An introduction to the Hodge conjecture for abelian varieties

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1 Introduction

1.1 In this lecture we give a brief introduction to the Hodge conjecture for abelian varieties. We describe in some detail the abelian varieties of Weil-type. These are examples due to A. Weil of abelian varieties for which the Hodge conjecture is still open in general.

The Mumford-Tate groups are a very usefull tool for finding the Hodge classes in the cohomology of an abelian variety. We recall their main properties and illustrate it with an example.

Finally we discuss recent results on the Hodge conjecture for abelian fourfolds. Most of this material is well known, and we just hope to provide an easy going introduction.

I am indebted to F. Bardelli for his invitation to give this talk and for stimulating discussions on the Hodge conjecture and Mumford-Tate groups.

2 The Hodge (p, p)-conjecture for abelian varieties

2.1 Let X be a smooth, projective variety over the complex numbers. We denote by

$$Z^p(X)_Q := Z^p(X) \otimes_Z Q$$

the group of codimension p cycles on X with rational coefficients. The group of Hodge classes (of codimension p) is:

$$B^p(X) := H^{2p}(X, Q) \cap H^{p,p}(X) \quad (\subset H^{2p}(X, C)).$$

The cycle class map $Z^p(X)_Q \to H^{2p}(X, Q)$ factors over $B^p(X)$ and defines:

$$\Psi : Z^p(X)_Q \longrightarrow B^p(X),$$

we will usually write $[Z]$, the cohomology class of the cycle Z, for $\Psi(Z)$.

2.2 Hodge (p, p)-conjecture: The map Ψ is surjective.

2.3 The Hodge (p, p)-conjecture is known in case $p = 0, 1$ and thus also in case $p = d - 1, d = \dim X$. In fact, $B^0(X)$ is spanned by $[X]$ and for $p = 1$ the conjecture is proven using the exponential sequence (cf. [V]). However, for the other p's very little is known, [Mu]. It is not easy to determine the groups $B^p(X)$ for a given X. If X varies in a family, the dimension of $B^p(X)$ may change for example.

2.4 One can exploit the ring structure (cup-product) on

$$B^{\cdot} := \oplus_p B^p(X)$$

to obtain information on the Hodge conjecture as follows. Let $D^{\cdot} \subset B^{\cdot}$ be the subring generated by $B^0(X)$ and $B^1(X)$. Then D^p is spanned by:

$$[D_1] \cup [D_2] \cup \ldots \cup [D_p] = [D_1 \cdot D_2 \cdot \ldots \cdot D_p], \qquad D_i \in Z^1(X),$$

with $D_i \cdot D_j$ the intersection product of cycles. Therefore we have the inclusions:

$$D^p \subset Im(\Psi) \subset B^p.$$

In particular, if $D^p = B^p$, then the Hodge (p,p)-conjecture is true for X.

2.5 Definition. An exceptional Hodge class (of codimension p) is an element of B^p which is not in D^p.

2.6 Example. Let $Q \subset P^{2n+1}$ be a smooth quadric. Then one has $H^{p,q}(Q) = 0$ for $p \neq q$ and:

$$B^p(Q) = H^{2p}(Q, X) = \begin{cases} Q & p \neq n, \\ Q^2 & p = n. \end{cases}$$

Thus if $n > 1$ we have $B^1 = D^1 = Q$ and thus $D^p = Q$ for all p. Therefore Q has exceptional Hodge classes in codimension n. It is well known however that the Hodge conjecture is true for Q, in fact such a Q has two rulings (= families of P^n's on it), the cohomology classes of the P^n's span B^n.

3 Abelian varieties

3.1 Let X be an abelian variety over the complex numbers, that is $X \cong C^g/\Lambda$ for some lattice Λ, and X is a projective variety. Note that $\Lambda = \pi_1(X) = H_1(X, Z)$ and C^g is the universal cover of X; also $C^g = T_0X$, the tangent space to X at the origin. Thus $H_1(X, R) = H_1(X, Z) \otimes_Z R \cong T_0X$ and multiplication by $i \in C$ on T_0X corresponds to an R-linear map:

$$J : H_1(X, R) \to H_1(X, R) \qquad \text{with} \quad J^2 = -I.$$

The map J allows us to recover this structure of complex vector space on $H_1(X, R)$; multiplication by $a + bi \in C$, with $a, b \in R$, is given by the linear map $aI + bJ$: $H_1(X, R) \to H_1(X, R)$. Thus

$$T_0X \cong (H_1(X, R), J).$$

3.2 Any embedding $\theta : X \hookrightarrow P^n$ defines a polarization $E := c_1(\theta^*\mathcal{O}(1)) \in B^1(X) \subset H^2(X, Q)$. By the duality, E defines a map, denoted by the same name:

$$E : \wedge^2 H_1(X, Q) \longrightarrow Q$$

and this map satisfies the Riemann Relations (here we extend E R-linearly):

$$E(Jx, Jy) = E(x, y), \qquad E(x, Jx) > 0$$

for all $x, y \in H_1(X, R)$, with $x \neq 0$ for the last condition. That condition also implies that E is non-degenerate.

Conversely, C^g/Λ is an abelian variety iff there exists an $E : \wedge^2\Lambda \to Q$ satisfying the Riemann Relations.

3.3 The cohomology of X and the Hodge structure on it is completely determined by $H^1(X, Q)$ and its Hodge structure:

$$H^p(X, Q) = \wedge^p H^1(X, Q), \qquad H^{p,q}(X) = (\wedge^p H^{1,0}(X)) \otimes (\wedge^q H^{0,1}(X)).$$

This remarkable fact can be exploited to determine the B^p's, see section 6.

3.4 One can (almost) recover X from the Hodge structure on $H^1(X, Q)$ and the polarization E. In fact, using the Hodge decomposition:

$$H^1(X, R) \hookrightarrow H^1(X, C) = H^{1,0}(X) \oplus H^{0,1}(X)$$

one defines a C-linear map $H^1(X, C) \to H^1(X, C)$ by defining it to be multiplying by $+i$ on $H^{1,0}$ and by $-i$ on $H^{0,1}$. This map restricts to an R-linear map:

$$J' : H^1(X, R) \longrightarrow H^1(X, R), \qquad \text{with} \quad (J')^2 = -I.$$

(see [G]). Using the duality: $H^1(X, R) \xrightarrow{\cong} H_1(X, R)^*$ the map J' defines a dual map:

$$J := (J')^* : H_1(X, R) \to H_1(X, R) \qquad \text{with} \quad J^2 = -I.$$

To obtain X, we have to take the quotient of the complex vector space $(H_1(X, R), J)$ by a lattice $\Lambda \subset H_1(X, Q) \subset H_1(X, R)$. Note that $H_1(X, Q)$ is just the dual of $H^1(X, Q)$, but since we are not given $H^1(X, Z)$ we cannot reconstruct $H_1(X, Z)$, that is, we don't know which lattice $\Lambda \subset H_1(X, Q)$ to choose. This leads to the following definitions.

3.5 Definition. Abelian varieties X and Y are said to be isogeneous, $X \approx_{isog} Y$, if there is a finite, surjective map (an isogeny) $\phi : Y \to X$. (If $X \approx_{isog} Y$ then there is actually also a finite, surjective map $X \to Y$.)

An abelian variety is said to be simple if X is not isogeneous to a product of abelian varieties (of dimension > 0).

