(-D+p-q)) & \xrightarrow{\delta_{p,q}} & H^0(\omega(-D'-p+q))^* \\ \cong\uparrow & & \downarrow\cong \\ H^0(\omega(-D-q)) & & H^0(\omega(-D'-p))^* \\ \downarrow & & \uparrow \\ H^0(\omega(-D)) & \xrightarrow{\delta_\epsilon} & H^0(\omega(-D'))^*\ . \end{array}$$

From this one can analyze when $\delta_{p,q}$ is zero: the condition is that $\delta_\epsilon(H^0(\omega(-D-q)))$ lies in the kernel of the map $H^0(\omega(-D'))^* \to H^0(\omega(-D'-p)^*$. However, since q is general, the image of $H^0(\omega(-D-q))$ is a general line in $H^0(\omega(-D))$. The image of this line by δ_ϵ will not be contained in the kernel of $H^0(\omega(-D'))^* \to H^0(\omega(-D'-p)^*$ for a fixed general p. Therefore a general (p,q) is not in $\alpha_\xi^{-1}(D_{E_\epsilon})$. $\square$

Therefore by Lemma 3.2 we have exact sequences

$$0 \longrightarrow H^0(E_\epsilon \otimes \xi) \otimes \mathcal{O} \longrightarrow E_\epsilon \otimes \xi \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

$$0 \longrightarrow H^0(E_\epsilon \otimes \xi^{-1}) \otimes \mathcal{O} \longrightarrow E_\epsilon \otimes \xi^{-1} \longrightarrow \mathcal{O}_{Z'} \longrightarrow 0$$

where $(Z, Z') \in |\omega \otimes \xi^{\otimes 2}| \times |\omega \otimes \xi^{\otimes -2}|$. It is immediate now that for the above choice of (Z, Z') and $E = E_\epsilon$ the claim is true. $\square$

3.4. Let Z be a general element of $|\omega \otimes \xi^{\otimes 2}|$. The space of maps $\omega^{\oplus 2} \xrightarrow{\nu_Z} \mathcal{O}_Z$ has dimension $2(2g-2) = 4g-4$ and a general such map is surjective. Consider a general such map ν_Z and let $E \otimes \xi^{-1}$ be its kernel. Apply the functor $Hom(.,\omega)$ to the sequence

$$0 \longrightarrow E \otimes \xi^{-1} \longrightarrow \omega^{\oplus 2} \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

to obtain

$$0 \longrightarrow \mathcal{O}^{\oplus 2} \longrightarrow E \otimes \xi \longrightarrow \omega_Z \longrightarrow 0\ .$$

As Z varies in $|\omega \otimes \xi^2|$, the spaces of surjective maps $\omega^{\oplus 2} \xrightarrow{\nu_Z} \mathcal{O}_Z$ form an open subset of a vector bundle over $|\omega \otimes \xi^2|$. In particular, the space of such maps is irreducible. Therefore, by 3.3, for general Z and a generic map $\omega^{\oplus 2} \xrightarrow{\nu_Z} \mathcal{O}_Z$, the vector bundle E is semi-stable, $h^0(E \otimes \xi) = h^0(E \otimes \xi) = 2$ and $C - C + \xi \not\subset D_E$. So, by Lemma 3.2, we have $\alpha_\xi^{-1}(D_E) - \Delta = C \times Z + Z' \times C$ where Z' is as in Lemma 3.2. Sending ν_Z to Z' defines a map to $|\omega \otimes \xi^{-2}|$ which identifies the space of maps $\nu_Z : \omega^{\oplus 2} \twoheadrightarrow \mathcal{O}_Z$ with the space of maps $\nu_{Z'} : \omega^{\oplus 2} \twoheadrightarrow \mathcal{O}_{Z'}$ since the construction is symmetric in Z and Z'.

3.5. Let $Grass := Grass(2, H^0(\omega^{\oplus 2}))$ be the grassmannian of two-dimensional subvector spaces of $H^0(\omega^{\oplus 2})$. Let $U \subset Grass$ be the subset parametrizing two-dimensional subvector spaces V such that V generates a rank 2 subsheaf of $\omega^{\oplus 2}$ and the cokernel of the evaluation map $V \otimes \mathcal{O} \longrightarrow \omega^{\oplus 2}$ is the structure sheaf of a reduced divisor. In other words, we have the exact sequence

$$0 \longrightarrow V \otimes \mathcal{O} \longrightarrow \omega^{\oplus 2} \longrightarrow \mathcal{O}_D \longrightarrow 0$$

where D is a reduced divisor on C. It is immediate that $D \in |\omega^{\otimes 2}|$. Sending V to D defines a morphism $\mu : U \to |\omega^{\otimes 2}|$.

3.6. **Lemma.** The image of μ is a non-empty open subset of $|\omega^{\otimes 2}|$.

Proof. Consider a divisor $D = \sum_{i=1}^{4g-4} p_i \in |\omega^{\otimes 2}|$ where the points p_i are distinct. For any i, let $C_i \subset H^0(\omega^{\oplus 2})$ be the space of sections vanishing at p_i. Requiring that the length of the cokernel of $V \otimes \mathcal{O} \to \omega^{\oplus 2}$ at p_i be at least 1 is equivalent to requiring $V \cap C_i \neq \{0\}$. Note that since the total length of the cokernel of $V \otimes \mathcal{O} \to \omega^{\oplus 2}$ is $4g-4$, if the length at each p_i is at least 1, then the length at p_i is exactly 1.

Let then M_i be the closed subset of $Grass$ consisting of those V such that $V \cap C_i \neq \{0\}$. Then D is in the image of μ if and only if $(\cap_{i=1}^{4g-4} M_i) \bigcap U \neq \emptyset$. This is clearly an open condition on D hence the image of μ is open. It follows from 3.4 that the image of μ is also non-empty hence it is a dense open subset of $|\omega^{\otimes 2}|$. $\square$

3.7. **Proposition.** For a general element $(Z, Z') \in |\omega \otimes \xi^2| \times |\omega \otimes \xi^{-2}|$, there is a semi-stable vector bundle E with $\alpha_\xi^{-1}(D_E) - 2\Delta_C = Z \times C + C \times Z'$.

Proof. By 3.4 the image of μ intersects the image of $|\omega \otimes \xi^2| \times |\omega \otimes \xi^{-2}|$. Hence by Lemma 3.6 for a general pair (Z, Z') there is a two-dimensional vector space $V \in U$ such that we have the exact sequence

$$0 \longrightarrow V \otimes \mathcal{O} \longrightarrow \omega^{\oplus 2} \longrightarrow \mathcal{O}_{Z+Z'} \longrightarrow 0 .$$

Define $E \otimes \xi^{-1}$ to be the kernel of the composition $\omega^{\oplus 2} \twoheadrightarrow \mathcal{O}_{Z+Z'} \twoheadrightarrow \mathcal{O}_Z$ so that we have the exact sequences

$$0 \longrightarrow E \otimes \xi \longrightarrow \omega^{\oplus 2} \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O}^{\oplus 2} \longrightarrow E \otimes \xi \longrightarrow \mathcal{O}_{Z'} \longrightarrow 0 .$$

Applying the functor $Hom(., \omega)$ to the first sequence, we obtain

$$0 \longrightarrow \mathcal{O}^{\oplus 2} \longrightarrow E \otimes \xi^{-1} \longrightarrow \omega_Z \longrightarrow 0 .$$

We saw that there are pairs (Z, Z') for which such an E exists, is semi-stable, $h^0(E \otimes \xi) = 2 = h^0(E \otimes \xi^{-1})$ and $C - C + \xi \not\subset D_E$. These are all open conditions on the space of maps $\omega^{\oplus 2} \twoheadrightarrow \mathcal{O}_Z$ (= space of maps $\omega^{\oplus 2} \twoheadrightarrow \mathcal{O}_{Z'}$) so they remain true for a general pair (Z, Z') and a general map $\omega^{\oplus 2} \twoheadrightarrow \mathcal{O}_{Z+Z'}$. $\square$

4. Bundles with three sections

4.1. Bundles with 3 independent global sections are essential in the study of $T_{\mathcal{O}^2}\mathcal{M}_{\mathcal{O}}$, in particular for understanding the $\wedge^3 H^1(\mathcal{O})$ quotient of this tangent space.

Let E be a vector bundle on C, generated by global sections, with

$$\mathrm{rank}(E) = 2, \qquad \det(E) := \wedge^2 E = \omega, \qquad h^0(E) = 3.$$

For such E the evaluation map $H^0(E) \otimes_{\mathbb{C}} \mathcal{O} \to E$ is surjective and thus we have the exact sequence:

$$0 \longrightarrow \omega^{-1} \longrightarrow H^0(E) \otimes \mathcal{O} \longrightarrow E \longrightarrow 0.$$

Dualizing this sequence we obtain the exact sequence:

$$0 \longrightarrow E^* \longrightarrow H^0(E)^* \otimes \mathcal{O} \xrightarrow{\pi} \omega \longrightarrow 0.$$

The map $\pi : H^0(E)^* \otimes \mathcal{O} \to \omega$ induces a map on global sections:

$$\pi^0 : H^0(E)^* \otimes H^0(\mathcal{O}) \cong H^0(E)^* \longrightarrow H^0(\omega); \qquad \text{put} \quad W_E := Im(\pi^0).$$

The vector space $\wedge^2 H^0(E)$ is three dimensional and there is a natural map

$$\alpha : \wedge^2 H^0(E) \longrightarrow H^0(\wedge^2(E)) = H^0(\omega).$$

Since $H^0(E)$ is three dimensional, $H^0(E)^* \cong \wedge^2 H^0(E)$. This isomorphism can also be obtained from the Lemma below. We will show that π^0 is injective, so $H^0(E)^* \cong_{\pi^0} W_E$, and that α gives an isomorphism $\wedge^2 H^0(E) \cong_\alpha W_E$.

4.2. **Lemma.** Let E be a rank two bundle with $\det(E) = \omega$, $h^0(E) = 3$ which is generated by its global sections. With the notation of 4.1 we have:

1. The map π^0 is injective, so $\dim W_E = 3$.
2. The map α is injective and
$$\alpha(\wedge^2 H^0(E)) = W_E.$$
3. For any subline bundle L of E we have $h^0(L) \leq 1$.
4. The bundle E is stable.

Proof. 1. The kernel of π^0 is $H^0(E^*)$. If $s \in H^0(E^*)$ is a non-zero section, let L be the subsheaf of E^* generated by s. Then $h^0(L) \geq 1$ and thus $deg(L) \geq 0$. The degree of $L \otimes \omega$ is at least $2g - 2$, hence $h^0(L \otimes \omega) \geq g - 1$. As $E^* \otimes \omega \cong E$, the line bundle $L \otimes \omega$ maps into E so $h^0(E) \geq g - 1$. If $g > 4$ this contradicts $h^0(E) = 3$. In case $g = 4$ one could still have $H^0(L \otimes \omega) = H^0(E)$, but this contradicts the fact that E is globally generated. We conclude that $H^0(E^*) = 0$ and π^0 is injective.
2. Since E is generated by global sections, for any $p \in C$ the map on the fibers at p, $(H^0(E) \otimes \mathcal{O})_p \to E_p$, is surjective. A basis of the one-dimensional kernel will be denoted s_p.

Now $s_p \in H^0(E)$ defines a two-dimensional subspace $V_p \subset H^0(E)^*$ which maps to a two-dimensional subspace $\pi^0(V_p) \subset H^0(\omega)$. It is easy to verify that

$$\pi^0(V_p) = \{\omega \in W_E = \pi^0 H^0(E)^* : \ \omega(p) = 0\,\}.$$

Therefore the map $p \mapsto \pi^0(V_p)$ is the composition of the canonical map $C \to |\omega|^*$ and the projection $|\omega|^* \to \mathbb{P}W_E^*$, with center $\mathbb{P}W_E^\perp$. Since the canonical curve spans $|\omega|^*$, the image of C under this composition spans $\mathbb{P}W_E^*$. Thus the sections s_p span $H^0(E)$.

For p, $q \in C$ general, we have $\alpha(s_p \wedge s_q) \neq 0$ since otherwise we would have $\alpha(s_p \wedge s_q) = 0$ for all p, $q \in C$ which would imply that the sections s_p for $p \in C$ only span a subsheaf of rank 1 of E, in contradiction with the fact that they span $H^0(E)$ and that E is globally generated.

Let p, $q \in C$ be two general points. Then $V_p \cap V_q$ is one dimensional, let ω_{pq} be a basis of $\pi^0(V_p \cap V_q)$:

$$\langle \omega_{pq} \rangle := \pi^0(V_p \cap V_q) = \{\omega \in W_E : \ \omega(p) = \omega(q) = 0\,\}.$$

Below we show that

$$\omega_{pq} = \alpha(s_p \wedge s_q).$$

It will follow from this that $W_E \subset \alpha(\wedge^2 H^0(E))$ and, since $\dim W_E = \dim \alpha(\wedge^2 H^0(E)) = 3$ we obtain $W_E = \alpha(\wedge^2 H^0(E))$.

Let $r \in C$ be a zero of ω_{pq}. Then $\omega_{pq} \in \pi^0(V_r)$, and thus $\omega_{pq} \in V_p \cap V_q \cap V_r$. Therefore the three sections s_p, s_q and s_r are $\mathbb{C}$-linearly dependent in $H^0(E)$. Since s_p and s_q are linearly independent, the section s_r is a linear combination of s_p and s_q. So $s_p \wedge s_q$ vanishes at r. In particular, the differential form $\alpha(s_p \wedge s_q)$ is

zero at r. We see that any zero of ω_{pq} is also a zero of $\alpha(s_p \wedge s_q)$ and conclude (using the general position of p and q) that the differentials are the same.
3. If L is a subline bundle with $h^0(L) \geq 2$, there are two independent sections $s, t \in H^0(E)$ (in the image of $H^0(L) \hookrightarrow H^0(E)$) whose values $s(p)$, $t(p)$ are dependent in each fiber E_p of E. Thus $s \wedge t \neq 0$ but $\alpha(s \wedge t) = 0$. This contradicts the injectivity of α.
4. We must show that any subline bundle L of E has $deg(L) < deg(E)/2 = g - 1$. Suppose L is a subbundle of maximal degree of E and suppose that $deg(L) \geq g - 1$. The Riemann-Roch theorem implies $h^0(L) \geq h^0(\omega \otimes L^{-1})$. The exact sequence

$$0 \longrightarrow L \longrightarrow E \longrightarrow \omega \otimes L^{-1} \longrightarrow 0$$

shows that $h^0(E) \leq h^0(L) + h^0(\omega \otimes L^{-1})$ thus $h^0(E) \leq 2h^0(L)$. Since $h^0(E) = 3$ we obtain $h^0(L) \geq 2$ which contradicts 3. $\square$

4.3. Let $W(3)$ ($\subset \mathcal{M}_\omega$) be the locus of stable bundles E of rank two on C with $det(E) = \omega$ and $h^0(E) = 3$. We denote by $W(3)_+$ ($\subset W(3)$) the locus of bundles in $W(3)$ which are generated by global sections.

To determine $W(3)_+$ we use the injectivity of $\pi^0 : H^0(E)^* \hookrightarrow H^0(\omega)$. For any three-dimensional subspace $W \subset H^0(\omega)$ we can define a rank two bundle E_W^* as

$$E_W^* := \ker(\pi : W \otimes \mathcal{O} \longrightarrow \omega)$$

and let $E_W := (E_W^*)^*$ be the dual bundle (here π is the evaluation map). The determinant of E_W is ω if and only if π is surjective. If π is surjective, we have an exact sequence:

$$0 \longrightarrow \omega^{-1} \longrightarrow W^* \otimes \mathcal{O} \longrightarrow E_W \longrightarrow 0 \qquad (*)$$

so E_W is generated by global sections.

