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A non-selfdual automorphic representation of GL_3 and a Galois representation

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1 Introduction

1.1 There is a well known procedure which associates to any cusp form f on the congruence subgroup $\Gamma_0(N)$ of $SL_2(\mathbb{Z})$, which is an eigenform for the Hecke algebra, a representation $\sigma(f)$ of $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ on a two dimensional vector space over a finite extension of $\mathbb{Q}_l$ (for any prime number l) [D]. Moreover, one has an equality of L-series:

$$L(f, s) = L(\sigma(f), s) .$$

In case the weight of f is 2, this Galois representation is in $H^1(X_0(N)_{\bar{\mathbb{Q}}}, \mathbb{Q}_l)$, the first etale cohomology group of the modular curve $X_0(N)$. The modular forms of weight two on $\Gamma_0(N)$ correspond to the cohomology classes in $H^1(\Gamma_0(N), \mathbb{C})$.

In this paper we give some evidence for the fact that a certain cohomology class $u \in H^3(\Gamma_0(128), \mathbb{C})$, with now $\Gamma_0(N)$ the subgroup of matrices in $SL_3(\mathbb{Z})$ with $a_{21} \equiv a_{31} \equiv 0 \bmod N$, is related to a (compatible system of λ-adic) three dimensional Galois representation(s) σ. Related means that the local L-factors of u and σ coincide for all primes p, $3 \leqq p \leqq 67$ (cf. Proposition 3.11). (Using faster programs/computers and/or more patience one could try to verify the equality for more primes.) In the next section we explain how the local L-factors of u are computed.

Such a relation between certain cohomology classes u and Galois representations had been conjectured by Langlands and Clozel, see [Cl, Conjecture 4.5]. (In fact, u should correspond to a cuspidal automorphic representation π_u of $GL_{3,\mathbb{Q}}$. In our case, π_u is not selfdual in the sense that $\tilde{\pi}_u \not\cong \pi_u \otimes (\psi \circ \det)$, with $\tilde{\pi}_u$ the contragredient of π_u and ψ a grossencharacter).

Some 10 years ago Ash already tried to find examples, lack of computer power at that time probably prevented him from finding the example below. In a subsequent paper we hope to discuss more examples.

1.2 Assume there is a number field $K \subset \mathbb{C}$, a prime λ in K and a λ-adic Galois representation corresponding to π_u. Then one expects it to be unramified outside $p=2$ (in general: unramified outside the primes dividing N) and l, with $\lambda \mid l$. To construct the compatible system of Galois representations we search for suitable subspaces $V_\lambda \subset H^i(S_{\bar{\mathbb{Q}}}, K_\lambda)$ for a certain algebraic variety S defined over $\mathbb{Q}$, with good reduction outside $p=2$. These subspaces should be motivically defined (that is, should be cut out by correspondences). In particular, one also has a $V_\infty \subset H^i(S(\mathbb{C}), K)$.

Then $V_{\infty,\mathbb{C}} := V_\infty \otimes_K \mathbb{C} \subset H^i(S(\mathbb{C}), \mathbb{C})$ and one defines the Hodge numbers of V_∞ to be $h^{p,q} := \dim V_{\infty,\mathbb{C}} \cap H^{p,q}(S(\mathbb{C}))$. These Hodge numbers ought to correspond to the infinity type of π_u. In this case they should be: $h^{2,0} = h^{1,1} = h^{0,2} = 1$ (cf. [AS, p. 216]).

This suggests looking at the H^2 of surfaces (in fact by the Lefschetz hyperplane theorem, the H^2 of a variety of dimension $\geqq 3$ maps injectively to the H^2 of a suitable surface in it). Since the coefficients of the local L-factors are in $\mathbb{Z}[i]$, we will look for 6-dimensional spaces $T_l \subset H^2(S_{\bar{\mathbb{Q}}}, \mathbb{Q}_l)$, such that the Hodge numbers of T_∞ are $h^{2,0} = h^{1,1} = h^{0,2} = 2$. Moreover we want an automorphism ϕ of order 4 on S, defined over $\mathbb{Q}$ (so on H^2, ϕ^* commutes with the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$) and which has the eigenvalues i and $-i$, each with multiplicity 3, on $T_l \otimes_{\mathbb{Q}_l} \bar{\mathbb{Q}}_l$. Then we get three dimensional $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-representations of the desired type on each of the two eigenspaces. The construction of a suitable S and ϕ is given in section 3. In the last section we discuss variants of our constructions and related questions.