3.6 Given an isogeny $\phi : Y \to X$, the group $\phi_*(\pi_1(Y))$ is a subgroup of finite index of $\pi_1(X)$, and thus $N\pi_1(X) \cong \pi_1(X) \subset \phi_*\pi_1(Y)$ for some integer N. Therefore one has a finite, surjective map $X \to Y$. The inclusion $\phi_* : H_1(Y, Z) \to H_1(X, Z)$ extended Q-linearly and dualized gives an isomorphism:

$$\phi^* : H^1(X, Q) \xrightarrow{\cong} H^1(Y, Q), \qquad \text{and} \quad \phi_C^*(H^{1,0}(Y)) \subseteq H^{1,0}(X).$$

(The line above is equivalent to saying that ϕ is an isomorphism of Hodge structures.) Conversely, the existence of such a map ϕ^* implies that the abelian varieties X and Y are isogeneous.

An isogeny $\phi : Y \to X$ thus induces isomorphisms $B^p(X) \to B^p(Y)$. Moreover, using pull-back and push-forward of cycles we have the following consequence.

3.7 Lemma. Let $X \approx_{isog} Y$. Then the Hodge (p,p)-conjecture for X is true if and only the Hodge (p,p)-conjecture is true for Y.

4 Overview of results

4.1 With the definitons of the previous sections we can now state some of the results on the Hodge (p,p)-conjecture for abelian varieties. In the later sections we will discuss aspects of the proofs.

4.2 Theorem. (Mattuck, [Ma]) For a general abelian variety one has:

$$B^p(X) = D^p(X) = Q \qquad \text{for all } p,$$

and thus the Hodge (p,p)-conjecture is true for X and all p.

4.3 Theorem. (Tate, [Tat]) For an abelian variety X which is isogeneous to a product of elliptic curves (one dimensional abelian varieties), one has:

$$B^p(X) = D^p(X) \qquad \text{for all } p,$$

and thus the Hodge (p,p)-conjecture is true for X and all p.

4.4 These two theorems deal with rather extreme cases and one could wonder whether one has $B^{\cdot} = D^{\cdot}$ for any abelian variety. This is not the case, but it still happens quite often. Theorem 4.6 is proven using Mumford-Tate groups.

4.5 Theorem. (Mumford, [Po]) There exist simple four dimensional abelian varieties with $B^2 \neq D^2$.

4.6 Theorem. (Tankeev, [Tan], [R]) For a simple abelian variety X whose dimension is a prime number one has:

$$B^p(X) = D^p(X) \qquad \text{for all } p,$$

and thus the Hodge (p,p)-conjecture is true for X and all p.

4.7 The example of Mumford concerned abelian varieties with a large endomorphism algebra (in fact a CM field L with $[L:Q] = 2\dim X$). Weil observed that the field was a composite of a totally real field and an (arbitrary) imaginary quadratic field K. He found that the imaginary quadratic field was 'responsible' for the exceptional Hodge cycles.

4.8 An endomorphism $f : X \to X$ maps the origin $0 \in X$ to itself, and therefore induces a linear map $df_0 : T_0X \to T_0X$. This extends Q-linearly to a ringhomomorphism:

$$t : \mathrm{End}(X)_Q \longrightarrow \mathrm{End}(T_0X), \qquad f \otimes 1 \mapsto t(f \otimes 1) := df_0,$$

here $\mathrm{End}(X)_Q := \mathrm{End}(X) \otimes_Z Q$ is the endomorphism algebra of X.

Similarly, using the maps f^* and f_*, the algebra $\mathrm{End}(X)_Q$ acts on $H^1(X,Q)$ and $H_1(X,Q)$ respectively.

4.9 Definition. An abelian variety of Weil-type of dimension $2n$ is a pair (X, K) with X a $2n$ dimensional abelian variety and $K \hookrightarrow \mathrm{End}(X) \otimes Q$ is an imaginary quadratic field such that for all $x \in K$ the endomorphism $t(x)$ has n eigenvalues x and n eigenvalues $\overline{x}$:

$$t(x) \sim diag(x, \dots, x, \overline{x}, \dots, \overline{x})$$

(here we fix an embedding $K \subset C$).

is an has

The space of Weil-Hodge cycles of (X, K) is defined to be the two dimensional Q-vector space

$$\wedge_K^{2n} H^1(X, Q) \hookrightarrow B^n(X) \subset H^{2n}(X, Q) = \wedge_Q^{2n} H^1(X, Q),$$

where the K-vector space structure on $H^1(X, Q)$ is obtained via f^*, $f \in K \subset \mathrm{End}(X)_Q$. (Note $\dim_Q H^1(X, Q) = 4n$, $\dim_K H^1(X, Q) = 2n$).

A polarized abelian variety of Weil-type is a triple (X, K, E) where (X, K) is an abelian variety of Weil-type and where E is a polarization on X with $(\sqrt{-d})^* E = dE$ and $K = Q(\sqrt{-d})$.

4.10 That $\wedge_K^{2n} H^1(X, Q) \hookrightarrow B^n(X)$ follows from the condition 4.9 on $t(x)$, see 5.2.6. Any abelian variety of Weil type (X, K) with field $K = Q(\sqrt{-d})$ has a polarization E with $(\sqrt{-d})^* E = dE$, see 5.2.1.

In the next section we recall that any $2n$-dimensional polarized abelian variety of Weil-type is a member of a n^2-dimensional family of polarized abelian varieties of Weil-type. The 'general' in the next Theorems refers to the general member of such a family.

Theorem 4.12 shows that in dimension four, the abelian varieties of Weil-type are the only simple ones for which the Hodge conjecture needs to be verified. The proof depends on a detailed study of endomorphism algebras of abelian fourfolds and their Mumford-Tate groups.

4.11 Theorem. (Weil, [W]; cf. Thm 6.12) For a general $2n$-dimensional abelian variety X of Weil-type (with $n > 1$) one has $B^1(X) = Q$ (and thus $D^p(X) = Q$ for all p) but $B^n(X) = Q^3$, the direct sum of $D^n(X)$ and the space of Weil-Hodge cycles. Therefore:

$$B^n(X) \neq D^n(X).$$

4.12 Theorem. (Moonen-Zarhin, [MoZ]) Let X be a simple abelian variety of dimension 4 with $B^2(X) \neq D^2(X)$. Then X is of Weil-type.

4.13 Other examples of abelian varieties with $B^p \neq D^p$ are the abelan varieties with an endomorphism algebra of type III, see [Mur], [H]. In these papers also non-simple abelian varieties are considered. A simple abelian variety of dimension 4 of type III actually has $\dim B^1 = 1$, $\dim B^2 = 6$, however, its endomorphism algebra contains infinitely many imaginary quadratic fields and B^2 is spanned by D^2 and the spaces of Weil-Hodge cycles for these fields ([MoZ], 7.2). In higher dimensions there exist abelian varieties X with $End(X) = Z$ but $B^p \neq D^p$ (cf. [MoZ], 7.5).

4.14 The following theorem provides some examples of four dimensional abelian varieties of Weil-type for which the Hodge conjecture has been verified. In 5.2.3 we will define a discrete invariant

$$det\, H \in Q^*/Nm(K^*)$$

associated to (the isogeny class of) a polarized abelian variety of Weil-type (X, K, E) (here $Nm : K^* \mapsto Q_{>0},\ a \mapsto a\bar{a}$ is the norm map).

The group on the right is an infinite 2-torsion group, and for any $x \in Q^*/Nm(K^*)$ with $(-1)^n x > 0$ we will construct an n^2-dimensional family of abelian varieties of Weil-type with $det\, H = x$.