4.4. Denote the grassmannian of three-dimensional subspaces of $H^0(\omega)$ by $Gr(3, H^0(\omega))$. It has dimension $3(g-3)$ and its Picard group is generated by an ample line bundle which we denote as $\mathcal{O}(1)$. The Plücker embedding is the natural map

$$Gr(3, H^0(\omega)) \longrightarrow \mathbb{P}H^0(Gr(3, H^0(\omega)), \mathcal{O}(1))^* \cong \mathbb{P} \wedge^3 H^0(\omega).$$

Let

$$B_C \subset Gr(3, H^0(\omega))$$

be the locus of $W \in Gr(3, H^0(\omega))$ such that the two-dimensional linear system $|W|$ ($\subset |\omega|$) has a base point. We will denote by

$$D_C \subset Gr(3, H^0(\omega))$$

the locus of $W \in Gr(3, H^0(\omega))$ such that the multiplication map

$$m_W : W \otimes H^0(\omega) \longrightarrow H^0(\omega^{\otimes 2})$$

is not surjective. Note that we have $B_C \subset D_C$ since $W \in B_C$ implies that the quadratic differentials in the image of m_W all have a zero at a base point of $|W|$.

4.5. **Proposition.**

1. The map
$$S : W(3)_+ \longrightarrow Gr(3, H^0(\omega)) - D_C, \qquad E \longmapsto W_E$$
is an isomorphism, its inverse is the map
$$\beta : Gr(3, H^0(\omega)) - D_C \longrightarrow \mathcal{M}_\omega, \qquad W \longmapsto E_W.$$
2. The locus D_C has codimension one in $Gr(3, H^0(\omega))$ and D_C is the divisor of a global section of $\mathcal{O}(g-2)$.

Proof. 1. First we show that $Im(S) \subset Gr(3, H^0(\omega)) - D_C$. From the cohomology of the exact sequence $(*)$ in 4.3 we see that $h^0(E) = 3$ implies that the map

$$H^1(\omega^{-1}) \longrightarrow W_E^* \otimes H^1(\mathcal{O})$$

is injective. Thus the dual of this map, which is the multiplication map m_{W_E}, is surjective.

From the constructions it is clear that, if β is well-defined, the morphisms S and β are each other's inverses. To prove that β is well-defined, consider an element W of $Gr(3, H^0(\omega)) - D_C$. Since the multiplication map m_W is surjective, for each $p \in C$ there is an $\omega \in W$ with $\omega(p) \neq 0$. Then the evaluation map $e_W : W \otimes \mathcal{O} \longrightarrow \omega$ is surjective so E_W has determinant ω and is generated by global sections. The cohomology of $(*)$ and the fact that m_W is surjective show that $h^0(E_W) = 3$. From Lemma 4.2 we see that E_W is stable.

2. Lazarsfeld (Theorem 1.1 of [Gi]) and Beauville (private communication) proved that for any non-hyperelliptic curve there is a three-dimensional subspace W of $H^0(\omega)$ such that the multiplication map m_W is surjective. Thus D_C is a Zariski closed subset of codimension ≥ 1.

For non-hyperelliptic curves the multiplication map: $m : S^2H^0(\omega) \longrightarrow H^0(\omega^{\otimes 2})$ is surjective, so its kernel, which we denote by I_2, has dimension $\frac{1}{2}(g-2)(g-3)$. Let $S_W := Im(W \otimes H^0(\omega) \to S^2H^0(\omega))$. Then D_C is the locus of $W \in Gr(3, H^0(\omega))$ such that $I_2 \cap S_W \neq \emptyset$. Since the kernel of $W \otimes H^0(\omega) \to S^2H^0(\omega)$ is $(W \otimes H^0(\omega)) \cap \wedge^2 H^0(\omega) = \wedge^2 W$, we have $\dim S_W = 3g - 3$. Therefore I_2 and S_W have complementary dimensions and D_C is the divisor of zeros of the pull-back of a section of the Plücker bundle on $Gr(3g-3, S^2H^0(\omega))$ under the map:

$$Gr(3, H^0(\omega)) \longrightarrow Gr(3g-3, S^2H^0(\omega)), \qquad W \longmapsto S_W = (W \otimes H^0(\omega))/\wedge^2 W.$$

Recall that the Plücker bundle is the determinant of the universal quotient bundle over the Grassmanian. It is also the dual of the determinant of the universal subbundle. Let $\mathbf{W}$ be the universal subbundle on $Gr(3, H^0(\omega))$. The vector bundle $\mathbf{W} \otimes H^0(\omega)$ (where we view $H^0(\omega)$ as a trivial bundle of rank g over the Grassmanian) has determinant $det(\mathbf{W})^g = \mathcal{O}(-g)$. Define a bundle $\mathbf{S}$ on $Gr(3, H^0(\omega))$ by the exact sequence:

$$0 \longrightarrow \wedge^2\mathbf{W} \longrightarrow \mathbf{W} \otimes H^0(\omega) \longrightarrow \mathbf{S} \longrightarrow 0.$$

Then the fiber of $\mathbf{S}$ over $W \in Gr(3, H^0(\omega))$ is the vector space S_W and $\mathbf{S}$ is the pull-back of the universal subbundle on $Gr(3g-3, S^2H^0(\omega))$. The dual of the determinant of $\mathbf{S}$ is $det(\mathbf{W} \otimes H^0(\omega))^{-1} \otimes det(\wedge^2\mathbf{W})$. For a bundle F of rank k, the determinant of $\wedge^2 F$ is $det(F)^{\otimes(k-1)}$ (use the splitting principle for example). Thus $det(\wedge^2\mathbf{W}) = det(\mathbf{W})^{\otimes 2} = \mathcal{O}(-2)$ and $det(\mathbf{S})^{-1} = \mathcal{O}(g-2)$. □

4.6. **Remark.** Let $W_3 \subset \mathcal{M}_\omega$ be locus of all stable bundles E with $h^0(E) \geq 3$. From the results of [CP] one has $\dim W_3 = 3(g-3)$ $(= \dim Gr(3, H^0(\omega)))$. In fact their results imply that the closure of $W(3)_+$ is the only component of W_3 of maximal dimension.

4.7. **Examples.**

1. In case $g = 3$, $Gr(3, H^0(\omega))$ is a point and $S^2H^0(\omega) \longrightarrow H^0(\omega^{\otimes 2})$ is an isomorphism. The corresponding bundle E was discovered by Laszlo [L1] who also showed that $W(3) = \{E\}$, thus for a (non-hyperelliptic) genus three curve $W(3) = W(3)_+$.
2. In case $g = 4$, $Gr(3, H^0(\omega)) \cong |\omega|^*$, $W \mapsto W^\perp$. The canonical curve C_{can} lies on a unique quadric Q which is thus D_C and $B_C = C_{can}$. From [OPP] we know that β is the rational map

   $$\beta : |\omega|^* \longrightarrow |2\Theta|^*, \qquad x = (X_0 : \ldots : X_4) \longmapsto (X_0Q(x) : \ldots : X_3Q(x) : R(x)) \quad (\subset \mathbb{P}^4 \subset |2\Theta|^*)$$

   where R is any cubic such that C_{can} is defined by Q and R. The image of β is a cubic threefold X with one node in $\mathbb{P}^4$. The inverse of β is projection from the node $X \to \mathbb{P}^3 \cong |\omega|^*$.

In case $W^\perp \in Q \setminus C_{can}$, the bundle E_W is in the S-equivalence class of the semi-stable non-stable bundle $g^1_3 \oplus h^1_3$ where $\{g^1_3,\ h^1_3\}$ is the set of line bundles L of degree 3 such that $h^0(L) = 2$. These bundles E_W map to the node of X.

In case $W^\perp = p \in C_{can}$, the bundle E_W has determinant $\omega(-p)$. The Hecke transforms of E_W at p, tensored by $\mathcal{O}(p)$, have determinant ω. In this way each point of C determines a line in $X \subset \mathcal{M}_\omega$. These lines all pass through the node of X and under the projection from the node they are contracted to the corresponding points on C.

3. In case $g = 6$ and C is general, the locus W_3 has 6 irreducible components, one is the closure of the image of β. The other 5 are threefolds (cones over $\mathbb{P}^2$), one for each of the 5 g^1_4's on the curve, any bundle in such a component has a subbundle (the g^1_4) with 2 sections ([OPP]).

4.8. The following lemma will be used in the proof of Theorem 5.

4.9. **Lemma.** Let $E \in W(3)_+$ and let $p,\ q \in C$. Then:

$$h^0(E(p)) = 3 \iff \dim(W_E \cap H^0(\omega(-p))) = 2.$$

Moreover, $h^0(E(p+q)) \geq 4$ and

$$h^0(E(p+q)) = 4 \iff \dim(W_E \cap H^0(\omega(-p-q))) = 1.$$

The intersections take place in $H^0(\omega)$.

Furthermore, if $h^0(E(p+q)) = 4$, then $h^0(E(p)) = h^0(E(q)) = 3$ and the bundle $E(p+q)$ has a global section which is non-zero in p and in q.

Proof. Let $D = p$. Since $\chi(E(D)) = deg(E(D)) + 2(1-g) = 2g + 2(1-g) = 2$, we have $h^0(E(D)) = 2 + h^1(E(D))$. So $h^0(E(D)) = 3 \Leftrightarrow h^1(E(D)) = 1$. We have the exact sequence

$$0 \longrightarrow \omega^{-1}(D) \longrightarrow H^0(E) \otimes \mathcal{O}(D) \longrightarrow E(D) \longrightarrow 0$$

with cohomology sequence

$$0 \longrightarrow H^0(E) \longrightarrow H^0(E(D)) \longrightarrow H^1(\omega^{-1}(D)) \longrightarrow H^0(E) \otimes H^1(D) \longrightarrow H^1(E(D)) \longrightarrow 0.$$

Dualizing part of the cohomology sequence and using $H^0(E)^* \cong W_E$ $(\subset H^0(\omega))$ we see:

$$H^1(E(D))^* = \ker\left(m_D : W_E \otimes H^0(\omega(-D)) \longrightarrow H^0(\omega^{\otimes 2}(-D))\right),$$

where m_D is the multiplication map. Using a nonzero global section of $\mathcal{O}(D)$ we obtain inclusions $H^0(\omega(-D)) \subset H^0(\omega)$ and $H^0(\omega^{\otimes 2}(-D)) \subset H^0(\omega^{\otimes 2})$. The map $W_E \otimes H^0(\omega) \longrightarrow H^0(\omega^{\otimes 2})$ is surjective with kernel $\wedge^2 W_E$. Since $\dim W_E = 3$ and $h^0(\omega(-D)) = g-1$ we obtain $\dim W_E \cap H^0(\omega(-D)) \geq 2$. If $\dim(W_E \cap H^0(\omega(-D))) > 2$, we have $W_E \subset H^0(\omega(-D))$ and thus $\wedge^2 W_E \subset \ker(m_D)$. So $h^1(E(D)) \geq 3$. If $\dim(W_E \cap H^0(\omega(-D))) = 2$, and $s,\ t \in W_E \cap H^0((\omega(-D))$ are independent, we have $\langle s \wedge t \rangle = (\ker(m_D)) = 1$ and thus $h^1(E(D)) = 1$.

Now let $D = p + q$, since C is not hyperelliptic, $h^0(D) = 1$ and the proof is similar: Since $\chi(E(D)) = 2g + 2 + 2(1-g) = 4$, we have $h^0(E(D)) = 4 + h^1(E(D))$. Now $\dim W_E = 3$, $\dim H^0(\omega(-D)) = g-2$ so $\dim W_E \cap H^0((\omega(-D)) \geq 1$. If $\dim W_E \cap H^0(\omega(-D)) > 1$, and $s,\ t \in W_E \cap H^0((\omega(-D))$ are independent, we have $s \wedge t \in \ker(m_D)$ and thus $h^0(E(D)) > 4$. If $\dim W_E \cap H^0(\omega(-D)) = 1$ the map m_D is injective and thus $h^0(E(D)) = 4$.

Finally we observe that if $h^0(E(p+q)) = 4$ we have $\dim W_E \cap H^0(\omega(-p-q)) = 1$ and thus $\dim W_E \cap H^0(\omega(-p)) = 2$, so $h^0(E(p)) = 3$. Similarly, $h^0(E(q)) = 3$. Thus there is a $t \in H^0(E(p+q))$ with $t(p) \neq 0$ and $t(q) \neq 0$. $\square$

5. Bertram's map in degree two

5.1. In this section we study Bertram's extension map in the case $deg(D) = 2$ and D effective. The main result is Proposition 5.6 which we will use in the next section. Write $D = p + q$ with $p,\ q \in C$. We recall some results of [Ber], using the notation of section 2.

Let $\epsilon \in H^1(\mathcal{O}(-2D))$. An extension F_ϵ is stable if ϵ does not lie on any secant line of C_D. The extension is semi-stable, but not stable, if ϵ lies on a secant line $\langle r, s\rangle$ of C_D but $\epsilon \notin C_D$. In this case F_ϵ is Sheshadri equivalent to $\mathcal{O}(p+q-r-s) \oplus \mathcal{O}(r+s-p-q)$. In case $\epsilon \in C_D$, F_ϵ is unstable and ϕ_D is not defined on the curve C_D.

From [Ber], Theorem 1 (and p. 451-452), we know that ϕ_D lifts to a morphism

$$\widetilde{\phi}_D : \widetilde{\mathbb{P}}^{g+2} \longrightarrow \mathcal{M}_{\mathcal{O}}$$

where $\widetilde{\mathbb{P}}^{g+2}$ is the blow-up of $\mathbb{P}H^1(\mathcal{O}(-2D))$ along C_D. Composing the map $\widetilde{\phi}_D$ with Δ gives a morphism:

$$\widetilde{\psi}_D := \Delta\widetilde{\phi}_D : \widetilde{\mathbb{P}}^{g+2} \longrightarrow |2\Theta|.$$

Let

$$\mathbb{P}_D := \langle \widetilde{\psi}_D(\widetilde{\mathbb{P}}^{g+2})\rangle$$

be the span of the image of this morphism. The fiber over $r \in C$ of the blow-up morphism $\widetilde{\mathbb{P}}^{g+2} \to \mathbb{P}^{g+2}$ is $|\omega(2r)|^*$ and the image in $|2\Theta|$ of $|\omega(2r)|^*$ is the linear space $\mathbb{P}^g_r$ (as in section 2.3).

5.2. To determine the coordinate functions of $\widetilde{\psi}_D$ we use the following observation. As we saw above, given two points $r,\ s \in C$, the secant line $< r, s > \in \mathbb{P}H^1(\mathcal{O}(-2D))$ is mapped to the bundle $\mathcal{O}(r + s - D) \oplus \mathcal{O}(D - r - s)$. Thus the image of $\widetilde{\psi}_D$ contains the image of the surface $C^{(2)}$ (symmetric product) under the composition of the abel-jacobi map

$$\alpha_D : C^{(2)} \longrightarrow Pic^0(C), \qquad r + s \longmapsto r + s - D$$

with the map $Pic^0 C \to K^0(C) \subset \mathcal{M}_{\mathcal{O}}$ and finally $\Delta : \mathcal{M}_{\mathcal{O}} \to |2\Theta|$. We simply write

$$\Delta\alpha_D : C^{(2)} \longrightarrow |2\Theta|, \qquad r + s \longmapsto \Theta_{D-(r+s)} + \Theta_{(r+s)-D}$$

for this composition. Let M_D be the pull-back of $\mathcal{O}_{|2\Theta|}(1)$ to $C^{(2)}$, then M_D is also $\alpha_D^*(\mathcal{O}(2\Theta_0))$ where Θ_0 is a symmetric theta divisor on $Pic^0 C$. We are going to show that $\Delta\alpha_D(C^{(2)})$ spans $\mathbb{P}_D$ and this, combined with results of Bertram, allows us to determine $\widetilde{\psi}_D$ in Proposition 5.6. Lemma 5.4 and Proposition 5.5 below are due to C. Pauly.