Finally we would like to mention that in [APT] a (unique) Galois representation ρ: $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_3(\mathbb{F}_3)$ is constructed which has the same L-factors (mod $\sqrt{-3}$) for small primes as a certain automorphic representation on GL_3, this provided the first evidence that there might be Galois representations on λ-adic vector spaces associated to such automorphic representations.

2 The automorphic representation

2.1 The automorphic representation of GL_3 we consider is defined by a cuspidal cohomology class in $H^3(\Gamma_0(128), \mathbb{C})$. The subspace of all cuspidal classes is denoted by $H_!^3(\Gamma_0(N), \mathbb{C})$ (cf. [AGG, Sect. 2 and the references given there]). There is a surjective map [AGG, Lemma 3.5]:

$$H_3(\Gamma_0(N), \mathbb{C}) \to H_!^3(\Gamma_0(N), \mathbb{C})^*$$

(with a 'known' kernel). The space $H_3(\Gamma_0(N), \mathbb{C})$ can be computed explicitly using results of [AGG]:

2.2 Proposition. *For any integer N, we define a complex vector space*

$$W(\Gamma_0(N)) := \left\{ \mathbb{P}^2(\mathbb{Z}/N\mathbb{Z}) \xrightarrow{f} \mathbb{C} : \begin{array}{c} f(x,y,z)=f(z,x,y)=-f(-y,x,z), \\ f(x,y,z)+f(-y,x-y,z)+f(y-x,-x,z)=0 \end{array} \right\}.$$

Then there is a natural isomorphism

$$\Phi: W(\Gamma_0(N)) \to H_3(\Gamma_0(N), \mathbb{C}).$$

2.3 The (commutative) Hecke algebra $\mathcal{T}_N$ is generated by linear maps

$$E_p, F_p: H^3(\Gamma_0(N), \mathbb{C}) \to H^3(\Gamma_0(N), \mathbb{C})$$

for primes p, $(p \nmid N)$ [AGG, Proposition 4.1]. The action on $H_3(\Gamma_0(N), \mathbb{C})$ of the adjoint of a Hecke operator can be computed, using Proposition 2.2, with modular symbols [AGG, Sect. 4, 6B].

2.4 Let $u \in H_!^3(\Gamma_0(N), \mathbb{C})$ be an eigenclass for all Hecke operators, and define complex numbers (actually algebraic integers) by

$$E_p u = a_p u, \quad \text{then} \quad L_p(\pi_u, s) = (1 - a_p p^{-s} + \overline{a}_p p^{1-2s} - p^{3-3s})^{-1}$$

is the L-factor at p of the cuspidal automorphic representation π_u corresponding to u (and $F_p u = \overline{a}_p u$).

2.5 By explicit (computer) computation we determined in this way the existence of a unique (up to scalar multiples) eigenclass $u \in H^3(\Gamma_0(128), \mathbb{C})$ which satisfied:

$$E_p u = a_p u, \quad \mathbb{Q}(\ldots, a_p, \ldots) = \mathbb{Q}(i).$$

The explicit values we found are listed below (the a_p are all in $\mathbb{Z}[2i]$):

a_3	a_5	a_7	a_{11}	a_{13}	a_{17}	a_{19}	a_{23}	a_{29}
$1+2i$	$-1-4i$	$1+4i$	$-7-10i$	$-1+4i$	7	$1-14i$	$17-4i$	$-9-12i$

a_{31}	a_{37}	a_{41}	a_{43}	a_{47}	a_{53}	a_{59}	a_{61}	a_{67}
1	$-25+28i$	-5	$-7+30i$	$17+40i$	$23-20i$	$-39+22i$	$63+20i$	$65-22i$

(π_u is not selfdual since there is no Dirichlet character χ with $a_p = \chi(p)\overline{a_p}$ for all p.)

3 The Galois representation

3.1 A minimal surface S with $h^{2,0} = \dim H^0(S, \Omega_S^2) = 2$ has a (rational) canonical map

$$\kappa: S \to \mathbb{P}H^0(S, \Omega_S^2) \cong \mathbb{P}^1.$$

The fibers of κ are the effective canonical divisors on S. If F is such a fiber, the adjunction formula shows that $F^2=p_a(F)-1$, with $p_a(F)$ the arithmetic genus of F [BPV, II.11). If the fibers are elliptic curves, $F^2=0$ and so the canonical pencil is base point free. In that case κ is thus actually a morphism.