4.15 Theorem. (Schoen, [S]) The Hodge (2,2)-conjecture is true for the general four dimensional abelian varieties of Weil-type $(X, Q(\sqrt{-3})),\ (X, Q(i))$ with $det\, H = 1$.

of

4.16 The method of Schoen uses the theory of Prym varieties, and can also be used to find cycles on certain abelian varieties of Weil-type also in higher dimensions. In [vG] we use theta functions to give another proof of Theorem 4.15 for the field $Q(i)$.

We will sketch the proof of the last theorem in section 7. The theorem implies easily that for any four dimensional abelian variety (X, K) of Weil-type with field $K = Q(\sqrt{-3})$ or $Q(i)$ and $det\, H = 1$, the space of Weil-Hodge cycles of (X, K) is spanned by cohomology classes of algebraic cycles.

5 Abelian varieties of Weil-type

5.1 As we saw in the previous section, the abelian varieties of Weil type provide an interesting test case for the Hodge conjecture. In this section we first study these abelian varieties (Lemma 5.2) and then we will construct families of such abelian varieties.

Let (X, K) be an abelian variety of Weil type. The action of K on $H_1(X, Q)$ gives $H_1(X, Q)$ the structure of a K-vector space. The space $H_1(X, R)$ has the structure of complex vector space via its identification as $T_0X = (H_1(X, R), J)$ (see 3.1).

5.2 Lemma. Let (X, K), with $K = Q(\sqrt{-d}) \subset C$ be an abelian variety of Weil-type of dimension $2n$.

1. There exists a polarization E on X such that (X, K, E) is a polarized abelian variety of Weil-type.

2. Let (X, K, E) be a polarized abelian variety of Weil-type. Then the map:

   $$H : H_1(X, Q) \times H_1(X, Q) \longrightarrow K, \qquad H(x, y) := E(x, (\sqrt{-d})_* y) + \sqrt{-d} E(x, y)$$

   is a non-degenerate Hermitian form on the K-vectorspace $H_1(X, Q)$. (Hermitian means that H is K-linear in the second factor and $H(y, x) = \overline{H(x, y)}$).

3. Let $\Psi \in M_{2n}(K)$ be the Hermitian matrix which defines H w.r.t. some K-basis of $H_1(X, Q)$. Then
$$det\, \Psi \in Q^*/Nm(K^*)$$
does not depend on the choice of the K-basis, nor the lattice defining X. Thus $det\, \Psi$ is an isogeny invariant of (X, K, E) and will be denoted by $det\, H$ (cf. 4.14).

4. The signature of the Hermitian form H is (n, n).

5. Let $W \subset T_0(X)$ be the n-dimensional complex subspace on which K acts via scalar multiplication by $x \in K \subset C$. Then
$$H_{|W} > 0,$$
where H is extended R-linearly to $T_0X = (H_1(X, R), J)$.

6. $$\wedge_K^{2n} H^1(X, Q) \hookrightarrow B^n(X) = H^{2n}(X, Q) \cap H^{n,n}(X).$$

Proof. For the first statement we observe that $(\sqrt{-d})^*$ acts on $B^1(X)$, with eigenvalues in $\{-d,\, d\}$ (use that $B^1(X) \subset H^{1,1}(X) = H^{1,0}(X) \otimes H^{0,1}(X)$). Thus any polarization can be written as $E = E_+ + E_-$ with $E_\pm \in B^1(X)$ and $(\sqrt{-d})^* E_\pm = \pm d E_\pm$. We claim that E_+ is in fact a polarization. Since $E_+ \in B^1(X)$, the first Riemann condition is satisfied. The second one follows from adding the inequalities $dE(x, Jx) > 0$ and $E((\sqrt{-d})_* x, J(\sqrt{-d})_* x) = dE_+(x, Jx) - dE_-(x, Jx) > 0$ shows that $2dE_+(x, Jx) > 0$ for $x \neq 0$ (note $(\sqrt{-d})_*$ commutes with J since it is an endomorphism of X). Note that for the general abelian variety of Weil-type X one has $B^1(X) = Q$ (see Theorem 4.11), so the polarization is unique and must be of Weil-type.

That H is Hermitian is an easy computation, using $E((\sqrt{-d})_* x, (\sqrt{-d})_* y) = dE(x, y)$.

The form H is thus given by $H(x, y) = {}^t\bar{x}\Psi y$ and ${}^t\Psi = \overline{\Psi}$. Changing the K-basis by linear map A changes Ψ to ${}^t\overline{A}\Psi A$ and thus $det\, \Psi$ changes to $Nm(a) \cdot det\, \Psi$ where $a = det\, A$. Let $\phi : Y \to X$ with (X, K, E) of Weil-type be an isogeny. Since isogenies are isomorphisms on $(H_1)_Q$ which preserve End_Q, also $(Y, K, \phi^* E)$ must be of Weil-type, and the map $\phi^* : H^1(X, Q) \to H^1(Y, Q)$ is an isomorphism of K-vector spaces. Thus $det\, H_E = det\, H_{\phi^* E}$.

The map $(\sqrt{-d})_*$ is a C-linear map on $T_0(X) = (H_1(X, R), J)$ (since it commutes with J), and has two eigenspaces $W_\pm$, each of dimension n, on which it acts as $\pm\sqrt{-d} = \pm\sqrt{d}J$. Thus restricted to $W_\pm$ we have $H(x, x) := E(x, \sqrt{-d}x) = \pm\sqrt{d}E(x, Jx)$. Since $E(x, Jx) > 0$ for $x \neq 0$ by the second Riemann condition, H is positive definite on W_+ (and negative definite on W_-). Note that W_+ and W_- are perpendicular w.r.t. H since
$$dH(x_+, x_-) = H((\sqrt{-d})_* x_+, (\sqrt{-d})_* x_-) = H(\sqrt{d}Jx_+, -\sqrt{d}Jx_-) = -dH(x_+, x_-)$$
with $x_\pm \in W_\pm$ (we use that $H(Jx, Jy) = H(x, y)$ which follows from the fact that J and $(\sqrt{-d})_*$ commute and the first Riemann condition).

First we show there is an inclusion $\wedge_K^{2n} H^1(X,Q) \subset \wedge_Q^{2n} H^1(X,Q) = H^{2n}(X,Q)$. Let V be a finite dimensional K-vector space and let $W = V^* := \mathrm{Hom}_K(V,K)$ be its dual, note $V = W^*$. Let $Tr : K \to Q,\ z \mapsto z + \bar{z}$ be the trace map, then:

$$W^* = \mathrm{Hom}_K(W,K) \xrightarrow{\cong} W^{*Q} := \mathrm{Hom}_Q(W,Q), \qquad f \mapsto Tr \circ f$$

is an isomorphism of Q-vector spaces (reduce to $W \cong K^n$ and then $W = K$ (and $f(z) = az$), now use that for $a \in K$ one has: $Tr(az) = 0$ for all $z \in K$ iff $z = 0$, which proves injectivity, surjectivity follows by comparing dimensions).