5.3. **Lemma.** For any $D \in C^{(2)}$ we have:

$$\dim H^0(C^{(2)}, M_D) = 1 + g(g + 1)/2.$$

Proof. See [BV], Prop. 4.9 (and also [OP], Prop. 10.1). □

5.4. **Lemma. (C. Pauly).** Let $p_0, \ldots , p_g$ be $g + 1$ general points in C. Then the images of the $p_i + p_j$, $0 \leq i < j \leq g$ span a hyperplane in the space $|M_D|^*$.

Proof. Let H_{ij} $(\subset |M_D|)$ be the hyperplane defined by the point $p_i + p_j$. Proving the lemma is equivalent to showing that $\cap_{i<j} H_{ij}$ is one point. For dimension reasons, it is at least one point. So we need to show it is at most one point.

We use the canonical isomorphism from [OP], prop. 10.2:

$$H^0(C^{(2)}, M_D) \ \stackrel{\cong}{\longrightarrow}\ I_C(2) := H^0(|\omega(2D)|^*, \mathcal{I}_C(2))$$

where $\mathcal{I}_C$ is the ideal sheaf of C_D in in $|\omega(2D)|^*$. The hyperplane H_{ij} $(\subset |M_D|)$ corresponds to the hyperplane of quadrics in $|\omega(2D)|^*$ containing the curve and the secant line $< p_i, p_j >$. Therefore $\cap_{i<j} H_{ij}$ consists of the quadrics in $|\omega(2D)|^*$ which contain C_D and the span of $p_0, \dots, p_g$. We claim there is at most one such quadric:

Consider a general hyperplane $H \subset |\omega(2D)|^*$ $(\cong \mathbb{P}^{g+2})$ containing the span of $p_0, \dots, p_g$ and let $q_1, \dots, q_{g+1}$ be the residual points of intersection of H with C. Since the curve spans $\mathbb{P}^{g+2}$, any quadric $Q \in I_C(2)$ is irreducible and so H cannot be a component of Q. Thus the restriction map $I_C(2) \to H^0(H, \mathcal{O}_H(2))$ is injective. Any quadric containing the curve and the span of the p_i's intersects H in a quadric which contains the q_i's and the span of the p_i's. Hence it is enough to show that there is only one such quadric Q in H. Since each secant line $< q_i, q_j >$ is contained in H, it intersects the span of the p_i. Hence $< q_i, q_j >$ meets Q in at least 3 points and is thus contained in Q. Since Q contains all the $< q_i, q_j >$ it also contains the span of the q_i which is a $\mathbb{P}^g$. Thus $Q \cap H \supset \langle p_0, \dots, p_g \rangle \cup \langle q_1, \dots, q_{g+1} \rangle$ and we have equality since the two sides have the same degree and dimension and the right-hand side is reduced. $\square$

5.5. **Proposition. (C. Pauly).** Let $D \in Pic^2(C)$ and let

$$\alpha_D : C^{(2)} \longrightarrow Pic^0 C, \qquad r + s \longmapsto r + s - D.$$

Then the pull-back map

$$\alpha_D^* : H^0(Pic^0 C, \mathcal{O}(2\Theta_0)) \longrightarrow H^0(C^{(2)}, M_D)$$

is surjective. Hence $\dim \langle \Delta \alpha_D(C^{(2)}) \rangle = g(g+1)/2$.

Proof. By Lemma 5.4 and with the notation there, the intersection of the $g(g+1)/2$ hyperplanes H_{ij} in $|M_D|^*$ is one point. This intersection is $\alpha_D^*(\Theta_\xi + \Theta_{\omega\otimes\xi^{-1}})$, with $\xi := \mathcal{O}(p_0 + \ldots + p_g - D) \in Pic^{g-1}C$ because:

1. $\alpha_D(C^{(2)}) \not\subset \Theta_\xi + \Theta_{\omega\otimes\xi^{-1}}$ since for general $r, s \in C$: $h^0(\xi(r+s-D)) = h^0(p_0+\ldots+p_g+r+s-2D) = 0$ (the p_i's are also general) and $h^0(\omega\otimes\xi^{-1}(r+s-D)) = h^0(\xi(D-r-s)) = h^0(p_0+\ldots+p_g-r-s) = 0$.
2. for all $i < j$, we have $h^0(\xi(D-p_i-p_j)) > 0$, hence $h^0(\omega\otimes\xi^{-1}(p_i+p_j-D)) > 0$, hence $\mathcal{O}(p_i+p_j-D) \in \Theta_{\omega\otimes\xi^{-1}}$ which implies $\alpha_D^*(\Theta_\xi + \Theta_{\omega\otimes\xi^{-1}}) \in H_{ij}$.

Next we construct bundles $E_{ij} \in \mathcal{M}_\omega$ with $0 \le i < j \le g$ whose divisors $D_{ij} := \Delta(E_{ij}) = D_{E_{ij}}$ $(\subset Pic^0 C)$ (see 1.3) have the property:

$$\mathcal{O}(p_i + p_j - D) \notin D_{ij}, \qquad \mathcal{O}(p_k + p_l - D) \in D_{ij} \quad \text{if } \{k,l\} \ne \{i,j\}.$$

These conditions are equivalent to:

$$h^0(E_{ij}(p_i + p_j - D)) = 0, \qquad h^0(E_{ij}(p_k + p_l - D)) > 0 \quad \text{if } \{i,j\} \ne \{k,l\},$$

or, equivalently, using Serre Duality, Riemann-Roch and $det(E_{ij}) \cong \omega$,

$$h^0(E_{ij}(D - p_i - p_j)) = 0, \qquad h^0(E_{ij}(D - p_k - p_l)) > 0 \quad \text{if } \{i,j\} \ne \{k,l\}.$$

For simplicity we take the indices to be $i = 0$, $j = 1$. We consider extensions:

$$0 \longrightarrow y \longrightarrow E \longrightarrow \omega \otimes y^{-1} \longrightarrow 0, \qquad y = \mathcal{O}(p_1 + \ldots + p_g - D) \ (\in Pic^{g-2}C).$$

Then E is stable since $deg(\omega \otimes y^{-2}) = 2$ (cf. 2.3). From the exact sequence it is clear that

$$h^0(E(D - p_i - p_j)) \ge h^0(p_1 + \ldots + \hat{p}_i + \ldots + \hat{p}_j + \ldots + p_g) > 0 \qquad \text{for } 1 \le i < j \le g,$$

hence $\mathcal{O}(p_i + p_j - D) \in \Delta(E)$ if $1 \le i < j \le g$.

It remains to consider the points $p_0 + p_i$. Put $\lambda_i = \mathcal{O}(D - p_0 - p_i)$ for $1 \leq i \leq g$. Then, for general p_i's, $h^0(y \otimes \lambda_i) = 0$ and $h^0(\omega \otimes y^{-1} \otimes \lambda_i) = 1$. The long exact sequence

$$0 \longrightarrow H^0(y \otimes \lambda_i) \longrightarrow H^0(E \otimes \lambda_i) \longrightarrow H^0(\omega \otimes y^{-1} \otimes \lambda_i) \xrightarrow{\delta(\epsilon)} H^1(y \otimes \lambda_i) \ldots$$

shows that $h^0(E \otimes \lambda_i) > 0$ iff $\delta(\epsilon) = 0$ where $\epsilon \in H^0(\omega^2 \otimes y^{-2})^*$ is the extension class defining E. This coboundary is zero exactly when the image of the multiplication map

$$m_i : H^0(\omega \otimes y^{-1} \otimes \lambda_i) \otimes H^0(\omega \otimes y^{-1} \otimes \lambda_i^{-1}) \longrightarrow H^0(\omega^2 \otimes y^{-2})$$

is contained in the hyperplane $H_\epsilon \subset H^0(\omega^2 \otimes y^{-2})$ defined by ϵ.

5.5.1. *Claim.* For general p_i's the images of the m_i's are independent.

Proof. We have $\omega \otimes y^{-1} \otimes \lambda_i = \omega(2D - p_0 - \ldots - 2p_i - \ldots - p_g)$ and $\omega \otimes y^{-1} \otimes \lambda_i^{-1} = \omega(p_0 - p_1 \ldots - \hat{p}_i - \ldots - p_g)$. Since the $p_i's$ are general, all these sheaves have exactly one non-zero global section. For $1 \leq i \leq g$, let s_i be a section of ω vanishing at p_j for $1 \leq j \leq g, j \neq i$ and let t_i be a section of $\omega(2D - p_0 \ldots - p_g)$ vanishing at p_i. Then, since the p_i's are general, the sections s_i generate $H^0(C, \omega)$ and the sections t_i generate $H^0(C, \omega(2D - p_0 \ldots - p_g))$. By the base-point-free-pencil-trick, the multiplication map $H^0(C, \omega) \otimes H^0(C, \omega(2D - p_0 \ldots - p_g)) \to H^0(C, \omega^2(2D - p_0 \ldots - p_g))$ is injective. Since the $s_i \otimes t_i$ are independent, so are their images $s_i t_i \in H^0(C, \omega^2(2D - p_0 \ldots - p_g))$. Now, since p_0 is the base point of $|\omega(p_0)|$, the image of $H^0(\omega \otimes y^{-1} \otimes \lambda_i) \otimes H^0(\omega \otimes y^{-1} \otimes \lambda_i^{-1})$ in $H^0(\omega^2 \otimes y^{-2})$ is generated by $s_i t_i s_0$ where s_0 is a non-zero section of $\mathcal{O}(p_0)$. Since the $s_i t_i$ are independent, so are the $s_i t_i s_0$. □

Thus we can find an ϵ with

$$\operatorname{im}(m_1) \notin H_\epsilon, \qquad \operatorname{im}(m_i) \in H_\epsilon \quad \text{for} \quad i \geq 2.$$

The bundle E defined by such an ϵ has the properties required of E_{01}.

We conclude that the family

$$\{\alpha_D^*(\Theta_\xi + \Theta_{\omega \otimes \xi^{-1}})\} \cup \{\alpha_D^*(D_{ij})\}_{0 \leq i < j \leq g}$$

generates $H^0(C^{(2)}, M_D)$, hence α_D^* is surjective. □

5.6. **Corollary.** The composition (a rational map)

$$\psi_D := \Delta \phi_D : \mathbb{P}H^1(\mathcal{O}(-2D)) \longrightarrow |2\Theta|$$

is given by the linear system of all quadrics containing $C_D \subset |\omega(2D)|^*$. The dimension of $\mathbb{P}_D$ is $\frac{1}{2}g(g+1)$.

Proof. From Theorems 1 and 2 of [Ber] one deduces that the rational map ψ_D is given by a subspace of $I_C(2) = H^0(\mathbb{P}H^1(\mathcal{O}(-2D)), \mathcal{I}_C(2))$ (notation as in the proof of Lemma 5.4). In particular, $\dim \mathbb{P}_D \leq \dim |I_C(2)|$ and we have equality only if ψ_D is given by all quadrics in $I_C(2)$. Since $I_C(2)$ is the kernel of the (surjective) multiplication map

$$S^2 H^0(\omega(2D)) \longrightarrow H^0(\omega^{\otimes 2}(4D)),$$

we get from Riemann-Roch that $\dim |I_C(2)| = g(g+1)/2$. Since $\Delta \alpha_D(C^{(2)})$ lies in $\mathbb{P}_D$, we get $\dim \mathbb{P}_D \geq g(g+1)/2$ from 5.5. The Proposition follows. □

6. The moduli space near the trivial bundle

6.1. We use Bertram's extension maps (see 2.2), restricted to certain three-dimensional projective spaces, to study $\mathcal{M}_{\mathcal{O}}$ near the (very) singular point $\mathcal{O}^2$. The image of such a $\mathbb{P}^3$ in $|2\Theta|$ spans a $\mathbb{P}^4$ which contains $\Delta(\mathcal{O}^2)$ and which lies in $\mathbb{T}$. This $\mathbb{P}^4$ intersects $\mathbb{T}_0$ in a three-dimensional space and thus defines a point in $\mathbb{T}/\mathbb{T}_0$. An investigation of the intersection of such a $\mathbb{P}^4$ with the hyperplanes H_W, with $W \in Gr(3, H^0(\omega))$ leads to the proofs of Theorems 2 and 5.

6.2. Choose a divisor $D = p + q$ ($\in C^{(2)}$). Since C is not hyperelliptic, we have $h^0(p+q) = 1$. Using the square of a non-zero section $s \in H^0(\mathcal{O}(D))$ we obtain a natural inclusion

$$H^0(\omega) \hookrightarrow H^0(\omega(2D)) .$$

Let

$$\pi_D : H^1(\mathcal{O}(-2D)) \twoheadrightarrow H^1(\mathcal{O}),$$

be the dual map. Then the kernel of π_D has dimension 3. Thus $\mathbb{P}\ker(\pi_D) \cong \mathbb{P}^2 \hookrightarrow |\omega(2D)|^*$. By Proposition 6.3 below, $\mathbb{P}\ker(\pi_D) = \langle 2p + 2q \rangle$. Let U' be the open subset of $C^{(2)} \times C$ parametrizing points $(p+q, r)$ such that p, q, r are distinct and $r \notin \langle 2p + 2q \rangle$. Then, for $(p+q, r) \in U'$,

$$\langle 2p + 2q + r \rangle = \langle r,\ \mathbb{P}\ker(\pi_D) \rangle \qquad \subset |\omega(2D)|^*.$$

Proposition 6.3 determines the restriction of ψ_D to the three-dimensional projective space $\langle 2p + 2q + r \rangle$.