If the fibration is not isotrivial (that is, the period map is not constant), then there are no non-constant maps from S to an abelian variety X and thus

$$0=\dim \text{Albanese}(S)=\dim H^0(S, \Omega_S^1)=h^{1,0}, \quad \text{so } h^1(S)=h^3(S)=0\ .$$

(A map $S\to X$ must factor over a map $\mathbb{P}^1\to X$, which is constant; otherwise either X is isogeneous to $\kappa^{-1}(t)\times X'$, but as the moduli of the $\kappa^{-1}(t)$ are non-constant this is impossible or $\kappa^{-1}(t)$ maps to a curve of geometric genus 0, but these always map to a point in an abelian variety (consider the induced map on the Picard varieties).) Thus we have $\chi(\mathcal{O}_S)=1-0+2=3$, and the Noether formula $\chi(\mathcal{O}_S)=(1/12)(c_1^2(S)+c_2(S))$ (cf. [GH, Chap. 4.1]) (with $c_1^2(S)=0$ since $K^2=0$ and $c_2(S)=\chi(S)$ since S is a surface (Gauss–Bonnet formula, [GH, Chap. 3.3])) we find:

$$\chi(S)=36, \quad \text{so} \quad h^2(S)=34\ .$$

If S is such a fibration with a section, there is an involution on S, fiberwise -1 on the elliptic curve, and S modulo that involution is a rational surface. Examples might thus be found among double covers of $\mathbb{P}^2$ ramified along a curve of degree at least 8 (to get $h^{2,0}>1$) and suitable singularities (to get $h^{2,0}<3$).

3.2 The following surfaces were first studied by Ash and Grayson (actually we have a slightly modified form). Let S_a be the (projective) minimal model of the $2:1$ cover of $\mathbb{P}^2$ defined by the (affine) equation:

$$t^2=xy(x^2-1)(y^2-1)(x^2-y^2+axy) \quad (a\in\mathbb{Z}-\{0\})\ .$$

Over $\bar{\mathbb{Q}}$, the branch curve B is a union of 8 lines, it has 16 double points, 2 triple points $((1:0:0),(0:1:0))$ and one fourfold point $(P:=(0:0:1))$. The fourfold point imposes one adjunction condition, so $h^{2,0}(S_a)=2$, and the (rational) canonical map is given by the pencil of lines through P. The fibers of the canonical map are thus elliptic curves, so $h^2(S_a)=34$, $h^1(S_a)=0$.

Let $\bar{S}_a$ be the (singular) double cover of $\mathbb{P}^2$ defined by the affine equation above. Over the double and triple points of B it has singularities of type A_1 and D_4 respectively, resolving them gives a $\mathbb{P}^1$ respectively 4 $\mathbb{P}^1$'s in S_a over such a point (if we call these $\mathbb{P}^1$'s $D_0,\ldots,D_3$ then $D_0\cdot D_i=1$ and $D_i\cdot D_j=0$ if $1\leqq i,j\leqq 3$), see [BPV, III.7]. Using the method of desingularization described there one obtains an elliptic curve E_P over the fourfold point. The equation of E_P is easily found since E_P maps to the exceptional fiber D over P in the blow up of $\mathbb{P}^2$ in P and ramifies over the four points where D meets the strict transform of B.

3.3 The Neron–Severi group $NS(S_a)$ of S_a contains, over $\bar{\mathbb{Q}}$, a 28-dimensional space $N_{\bar{\mathbb{Q}}}$ spanned by the following divisors: the pull-back of a line on $\mathbb{P}^2$, the 16 $\mathbb{P}^1$'s mapping to the 16 double points of B, the $2 \cdot 4$ $\mathbb{P}^1$'s mapping to the 2 triple points, the fiber E_P over the fourfold point, a $\mathbb{P}^1$ mapping to the diagonal $x=y$ and a $\mathbb{P}^1$ mapping to the 'anti-diagonal' $x=-y$ (note that the inverse image of the diagonal and 'anti-diagonal' are reducible in S_a). That these divisor classes are independent in $NS(S_a)\otimes_{\mathbb{Z}}\mathbb{Q}$ follows from the fact that the matrix of their intersection numbers has rank 28, which is not hard to verify.