Using that $\wedge_K^n W^* = (\wedge_K^n W)^*$ and that K-linear maps are in particular Q-linear we obtain:

$$\wedge_K^n V = \mathrm{Hom}_K(\wedge_K^n W, K) \longrightarrow \mathrm{Hom}_Q(\wedge_Q^n W, K) \xrightarrow{Tr\circ} \mathrm{Hom}_Q(\wedge_Q^n W, Q),$$

where we compose with the trace in the last map. The space on the right is $(\wedge_Q^n W)^{*Q} \cong \wedge_Q^n (W^{*Q})$ and is thus isomorphic to $\wedge_Q^n W^*$, so

$$\wedge_K^n V \longrightarrow \mathrm{Hom}_Q(\wedge_Q^n W, Q) \cong \wedge_Q^n W^* = \wedge_Q^n V.$$

Actually $\wedge_K^n V$ is naturally a direct summand of $\wedge_Q^n V$, since the Q-linear, alternating map $V^n \to \wedge_K^n V,\ (v_1, \ldots, v_n) \mapsto v_1 \wedge_K \ldots \wedge_K v_n$ factors over a map $\wedge_Q^n V \to \wedge_K^n V$.

To get the Hodge type, we tensor by R (note $K \otimes_Q R = C$) and consider the space $\wedge_C^{2n} H^1(X,R) \subset \wedge_R^{2n} H^1(X,R)$. The eigenspaces $W'_\pm \subset H_1(X,R)$ of $(\sqrt{-d})^*$ (the duals of the $W_\pm$'s) are $K \otimes_Q R$ stable, thus $\wedge_C^{2n} H^1(X,R) = \wedge_C^{2n} W'_+ \oplus \wedge_C^{2n} W'_-$. Since $W'_\pm$ are also stable under J', and the eigenspaces of J' in $H^1(X,C)$ are $H^{1,0}$ and $H^{0,1}$ (with eigenvalues i and $-i$), we have $\dim W'_\pm \otimes_R C \cap H^{1,0} = n$ and thus $(\wedge_C^{2n} W'_\pm) \otimes_R C \subset H^{n,n}(X)$.

□

5.3 In the remainder of this section we will construct and investigate n^2 dimensional families of polarized abelian varieties of Weil type.

Such a family is constructed from data

$$(V, K, H, \Lambda).$$

Here V is a vector space over an imaginary quadratic field $K = Q(\sqrt{-d}) \subset C$ with $\dim_K V = 2n$. Furthermore $\Lambda \subset V$ is a lattice in $V_R := V \otimes_Q R$. Finally H is a Hermitian form on V, with signature (n,n).

Note that a polarized abelian variety of Weil-type of dimension $2n$ (X, K, E) provides such data. In fact one takes $V = H_1(X,Q)$, $K = K$, H as in Lemma 5.2 and $\Lambda = H_1(X,Z)$. If, in the construction below, one puts $V_+ := W$, with W as in Lemma 5.2.5, one obtains again (X,K). In particular, any (X,K,E) is a member of an n^2 dimensional family of polarized abelian varieties of Weil-type.

5.4 Given the Hermitian form

$$H : V \times V \longrightarrow K$$

of signature (n,n), there exists a K-basis of V, on which H is given by:

$$H(z,w) = a\bar{z}_1 w_1 + \ldots + \bar{z}_n w_n - (\bar{z}_{n+1} w_{n+1} + \ldots + \bar{z}_{2n} w_{2n}), \qquad (5.4.1)$$

with $a \in Q_{>0}$ (see [L]). Conversely, taking $V = K^{2n}$ and defining H by this formula with $a \in Q_{>0}$ we obtain a Hermitian form on V of signature (n,n) with $det\, H = (-1)^n a$.

5.5 Let $K = Q(\sqrt{-d}) \subset C$, so $K \otimes_Q R$ is naturally identified with C and thus $V_R := V \otimes_Q R \cong C^{2n}$. More precisely, multiplication by:

$$i := \sqrt{-d} \otimes (1/\sqrt{d}) : V_R \longrightarrow V_R$$

defines structure of complex vector space (V_R, i) on V_R.

The abelian varieties which we construct are all obtained as V_R/Λ. What changes will be the complex structure on V_R. Such a complex structure is just an R-linear map $J : V_R \longrightarrow V_R$. The polarization E will also be fixed, it will be given by the imaginary part of H:

$$E := Im\, H : V \times V \longrightarrow Q$$

which is an alternating map since H is Hermitian.

To define the complex structures, we choose a complex subspace $V_+ \subset V_R$ (with its complex structure (V_R, i)) with $\dim_C V_+ = n$ such that:

$$H_{|V_+} > 0, \qquad \text{define} \quad V_- := V_+^{\perp},$$

here we extend H R-linearly from V to V_R. Thus H is positive definite on V_+ and therefore negative definite on V_-, the perpendicular w.r.t. H of V_+. Then we have:

$$V_R = V_+ \oplus V_-.$$

Now we define the complex structure on V_R corresponding to V_+ as follows:

$$J = J_{V_+} : V_R \longrightarrow V_R, \qquad Jv_+ = iv_+, \quad Jv_- = -iv_-$$

for all $v_\pm \in V_\pm$. Clearly $J^2 = -I$ and thus we obtain a complex vector space (V_R, J). We note that J is in fact C-linear on (V_R, i), since J and i commute. Therefore J also commutes with the action of K on V_R (via multiplication on the left on $V_R = V \otimes_Q R$).

5.6 With this choice of complex structure on V_R, it remains to show that E satisfies the Riemann Relations, i.e. that

$$E(Jx, Jy) = E(x, y), \qquad E(x, Jx) > 0 \quad \text{for } x \neq 0.$$

The first condition is clear when we write $x = x_+ + x_-$ etc. with $x_\pm \in V_\pm$, use that $Jx_\pm = \pm iv_\pm$, and that $E = Im\, H$ for a Hermitian form H for which $V_+ \perp V_-$. For the second we recall that given $E = Im\, H$ one recovers (the R-linear extension of) H as $H(x, y) = E(x, iy) + iE(x, y)$ (just write $H(x, y) = A(x, y) + iE(x, y)$ with R-valued forms A, E and expand both sides of $iH(x, y) = H(x, iy)$). Writing x as before we get

$$E(x, Jx) = E(x_+, ix_+) - E(x_-, ix_-) = H(x_+, x_+) - H(x_-, x_-) \ > 0$$

for $x \neq 0$ since H is positive definite on V_+ and negative definite on V_-.

5.7 Thus from the data (V, K, H, Λ) and a $V_+ \subset V_R$ we have constructed a polarized abelian variety $(X := V/\Lambda, E)$, with complex structure J on V. E

Moreover, since the complex structure J commutes with the action of K, we have $K \subset \mathrm{End}(X)_Q$. The complex vector space (V_R, J) is the tangent space at the origin of X. The subspaces $V_\pm$ are stable under the action of J and are thus also complex subspaces of (V, J). Since $i = \pm J$ on $V_\pm$ and $i := \sqrt{-d} \otimes (1/\sqrt{d})$, the action of $x \in K$ on V_+ is scalar multiplication by x whereas on V_- it is scalar multiplication by $\overline{x}$. Thus (X, K, E) is a polarized abelian variety of Weil-type.

5.8 The triple (X, K, E) we constructed is thus determined by the choice of V_+ in the Hermitian vectorspace $((V_R, i), H)$. The only condition on V_+ is that H is positive definite on V_+, which is an open condition (in the analytic topology on the Grassmanian of n-dimensional subspaces in the $2n$ dimensional space (V_R, i)). Thus X is a member of an $n^2 = \dim Grass(n, 2n)$ dimensional family of abelian varieties of Weil-type.