6.3. **Proposition.** Suppose $(p+q, r) \in U'$. We have:

1. The projective plane $\mathbb{P}\ker(\pi_D)$ is spanned by the secant $l := \langle p,\ q \rangle$ and the tangent lines l_p, l_q at p and q to C_D ($\subset |\omega(2D)|^*$).
2. The map ψ_D restricted to $\mathbb{P}\ker(\pi_D)$ is given by the pencil of conics l^2 and $l_p l_q$.
3. The image under $\widetilde{\psi}_D$ of $\mathbb{P}\ker(\pi_D)$ is the line spanned by $\Delta(\mathcal{O}^{\oplus 2})$ and $\Delta(\mathcal{O}(p-q) \oplus \mathcal{O}(q-p))$. This line lies in $\mathbb{T}_0$.
4. Suppose $h^0(\mathcal{O}(2D)) = 1$ (this will be the case if $g \geq 4$ and $D \in C^{(2)}$ is general). Let ϵ be a point of $\langle 2p + 2q + r \rangle \setminus \mathbb{P}Ker(\pi_D)$ and let

$$0 \longrightarrow \mathcal{O}(-D) \longrightarrow F_\epsilon \longrightarrow \mathcal{O}(D) \longrightarrow 0$$

be the extension defined by ϵ. Then $h^0(F_\epsilon(D)) = 1$.
5. Choose coordinates x, y, z, t on $\langle 2p + 2q + r \rangle$ such that

$$\mathbb{P}\ker(\pi_D) = \{t = 0\}, \quad \langle 2p + r \rangle = \{x = 0\}, \quad \langle 2q + r \rangle = \{y = 0\}, \quad \langle p + q + r \rangle = \{z = 0\}.$$

Then the map ψ_D restricted to $\langle 2p + 2q + r \rangle$ factors through a 4-dimensional linear subspace $\mathbb{P}^4_{p+q,r}$ of $|2\Theta|$ and is given by:

$$\psi_D : \langle 2p + 2q + r \rangle \longrightarrow \mathbb{P}^4_{p+q,r} \hookrightarrow |2\Theta|, \qquad (x : y : z : t) \longmapsto (xy : z^2 : xt : yt : zt).$$

6. The image $Y_{p+q,r}$ of $\langle 2p + 2q + r \rangle$ by $\widetilde{\psi}_D$ is a cubic threefold in $\mathbb{P}^4_{p+q,r}$ and for a good choice of coordinates:

$$Y_{p+q,r} = \widetilde{\psi}_D(\langle 2p + 2q + r \rangle) = \quad \{X_1 X_2 X_3 - X_0 X_4^2 = 0\}.$$

The threefold $Y_{p+q,r}$ contains the points (we write $\Delta(D)$ for $\Delta(\mathcal{O}(D) \oplus \mathcal{O}(-D))$):

$$\begin{aligned} \Delta(\mathcal{O}) &= (1:0:0:0:0), & \Delta(p-q) &= (0:1:0:0:0), \\ \Delta(p-r) &= (0:0:1:0:0), & \Delta(q-r) &= (0:0:0:1:0). \end{aligned}$$

Proof. 1. From the definition of $\mathbb{P}\ker(\pi_D)$ we see that its defining (linear) equations are the elements of $H^0(\omega)$ ($\subset H^0(\omega(2D))$). Since $H^0(\omega) \subset H^0(\omega(2p)) \subset H^0(\omega(2D))$ we see that $l_q \subset \mathbb{P}\ker(\pi_D)$, similarly $l_p \subset \mathbb{P}\ker(\pi_D)$ and these two distinct lines already span $\mathbb{P}\ker(\pi_D)$. Similarly $H^0(\omega) \subset H^0(\omega(p+q)) \subset H^0(\omega(2D))$ verifies that the line l lies in $\mathbb{P}\ker(\pi_D)$.
2. The map ψ_D on $\mathbb{P}\ker(\pi_D)$ is given by the restriction of elements of $I_C(2)$ to the plane $\mathbb{P}\ker(\pi_D)$. These are conics passing through p, q ($\in \mathbb{P}\ker(\pi_D)$), tangent to l_p at p and tangent to l_q at q. There is a pencil of conics with these properties and since it contains l^2 and l_pl_q, it is spanned by them.
3. It follows from section 5.2 that ψ_D contracts $l =< p, q >$ to the point $\Delta(\mathcal{O}(p+q-D)\oplus\mathcal{O}(D-(p+q)) = \Delta(\mathcal{O}^2)$. Similarly:

$$\psi_D(l_p) = \Delta(\mathcal{O}(2p-D)\oplus\mathcal{O}(D-2p)) = \Delta(\mathcal{O}(p-q)\oplus\mathcal{O}(q-p)) = \Delta(\mathcal{O}(2q-D)\oplus\mathcal{O}(D-2q)) = \psi_D(l_q).$$

By 2, the image of $\mathbb{P}Ker(\pi_D)$ is a line which therefore is $\langle\Delta(\mathcal{O}^{\oplus 2}), \Delta(\mathcal{O}(p-q)\oplus\mathcal{O}(q-p))\rangle \subset \langle C-C\rangle = \mathbb{T}_0$.
4. Tensor the defining sequence of F_ϵ by $\mathcal{O}(D)$, we obtain

$$0 \longrightarrow \mathcal{O} \longrightarrow F_\epsilon(D) \longrightarrow \mathcal{O}(2D) \longrightarrow 0.$$

Now $h^0(F_\epsilon(D)) \geq 2$ iff the coboundary map $\delta_\epsilon : H^0(\mathcal{O}(2D)) \to H^1(\mathcal{O})$ is not injective. The dual of the map

$$H^1(\mathcal{O}(-2D)) \longrightarrow \mathrm{Hom}(H^0(\mathcal{O}(2D)), H^1(\mathcal{O})) \cong H^0(\mathcal{O}(2D))^* \otimes H^0(\omega)^*, \qquad \epsilon \longmapsto \delta_\epsilon$$

is the multiplication map $m : H^0(\mathcal{O}(2D)) \otimes H^0(\omega) \to H^0(\omega(2D))$. Since $h^0(\mathcal{O}(2D)) = 1$, the map m is injective and its dual can be identified with π_D. Therefore the map δ_ϵ fails to be injective exactly when ϵ is in $Ker(\pi_D)$.
5. On $\langle 2p+2q+r\rangle$ the map ψ_D is given by quadrics passing through p, q, r and whose restriction to $\mathbb{P}\ker(\pi_D)$ lies in $\langle xy, z^2\rangle$, thus they lie in the space $\langle xy,\, z^2,\, xt,\, yt,\, zt\rangle$. For $\epsilon \in \langle 2p+2q+r\rangle \setminus \langle 2p+2q\rangle$, we have $h^0(F_\epsilon(D)) = 1$ by 4. This implies that, up to multiplication by a scalar, there is a unique nonzero map $\mathcal{O}(-D) \to F_\epsilon$ and gives the exact sequence defining the extension. Hence the restriction of ψ_D to the plane $\langle p+q+r\rangle$ is birational to its image. The conics in this plane passing through p, q, r span $\langle yt,\, xt,\, xy\rangle$. Since the map is birational, it is given by these coordinate functions. Thus xy, z^2, xt, yt are coordinate functions. The only other possible coordinate function is zt, if it is absent the map $\psi_D|_{\langle 2p+2q+r\rangle}$ is 2:1 (due to the z^2), hence zt is also a coordinate function.
6. The coordinate functions obviously satisfy the cubic equation, since the equation is irreducible and the image of $\widetilde{\psi}_D$ is reduced ([H] page 92) and has dimension three it must be the defining equation of the image. □

6.4. We now determine the relation between the linear subspaces $\mathbb{P}^4_{p+q,r}$ and the hyperplanes H_L and H_W which we defined in the introduction. We let U_0 be the open subset of U' parametrizing points $(p+q, r)$ such that the images of p, q, r in $|\omega|^*$ are not colinear.

6.5. **Proposition.** Suppose $(p+q, r) \in U'$.

1. Suppose $L \in Sing(\Theta)$. Then:
$$\mathbb{P}^4_{p+q,r} \subset H_L.$$
2. Let W be an element of $Gr(3, H^0(\omega)) \setminus D_C$ and let $H_W \in |2\Theta|^*$ be the corresponding hyperplane in $|2\Theta|$. Then, for $(p+q, r) \in U_0$,
$$\mathbb{P}^4_{p+q,r} \subset H_W \iff \langle \wedge^3 W \cdot p \wedge q \wedge r\rangle = 0.$$

Proof. 1. Since $\mathbb{P}^4_{p+q,r}$ is the span of $\psi_D(\langle 2p+2q+r\rangle)$, we must show that $F_\epsilon = \psi_D(\epsilon)$ lies in H_L for all $\epsilon \in \langle 2p+2q+r\rangle$. So we must show that $h^0(F_\epsilon \otimes L) \neq 0$. Tensoring the sequence defining F_ϵ by L we see that $L(-D) \hookrightarrow F_\epsilon \otimes L$, so we are done if $h^0(L(-D)) \neq 0$. Suppose therefore that $h^0(L(-D)) = 0$, i.e.,

$h^0(\omega\otimes L^{-1}(D))=2$. In particular, we have $h^0(L)=h^0(\omega\otimes L^{-1})=2$. Since $H_L=H_{\omega\otimes L^{-1}}$ we may also assume that $h^0(\omega\otimes L^{-1}(-D))=0$, i.e., $h^0(L(D))=2$.

It remains to show that the coboundary map

$$\delta_\epsilon : H^0(L(D))\longrightarrow H^1(L(-D))\cong H^0(\omega\otimes L^{-1}(D))^*,$$

with $L\in Sing(\Theta),\ h^0(L(D))=h^0(\omega\otimes L^{-1}(D))=2$, is not injective.

The dual of the map $\epsilon\mapsto\delta_\epsilon$ is again the multiplication map:

$$m : H^0(\omega\otimes L^{-1}(D))\otimes H^0(L(D))\longrightarrow H^0(\omega(2D))=H^1(\mathcal{O}(-2D))^*.$$

Choose bases $\{s_1',\, s_2'\}$ of $H^0(L)$ and $\{t_1',\, t_2'\}$ of $H^0(\omega\otimes L^{-1})$ and let $\{s\}$ be a basis of $H^0(\mathcal{O}(D))$. Let $h_{ij}:=m(ss_i'\otimes st_j')\ \in H^1(\mathcal{O}(-2D))^*$. Then δ_ϵ is not injective iff $Q(\epsilon)=0$ with $Q:=h_{11}h_{22}-h_{12}h_{21}$. Note that $Q=s^4\overline{Q}$ where $\overline{Q}:=g_{11}g_{22}-g_{12}g_{21}$ with $g_{ij}:=h_{ij}/s^2$. So $\overline{Q}$ is the equation of the tangent cone to Θ at the singular point L ([ACGH] page 240). Therefore $\overline{Q}$ defines a quadric $\overline{q}$ in $|\omega|^*$ which contains C ([ACGH] page 241) and the inverse image of $\overline{q}$ in $|\omega(2D)|^*$ is the quadric q of equation Q which contains C_D and whose vertex contains $\mathbb{P}Ker(\pi_D)=\langle 2p+2q\rangle$. It follows that q contains $\langle 2p+2q+r\rangle$ for all $r\in C$ and hence $\mathbb{P}^4_{p+q,r}\subset H_L$.

2. The assumption on the points implies that $(p\wedge q\wedge r)^\perp=H^0(\omega(-p-q-r))\ \subset H^0(\omega)$ has dimension $g-3$ and $\langle\wedge^3W\cdot p\wedge q\wedge r\rangle=0$ is equivalent to $W\cap H^0(\omega(-p-q-r))\neq\{0\}$. We will write $D=p+q$. A result of Beauville ([Bea2] page 268) on the intersection of $H_E:=\Delta_\omega^*(E)$ and $\mathcal{M}_{\mathcal{O}}$ is:

$$\Delta(F)\in H_E\Longleftrightarrow h^0(E\otimes F)\neq 0\qquad\qquad F\in\mathcal{M}_{\mathcal{O}},\ E\in\mathcal{M}_\omega.$$

Thus we have to show:

$$(h^0(E_W\otimes F_\epsilon)\neq 0\quad\forall\epsilon\in\langle 2p+2q+r\rangle)\iff(W\cap H^0(\omega(-p-q-r))\neq\{0\}).$$

We will write E for E_W. The defining sequence of F_ϵ tensored by E shows $H^0(E(-D))\hookrightarrow H^0(E\otimes F_\epsilon)$. If $h^0(E(-D))>0$, then $h^0(E\otimes F_\epsilon)\neq 0$ for all $\epsilon\in\langle 2p+2q+r\rangle$. On the other hand $h^0(E(-D))=h^0(\omega\otimes E^*(-D))=h^1(E(D))=h^0(E(D))-4$ by Riemann-Roch, Serre duality and the fact that $E\cong\omega\otimes E^*$. Thus $h^0(E(-D))>0\Leftrightarrow h^0(E(D))>4$. By Lemma 4.9, $h^0(E(D))\geq 5$ implies $\dim W_E\cap H^0(\omega(-p-q))>1$ and thus $\dim W_E\cap H^0(\omega(-p-q-r))>0$.

It remains to consider the case where $h^0(E(-D))=0$ and $h^0(E(D))=4$. Now $h^0(E\otimes F_\epsilon)\neq 0$ iff the coboundary map

$$\delta_\epsilon : H^0(E(D))\longrightarrow H^1(E(-D))\cong H^0(E(D))^*$$

is not injective.

Let $\{t_1,\ldots,t_4\}$ be a basis of $H^0(E(D))$. Let $\alpha:\wedge^2H^0(E(D))\to H^0(\wedge^2E(D))=H^0(\omega(2D))\cong H^1(\mathcal{O}(-2D))^*$ be the evaluation map and define $h_{ij}:=\alpha(t_i\wedge t_j)$ ($\in H^1(\mathcal{O}(-2D))^*$). Let P_E be the pfaffian (a square root of the determinant) of the 4×4 alternating matrix (h_{ij}). Thus P_E is an element of $S^2H^0(\omega(2D))$ and defines a quadratic form on $H^1(\mathcal{O}(-2D))$ also denoted by P_E. By construction $\Delta(F_\epsilon)\in H_E$ iff $P_E(\epsilon)=0$. Thus we must show:

$$P_E|_{\langle 2p+2q+r\rangle}=0\iff W\cap H^0(\omega(-p-q-r))\neq\{0\}.$$

We now choose the basis $\{t_i\}$ of $H^0(E(D))$ with some more care. Let $s_1\in H^0(E)$ be a generator of the (one-dimensional) kernel of the evaluation map $H^0(E)\to E_p$. Then $s_1(q)\neq 0$ because $h^0(E(-D))=0$. Let $s_2\in H^0(E)$ be a generator of $Ker(H^0(E)\to E_q)$ and let s_3 be a third section such that $\{s_1,s_2,s_3\}$ is a basis of $H^0(E)$. Then s_3 is not zero at p nor q. Put

$$t_i:=ss_i\quad\in H^0(E(D)),$$

where $\{s\}$ is a basis of $H^0(D)$.

Next let $W = \{t \in H^0(E(D)) : t(r) = 0\}$, a vector space of dimension at least 2. Since $r \notin supp(D)$ and E is generated by global sections, the image of $H^0(E) \overset{s}{\hookrightarrow} H^0(E(D))$ intersected with W has dimension at most one. Hence there exits a $t_4 \in H^0(E(D))$ with $t_4(r) = 0$ and $t_4 \neq st$ for any $t \in H^0(E)$.

Now we observe that $\alpha(t_1 \wedge t_4) = \alpha(ss_1 \wedge t_4)$ is zero in p with multiplicity ≥ 2 (since $s(p) = 0$ and $s_1(p) = 0$) and is also zero in q ($s(q) = 0$) and r ($t_4(r) = 0$). Therefore this element of $H^0(\omega(2D))$ must be zero on the subspace $\langle 2p + 2q + r\rangle$ (which is spanned by q, r and $l_p = \langle 2p\rangle$ the tangent line to C_D at p). Similarly $\alpha(t_2 \wedge t_4)$ must be zero at $p + 2q + r$ and thus it is zero on the subspace $\langle 2p + 2q + r\rangle$. The pfaffian P_E, restricted to $\langle 2p + 2q + r\rangle$ is then easy to compute:

$$P_E|_{\langle 2p+2q+r\rangle} = \alpha(t_1 \wedge t_2) \odot \alpha(t_3 \wedge t_4) \qquad \in S^2H^0(\omega(2D)).$$

We claim:

$$\alpha(t_1 \wedge t_2)|_{\langle 2p+2q+r\rangle} = 0 \iff \dim W_E \cap H^0(\omega(-p - q - r)) = 1$$

and

$$\alpha(t_1 \wedge t_2)|_{\langle 2p+2q+r\rangle} \neq 0 \implies \alpha(t_3 \wedge t_4)|_{\langle 2p+2q+r\rangle} \neq 0.$$

This will prove 2.