For any l, we define a $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-subrepresentation of $H^2(S_{a,\bar{\mathbb{Q}}}, \mathbb{Q}_l)$ by:

$$N_{\mathbb{Q}_l} := \mathrm{Image}(N_{\bar{\mathbb{Q}}} \otimes_{\mathbb{Z}} \mathbb{Q}_l \to H^2(S_{a,\bar{\mathbb{Q}}}, \mathbb{Q}_l)) .$$

Let $T_{\mathbb{Q}_l}$ be the orthogonal complement of $N_{\mathbb{Q}_l}$ w.r.t. the intersection form, then:

$$H^2(S_{a,\bar{\mathbb{Q}}}, \mathbb{Q}_l) = T_{\mathbb{Q}_l} \oplus N_{\mathbb{Q}_l} ,$$

a direct sum of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-representations, in particular $\dim T_{\mathbb{Q}_l} = 6$.

3.4 The map:

$$(x, y, t) \mapsto (y, -x, t) \quad \text{induces} \quad \phi: S_a \to S_a ,$$

an automorphism of order 4, defined over $\mathbb{Q}$. Let T_∞ be the orthogonal complement of $NS_{\bar{\mathbb{Q}}}$ in $H^2(S_a(\mathbb{C}), \mathbb{Q})$, then its Hodge numbers are $h^{2,0} = h^{1,1} = h^{0,2} = 2$, in particular $H^{2,0}(S_a) \subset T_{\infty,\mathbb{C}}$.

Since

$$H^{2,0}(S_a) \cong H^0(S_a, \Omega^2_{S_a}) \cong H^0(\mathbb{P}^2, \mathcal{O}(1)(-P)) = \langle x, y \rangle$$

(cf. [BPV, III, Theorem 7.2) the map ϕ has eigenvalues i, $-i$ on $H^{2,0}(S_a)$) (and thus also on $H^{0,2}(S_a)$). To show that the remaining two eigenvalues of ϕ on T_∞ are also i and $-i$ it suffices to show that $\phi^2 = -I$ on T_∞. Note that the fixed point set R of ϕ^2 on S_a consists of the (disjoint) union of E_P, E_∞ and two $\mathbb{P}^1$'s, one over each triple point. Thus $\chi(R) = 4$ and since $\chi(S) = 2\chi(X) - \chi(R)$ (with $X = S/\phi^2$) we have $\chi(X) = 20$. Since $H^i(X, \mathbb{Q}) = H^i(S, \mathbb{Q})^{\phi^2 = I}$, ϕ^2 has 18 eigenvalues $+1$ and thus 16 eigenvalues -1 on $H^2(S, \mathbb{Q})$. It is easy to check that ϕ^2 has trace 8 on $N_{\bar{\mathbb{Q}}}$ (in fact, the only contributions come from the pull-back of a line, two $\mathbb{P}^1$'s (only one of which is pointwise fixed by ϕ^2) over each triple point, E_P, the $\mathbb{P}^1$ over a diagonal and the $\mathbb{P}^1$ over an anti-diagonal). Thus there are 18 eigenvalues $+1$ and 10 eigenvalues -1 on $N_{\bar{\mathbb{Q}}}$. Therefore we have $\phi^2 = -I$ on T_∞.

3.5 Using the basis of $N_{\bar{\mathbb{Q}}}$ and $h^1 = 0$ it is easy to find a formula for the trace of Frobenius acting on $T_{\mathbb{Q}_l}$ using the Lefschetz trace formula. Let E_∞: $w^2 = v(v^2 + av - 1)$, it is the curve in S_a over the line at infinity in $\mathbb{P}^2$, moreover $E_P \cong_{\mathbb{Q}} E_\infty$. Then we have:

3.6 Proposition. *Let* $F_p \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ *be a Frobenius element at* p, *with* $p \nmid 2a(a^2+4)$. Then:

$$\mathrm{Trace}(F_p^n \mid T_{\mathbb{Q}_l}) = N_q(S_a) + 2N_q(E_\infty) - q^2 - 2q(1+\delta^n)\,,$$

with $q = p^n$, *the Legendre symbol* $\delta := \left(\frac{a}{p}\right)$ *and*

$$N_q(S_a) := \#\{(x, y, t) \in \mathbb{F}_q^3 \colon t^2 = xy(x^2-1)(y^2-1)(x^2-y^2+axy)\}\,,$$

$$N_q(E_\infty) := \#\{(v, w) \in \mathbb{F}_q^2 \colon w^2 = v(v^2+av-1)\}\,.$$

Proof. We will calculate the number of $\mathbb{F}_{p^n}$-rational points on S_a in two ways. Firstly, the Lefschetz trace formula implies that it equals

$$1 + p^{2n} + \mathrm{Trace}(F_p^n \mid N_{\mathbb{Q}_l}) + \mathrm{Trace}(F_p^n \mid T_{\mathbb{Q}_l})\,.$$