5.9 The global structure of this family can also be derived easily. Let

$$SU(n,n) := \{A \in GL((V_R, i)) \cong GL(2n, C) : \; H(x,y) = H(Ax, Ay), \; det(A) = 1\}$$

for all x, $y \in V_R$ (we supress H from the notation, a more accurate notation would be $SU_H(R)$, see the next section).

5.10 Lemma. The group $SU(n,n)$ acts transitively on the set

$$H_n := \{W \subset (V_R, i) \cong C^{2n} : \; W \cong C^n, \; H_{|W} > 0\}.$$

The stabilizer of a $W \in H_n$ is isomorphic to the group $S(U(n) \times U(n))$ (pairs of unitairy matrices with product of the determinants equal to one). Thus:

$$H_n \cong SU(n,n)/S(U(n) \times U(n)).$$

Proof. Given any two n-dimensional subspaces V_+, W_+ on which H is positive definite, we can choose orthonormal bases (w.r.t. to H) $e_1, \ldots, e_n$ of V_+ and $f_1, \ldots, f_n$ of W_+ which can be extended with orthonormal bases $e_{n+1}, \ldots, e_{2n}$ of W_- and $f_{n+1}, \ldots,$ f_{2n} of W_- to C-bases of (V_R, i). On each of these bases of (V_R, i), the form H is then given by the formula 5.4.1 with $a = 1$. Thus the matrix relating the e_i and the f_i preserves H and, after multiplying say f_1 by a suitable $\lambda \in C, \; |\lambda| = 1$, the matrix will be in $SU(n,n)$. Thus the group $SU(n,n)$ acts transitively on the set of n-dimensional subspaces on which H is positive definite.

The stabilizer of V_+ in $SU(n,n)$ consists of maps mapping V_+, and thus also $V_- = V_+^{\perp}$, into itself and preserving the restriction of H on these subspaces, which is definite on each of them. Thus the stabilizer is isomorphic to $S(U(n) \times U(n))$. □

5.11 The set H_n actually has the structure of complex manifold on which the group $SU(n,n)$ acts by holomorphic maps, in fact it is a bounded Hermitian domain. The construction of Weil-type abelian varieties shows that there exist embeddings $H_n \hookrightarrow S_{2n}$, the Siegel space of positive definite $2n \times 2n$ period matrices, see [Sh] and [Mum].

Using the lattice $\Lambda \subset V \subset V_R$ we define a group by:

$$\Gamma = \Gamma_\Lambda = \{A \in SU(n,n) : \; A\Lambda \subset \Lambda\}.$$

Then $\mathcal{H} := H_n/\Gamma$ is a quasi-projective variety and it parametrizes abelian varietes of Weil-type (n,n); it is an example of a Shimura variety (although that name is nowadays in fact reserved for a more sophisticated but related object).

In the case that E is a principal polarization, one has $H_n \subset S_{2n}$ and $\Gamma := Sp(2g, Z) \cap SU(n,n)$, and one obtains an algebraic subvariety $\mathcal{H} \subset \mathcal{A}_{2n}$, the moduli space of principally polarized, $2n$-dimensional, abelian varieties. For a general polarization one has to make the obvious changes.

5.12 Example. Usually one describes (principally polarized) abelian varieties via their period matrix. Here is an example of principally polarized abelian varieties of Weil-type with $K = Q(i)$ and $det\, H = 1$.

Any principally polarized abelian variety of even dimension can be obtained as C^{2n}/Λ with the inclusion $\Lambda \cong Z^{4n} \hookrightarrow C^{2n}$ given by the 'period matrix'

$$\Omega := (I\,\tau) = \begin{pmatrix} I & 0 & \tau_{11} & \tau_{21} \\ 0 & I & \tau_{21} & \tau_{22} \end{pmatrix},$$

where each block on the right is an $n \times n$ matrix. The polarization $\Lambda \times \Lambda \to Z$ is given by the alternating matrix (with $2n \times 2n$ blocks):

$$E := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

The Riemann conditions on E are equivalent to $\tau \in S_{2n}$ (the Siegel space), that is, τ satisfies ${}^t\tau = \tau$ and $Im\,\tau > 0$.

To give an endomorphism of an abelian variety $X \cong C^{2n}/\Lambda$ we must give a C-linear map $A : C^{2n} \to C^{2n}$ which satisfies $A(\Lambda) \subset \Lambda$. Define

$$A := \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \qquad \text{then}: \quad A^2 = -I.$$

In particular, A has n eigenvalues i and n eigenvalues $-i$. Define

$$H_A := \{\tau \in S_{2n} : \ \tau_{11} = \tau_{22},\ \tau_{21} = -\tau_{12}\}.$$

Note that we have $(1/2)n(n+1)$ parameters for τ_{11} and $(1/2)n(n-1)$ parameters for τ_{12} since $\tau_{12} = -\tau_{21} = -{}^t\tau_{12}$, thus $\dim H_A = n^2$. For $\tau \in H_A$ one has:

$$A\Omega = \Omega B \qquad \text{with} \quad B := \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix},$$

which shows that A preserves the lattice. The automorphism B it induces on the lattice preserves E, so A defines an automorphism of X/Λ which preserves the polarization.

Thus we have an n^2 dimensional family of polarized abelian varieties of Weil type, with $K = Q(i)$, over H_A. An easy computation shows $det\, H = 1$.

6 Mumford-Tate groups

6.1 The Mumford-Tate groups were introduced in [Mum]. In general, they are associated to Q-Hodge structures, cf. [DMOS]. Here we restrict ourselves to the case of abelian varieties and do not discuss nor give general definitions. We use Mumford-Tate groups mainly to find the (dimensions of the) spaces of Hodge cycles B^p.

6.2 Let X be an abelian variety. In this section we write:

$$V = V_Q = H^1(X, Q), \qquad V_L := V \otimes_Q L = H^1(X, L),$$

for any field $L \supset Q$. In 3.4 we defined the structure of a complex vectorspace on V_R using a map J' with $(J')^2 = -I$. We define a group:

$$S^1 := \{z \in C^* : \ |z| = 1\}$$

and a representation of this group on V_R:

$$h = h_X : S^1 \longrightarrow SL(V_R), \qquad h(a+bi)v := (aI + bJ')v,$$

with $a, b \in R$ and $v \in V_R$. So h just gives the scalar multiplication by complex numbers of length one on the complex vector space (V_R, J').

That $det(h(z)) = 1$ follows from the fact that J' acts as i on $H^{1,0}$, so $h(z)$ is scalar multiplication by z and $\bar{z}$ on $H^{1,0}$ and $H^{0,1}$ respectively. We also have representations

$$\wedge^n h : S^1 \longrightarrow GL(\wedge^n H^1(X, R)) = GL(H^n(X, R))$$

and, by C-linear extension, on $H^n(X, C)$. The Hodge decomposition on $H^n(X, C)$ can be recovered from $\wedge^n h$ since the action of $(\wedge^n h)(z)$ on $H^{p,q}(X)$, with $p + q = n$, is scalar multiplication by $z^p \bar{z}^q$.

6.3 Recall that an algebraic group defined over a field L is a quasi-projective variety over L whose group laws are given by morphisms defined over L. For example, the group $GL(n)_L$, with a field L, is an algebraic group (as a variety it is defined by $t \cdot det(A) - 1 = 0$, which is a polynomial equation in $L[\ldots, a_{ij}, \ldots, t]$, and the group laws are given by polynomials in the a_{ij} and $t \, (= (detA)^{-1})$ with coefficients in L).