Since $h^0(E(D)) = 4$ we have $\dim W_E \cap H^0(\omega(-p - q)) = 1$ by Lemma 4.9. By construction $0 \neq \alpha(s_1 \wedge s_2) \in H^0(\omega)$ is zero in p and q. Thus $\langle\alpha(s_1 \wedge s_2)\rangle = W_E \cap H^0(\omega(-p - q))$. This implies:

$$\dim W_E \cap H^0(\omega(-p - q - r)) = 1 \iff \alpha(s_1 \wedge s_2)(r) = 0.$$

By construction, $\alpha(t_1 \wedge t_2)$ has double zeros in p and q; if it also has a zero in r then it vanishes on $\langle 2p + 2q + r\rangle$. Conversely, if $\alpha(t_1 \wedge t_2)|_{\langle 2p+2q+r\rangle} = 0$, then $\alpha(t_1 \wedge t_2) = \alpha(ss_1 \wedge ss_2)$ must vanish in r. Since $s(r) \neq 0$, we have $\alpha(ss_1 \wedge ss_2)(r) = 0 \Leftrightarrow \alpha(s_1 \wedge s_2)(r) = 0$.

It remains to show that the restriction of $\alpha(t_3 \wedge t_4)$ to $\langle 2p+2q+r\rangle$ is non-zero if $\alpha(t_1 \wedge t_2)|_{\langle 2p+2q+r\rangle} \neq 0$. So assume $\alpha(t_1 \wedge t_2)|_{\langle 2p+2q+r\rangle} \neq 0$. Then $s_1 \wedge s_2$ is not zero at r and we can take s_3 to be a basis of $Ker(H^0(E) \longrightarrow E_r)$. The differential $\alpha(t_3 \wedge t_4)$ is zero in p and q (since $t_3 = ss_3$ and s is zero there) and also in r, thus we must show it does not have a double zero at q (or, equivalently, at p).

Let F_x be the stalk of a sheaf F on C at $x \in C$. From the choice of the $s_i \in H^0(E)$, we can define an isomorphism $\phi : E_q \cong \mathcal{O}_q^{\oplus 2}$ such that, with z a local parameter at q:

$$\phi : \quad s_1 \longmapsto (1, 0) + z(\dots), \qquad s_2 \longmapsto z(a, b) + z^2(\dots), \qquad s_3 \longmapsto (0, 1) + z(\dots).$$

Also, we can define an isomorphism $\psi : E(D)_q \cong \mathcal{O}_q^{\oplus 2}$, such that $\psi(t_i) = z\phi(s_i)$ for $i \in \{1, 2, 3\}$. The section t_4 is not zero at q, thus we can write

$$\psi(t_4) = (c, d) + z(\dots) \in \mathcal{O}_x^{\oplus 2} \qquad \text{with} \quad (c, d) \neq (0, 0) \quad \in \mathbb{C}^2.$$

Now we recall that $\alpha(t_1 \wedge t_4)$ has a double zero in q. In q the local expansion of $t_1 \wedge t_4$ is:

$$(t_1 \wedge t_4)_q = \Big(z(1, 0) + z^2(\dots)\Big) \wedge ((c, d) + z(\dots)) = dz + z^2(\dots)$$

thus we must have $d = 0$ and we see that $c \neq 0$. Therefore:

$$(t_3 \wedge t_4)_q = \Big(z(0, 1) + z^2 \dots)\Big) \wedge ((c, 0) + z(\dots)) = cz + z^2(\dots).$$

We conclude that $\alpha(t_3 \wedge t_4)$ has a zero of order exactly one at q. This proves the claim and completes the proof of the proposition. □

6.6. **Corollary.** For any $(p+q,r) \in U'$ and $W \in Gr(3, H^0(\omega)) \setminus D_C$ we have:

$$\mathbb{P}^4_{p+q,r} \subset \mathbb{T}$$

and

$$\mathbb{T}_0 \subset H_W.$$

If, moreover, $(p+q,r) \in U_0$,

$$\dim \mathbb{P}^4_{p+q,r} \cap \mathbb{T}_0 = 3.$$

For $W \in Gr(3, H^0(\omega)) \setminus D_C$,

$$\mathbb{T} \not\subset H_W.$$

Proof. The intersection of $\mathbb{P}^4_{p+q,r}$ with $\Delta(\mathcal{M}_{\mathcal{O}})$ contains the cubic threefold determined in 6.3. Since this threefold is singular at $\Delta(\mathcal{O}^{\oplus 2})$, its embedded tangent space at $\Delta(\mathcal{O}^{\oplus 2})$ is all of $\mathbb{P}^4_{p+q,r}$. Thus $\mathbb{P}^4_{p+q,r}$ is contained in $\mathbb{T}$.

Since $h^0(E_W) = 3$, $h^0(E(-p)) \geq 1$ for all $p \in C$. Therefore $C - C \subset Pic^0 C$ is contained in $\Delta_\omega(E_W)$ and so its span, which is $\mathbb{T}_0$ (cf. 1.5), is contained in $H_E = H_W$.

Since $\Delta(\mathcal{O})$, $\Delta(p-q)$, $\Delta(p-r)$, $\Delta(q-r)$ are in $\mathbb{P}^4_{p+q,r}$ (see 6.3) and in the span of $C - C$, they lie in $\mathbb{T}_0 \cap \mathbb{P}^4_{p+q,r}$. As they are independent (cf. 6.3) we obtain $\dim \mathbb{T}_0 \cap \mathbb{P}^4_{p+q,r} \geq 3$. For $p, q, r \in C_{can}$ distinct and non-colinear, the subvariety $\langle p+q+r \rangle^\perp$ of $Gr(3, H^0(\omega))$ consisting of those $W \in Gr(3, H^0(\omega))$ such that $\langle \wedge^3 W, p \wedge q \wedge r \rangle = 0$ is a divisor in the Plücker system. Choosing $W \in Gr(3, H^0(\omega)) \setminus \left(D_C \cup \langle p+q+r \rangle^\perp \right)$, we have $\langle \wedge^3 W, p \wedge q \wedge r \rangle \neq 0$. Then $\mathbb{P}^4_{p+q,r} \not\subset H_W$ by Proposition 6.5. However, $\mathbb{T}_0 \subset H_W$ hence $\mathbb{P}^4_{p+q,r} \not\subset \mathbb{T}_0$.

As p, q and r vary, the divisors $\langle p+q+r \rangle^\perp$ span the Plücker linear system on $Gr(3, H^0(\omega))$ which has no base points. Hence, given $W \in Gr(3, H^0(\omega)) \setminus D_C$, we can find p, q, r with $\langle \wedge^3 W, p \wedge q \wedge r \rangle \neq 0$. Then, by Proposition 6.5, we have $\mathbb{P}^4_{p+q,r} \not\subset H_W$. Since $\mathbb{P}^4_{p+q,r} \subset \mathbb{T}$ we obtain $\mathbb{T} \not\subset H_W$. $\square$

Let U'' be the open subset of $C^{(3)}$ parametrizing divisors E such that $h^0(E) = 1$. Note that, since C is not hyperelliptic, either $U'' = C^{(3)}$ or the complement of U'' has codimension two in $C^{(3)}$. We have

6.7. **Proposition.** The morphism

$$\rho : U_0 \longrightarrow \mathbb{T}/\mathbb{T}_0, \qquad (p+q,r) \longmapsto \mathbb{P}^4_{p+q,r}/(\mathbb{P}^4_{p+q,r} \cap \mathbb{T}_0)$$

is the composition of the morphism $\pi : U_0 \hookrightarrow C^{(2)} \times C \to C^{(3)}$, $\quad (p+q,r) \mapsto p+q+r$ with the morphism:

$$\begin{array}{cccccc} P: & U'' & \longrightarrow & Gr(3, H^1(\mathcal{O})) & \longrightarrow & \mathbb{P}(\wedge^3 H^1(\mathcal{O})), \\ & p+q+r & \longmapsto & \langle p+q+r \rangle & \longmapsto & \wedge^3 \langle p+q+r \rangle, \end{array}$$

and with a linear embedding $\mathbb{P}(\wedge^3 H^1(\mathcal{O})) \hookrightarrow \mathbb{T}/\mathbb{T}_0$.

In particular we have:

$$\dim \langle \cup \mathbb{P}^4_{p+q,r} \rangle = \tfrac{1}{2} g(g+1) + \binom{g}{3}.$$

Furthermore, the morphism

$$Gr(3, H^0(\omega)) \setminus D_C \longrightarrow (\mathbb{T}/\mathbb{T}_0)^*, \qquad W \longmapsto \bar{H}_W$$

is the composition of the Plücker map with a linear embedding $\mathbb{P}(\wedge^3 H^0(\omega)) \hookrightarrow (\mathbb{T}/\mathbb{T}_0)^*$.

Proof. By Corollary 6.6 any $W \in Gr(3, H^0(\omega)) \setminus D_C$ gives a hyperplane $\bar{H}_W := (H_W \cap \mathbb{T})/\mathbb{T}_0$ in $\mathbb{T}/\mathbb{T}_0$. We first want to determine the divisor $D_W := \rho^* \bar{H}_W$. From 6.5.2 we have the following equality of sets:

$$D_W = \{ (p+q,r) : \ \langle \wedge^3 W, p \wedge q \wedge r \rangle = 0 \ \}.$$

On the other hand, let D'_W be the inverse image under P of the hyperplane $(\wedge^3 W)^\perp \subset \mathbb{P}(\wedge^3 H^1(\mathcal{O}))$, i.e., $D'_W := P^*(\wedge^3 W)^\perp$. Then, again as sets,

$$D'_W = P^{-1}((\wedge^3 W)^\perp) = \{p+q+r : \langle \wedge^3 W, p\wedge q\wedge r\rangle = 0\ \}.$$

Hence $D_W = \pi^{-1} D'_W$ as sets.

In Lemma 6.8 below we prove that the divisors D'_W and their inverse images in C^3 are reduced and irreducible for general W. Hence $\pi^*(D'_W) \subset U_0$ is reduced and irreducible. Thus $D_W = n\pi^* D'_W$ as divisors for some positive integer n. The divisor D'_W is the zero locus of $s_W \in H^0(U'', P^*\mathcal{O}(1)) = H^0(C^{(3)}, P^*\mathcal{O}(1))$. The map $\pi^* : H^0(U'', P^*\mathcal{O}(1)) \to H^0(U_0, \rho^*\mathcal{O}(1))$ is linear on the one hand and maps $s_W \mapsto (\pi^*(s_W))^{\otimes n}$ on the other hand. We conclude that $n = 1$ and that ρ is given by sections of the Plücker line bundle.

Now we need to prove that ρ is given by the full Plücker linear system. For this, we need to produce $\binom{g}{3}$ elments of U_0 whose images by ρ are linearly independent. Since $P(\pi(U_0))$ spans $\mathbb{P}(\wedge^3 H^1(\mathcal{O}))$ (just take g general points on $C_{can} \subset \mathbb{P}H^1(\mathcal{O})$ then the wedges of three distinct points are independent), we can find $D_i \in \pi(U_0)$ and $W_i \in Gr(3, H^0(\omega)) \setminus D_C$ (for $i = 1, \dots, \binom{g}{3}$) such that $P(D_i) \notin (\wedge^3 W_i)^\perp$ and $P(D_i) \in (\wedge^3 W_j)^\perp$ if $j \neq i$. For any choice of $(p_i+q_i, r_i) \in \pi^{-1}(D_i)$, the images $\rho(p_i+q_i, r_i)$ are independent.

The last statement follows from the natural duality between $G(3, H^1(\mathcal{O}))$ and $G(3, H^0(\omega))$ and the fact that ρ is the composition of π with the Plücker map. $\square$

6.8. **Lemma.** Let W be a general three-dimensional subvector space of $H^0(\omega)$. Then the divisor $D'_W := \{p+q+r : \langle \wedge^3 W, p\wedge q\wedge r\rangle = 0\ \}$ in U'' and its inverse image in C^3 are irreducible and reduced.

Proof. Set-theoretically, the divisor D'_W parametrizes the divisors $D \in U''$ such that there exists a divisor $D' \in |W| \subset |\omega|$ with $D \leq D'$ (i.e., the divisor $D' - D$ is effective). The cohomology class of the closure of this subset in $C^{(3)}$ is the coefficient of t in the expression ([Ma], 16.2, p. 338)

$$(1+\eta.t)^{g-4}\prod_{i=1}^{g}(1+A_iB_i.t),$$

i.e.,

$$(g-4)\eta + \sum_{i=1}^{g} A_iB_i. \tag{1}$$

To determine the cohomology class of D'_W we proceed as follows. Let $D \subset C^{(3)} \times C$ be the universal divisor, then we have an exact sequence:

$$0 \longrightarrow \mathcal{O}_{C^{(3)}\times C} \longrightarrow \mathcal{O}_{C^{(3)}\times C}(D) \longrightarrow \mathcal{O}_D(D) \longrightarrow 0.$$

Let $p : C^{(3)} \times C \to C^{(3)}$ be the first projection and consider p_* of the sequence above. It is easy to see that $p_*\mathcal{O}_{C^{(3)}\times C} \cong p_*\mathcal{O}_{C^{(3)}\times C}(D) \cong \mathcal{O}_{C^{(3)}}$ and, by the general theory of Hilbert schemes ([Gro]), we have $p_*\mathcal{O}_D(D) \cong T_{C^{(3)}}$. Furthermore, since $p|_D$ is finite, all higher direct images by p of $\mathcal{O}_D(D)$ are zero. So we have the short exact sequence of sheaves (use $p_*\mathcal{O}_D(D) \cong T_{C^{(3)}}$ [Gro]):

$$0 \longrightarrow T_{C^{(3)}} \longrightarrow H^1(\mathcal{O}) \otimes \mathcal{O} \longrightarrow Q \longrightarrow 0$$

where we have put $Q := R^1 p_*\mathcal{O}_{C^{(3)}\times C}(D)$. For every $E \in U''$, the image of the fiber $(p_*\mathcal{O}_D(D))_E$ in $\mathbb{P}H^1(\mathcal{O})$ is $\langle E\rangle$, hence $T_{C^{(3)}}|_{U''} = \rho^*\mathbf{W}$ where $\mathbf{W}$ is the universal subbundle on $Gr(3, H^1(\mathcal{O}))$. Since the Plücker map is given by $\wedge^3\mathbf{W}^*$, the cohomology class of D'_W is $-c_1(\wedge^3 T_{C^{(3)}}) = c_1(\omega_{C^{(3)}})$. The class $c_1(\omega_{C^{(3)}})$ is determined in [Ma], (14.9), p. 334 and [ACGH], VII, (5.4) and is equal to the class (1) above.

Since the two cohomology classes are equal, we deduce that the two divisors are equal as schemes and D'_W is reduced. Hence its inverse image in C^3 is also reduced. The irreducibility of the inverse image of D'_W in C^3 (and hence the irreducibility of D'_W) follows from the Lemma on page 111 of [ACGH] (the argument there works for $r = 2$ as well, there is a misprint in the statement of the Uniform Position Theorem on page 112). $\square$

6.9. **Proof of Theorem 2.** It remains (cf. Corollary 2.9) to consider the case $\xi^{\otimes 2} \cong \mathcal{O}$. Since the points of order two of $Pic^0 C$ act on $\mathcal{M}_{\mathcal{O}}$ (by $F \mapsto F \otimes \xi$) and on $|2\Theta|$ ($D \mapsto D + \xi$) it suffices to consider the case $\xi = \mathcal{O}$. Thus we need to show that the differential $(\mathrm{d}\Delta)_{\mathcal{O}^{\oplus 2}}$ of $\Delta : \mathcal{M}_{\mathcal{O}} \to |2\Theta|$ at $\mathcal{O}^{\oplus 2}$ is injective.