The 8 elements in $N_{\mathbb{Q}_l}$ coming from the $\mathbb{P}^1$'s over $(x, y) = (\pm 1, \pm 1), (\pm 1, 0), (0, \pm 1)$ each contribute p^n to this trace while the other 8 $\mathbb{P}^1$'s over double points bring in a contribution $4p^n\left(1+\left(\frac{a^2+4}{p}\right)^n\right)$. One pair of such double points is for example $Q_\pm = (\pm\sqrt{a^2+4}, 1)$; thus the $\mathbb{P}^1$'s over them are interchanged by F_p^n if a^2+4 is not a square in $\mathbb{F}_{p^n}$, and then their contribution to the trace on $N_{\mathbb{Q}_l}$ is zero. In case a^2+4 is a square, the $\mathbb{P}^1$'s over them are defined over $\mathbb{F}_{p^n}$ and thus their classes are multiplied by p^n by F_p^n. (Note that if a^2+4 is not a square in $\mathbb{F}_q$, then there are no $\mathbb{F}_q$-rational points over $Q_\pm$.) Next we obtain $8p^n$ from the $\mathbb{P}^1$'s over the two triple points and p^n from the curve E_P. The contribution from the remaining three cycles is $\left(1+2\left(\frac{a}{p}\right)^n\right)p^n$ (the two components C_1, C_2 over a (anti)diagonal being rational iff a is a square in $\mathbb{F}_{p^n}$, and the sum of the two components is the class H of the pull-back of a line; thus if two such components are interchanged, $F_p^n C_1 = p^n C_2 = p^n(-C_1+H)$ and the contribution to the trace is $-p^n$). Hence the number of rational points is:

$$1 + p^{2n} + 18p^n + 4\left(1+\left(\frac{a^2+4}{p}\right)^n\right)p^n + 2\left(\frac{a}{p}\right)^n p^n + \mathrm{Trace}(F_p^n \mid T_{\mathbb{Q}_l})\,.$$

Now we simply try to count points. On the elliptic curve over infinity we find $N_{p^n}(E_\infty)+1$ of them. Each of the configurations over the triple points yields $4(p^n+1)-3$ points (the points on the 4 D_i's, minus the points counted double). On the affine part minus what is above the (rational) double points and the fourfold point of the configuration one finds $N_{p^n}(S_a) - 9 - 4\left(1+\left(\frac{a^2+4}{p}\right)^n\right)$ points. Then there are the contributions one obtains over the double points, which add up to $8(p^n+1) + 4\left(1+\left(\frac{a^2+4}{p}\right)^n\right)(p^n+1)$. Lastly, on the elliptic curve E_P there are $N_{p^n}(E_\infty)+1$ points. Adding everything yields as a second expression for the number of $\mathbb{F}_{p^n}$-rational points on S_a

$$2N_{p^n}(E_\infty) + N_{p^n}(S_a) + 16p^n + 4\left(1+\left(\frac{a^2+4}{p}\right)^n\right)p^n + 1\,.$$

Combining the two formulas proves the proposition.

3.7 To determine the eigenvalue polynomial of F_p on $T_{\mathbb{Q}_l}$ one would have to compute the number of points over $\mathbb{F}_{p^i}$ for $i=1,\ldots,6$, but in fact one can take $i\leqq 3$. Indeed, if α is an eigenvalue of F_p, then so is $\bar{\alpha}$ and $\alpha\bar{\alpha}=p^2$. Moreover $\mathrm{Det}(F_p|T_{\mathbb{Q}_l})=p^6$, in fact by the previous remark it remains to show that the number of eigenvalues $-p$ is even. Take a prime l with $\sqrt{-1}\notin\mathbb{Q}_l$. Then $T_{\mathbb{Q}_l}$ is a three dimensional $\mathbb{Q}_l(\phi)$ vector space since $\phi^2=-I$ on $T_{\mathbb{Q}_l}$. As ϕ and F_p commute, the kernel of F_p+p is also a $\mathbb{Q}_l(\phi)$ subspace and thus has even dimension as $\mathbb{Q}_l$ vector space.