Let G be an algebraic subgroup of $SL(n)_Q$, defined over Q (so its ideal $I(G)$ is generated by polynomials in $Q[\ldots, a_{ij}, \ldots]$). The ring of polynomial functions on G is, as usual,

$$Q[G] := Q[\ldots, a_{ij}, \ldots]/I(G).$$

For a Q-algebra L we define

$$G(L) := \mathrm{Hom}_{Q-algebra}(Q[G], L),$$

the set (in fact group) of L-valued points of G. (If $\phi \in G(L)$ maps $a_{ij} \in Q[G]$ to $l_{ij} \in L$, then ϕ defines a matrix with coefficients l_{ij} which satisfies the defining equations of G; conversely, such a matrix defines a $\phi \in G(L)$. Thus $G(L)$ is just the set of matrices with determinant one with coefficients in L which satisfy the equations defining G. The group law on $G(L)$ is just the matrix product of matrices with coefficients in L).

6.4 Definition. The Special Mumford-Tate group G (sometimes also called Hodge group) of the abelian variety X is the smallest algebraic subgroup $G \subset SL(V_Q)$, which is defined over Q, such that:

$$h(S^1) \subset G(R).$$

(The Mumford Tate group itself is $G_m \cdot G$, i.e. one also allows scalar multiples of the identity.)

6.5 In this definition, note that $h(S^1) \subset SL(V_R)$ (thus $G \subset SL(V_Q)$ and that the intersection of two algebraic subgroups is again an algebraic subgroup, so the definition makes sense. Since G acts on V_Q it also acts on $\wedge^k V_Q = H^k(X, Q)$ for all k.

Let $Sp(E)$ be the algebraic subgroup of $SL(V_Q)$ which fixes a polarization $E \in \wedge^2 V_Q$ of X. Note that $Sp(E)$ is defined over Q and that $h(S^1) \subset Sp(E)(R)$, in fact, this is equivalent to $E(h^*(z)x, h^*(z)y) = E(x, y)$ for all $x, y \in H_1(X, R)$ (with $h^*(z) = aI + bJ$ the dual representation of h), which follows from $J^2 = -I$ and the Riemann Relation $E(x, y) = E(Jx, Jy) = E(x, y)$.

Therefore:

$$G \subset Sp(E).$$

A generalization of this argument gives the main result:

6.6 Theorem. For all p, the space of Hodge cycles is the subspace of G-invariants in $H^{2p}(X, Q)$:

$$B^p(X) = H^{2p}(X, Q)^G.$$

Proof. (Sketch.) First we show $B^p(X) \subset H^{2p}(X, Q)^G$. Let $\rho_k : GL(V) \to GL(\wedge^k V)$ be the k^{th}-exterior power of the standard representation ρ_1 of $GL(V)$. Since $B^p(X)$ is a Q-subspace of $H^{2p}(X, Q)$, the subgroup $P = P_p \subset SL(V_Q)$ which via ρ_{2p} acts as the identity on $B^p(X)$, is an algebraic subgroup defined over Q. Because $(\wedge^{2p} h)(z)$ acts as $z^p \bar{z}^p$, that is, as the identity on $H^{p,p} \supset B^p$, we have $(\wedge^{2p} h)(S^1) = \rho_{2p}(h(S^1)) \subset \rho_{2p}(P(R))$ and thus $h(S^1) \subset P(R)$. Therefore $G \subset P$ and thus $B^p \subset H^{2p}(X, Q)^P \subset H^{2p}(X, Q)^G$.

To show $B^p(X) \supset H^{2p}(X, Q)^G$ we must show $H^{p,p} \supset H^{2p}(X, Q)^G$ and $H^{2p}(X, Q) \supset H^{2p}(X, Q)^G$. The last is trivial, the first follows from the fact that $h(S^1) \subset G(R)$, so $G(R)$-invariants must be $h(S^1)$-invariants. Since $h(S^1)$ acts as $z^a \bar{z}^b$ on $H^{a,b}$, the space of $h(S^1)$-invariants in $H^{2p}(X, C)$ is just $H^{p,p}(X)$. □

6.7 The Special Mumford-Tate group itself can be somewhat subtle, however one has

$$B^p(X) \otimes_Q C = (H^{2p}(X, Q)^G) \otimes C = H^{2p}(X, C)^{G(C)}.$$

The group $G(C)$, a complex Lie group, is known to be a connected reductive Lie group [DMOS] (here one uses the polarization in an essential way) and thus it, and especially its Lie algebra, is well understood. To compute the space of Hodge cycles one determines the representation of $G(C)$ (or its Lie algebra) on V_C and then, using representation theory, one tries to find the invariants in $\wedge^{2p} V_C$.

6.8 Example. For a general abelian variety of dimension g one has $G = Sp(E)$, the proof is similar to the one sketched in 6.11. Then $G(C) \cong Sp(2g, C)$ and the representation of $Sp(2g, C)$ on V_C is just the standard representation. It is well known that the subspace of invariants of $Sp(2g, C)$ in $\wedge^{2p} V_C$ is one dimensional and is spanned by $\wedge^p E$ (where $E \in H^2(X, Q)$ is the polarization), cf. Thm 17.5 from [Fu]. Thus we obtain another proof of Mattuck's result 4.2.

6.9 Example. Let now (X, K, E) be a $2n$-dimensional polarized abelian variety of Weil-type with $K = Q(\sqrt{-d})$. Then $V = H^1(X, Q)$ has the structure of a Hermitian K-vector space, with H the Hermitian form associated to E. We will construct algebraic groups U_H and SU_H, defined over Q. The group SU_H will be the Special Mumford Tate group of a general abelian variety of Weil-type.

For convenience, let $B_K := \{e_1, \ldots, e_{2n}\}$ be a K-basis of V for which H is given by a diagonal Hermitian matrix Ψ, so:

$$\Psi = {}^t\Psi \in M_{2n}(Q).$$

The condition that a $2n \times 2n$ matrix $A = (a_{ij})$ is unitary (that is ${}^t\overline{A}\Psi A = \Psi$) is *not* given by polynomials in $K[\ldots, a_{ij}, \ldots]$, since conjugation $x \mapsto \bar{x}$ is not given by a polynomial in $K[X]$. However, viewing the K-vector space V as a Q-vectorspace we can define an algebraic group U_H over Q with the property that $U_H(Q)$ is isomorphic to the K-linear maps on V which preserve H.

A Q-basis of V is given by $B_Q := \{e_1, \ldots, e_{4n}\}$, with $e_{2n+j} := (\sqrt{-d})^* e_j$. Since the K-linear maps are just the Q-linear maps which commute with $(\sqrt{-d})^*$, we consider the algebraic group R of such (invertible) maps:

$$R := \left\{ r_{B,C} := \begin{pmatrix} B & -dC \\ C & B \end{pmatrix} \in GL(V_Q) \right\},$$

where each block is an $2n \times 2n$ matrix (in fact $R = Res_{K/Q}(GL(V_K))$). Note that on the basis B_Q, the map $(\sqrt{-d})^*$ is given by $r_{0,I} \in R(Q)$. One easily verifies, for any Q-algebra L, that:

$$R(L) = \{A \in GL(4n, L) : \; A r_{0,I} = r_{0,I} A\},$$

thus $R(L)$ consists of the invertible L-linear maps commuting with $(\sqrt{-d})^* \otimes 1$, i.e. $(\sqrt{-d})^* \otimes 1$-linear maps, on $V_Q \otimes_Q L$. In particular:

$$\kappa : R(Q) \xrightarrow{\cong} GL(V_K)(K) \cong GL(2n, K) \qquad r_{B,C} \mapsto B + \sqrt{-d}C.$$

Next we observe that (in $Q[\ldots, b_{ij}, \ldots, c_{ij}, \ldots]$):

$${}^t(B - \sqrt{-d}C)\Psi(B + \sqrt{-d}C) = \Psi \quad \Longleftrightarrow \left\{ \begin{array}{ll} {}^tB\Psi B + d\,{}^tC\Psi C & = \Psi \\ {}^tB\Psi C - d\,{}^tC\Psi B & = 0. \end{array} \right.$$

This shows that the equations on the right define an algebraic subgroup of R, denoted by U_H, defined over Q, with the property that $\kappa(U_H(Q))$ are the K-linear maps preserving H.