Choose $(p+q, r) \in U_0$. Recall that $Y_{p+q,r} \subset \mathbb{P}^4_{p+q,r}$ is the image of $\langle 2p + 2q + r\rangle \cong \mathbb{P}^3$ by $\widetilde{\psi}_{p+q} := \Delta\widetilde{\phi}_{p+q}$. By Proposition 6.3.6, there is a choice of coordinates $X_0, \dots, X_4$ on $\mathbb{P}^4_{p+q,r}$ such that $Y_{p+q,r}$ has equation $X_1X_2X_3 - X_0X_4^2 = 0$. A simple computation shows that $Y_{p+q,r}$ is nonsingular in codimension 1. Since $Y_{p+q,r}$ is Cohen-Macaulay (complete intersection in a smooth scheme), it follows that $Y_{p+q,r}$ is normal. Put $X_{p+q,r} := \widetilde{\phi}(\langle 2p + 2q + r\rangle) \subset \mathcal{M}_{\mathcal{O}}$. Then $X_{p+q,r}$ is reduced ([H] page 92). The morphism Δ is injective ([BV]), thus $\Delta|_{X_{p+q,r}} : X_{p+q,r} \to Y_{p+q,r}$ is bijective. As $Y_{p+q,r}$ is normal, the morphism $\Delta|_{X_{p+q,r}}$ is an isomorphism.

Thus $(\mathrm{d}\Delta)_{\mathcal{O}^{\oplus 2}}$ induces an isomorphism between $T_{p+q,r} := T_{\mathcal{O}^{\oplus 2}} X_{p+q,r}$ (a subspace of $T_{\mathcal{O}^{\oplus 2}}\mathcal{M}_{\mathcal{O}}$) and $T_{\Delta(\mathcal{O}^{\oplus 2})} Y_{p+q,r}$, a 4-dimensional vector space. Hence $(\mathrm{d}\Delta)_{\mathcal{O}^{\oplus 2}}(T_{p+q,r}) = T_{\Delta(\mathcal{O}^{\oplus 2})}\mathbb{P}^4_{p+q,r}$. Proposition 6.7 shows $\dim\langle\cup\mathbb{P}^4_{p+q,r}\rangle = \dim T_{\mathcal{O}^2}\mathcal{M}_{\mathcal{O}}$. Therefore the dimension of the image of $(\mathrm{d}\Delta)_{\mathcal{O}^{\oplus 2}}$ is at least $\dim T_{\mathcal{O}^{\oplus 2}}\mathcal{M}_{\mathcal{O}}$ which implies the injectivity of $(\mathrm{d}\Delta)_{\mathcal{O}^{\oplus 2}}$. $\square$

6.10. **Corollary.** We have

$$\dim \mathbb{T} = \dim T_{\mathcal{O}^{\oplus 2}}\mathcal{M}_{\mathcal{O}} = \tfrac{1}{2}g(g+1) + \binom{g}{3} \qquad \text{and} \quad \langle \cup \mathbb{P}^4_{p+q,r}\rangle = \mathbb{T} \ \subset \ \langle Sing(\Theta)\rangle^{\perp}.$$

Proof. The first equalities follow from the fact that Δ is an embedding. Next, by Theorem 2, we have that

$$\dim \mathbb{T} = \dim T_{\mathcal{O}^{\oplus 2}}\mathcal{M}_{\mathcal{O}} = \tfrac{1}{2}g(g+1) + \binom{g}{3}.$$

This is the same as the dimension of $\langle\cup\mathbb{P}^4_{p+q,r}\rangle$ by Proposition 6.7. Since each $\mathbb{P}^4_{p+q,r} \subset \mathbb{T}$ (cf. 6.6), we obtain $\langle\cup\mathbb{P}^4_{p+q,r}\rangle = \mathbb{T}$. The inclusion $\mathbb{T} \subset \langle Sing(\Theta)\rangle^{\perp}$ follows from the fact that each $\mathbb{P}^4_{p+q,r} \subset H_L$ for any $L \in Sing(\Theta)$ (cf. 6.5.1), hence $\mathbb{T} \subset \cap H_L = \langle Sing(\Theta)\rangle^{\perp}$. $\square$

6.11. **Proof of Theorem 5.** By Proposition 6.7, the map:

$$Gr(3, H^0(\omega)) \setminus D_C \longrightarrow (\mathbb{T}/\mathbb{T}_0)^*, \qquad W \longmapsto \bar{H}_W$$

where $\bar{H}_W \subset \mathbb{T}/\mathbb{T}_0$ is the image of $H_W \cap \mathbb{T}$, is the Plücker map. For dimension reasons we then get the isomorphism $\mathbb{P}(\wedge^3 H^0(\omega)) \cong (\mathbb{T}/\mathbb{T}_0)^*$. $\square$

7. The span of $Sing(\Theta_\xi)$

7.1. In this section we prove Theorem 4. We have already seen in 6.10 and 2.9 that

$$\mathbb{T}_\xi \ \subset \ \langle Sing(\Theta_\xi)\rangle^{\perp},$$

and that the dimensions of the embedded tangent spaces are:

$$\dim \mathbb{T}_\xi = \tfrac{1}{2}g(g+1) + \binom{g}{3} \ \text{ if } \xi^{\otimes 2} \cong \mathcal{O}, \qquad \dim \mathbb{T}_\xi = g^2 - g + 1 \ \text{ if } \xi^{\otimes 2} \not\cong \mathcal{O}.$$

Thus to prove Theorem 4 it suffices to prove that the dimension of the linear subspace $\langle Sing(\Theta_\xi)\rangle^{\perp}$ of $|2\Theta|$ is equal to the dimension of $\mathbb{T}_\xi$. This subspace is the linear system of 2Θ-divisors which contain $Sing(\Theta_\xi)$.

By translation by ξ, it can be identified with the linear system of $2\Theta_\xi$-divisors which contain $Sing(\Theta)$, i.e., the projectivization of $H^0(\mathcal{I}\otimes\mathcal{O}_{Pic^{g-1}C}(2\Theta_\xi))$ where $\mathcal{I}$ is the ideal sheaf of $Sing(\Theta)$ in $Pic^{g-1}C$. Put

$$\langle Sing(\Theta)\rangle_\xi^\perp := \{D\in|2\Theta_\xi| :\ Sing(\Theta)\subset D\} = \mathbb{P}H^0(Pic^{g-1}C,\mathcal{I}\otimes\mathcal{O}_{Pic^{g-1}C}(2\Theta_\xi)).$$

We have the usual exact sequence

$$0\longrightarrow\mathcal{O}_{Pic^{g-1}C}(2\Theta_\xi-\Theta)\longrightarrow\mathcal{O}_{Pic^{g-1}C}(2\Theta_\xi)\longrightarrow\mathcal{O}_\Theta(2\Theta_\xi)\longrightarrow 0\,.$$

By the theorem of the square,

$$\mathcal{O}_{Pic^{g-1}C}(2\Theta_\xi)\cong\mathcal{O}_{Pic^{g-1}C}(\Theta_{\xi^{\otimes 2}}+\Theta).$$

Hence the above exact sequence becomes

$$0\longrightarrow\mathcal{O}_{Pic^{g-1}C}(\Theta_{\xi^{\otimes 2}})\longrightarrow\mathcal{O}_{Pic^{g-1}C}(2\Theta_\xi)\longrightarrow\mathcal{O}_\Theta(2\Theta_\xi)\longrightarrow 0\,.$$

As $h^1(\mathcal{O}_{Pic^{g-1}C}(\Theta_{\xi^{\otimes 2}}))=0$, we have $h^0(\mathcal{O}_\Theta(2\Theta_\xi))=h^0(\mathcal{O}_{Pic^{g-1}C}(2\Theta_\xi))-1$.

Since the restriction of $\mathcal{I}$ to Θ is the ideal sheaf of $Sing(\Theta)$ in Θ, we need to prove that $h^0(\Theta,\mathcal{I}\otimes\mathcal{O}_\Theta(2\Theta_\xi))=\dim T_{\xi\oplus\xi^{-1}}\mathcal{M}_\mathcal{O}$.

7.2. To determine $H^0(\Theta,\mathcal{I}\otimes\mathcal{O}_\Theta(2\Theta_\xi))$ we use the resolution of singularities of Θ obtained from the natural morphism $C^{(g-1)}\to Pic^{g-1}C$:

$$\rho: C^{(g-1)}\longrightarrow\Theta,\qquad Z:=\rho^{-1}(Sing(\Theta))\quad\subset C^{(g-1)},\qquad L_\xi:=\rho^*\mathcal{O}_\Theta(\Theta_\xi),\qquad L:=L_\mathcal{O}.$$

Let $\mathcal{I}_Z$ be the ideal sheaf of Z in $C^{(g-1)}$. Then $\mathcal{I}_Z=\rho^*\mathcal{I}$ (see [ACGH] Proposition 3.4 page 181). We have

7.3. **Lemma.** The map ρ induces an isomorphism:

$$\rho^*: H^0(\Theta,\mathcal{I}\otimes\mathcal{O}_\Theta(2\Theta_\xi))\longrightarrow H^0(C^{(g-1)},\mathcal{I}_Z\otimes L_\xi^{\otimes 2}).$$

Proof. Clearly ρ^* is injective. The two spaces have the same dimension since $H^0(C^{(g-1)},\mathcal{I}_Z\otimes L_\xi^{\otimes 2})\cong H^0(\Theta,\rho_*(\mathcal{I}_Z\otimes L_\xi^{\otimes 2}))\cong H^0(\Theta,\mathcal{I}\otimes\mathcal{O}_\Theta(2\Theta_\xi)\otimes\rho_*\mathcal{O}_{C^{(g-1)}})$ (the last isomorphism is obtained from the projection formula) and $\rho_*\mathcal{O}_{C^{(g-1)}}=\mathcal{O}_\Theta$ because the fibers of ρ are connected. □

7.4. Therefore we need to determine the dimension of $H^0(C^{(g-1)},\mathcal{I}_Z\otimes L_\xi^{\otimes 2})$. The ideal sheaf $\mathcal{I}_Z$ has the resolution (see [Gre] page 88, [ACGH], VI.4, p. 258)

$$0\longrightarrow T_{C^{(g-1)}}\otimes L^{-1}\longrightarrow H^1(C,\mathcal{O})\otimes L^{-1}\longrightarrow\mathcal{I}_Z\longrightarrow 0.$$

Tensoring the sequence with $L_\xi^{\otimes 2}$ and using the theorem of the square, we obtain

$$0\longrightarrow T_{C^{(g-1)}}\otimes L_{\xi^{\otimes 2}}\longrightarrow H^1(C,\mathcal{O}_C)\otimes L_{\xi^{\otimes 2}}\longrightarrow\mathcal{I}_Z\otimes L_\xi^{\otimes 2}\longrightarrow 0 \tag{2}$$

7.5. **Theorem.**

1. Suppose that $\xi^{\otimes 2}\not\cong\mathcal{O}$. With the above notation we have the exact sequence

$$0\longrightarrow H^1(C,\mathcal{O})\longrightarrow H^0(C^{(g-1)},\mathcal{I}_Z\otimes L_\xi^{\otimes 2})\longrightarrow H^1(C^{(g-1)},T_{C^{(g-1)}}\otimes L_{\xi^{\otimes 2}})\longrightarrow 0$$

and $h^1(C^{(g-1)},T_{C^{(g-1)}}\otimes L_{\xi^{\otimes 2}})=(g-1)^2$. Thus

$$\dim\langle Sing(\Theta_\xi)\rangle^\perp = g+(g-1)^2=g^2-g+1,$$
$$\dim\langle Sing(\Theta_\xi)\rangle = 2^g-2-g^2+g$$

and $\mathbb{T}_\xi=\langle Sing(\Theta_\xi)\rangle^\perp$.

2. Suppose that $\xi^{\otimes 2} \cong \mathcal{O}$. We have the exact sequence
$$0 \longrightarrow S^2 H^1(C,\mathcal{O}) \longrightarrow H^0(C^{(g-1)}, \mathcal{I}_Z \otimes L_\xi^{\otimes 2}) \longrightarrow \Lambda^3 H^1(C,\mathcal{O}) \longrightarrow 0.$$
Thus
$$\dim\langle Sing(\Theta_\xi)\rangle^\perp = \binom{g+1}{2} + \binom{g}{3} = \sum_{i=1}^{3}\binom{g}{i},$$
$$\dim\langle Sing(\Theta_\xi)\rangle = 2^g - 2 - \left(\binom{g+1}{2} + \binom{g}{3}\right) = 2^g - 1 - \sum_{i=0}^{3}\binom{g}{i}$$
and $\mathbb{T}_\xi = \langle Sing(\Theta_\xi)\rangle^\perp$.

7.6. The exact sequences in the theorem are obtained from the cohomology sequence of sequence (2). Therefore, to prove the theorem, we will compute the cohomology vector spaces of the sheaves appearing in sequence (2).

7.7. **Lemma.**
1. Suppose that $\xi^{\otimes 2} \not\cong \mathcal{O}$. Then, for all $i > 0$,
$$H^i(C^{(g-1)}, L_{\xi^{\otimes 2}}) = 0$$
and $H^0(C^{(g-1)}, L_{\xi^{\otimes 2}}) \cong H^0(Pic^{g-1}C, \mathcal{O}_{Pic^{g-1}C}(\Theta_{\xi^{\otimes 2}})) \cong \mathbb{C}$.
2. For all $i \geq 0$
$$H^i(C^{(g-1)}, L) \cong H^i(\Theta, \mathcal{O}_\Theta(\Theta)) \cong H^{i+1}(Pic^{g-1}C, \mathcal{O}_{Pic^{g-1}C}) \cong \Lambda^{i+1}H^1(C,\mathcal{O}).$$

Proof. Since the morphism $\rho : C^{(g-1)} \to \Theta$ is a rational resolution, we have
$$H^i(C^{(g-1)}, L_{\xi^{\otimes 2}}) \cong H^i(\Theta, \mathcal{O}_\Theta(\Theta_{\xi^{\otimes 2}}))$$
for all $i \geq 0$ and all ξ. Now the lemma follows from the cohomology sequence of the usual exact sequence
$$0 \longrightarrow \mathcal{O}_{Pic^{g-1}C}(\Theta_{\xi^{\otimes 2}} - \Theta) \longrightarrow \mathcal{O}_{Pic^{g-1}C}(\Theta_{\xi^{\otimes 2}}) \longrightarrow \mathcal{O}_\Theta(\Theta_{\xi^{\otimes 2}}) \longrightarrow 0\,.$$
$\square$

The above Lemma determines the cohomology of $H^1(C,\mathcal{O}) \otimes L_{\xi^{\otimes 2}}$. Our next step is

7.8. **Lemma.** We have
$$H^0(C^{(g-1)}, T_{C^{(g-1)}} \otimes L) \cong \Lambda^2 H^1(C,\mathcal{O})$$
and, if $\xi^{\otimes 2} \not\cong \mathcal{O}$, then
$$H^0(C^{(g-1)}, T_{C^{(g-1)}} \otimes L_{\xi^{\otimes 2}}) = 0.$$