Therefore the eigenvalue polynomial of F_p on $T_{\mathbb{Q}_l}$ looks like:

$$H_p=X^6-c_1X^5+c_2X^4-c_3X^3+p^2c_2X^2-p^4c_1X+p^6\ .$$

Since the Galois representation on $T_{\mathbb{Q}_l}$ is reducible (after adjoining an $i=\sqrt{-1}$ to $\mathbb{Q}_l$ if necessary), say

$$T_{\mathbb{Q}_l}=V_1\oplus V_2,\quad \text{let}\quad \sigma_1\colon \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})\to \mathrm{GL}(V_1)$$

be the corresponding Galois representation. Then we have a factorization (in $\mathbb{Z}[i][X]$):

$$H_p=(X^3-\chi(p)b_pX^2+p\chi(p)^2\bar{b}_pX-\chi(p)^3p^3)$$
$$\cdot(X^3-\overline{\chi(p)b_p}X^2+p\overline{\chi(p)}^2b_pX-\overline{\chi(p)}^3p^3)\ .$$

Here χ is a $\mathbb{Z}[i]^*$-valued Dirichlet character unramified outside $2a(a^2+4)$ (it is a Tate-twist of the determinant of the Galois representation on V_1) and we write $\chi(p)b_p$ rather than b_p to emphasize that we consider σ_1 to be a twist of the desired representation.

In the case $a=2$, computing points on S_a over $\mathbb{F}_{p^i}$ for $p=3,5$ and $i=1,2,3$ determined H_3 and H_5 and revealed that $\chi(3)=-1$, $\chi(5)=+1$. This suffices to determine χ (note that for $a=2$ it is ramified only at 2) and one has

$$\chi(p)=1\quad \text{if } p\equiv 1,3 \bmod 8,\quad \chi(p)=-1\quad \text{if } p\equiv 5,7 \bmod 8\ .$$

In particular, the number of points over $\mathbb{F}_p$ and $\mathbb{F}_{p^2}$ determines the b_p's for larger primes. For each p we then have a set $\{b_p, \overline{b_p}\}$, but a more careful analysis is necessary to determine which cubic factor of H_p is (minus) the eigenvalue polynomial on V_1. This will be done in 3.8–3.10 below.

3.8 To find out which factor of H_p is the eigenvalue polynomial of $\sigma_1(F_p)$, assume for simplicity that $\ell\equiv 1 \bmod 4$ and fix $i=\sqrt{-1}\in\mathbb{Q}_l$. (Without this assumption one needs to adjoin a square root of -1 to $\mathbb{Q}_l$; this doesn't change to following argument.) Note that $1-i\phi$ as a linear map on $T_{\mathbb{Q}_l}$ has as image V_1, and it acts by multiplication by 2 on V_1. Hence for a Frobenius element $F_p\in\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ at p one finds

$$2\,\mathrm{Trace}(F_p|V_1)=\mathrm{Trace}(F_p|(1-i\phi)T_{\mathbb{Q}_l})$$
$$=\mathrm{Trace}(F_p|T_{\mathbb{Q}_l})-i\,\mathrm{Trace}(F_p\phi\,|\,T_{\mathbb{Q}_l})\ .$$

One calculates the second term here by using a comparison isomorphism

$$\begin{array}{ccc} H^2(S_{a\bar{\mathbb{Q}}}, \mathbb{Q}_l) & \cong & H^2(S_{a\overline{\mathbb{F}_p}}, \mathbb{Q}_l) \\ \cup & & \cup \\ T_{\mathbb{Q}_l} & \xrightarrow{\sim} & \overline{T_{\mathbb{Q}_l}} . \end{array}$$

Take $f_p \in \mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$ the usual Frobenius, then

$$\mathrm{Tace}(F_p\phi \mid T_{\mathbb{Q}_l}) = \mathrm{Trace}(f_p\phi \mid \overline{T_{\mathbb{Q}_l}}) .$$

The latter trace is just the trace of Frobenius on a twist of $\overline{T_{\mathbb{Q}_l}}$, namely the one obtained by taking the quotient

$$S_a^{\mathrm{tw}} = S_a \times_{\mathrm{Spec}(\mathbb{F}_p)} \mathrm{Spec}(\mathbb{F}_{p^4}) / \langle \phi \times f_p \rangle$$

and taking the piece in $H^2(S^{\mathrm{tw}}_{a\overline{\mathbb{F}_p}}, \mathbb{Q}_l)$ corresponding to $\overline{T_{\mathbb{Q}_l}}$.