The conditions above are moreover equivalent to:

$${}^t r_{B,C} E_\Psi r_{B,C} = E_\Psi \qquad \text{with} \quad E_\Psi = \begin{pmatrix} 0 & \Psi \\ -\Psi & 0 \end{pmatrix}.$$

This is not so surprising, since the Hermitian form H is determined by its imaginary part E which is given by the alternating matrix E_Ψ on the basis B_Q. Thus we have:

$$U_H = R \cap Sp(E).$$

For any Q-algebra L, $U_H(L) \subset GL(V_Q)(L) \cong GL(4n, L)$ is thus the subgroup of matrices which commute with the action $(\sqrt{-d})^* \otimes 1$ on $V_L := V \otimes_Q L$, which preserve the L-linear extension of the Q-bilinear form E.

Finally we define SU_H to be the subgroup of U_H defined by the two polynomial equations ('real' and 'imaginary part') in $Q[\ldots, b_{ij}, \ldots, c_{ij}, \ldots]$ obtained from the condition:

$$det(B + \sqrt{-d}C) = 1.$$

Then one has $SU_H(R) = SU(n, n)$, as in 5.8.

6.10 Lemma. With the notation from above, we have:

$$SU_H(C) \cong SL(2n, C).$$

Moreover, the representation of $SU_H(C)$ on $V \otimes_Q C$ is isomorphic to the direct sum of the standard representation of $SL(2n, C)$ and its dual representation.

Proof. The action of $(\sqrt{-d})^*$ on V can be diagonalized on $V \otimes_Q C$:

$$V_C = W \oplus \overline{W}, \qquad \text{with} \quad W := \langle \ldots, e_i \otimes \sqrt{-d} + e_{i+2n} \otimes 1, \ldots \rangle_{i=1,\ldots,2n}$$

(use $e_{i+2n} := (\sqrt{-d})^* e_i$). Since the SU_H action on V commutes with $(\sqrt{-d})^*$, we see that both W and $\overline{W}$ are invariant subspaces of $SU_H(C)$. In terms of matrices one has, with $r_{B,C} \in SU_H \subset R$: $r_{B,C} S = SD$, with

$$D := \begin{pmatrix} B + \sqrt{-d}C & 0 \\ 0 & B - \sqrt{-d}C \end{pmatrix}, \qquad S := \begin{pmatrix} \sqrt{-d}I & -\sqrt{-d}I \\ I & I \end{pmatrix}.$$

Thus we have an injective homomorphism of groups:

$$\tau : SU_H(C) \longrightarrow SL(W) \times SL(\overline{W}) \cong SL(2n, C)^2, \qquad r_{B,C} \mapsto (B + \sqrt{-d}C,\; B - \sqrt{-d}C).$$

The spaces W and $\overline{W}$ are isotropic subspaces w.r.t. the C-linear extension of E. In fact, since (X, K, E) is of Weil-type we have $E((\sqrt{-d})^* x, (\sqrt{-d})^* y) = dE(x, y)$ and for $x \in W$ one has $(\sqrt{-d})^* x = \sqrt{-d}x$. Thus if $x, y \in W$ we have:

$$dE(x, y) = E((\sqrt{-d})^* x, (\sqrt{-d})^* y) = E(\sqrt{-d}x, \sqrt{-d}y) = -dE(x, y),$$

and so $H_{|W \times W} = 0$, the same with $\overline{W}$. Therefore E induces a duality between W and $\overline{W}$. Since E is invariant under $SU_H(C) \subset Sp(E)(C)$, the representations induced on W and $\overline{W}$ are dual.

Since $E_{|W \times W} = 0$, any C-linear map $Q : W \to W$ preserves the restriction of E to W. Using the duality of W and $\overline{W}$ defined by E, one gets a map $Q' : \overline{W} \to \overline{W}$ such that the pair $(Q, Q') \in GL(W) \times GL(\overline{W})$ is in (the image of) $U_H(C)$. Taking $Q \in SL(W)$ we get the isomorphism $SL(2n, C) \cong SU_H(C)$.

(To show the map $SU_H(C) \to SL(W)$ is surjective, we can use also a dimension argument. The dimension of $SL(2n, C)$ (as a complex manifold) is $(2n)^2 - 1$. The dimension of $SU(n, n)$ (as real manifold) is also $(2n)^2 - 1$, as is easily seen by a Lie algebra computation.)

Another way to prove the Theorem is to show that $su(n, n) \otimes_R C$, with $su(n, n)$ the Lie algebra of $SU(n, n)$, is isomorphic to the Lie algebra of $SL(2n, C)$, which is the vector space of matrices with trace zero. $\square$

6.11 Theorem. (Weil, [W]) The Special Mumford-Tate group of a general polarized abelian variety (X, K, E) of Weil-type is SU_H.

Proof. (Sketch.) First we show $h(S^1) \subset SU_H(R)$. Recall that $U_H(R) = R(R) \cap Sp(E)(R)$. Since the complex structure J' commutes with $(\sqrt{-d})^* \otimes 1$, we have $J' \in R(R)$ and then also $h(S^1) \in R(R)$. That $h(z)$ fixes the polarization E we have already seen. Finally, the fact that $h(z)$ has n eigenvalues z and n eigenvalues $\bar{z}$ on the complex vectorspace (V_R, i) with $i = (\sqrt{-d})^* \otimes (1/\sqrt{d})$ shows that $h(z) \in SU_H(R)$.

Next we must show that in general SU_H is the smallest algebraic subgroup of $GL(V)$ defined over Q containing $h(S^1)$ for the general X of Weil type. With $z = \cos\phi + i \sin\phi \in S^1$, we have (since $(J')^2 = -I$):

$$h(z) = (\cos\phi)I + (\sin\phi)J' = \ exp(\phi J'), \qquad \text{with} \quad exp : \mathrm{End}(V_R) \longrightarrow GL(V_R)$$

the exponential map ($exp(M) = \sum_{n=0}^{\infty} M^n/(n!)$). For an algebraic subgroup G' of $GL(V_Q)$, the group $G'(R)$ is a Lie group and we thus have:

$$h(S^1) \subset G'(R) \quad \Longleftrightarrow J' \in Lie(G')_R := Lie(G') \otimes_Q R.$$

The complex structure J' is determined by $V_+ \subset (V_R, i)$. For $g \in SU(n,n) = SU_H(R)$ the complex structure $gJ'g^{-1}$ is then determined by the subspace $g(V_+)$. The manifold $H_n = SU(n,n)/S(U(n) \times U(n))$ from 5.10, parametrizing $V_+ \subset (V_R, i)$, may thus be identified with the submanifold of $Lie(SU_H)_R$:

$$H_n = \{gJ'g^{-1} : \ g \in SU_H(R)\} \quad \subset Lie(SU_H)_R.$$

The Adjoint representation of $SU_H(R)$ on its Lie algebra $Lie(SU_H)_R \subset \mathrm{End}(V_R)$ (via $g \cdot M := gMg^{-1}$) is irreducible (one may use for example that $SU_H(R) \subset SU_H(C) \cong SL(2n, C)$ is Zariski dense and that the Adjoint representation of $SL(2n, C)$ is irreducible). Therefore H_n does not lie in any (proper) linear subspace of $Lie(SU_H)_R$.