Proof. From the exact sequence (2) (in the case $\xi \cong \mathcal{O}$) we obtain the exact sequence of cohomology
$$0 \longrightarrow H^0(C^{(g-1)}, T_{C^{(g-1)}} \otimes L) \longrightarrow H^1(C,\mathcal{O}) \otimes H^0(C^{(g-1)}, L) \longrightarrow H^0(C^{(g-1)}, \mathcal{I}_Z \otimes L^{\otimes 2}) \longrightarrow \ldots.$$
or, by Lemma 7.7,
$$0 \longrightarrow H^0(C^{(g-1)}, T_{C^{(g-1)}} \otimes L) \longrightarrow H^1(C,\mathcal{O})^{\otimes 2} \longrightarrow H^0(C^{(g-1)}, \mathcal{I}_Z \otimes L^{\otimes 2}) \longrightarrow \ldots. \tag{3}$$
The map $H^1(C,\mathcal{O})^{\otimes 2} \longrightarrow H^0(C^{(g-1)}, \mathcal{I}_Z \otimes L^{\otimes 2})$ can be described as follows. Choose a basis $\{D_1, ..., D_g\}$ of $H^1(C,\mathcal{O})$ and think of its elements as translation invariant vector fields on $Pic^{g-1}C$. Then the isomorphism $H^1(C,\mathcal{O}) \xrightarrow{\cong} H^0(C^{(g-1)}, L)$ is defined by $D_i \mapsto \rho^* D_i \theta$ where θ is a nonzero element of $H^0(Pic^{g-1}C, \mathcal{O}_{Pic^{g-1}C}(\Theta))$ (see [Gre] page 92). Hence the natural map $C^{(g-1)} \to |L|^*$ is the composition

of the rational resolution ρ with the Gauss map $\Theta \to \mathbb{P}T_0 Pic^{g-1}C \cong \mathbb{P}^{g-1}$, the map $H^1(C,\mathcal{O}) \otimes L^{-1} \to \mathcal{I}_Z$ is multiplication by global sections of L, the map $H^1(C,\mathcal{O})^{\otimes 2} \longrightarrow H^0(C^{(g-1)}, \mathcal{I}_Z \otimes L^{\otimes 2})$ is defined by $D_i \otimes D_j \mapsto \rho^*(D_i\theta.D_j\theta)$ and is equal to multiplication

$$H^0(C^{(g-1)}, L) \otimes H^0(C^{(g-1)}, L) \longrightarrow H^0(C^{(g-1)}, L^{\otimes 2})$$

which factors through $H^0(C^{(g-1)}, \mathcal{I}_Z \otimes L^{\otimes 2})$. This multiplication map is induced by the multiplication map

$$H^0(\mathbb{P}^{g-1}, \mathcal{O}_{\mathbb{P}^{g-1}}(1)) \otimes H^0(\mathbb{P}^{g-1}, \mathcal{O}_{\mathbb{P}^{g-1}}(1)) \longrightarrow H^0(\mathbb{P}^{g-1}, \mathcal{O}_{\mathbb{P}^{g-1}}(2))$$

via the Gauss map. Since the kernel of this map is $\wedge^2 H^0(\mathbb{P}^{g-1}, \mathcal{O}_{\mathbb{P}^{g-1}}(1))$ and the Gauss map is generically finite ([S] page 114), the kernel of $H^0(C^{(g-1)}, L) \otimes H^0(C^{(g-1)}, L) \longrightarrow H^0(C^{(g-1)}, L^{\otimes 2})$ is $\Lambda^2 H^0(C^{(g-1)}, L) = \Lambda^2 H^1(C,\mathcal{O})$. Hence $H^0(C^{(g-1)}, T_{C^{(g-1)}} \otimes L)$ is isomorphic to $\Lambda^2 H^1(C,\mathcal{O})$.

Now suppose that $\xi^{\otimes 2} \not\cong \mathcal{O}$. As above, the space $H^0(C^{(g-1)}, T_{C^{(g-1)}} \otimes L_{\xi^{\otimes 2}})$ is the kernel of the map

$$H^1(C,\mathcal{O}) \otimes H^0(C^{(g-1)}, L_{\xi^{\otimes 2}}) \longrightarrow H^0(C^{(g-1)}, \mathcal{I}_Z \otimes L_\xi^{\otimes 2})$$

which sends $D_i \otimes s$ to $\rho^* D_i\theta.s$. Using Lemma 7.7, this is immediately seen to be injective. □

7.9. To compute the higher cohomology vector spaces of $T_{C^{(g-1)}} \otimes L_{\xi^{\otimes 2}}$, we will first realize sequence (2) as a pushforward of an exact sequence of sheaves on $C^{(g-1)} \times C$ as follows: Let $D \subset C^{(g-1)} \times C$ be the universal divisor and let p, q be the projections of $C^{(g-1)} \times C$ on the first and second factors:

$$\begin{array}{ccccc} D \hookrightarrow & C^{(g-1)} \times C & \xrightarrow{q} & C \\ & {\scriptstyle p}\downarrow & & \\ & C^{(g-1)} & & \end{array}$$

Consider the exact sequence

$$0 \longrightarrow p^*L_{\xi^{\otimes 2}} \longrightarrow \mathcal{O}_{C^{(g-1)}\times C}(D) \otimes p^*L_{\xi^{\otimes 2}} \longrightarrow \mathcal{O}_D(D) \otimes p^*L_{\xi^{\otimes 2}} \longrightarrow 0. \tag{4}$$

Since $p|_D$ is finite, we have

7.10. **Lemma.** All the higher direct images $R^ip_*(\mathcal{O}_D(D) \otimes p^*L_{\xi^{\otimes 2}})$ are zero for $i > 0$.

7.11. Also, it is easily seen that $p_*\mathcal{O}_{C^{(g-1)}\times C} \cong p_*\mathcal{O}_{C^{(g-1)}\times C}(D) \cong \mathcal{O}_{C^{(g-1)}}$ so that

$$p_*p^*L_{\xi^{\otimes 2}} \cong p_*(\mathcal{O}_{C^{(g-1)}\times C}(D) \otimes p^*L_{\xi^{\otimes 2}}) \cong L_{\xi^{\otimes 2}}$$

by the projection formula. So we obtain the following exact sequence from the pushforward by p of sequence (4)

$$0 \longrightarrow p_*(\mathcal{O}_D(D) \otimes p^*L_{\xi^{\otimes 2}}) \longrightarrow R^1p_*p^*L_{\xi^{\otimes 2}} \longrightarrow R^1p_*(\mathcal{O}_{C^{(g-1)}\times C}(D) \otimes p^*L_{\xi^{\otimes 2}}) \longrightarrow 0,$$

or, again by the projection formula,

$$0 \longrightarrow p_*\mathcal{O}_D(D) \otimes L_{\xi^{\otimes 2}} \longrightarrow H^1(C,\mathcal{O}) \otimes L_{\xi^{\otimes 2}} \longrightarrow R^1p_*\mathcal{O}_{C^{(g-1)}\times C}(D) \otimes L_{\xi^{\otimes 2}} \longrightarrow 0\ .$$

By the general theory of Hilbert schemes ([Gro]), we have

$$p_*\mathcal{O}_D(D) \cong T_{C^{(g-1)}},$$

and the map $T_{C^{(g-1)}} \otimes L_{\xi^{\otimes 2}} \to H^1(C,\mathcal{O}) \otimes L_{\xi^{\otimes 2}}$ obtained by this isomorphism from the above sequence is equal to the first map in sequence (2) because both maps are obtained from the differential of the morphism $\rho : C^{(g-1)} \to Pic^{g-1}C$. Therefore the sequence above is equal to sequence (2) and $R^1p_*\mathcal{O}_{C^{(g-1)}\times C}(D) = \mathcal{I}_Z \otimes L$. So we have proved

7.12. **Lemma.** We have

$$\begin{array}{ccccccccc}
0 & \longrightarrow & p_*(\mathcal{O}_D(D)\otimes p^*L_{\xi^{\otimes 2}}) & \longrightarrow & R^1p_*p^*L_{\xi^{\otimes 2}} & \longrightarrow & R^1p_*(\mathcal{O}_{C^{(g-1)}\times C}(D)\otimes p^*L_{\xi^{\otimes 2}}) & \longrightarrow & 0 \\
\| & & \| & & \| & & \| & & \| \\
0 & \longrightarrow & T_{C^{(g-1)}}\otimes L_{\xi^{\otimes 2}} & \longrightarrow & H^1(C,\mathcal{O})\otimes L_{\xi^{\otimes 2}} & \longrightarrow & \mathcal{I}_Z\otimes L\otimes L_{\xi^{\otimes 2}} & \longrightarrow & 0
\end{array}$$

From Lemmas 7.10, 7.12 and the Leray spectral sequence we deduce

7.13. **Lemma.** There are isomorphisms

$$H^i(C^{(g-1)}, T_{C^{(g-1)}}\otimes L_{\xi^{\otimes 2}}) \xrightarrow{\cong} H^i(D, \mathcal{O}_D(D)\otimes p^*L_{\xi^{\otimes 2}})$$

for all $i\geq 0$.

Hence we need to determine the cohomology of $\mathcal{O}_D(D)\otimes p^*L_{\xi^{\otimes 2}}$. For this we will consider the push-forward by $q: C^{(g-1)}\times C\to C$ of sequence (4). We first prove

7.14. **Lemma.** For all $i\geq 1$,

$$R^iq_*(\mathcal{O}_{C^{(g-1)}\times C}(D)\otimes p^*L_{\xi^{\otimes 2}}) = 0\ .$$

Proof. It is sufficient to show that, for all $x\in C$ and all $i\geq 1$, we have $H^i(C^{(g-1)}\times\{x\}, \mathcal{O}_{C^{(g-1)}\times\{x\}}(D)\otimes p^*L_{\xi^{\otimes 2}}) = 0$. Now $\mathcal{O}_{C^{(g-1)}\times\{x\}}(D)\otimes p^*L_{\xi^{\otimes 2}}\cong\mathcal{O}_{C^{(g-1)}}(U_x)\otimes L_{\xi^{\otimes 2}}$ where $U_x\subset C^{(g-1)}$ is the image of $C^{(g-2)}$ by the morphism $D\mapsto D+x$. We have $\mathcal{O}_{C^{(g-1)}}(U_x)\otimes L_{\xi^{\otimes 2}}\cong\mathcal{O}_{C^{(g-1)}}(U_x)\otimes(L_{\xi^{\otimes 2}}\otimes L^{-1})\otimes L$. The invertible sheaf $\mathcal{O}_{C^{(g-1)}}(U_x)\otimes(L_{\xi^{\otimes 2}}\otimes L^{-1})$ is ample because U_x is ample and $\Theta_{\xi^{\otimes 2}}-\Theta$ is algebraically equivalent to 0. Since L is the canonical sheaf of $C^{(g-1)}$ (see [Gre] page 88), we have Kodaira vanishing

$$H^i(C^{(g-1)}, \mathcal{O}_{C^{(g-1)}}(U_x)\otimes L_{\xi^{\otimes 2}}) = H^i(C^{(g-1)}, \mathcal{O}_{C^{(g-1)}}(U_x)\otimes(L_{\xi^{\otimes 2}}\otimes L^{-1})\otimes L) = 0 \tag{5}$$

for $i>0$. $\square$

Therefore, by the Leray spectral sequence,

7.15. **Lemma.** For all $i\geq 0$,

$$H^i(C^{(g-1)}\times C, \mathcal{O}_{C^{(g-1)}\times C}(D)\otimes p^*L_{\xi^{\otimes 2}})\cong H^i(C, q_*(\mathcal{O}_{C^{(g-1)}\times C}(D)\otimes p^*L_{\xi^{\otimes 2}}))\ .$$

7.16. Also, pushing (4) forward by q, we obtain the exact sequence

$$0\longrightarrow q_*p^*L_{\xi^{\otimes 2}}\longrightarrow q_*(\mathcal{O}_{C^{(g-1)}\times C}(D)\otimes p^*L_{\xi^{\otimes 2}})\xrightarrow{\zeta} q_*(\mathcal{O}_D(D)\otimes p^*L_{\xi^{\otimes 2}})\longrightarrow R^1q_*p^*L_{\xi^{\otimes 2}}\longrightarrow 0 \tag{6}$$

and the isomorphisms

$$R^iq_*(\mathcal{O}_D(D)\otimes p^*L_{\xi^{\otimes 2}})\cong R^{i+1}q_*p^*L_{\xi^{\otimes 2}}$$

for $i>0$. We have

7.17. **Lemma.** Suppose that $\xi^{\otimes 2} \cong \mathcal{O}$. Then the map ζ in the exact sequence

$$0 \longrightarrow q_* p^* L \longrightarrow q_*(\mathcal{O}_{C^{(g-1)} \times C}(D) \otimes p^* L) \xrightarrow{\zeta} q_*(\mathcal{O}_D(D) \otimes p^* L) \longrightarrow R^1 q_* p^* L \longrightarrow 0$$

obtained from sequence (6) is zero.

Proof. Let x be an arbitrary element of C. Then we have the exact sequence

$$0 \longrightarrow L \longrightarrow \mathcal{O}_{C^{(g-1)}}(U_x) \otimes L \longrightarrow \mathcal{O}_{U_x}(U_x) \otimes L \longrightarrow 0\,.$$

It is sufficient to show that the map on global sections

$$H^0(C^{(g-1)}, \mathcal{O}_{C^{(g-1)}}(U_x) \otimes L) \longrightarrow H^0(U_x, \mathcal{O}_{U_x}(U_x) \otimes L)$$

obtained from the exact sequence is zero. For this, since $h^0(C^{(g-1)}, L) = g$ (7.7), it is enough to show that $h^0(C^{(g-1)}, \mathcal{O}_{C^{(g-1)}}(U_x) \otimes L) = g$ as well. Since all higher cohomology of $\mathcal{O}_{C^{(g-1)}}(U_x) \otimes L$ vanishes (see (5)), we need to show that $\chi(C^{(g-1)}, \mathcal{O}_{C^{(g-1)}}(U_x) \otimes L) = g$, or

$$\chi(C^{(g-1)}, L) + \chi(U_x, \mathcal{O}_{U_x}(U_x) \otimes L) = g\,.$$

By [Gre] page 88, the canonical sheaf of $C^{(g-1)}$ is L. By adjunction, the canonical sheaf of $C^{(g-2)} \cong U_x$ is $\mathcal{O}_{U_x}(U_x) \otimes L$. So we need to show

$$\chi(C^{(g-1)}, \omega_{C^{(g-1)}}) + \chi(C^{(g-2)}, \omega_{C^{(g-2)}}) = g$$

or

$$h^{g-1,0}(C^{(g-1)}) - h^{g-1,1}(C^{(g-1)}) + ... + (-1)^{g-1} h^{g-1,g-1}(C^{(g-1)}) +$$
$$+ h^{g-2,0}(C^{(g-2)}) - h^{g-2,1}(C^{(g-2)}) + ... + (-1)^{g-2} h^{g-2,g-2}(C^{(g-2)}) = g\,.$$

By the weak Lefschetz theorem

$$h^{g-1,i}(C^{(g-1)}) = h^{g-2,i-1}(C^{(g-2)})$$

for $i > 1$ because U_x is a smooth and ample divisor in $C^{(g-1)}$. Therefore we are reduced to showing

$$h^{g-1,0}(C^{(g-1)}) - h^{g-1,1}(C^{(g-1)}) + h^{g-2,0}(C^{(g-2)}) = g\,.$$

We have $H^{g-1,0}(C^{(g-1)}) \cong H^0(C^{(g-1)}, \omega_{C^{(g-1)}}) \cong H^0(C^{(g-1)}, L) \cong H^1(C, \mathcal{O})$ by Lemma 7.7. So we are reduced to showing

$$h^{g-1,1}(C^{(g-1)}) = h^{g-2,0}(C^{(g-2)})$$

or, by Serre duality,

$$h^{0,g-2}(C^{(g-1)}) = h^{0,g-2}(C^{(g-2)})$$

which is true by [Ma] (11.1) page 327. $\square$

Therefore we have

7.18. **Corollary.** For all $i \geq 0$,

$$R^i q_*(\mathcal{O}_D(D) \otimes p^* L) \cong R^{i+1} q_* p^* L$$

and

$$q_* p^* L \cong q_*(\mathcal{O}_{C^{(g-1)} \times C}(D) \otimes p^* L)\,.$$

Note that $q_* p^* L \cong H^0(C^{(g-1)}, L) \otimes \mathcal{O}$. Therefore, combining the above Corollary with Lemmas 7.7 and 7.15 we obtain

7.19. **Lemma.** The cohomology groups of the sheaf $\mathcal{O}_{C^{(g-1)}\times C}(D)\otimes p^*L$ on $C^{(g-1)}\times C$ are as follows.