Recall that an affine equation for S_a is given by

$$t^2 = xy(x^2 - y^2 + axy)(x^2 - 1)(y^2 - 1) .$$

A similar equation for S_a^{tw} is obtained as follows. Put $\mathbb{F}_{p^4} = \mathbb{F}_p(\alpha)$. Then $t, u := \alpha x + \alpha^p y - \alpha^{p^2} x - \alpha^{p^3} y = (\alpha - \alpha^{p^2})x + (\alpha^p - \alpha^{p^3})y$ and $v := \alpha y - \alpha^p x - \alpha^{p^2} y + \alpha^{p^3} x = -(\alpha^p - \alpha^{p^3})x + (\alpha - \alpha^{p^2})y$ generate the function field of S_a^{tw} over $\mathbb{F}_p$. Note that u, v are the $\mathbb{F}_{p^4}(x, y)/\mathbb{F}_{p^4}(x, y)^{\langle \phi \times f_p \rangle}$-traces of αx, αy, respectively. Since u, v are linear expressions in x, y, the equation for S_a yields by substitution the desired equation for S_a^{tw}, namely

$$t^2 = G(u, v)$$

with $G(U, V) \in \mathbb{F}_p[U, V]$ of degree 8. This leads to the following analogue of Proposition 3.6.

3.9 Proposition. *With notations as above, for primes $p \nmid 2a(a^2+4)$ one has*

$$\mathrm{Trace}(F_p\phi \mid T_{\mathbb{Q}_l}) = N_p(S_a^{\mathrm{tw}}) - p^2 .$$

Here $N_p(S_a^{\mathrm{tw}}) = \#\{(t, u, v) \in \mathbb{F}_p^{(3)} \mid t^2 = G(u, v)\}$.

Proof. We proceed as in the proof of Proposition 3.6. Apart from a change of the action of Frobenius, the configuration of lines in $\mathbb{P}^2$ is identical. Note that Frobenius permutes the 16 double points here in cycles of length 4 (both in the case that a^2+4 is, or is not a square in $\mathbb{F}_p^*$). Similarly Frobenius interchanges the two triple points. This implies that on the 24-dimensional space in H^2 generated by the cycles over these points Frobenius has trace 0. From the fact that in this twisted case Frobenius interchanges the pull backs of the lines $x=y$ and $x=-y$ one concludes that the trace on the 3-dimensional space generated by components of these pull backs we have trace p. There is also a contribution p from the elliptic curve E_P, hence the number of $\mathbb{F}_p$-rational points equals

$$1+p^2+2p+\mathrm{Trace}(F_p\phi \mid T_{\mathbb{Q}_l}) .$$

Now we count points in the more naive way. First observe that there is no contribution from any of the cycles over double or triple points. The contribution from the two elliptic curves E_∞ and E_P turns out to be remarkably easy here: using the transformation from (x, y) to (u, v) coordinates one finds that the two curves are not isomorphic as in the previous case, but they are non-trivial quadratic twists of each other. Hence the sum of their rational points (over $\mathbb{F}_p$!) equals $2p+2$. So a second formula for the number of $\mathbb{F}_p$-points is

$$N_p(S_a^{\mathrm{tw}})-1+2p+2=N_p(S_a^{\mathrm{tw}})+2p+1 .$$

This easily implies the proposition.

3.10 We consider again the representation σ_1 on $V_1 \subset T_{\mathbb{Q}_l}$, and let $\chi(p)b_p \in \mathbb{Z}[i]$ be the trace of $\sigma_1(F_p)$ as in (3.7). Then we just found:

$$\chi(p)b_p=\tfrac{1}{2}\mathrm{Trace}(F_p \mid T_{\mathbb{Q}_l})-\tfrac{1}{2}i\,\mathrm{Trace}(F_p\phi \mid T_{\mathbb{Q}_l}) ,$$

and the Propositions 3.6 and 3.9 provide easy formulas to compute these numbers explicitly at least for small primes p.

Since the L-factors of eigenclasses in $H^3(\Gamma_0(128), \mathbb{C})$ have a term $-p^{3-3s}$ and we want a Galois representation with the same L-factors, we define:

$$\varepsilon\colon \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gal}(\mathbb{Q}(\sqrt{-2})/\mathbb{Q}) \to \{\pm 1\}, \quad \text{thus:} \quad \varepsilon(F_p)=\chi(p)^{-1}(=\chi(p)) .$$

The eigenvalues of F_p on $T_{\mathbb{Q}_l} \otimes \varepsilon$ (note the abuse of notation) are the ones of F_p on $T_{\mathbb{Q}_l}$ multiplied by $\chi(p)^{-1}$. The 3-dimensional Galois representation to be considered is:

$$\sigma := \sigma_1 \otimes \varepsilon\colon \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}(V_1 \otimes \varepsilon) .$$

Let P_p be the eigenvalue polynomial of $\sigma(F_p)$ then

$$P_p(X)=X^3-b_pX^2+p\overline{b_p}X-p^3, \quad \text{and} \quad L_p(\sigma, s) := P_p(p^{-s})^{-1} .$$

(This twisted Galois representation occurs in the H^2 of the surface $S_a^{\#}$ whose equation is obtained by replacing t^2 by $-2t^2$ in the equation defining S_a with $a=2$.)

3.11 Proposition. *Let σ be the three dimensional Galois representation obtained from S_2 as above, let $u \in H^3(\Gamma_0(128), \mathbb{C})$ be as in 2.5 and let π_u be the corresponding automorphic representation of* $\mathrm{GL}_{3,\mathbb{Q}}$. *Then:*

$$L_p(\sigma, s)=L_p(\pi_u, s) \quad \textit{for} \quad 3 \leqq p \leqq 67 .$$

3.12 Remarks. 1. Arranging the points on S_a^{tw} in orbits under ϕ one finds using Proposition 3.9 that in fact $b_p \in \mathbb{Z}[2i]$ for all $p \nmid 2a(a^2+4)$.

2. In the case $a=\ell=2$ the representation σ_1 yields an example of a 3-dimensional, non-selfdual, irreducible Galois representation which is unramified outside 2. Indeed, for a general $a\in\mathbb{Z}$ the surface S_a has good reduction at every prime $p\nmid 2a(a^2+4)$. Hence an inertia group at p acts trivially on $H^2(S_{a\bar{\mathbb{Q}}},\mathbb{Q}_l)$ and on 'our' V_1, whenever $p\nmid 2\ell a(a^2+4)$. In particular in case $a=\ell=2$ one obtains a representation unramified outside 2. Moreover, in that case the characteristic polynomial of a Frobenius element at the prime $p=5$ is

$$P_5=X^3+(1+4i)X^2+5(-1+4i)X-125\ .$$

As remarked earlier, from this one concludes that the representation is non-selfdual, because $(-1-4i)/(-1+4i)$ cannot be the value at 5 of a Dirichlet character.

If the representation were reducible, then it would have a one dimensional quotient or a one dimensional subrepresentation. Now one dimensional $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-representations are up to a finite character given by a power of the cyclotomic character. However, one can verify that the zeroes of P_5 are of the form $-1-2(\zeta^4-\zeta^{-4}+\zeta^6+\zeta^{-6})-2i(1+\zeta+\zeta^{-1}+\zeta^5+\zeta^{-5})$, for a primitive 13th root of unity ζ. Such a zero generates a cubic extension of $\mathbb{Q}_2(i)$, and is not a power of 5 times a root of unity. This shows the (absolute) irreducibility of the representation.

4 Generalizations and problems

4.1 Besides the surfaces S_a we have various other families of surfaces which have sub Galois representations of the desired type in H^2. For example, let $\mathscr{E}_a$ be the Neron model of the elliptic surface

$$\mathscr{E}_a\colon\ Y^2=X(X^2+2a(t^2+1)X+1)\ .$$

It has 5 singular fibers, one of type I_8 and 4 of type I_1 and it has a section of infinite order. Next one pulls back $\mathscr{E}$ along a 3:1 or 4:1 Galois cover $\mathbb{P}^1\to\mathbb{P}^1_t$ branching over two of the I_1 fibers. The Neron–Severi group of the pull back has rank (at least) 28, and we can again define $T_{\mathbb{Q}_l}$'s as before. In this case however, the representations we find seem to be selfdual (we found $b_p=\bar{b}_p$ for small p, if true for all p this would imply $V_1\cong V_2$), and in one case we could actually find a 2-dimensional Galois representation ρ such that $\mathrm{Sym}^2(\rho)$ has the same b_p's for small p's.

4.2 Another example is provided by the Neron models of the elliptic surfaces

$$\mathscr{E}'_a\colon y^2=x^3+a(x+t^2(t-1))^2\ .$$

These have one I_6, one I_3 and three I_1 fibres as well as a section of infinite order. Pulling them back as in the previous example one again obtains 6-dimensional Galois representations, which split in two 3-dimensional pieces, which now are not selfdual in general (actually for the 4:1 cover, one has to modify the construction of T since the surface itself has $h^{2,0} = 3$ in that case). We hope to relate also these surfaces to automorphic representations of GL_3.

4.3 One can also try to generalize Serre's work and conjectures on mod p Galois representations and mod p modular forms from GL_2 to GL_3. Promising experimental evidence has been found by Ash and McConnell, [AM].

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