$$H^i(C^{(g-1)}\times C,\mathcal{O}_{C^{(g-1)}\times C}(D)\otimes p^*L)\cong\begin{cases} H^1(C,\mathcal{O}) & \text{if } i=0\\ H^1(C,\mathcal{O})^{\otimes 2} & \text{if } i=1\\ 0 & \text{if } i\geq 2\end{cases}$$

7.20. Now we will compute the cohomology of $\mathcal{O}_{C^{(g-1)}\times C}(D)\otimes p^*L_{\xi^{\otimes 2}}$ in the case where $\xi^{\otimes 2}\not\cong\mathcal{O}$. We have

7.21. **Lemma.** If $\xi^2\not\cong\mathcal{O}$, the cohomology groups of the sheaf $\mathcal{O}_{C^{(g-1)}\times C}(D)\otimes p^*L_{\xi^{\otimes 2}}$ on $C^{(g-1)}\times C$ are as follows.

$$H^i(C^{(g-1)}\times C,\mathcal{O}_{C^{(g-1)}\times C}(D)\otimes p^*L_{\xi^{\otimes 2}})\cong\begin{cases} H^0(\Theta,\mathcal{O}_\Theta(\Theta_{\xi^{\otimes 2}})\cong\mathbb{C} & \text{if } i=0\\ \mathbb{C}^{g^2-g+1} & \text{if } i=1\\ 0 & \text{if } i\geq 2\end{cases}$$

Proof. We have $H^0(C^{(g-1)}\times C,\mathcal{O}_{C^{(g-1)}\times C}(D)\otimes p^*L_{\xi^{\otimes 2}})\cong H^0(C^{(g-1)},p_*(\mathcal{O}_{C^{(g-1)}\times C}(D)\otimes p^*L_{\xi^{\otimes 2}}))$ and it can be easily seen that $L_{\xi^{\otimes 2}}\cong p_*(\mathcal{O}_{C^{(g-1)}\times C}(D)\otimes p^*L_{\xi^{\otimes 2}})$. Hence $H^0(C^{(g-1)}\times C,\mathcal{O}_{C^{(g-1)}\times C}(D)\otimes p^*L_{\xi^{\otimes 2}})\cong H^0(C^{(g-1)},L_{\xi^{\otimes 2}})$. By Lemma 7.7, the space $H^0(C^{(g-1)},L_{\xi^{\otimes 2}})$ is isomorphic to $H^0(\Theta,\mathcal{O}_\Theta(\Theta_{\xi^{\otimes 2}})\cong H^0(Pic^{g-1}C,\Theta_{\xi^{\otimes 2}})$ which has dimension 1.

The fact that $H^i(C^{(g-1)}\times C,\mathcal{O}_{C^{(g-1)}\times C}(D)\otimes p^*L_{\xi^{\otimes 2}})=0$ for $i\geq 2$ follows from Lemma 7.15.

To compute the dimension of $H^1(C^{(g-1)}\times C,\mathcal{O}_{C^{(g-1)}\times C}(D)\otimes p^*L_{\xi^{\otimes 2}})$ we use the following deformation argument:

Let

$$\phi: C^{(g-1)}\times C\times Pic^0C\longrightarrow Pic^{g-1}C$$

be the composition of $\rho\times id\times id$ with the morphism

$$\Theta\times C\times Pic^0C\longrightarrow Pic^{g-1}C,\quad (l,x,\xi)\longmapsto l\otimes\xi\ .$$

Let $p_{12}: C^{(g-1)}\times C\times Pic^0C\to C^{(g-1)}\times C$ be the projection. Then $p_{12}^*\mathcal{O}_{C^{(g-1)}\times C}(D)\otimes\phi^*\mathcal{O}_{Pic^{g-1}C}(\Theta)$ is a family of invertible sheaves on $C^{(g-1)}\times C$ parametrized by Pic^0C. At $\xi\in Pic^0C$, we have

$$(p_{12}^*\mathcal{O}_{C^{(g-1)}\times C}(D)\otimes\phi^*\mathcal{O}_{Pic^{g-1}C}(\Theta))\,|_{C^{(g-1)}\times C\times\{\xi\}}\cong\mathcal{O}_{C^{(g-1)}\times C}(D)\otimes p^*L_{\xi^{\otimes 2}}\ .$$

It follows that, for all $\xi\in Pic^0C$, we have $\chi(\mathcal{O}_{C^{(g-1)}\times C}(D)\otimes p^*L_{\xi^{\otimes 2}})=\chi(\mathcal{O}_{C^{(g-1)}\times C}(D)\otimes p^*L)$. Hence $\chi(\mathcal{O}_{C^{(g-1)}\times C}(D)\otimes p^*L_{\xi^{\otimes 2}})=g-g^2$ by Lemma 7.19. Therefore $h^0(C^{(g-1)}\times C,\mathcal{O}_{C^{(g-1)}\times C}(D)\otimes p^*L_{\xi^{\otimes 2}})-h^1(C^{(g-1)}\times C,\mathcal{O}_{C^{(g-1)}\times C}(D)\otimes p^*L_{\xi^{\otimes 2}})=g-g^2$ and $h^1(C^{(g-1)}\times C,\mathcal{O}_{C^{(g-1)}\times C}(D)\otimes p^*L_{\xi^{\otimes 2}})=g^2-g+1$. $\square$

We are now ready to compute the cohomology of the sheaf $\mathcal{O}_D(D)\otimes p^*L_{\xi^{\otimes 2}}$. We first consider the case where $\xi^{\otimes 2}\not\cong\mathcal{O}$. From Lemma 7.7 it follows

7.22. **Lemma.** If $\xi^{\otimes 2}\not\cong\mathcal{O}$, then

$$R^iq_*p^*L_{\xi^{\otimes 2}}=0$$

for $i\geq 1$.

7.23. So, if $\xi^{\otimes 2} \ncong \mathcal{O}$, from 7.16 we obtain the exact sequence

$$0 \longrightarrow q_* p^* L_{\xi^{\otimes 2}} \longrightarrow q_*(\mathcal{O}_{C^{(g-1)}\times C}(D) \otimes p^* L_{\xi^{\otimes 2}}) \xrightarrow{\zeta} q_*(\mathcal{O}_D(D) \otimes p^* L_{\xi^{\otimes 2}}) \longrightarrow 0 \tag{7}$$

and

$$R^i q_*(\mathcal{O}_D(D) \otimes p^* L_{\xi^{\otimes 2}}) = 0$$

for $i > 0$.

Therefore, again by the Leray spectral sequence,

7.24. **Lemma.** If $\xi^{\otimes 2} \ncong \mathcal{O}$, then, for $i \geq 0$,

$$H^i(C^{(g-1)} \times C, p^* L_{\xi^{\otimes 2}}) \cong H^i(C, q_* p^* L_{\xi^{\otimes 2}})$$

and

$$H^i(D, \mathcal{O}_D(D) \otimes p^* L_{\xi^{\otimes 2}}) \cong H^i(C, q_*(\mathcal{O}_D(D) \otimes p^* L_{\xi^{\otimes 2}}))\ .$$

In particular, $H^i(D, \mathcal{O}_D(D) \otimes p^* L_{\xi^{\otimes 2}}) = H^i(C^{(g-1)} \times C, p^* L_{\xi^{\otimes 2}}) = 0$ for $i \geq 2$.

7.25. Hence the exact sequence (7) gives the exact sequence of cohomology

$$\begin{aligned} 0 \longrightarrow H^0(C, q_* p^* L_{\xi^{\otimes 2}}) \longrightarrow H^0(C, q_*(\mathcal{O}_{C^{(g-1)}\times C}(D) \otimes p^* L_{\xi^{\otimes 2}})) \longrightarrow H^0(C, q_*(\mathcal{O}_D(D) \otimes p^* L_{\xi^{\otimes 2}})) \longrightarrow \\ \longrightarrow H^1(C, q_* p^* L_{\xi^{\otimes 2}}) \longrightarrow H^1(C, q_*(\mathcal{O}_{C^{(g-1)}\times C}(D) \otimes p^* L_{\xi^{\otimes 2}})) \longrightarrow H^1(C, q_*(\mathcal{O}_D(D) \otimes p^* L_{\xi^{\otimes 2}})) \longrightarrow 0\ . \end{aligned}$$

By Lemmas 7.8, 7.13 and 7.24, we have $H^0(C, q_*(\mathcal{O}_D(D) \otimes p^* L_{\xi^{\otimes 2}})) = H^0(D, \mathcal{O}_D(D) \otimes p^* L_{\xi^{\otimes 2}}) = 0$. Therefore, by Lemmas 7.15, and 7.24,

$$\begin{aligned} 0 \longrightarrow H^1(C^{(g-1)} \times C, p^* L_{\xi^{\otimes 2}}) \longrightarrow H^1(C^{(g-1)} \times C, \mathcal{O}_{C^{(g-1)}\times C}(D) \otimes p^* L_{\xi^{\otimes 2}}) \longrightarrow \\ \longrightarrow H^1(C^{(g-1)} \times C, \mathcal{O}_D(D) \otimes p^* L_{\xi^{\otimes 2}}) \longrightarrow 0\ . \end{aligned}$$

Therefore, from the above Lemma, we can deduce

7.26. **Lemma.** We have

$$h^1(D, \mathcal{O}_D(D) \otimes p^* L_{\xi^{\otimes 2}}) = (g-1)^2\ .$$

7.27. **Proof of part 1 of Theorem 7.5.** The exact sequence in part 1 of Theorem 7.5 is obtained from the long exact sequence of cohomology of sequence (2) by using Lemmas 7.7 and 7.8. Then it follows from Lemmas 7.13 and 7.26 that $h^1(C^{(g-1)}, T_{C^{(g-1)}} \otimes L_{\xi^{\otimes 2}}) = (g-1)^2$. □

7.28. From now on we will suppose that $\xi^{\otimes 2} \cong \mathcal{O}$. We will compute the higher cohomology of $\mathcal{O}_D(D) \otimes L$:

7.29. **Lemma.** There are isomorphisms:

$$H^i(D, \mathcal{O}_D(D) \otimes p^* L) \xrightarrow{\cong} H^{i+1}(C^{(g-1)} \times C, p^* L)$$

for all $i \geq 1$.

Proof. Consider the long exact sequence of cohomology of (4) and use Lemma 7.19 to obtain the exact sequence

$$\begin{aligned} 0 \longrightarrow H^0(C^{(g-1)} \times C, p^* L) \longrightarrow H^1(C, \mathcal{O}) \quad \longrightarrow H^0(D, \mathcal{O}_D(D) \otimes p^* L) \longrightarrow \\ \longrightarrow H^1(C^{(g-1)} \times C, p^* L) \longrightarrow H^1(C, \mathcal{O})^{\otimes 2} \longrightarrow H^1(D, \mathcal{O}_D(D) \otimes p^* L) \longrightarrow \\ \longrightarrow H^2(C^{(g-1)} \times C, p^* L) \longrightarrow 0 \end{aligned} \tag{8}$$

and the isomorphisms

$$H^i(D, \mathcal{O}_D(D) \otimes p^* L) \xrightarrow{\cong} H^{i+1}(C^{(g-1)} \times C, p^* L)$$

for all $i \geq 2$.

It remains to do the case $i = 1$. By Lemmas 7.8 and 7.13 we have:

$$H^0(D, \mathcal{O}_D(D) \otimes p^*L) \cong H^0(C^{(g-1)}, T_{C^{(g-1)}} \otimes L) \cong \Lambda^2 H^1(C, \mathcal{O}) .$$

By the Künneth isomorphism and Lemma 7.7,

$$H^0(C^{(g-1)} \times C, p^*L) \cong H^0(C^{(g-1)}, L) \cong H^1(C, \mathcal{O})$$

and

$$\begin{aligned} H^1(C^{(g-1)} \times C, p^*L) &\cong \left(H^0(C^{(g-1)}, L) \otimes H^1(C, \mathcal{O})\right) \oplus H^1(C^{(g-1)}, L) \\ &\cong H^1(C, \mathcal{O})^{\otimes 2} \oplus \Lambda^2 H^1(C, \mathcal{O}). \end{aligned}$$

Putting these results in sequence (8) we obtain

$$\begin{aligned} 0 \longrightarrow \Lambda^2 H^1(C, \mathcal{O}) \longrightarrow H^1(C, \mathcal{O})^{\otimes 2} \oplus \Lambda^2 H^1(C, \mathcal{O}) \longrightarrow H^1(C, \mathcal{O})^{\otimes 2} \longrightarrow \\ \longrightarrow H^1(D, \mathcal{O}_D(D) \otimes p^*L) \xrightarrow{\psi} H^2(C^{(g-1)} \times C, p^*L) \longrightarrow 0 . \end{aligned} \tag{9}$$

and it follows that ψ is an isomorphism. This concludes the proof of the Lemma. □

7.30. **Proof of part 2 of Theorem 7.5.** We take the cohomology of the exact sequence (2):

$$\begin{aligned} 0 \longrightarrow H^0(T_{C^{(g-1)}} \otimes L) \longrightarrow H^1(C, \mathcal{O}) \otimes H^0(C^{(g-1)}, L) \longrightarrow H^0(C^{(g-1)}, \mathcal{I}_Z \otimes L^{\otimes 2}) \longrightarrow \\ \longrightarrow H^1(T_{C^{(g-1)}} \otimes L) \longrightarrow H^1(C, \mathcal{O}) \otimes H^1(C^{(g-1)}, L) \longrightarrow \dots \qquad . \end{aligned} \tag{10}$$

7.31. **Lemma.** The map:

$$H^1(T_{C^{g-1}} \otimes L) \longrightarrow H^1(C, \mathcal{O}) \otimes H^1(C^{g-1}, L)$$

in sequence (10) is surjective.

Proof. Since the fibers of p are one dimensional, $R^2 p_* = 0$. The Leray spectral sequence then shows that we have a surjective map:

$$H^2(C^{(g-1)} \times C, p^*L) \twoheadrightarrow H^1(C^{(g-1)}, R^1 p_*(p^*L)) \cong H^1(C^{(g-1)}, (R^1 p_* \mathcal{O}) \otimes L) \cong H^1(C, \mathcal{O}) \otimes H^1(C^{(g-1)}, L),$$

where the isomorphisms are given by the projection formula and the fact that $R^1 p_* \mathcal{O} \cong \mathcal{O} \otimes H^1(C, \mathcal{O})$. By Lemmas 7.13 and 7.29 we have an isomorphism $H^1(T_{C^{(g-1)}} \otimes L) \cong H^2(C^{(g-1)} \times C, p^*L)$ and it is easily seen, using Lemma 7.12, that the composition $H^1(T_{C^{(g-1)}} \otimes L) \cong H^2(C^{(g-1)} \times C, p^*L) \twoheadrightarrow H^1(C, \mathcal{O}) \otimes H^1(C^{(g-1)}, L)$ coincides with the map in sequence (10). This concludes the proof. □

7.32. Using Lemmas 7.7, 7.8, 7.13 and Lemma 7.29 together with the Künneth isomorphism, we obtain the following exact sequence from sequence (10)

$$\begin{aligned} 0 \longrightarrow \Lambda^2 H^1(C, \mathcal{O}) \longrightarrow H^1(C, \mathcal{O})^{\otimes 2} \longrightarrow H^0(C^{(g-1)}, \mathcal{I}_Z \otimes L^{\otimes 2}) \longrightarrow \\ \longrightarrow (\Lambda^2 H^1(C, \mathcal{O}) \otimes H^1(C, \mathcal{O})) \oplus \Lambda^3 H^1(C, \mathcal{O}) \longrightarrow \Lambda^2 H^1(C, \mathcal{O}) \otimes H^1(C, \mathcal{O}) \longrightarrow 0 \end{aligned}$$

and Theorem 7.5 follows. □

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