Thus if $G' \not\subset SU_H$ (so $Lie(G')_R \not\subset Lie(SU_H)_R$), then $H_n \cap Lie(G')_R$ ($\not\subset H_n$) is a real analytic submanifold of H_n. The algebraic group SU_H has only countably many (connected) algebraic subgroups G' defined over Q (because these are determined by their Lie algebra, which is a Q-vector space in the finite dimensional Lie algebra of SU_H). Since H_n is not a countable union of lower dimensional submanifolds, the general $J' \in H_n$ defines an abelian variety with Special Mumford-Tate group equal to SU_H. $\square$

6.12 Theorem. (Weil [W]) Let (X, K) be an abelian variety of Weil-type of dimension $2n$. If the Special Mumford-Tate group of X is SU_H, then:

$$\dim B^p(X) = \begin{cases} 1 & p \neq n, \\ 3 & p = n, \end{cases} \qquad \text{and} \quad B^n(X) = D^n \oplus \wedge_K^{2n} H^1(X, Q).$$

Proof. Since $B^p \cong B^{2n-p}$, it suffices to consider the $p \leq n$. In view of the previous results we have:

$$\dim_Q B^p(X) = \dim_C((\wedge^{2p} V) \otimes_Q C)^{SL_{2n}(C)}.$$

Let W be the standard $2n$-dimensional representation of $SL_{2n}(C)$ and let W^* be its dual. Then:

$$V_C := V \otimes_Q C = W \oplus W^* \cong W \oplus \wedge^{2n-1} W$$

(use the pairing $\wedge^k W \times \wedge^{2n-k} W \to \wedge^{2n} W \cong C$ to identify $\wedge^k W^* \cong \wedge^{2n-k} W$). Thus

$$\begin{aligned}(\wedge^{2p} V) \otimes_Q C &\cong \wedge^{2p}(W \oplus W^*) \\ &= \oplus_{a=0}^{2p} (\wedge^{2p-a} W) \otimes (\wedge^{2n-a} W).\end{aligned}$$

Viewing W as (standard) $GL(2n, C)$ representation, the decomposition of $(\wedge^{2p-a} W) \otimes (\wedge^{2n-a} W)$ into irreducible $GL(2n, C)$ representations is given by formula (6.9) of [Fu]. The $SL(2n, C)$-invariants correspond to the one dimensional $GL(2n, C)$ representations. The formula (6.9) combined with Theorem 6.3.1 of [Fu] shows that:

$$\dim \left((\wedge^{2p-a} W) \otimes (\wedge^{2n-a} W) \right)^{SL(2n,C)} = 1 \qquad \text{iff} \quad n + p - a = kn \quad (k \in N),$$

and in the other cases there no invariants.

In fact, the irreducible representations in $\wedge^a W \otimes \wedge^b W$, with, say $a \geq b$, correspond to partitions of $a + b$ of the form $\lambda = (2, \ldots, 2, 1, \ldots, 1)$, with $\lambda_{a+1} \leq 1$ and with $\lambda_{2n+1} = 0$. The dimension of the corresponding representation is equal to one iff $\lambda_i = \lambda_j$ for all $1 \leq i, j \leq 2n$. The only such partitions are thus $\lambda = (1, \ldots, 1)$ (and $a + b = 2n$) or $\lambda = (2, \ldots, 2)$ (and $a = b = 2n$).

Thus, if $p < n$ we must have $a = p$, and (the dual of) the map $(\wedge^p W) \otimes (\wedge^{2n-p} W) \to \wedge^{2n} W \cong C$ provides the invariant. In case $p = n$ we can take $a = 0, n, 2n$. The cases $a = 0, 2n$ give the invariant subspaces $\wedge^{2n} W$ and $\wedge^{2n} W^*$, which span $(\wedge_K^{2n} H^1(X, Q)) \otimes_Q C$.

Since we have $B^n(X) \supset \wedge_K^{2n} H^1(X, Q) \oplus D^n(X)$ for any abelian variety of Weil-type the equality now follows for dimension reasons. □

7 Sketch of proofs of Theorem 4.15

7.1 The method of Schoen to verify the Hodge conjecture is based on a geometrical construction. First of all, the abelian varieties are constructed in a geometrical way.

A curve C_3 of genus 3 and a subgroup G_3 of order three of $Jac(C_3)$ define an unramified cyclic 3:1 covering:

$$\pi : C_7 \longrightarrow C_3$$

with C_7 a genus seven curve. The map π induces a map π_* on divisors which again induces the norm map $Nm : J(C_7) \longrightarrow J(C_3)$. The connected component of $0 \in J(C_7)$ of $\ker(Nm)$ is called the Prym variety $P = P(C_7/C_3)$ of the covering. Then:

$$J(C_7) \approx_{isog} J(C_3) \times P.$$

The Prym variety P is an abelian variety of dimension 4. Let $\alpha \in Aut(C_7)$ be a generator of the covering group of π. Then α induces an automorphism α^* of order three on P, and thus $Q(\sqrt{-3}) \subset \mathrm{End}(P)_Q$. Using the holomorphic Lefschetz trace formula, one finds that P is Weil-type $(2, 2)$. With the polarization E on P which is induced by the natural polarization on $J(C_7)$, $(P, Q(\sqrt{-3}), E)$ is a polarized abelian variety of Weil-type.

An explicit computation, using for example the description of the action of α on the homology of C_7 given in [Fay], chap. IV, shows that $det\, H = 1$ (this remark was omitted in Thm 3.2 of [S]).

The construction $(C_3, G_3) \mapsto (P, E)$ extends to a morphism, the Prym map:

$$\mathcal{P} : \mathcal{M}_{3,Z/3Z} \longrightarrow \mathcal{A}_{4,E},$$

where $\mathcal{M}_{3,Z/3Z}$ is the (6 dimensional) moduli space of genus 3 curves with a subgroup of order three of $J(C_3)$ and $\mathcal{A}_{4,E}$ is moduli space of four dimensional abelian varieties with a polarization like the one on P.

From section 5.11 we know that the image of $\mathcal{P}$ lies in a $n^2 = 4$ dimensional subvariety $\mathcal{H}$ of $\mathcal{A}_{4,E}$, and Schoen proves that $\mathcal{P}$ has a Zariski dense image in $\mathcal{H}$. Thus the general abelian variety of Weil-type with $K = Q(\sqrt{-3})$ and $det\, H = 1$ is isogeneous to such a Prym variety.

7.2 Schoen explictly constructs cycles on these Prym varieties with cycle classes that span the space of Weil-Hodge cycles. The construction is as follows.

$$\begin{array}{ccc} \cup\tilde{S}_i & \subset & S^4C_7 \\ \downarrow & & \downarrow \\ \cup S_i & \subset & T \\ \downarrow & & {\scriptstyle\tau}\downarrow{\scriptstyle 3:1} \\ |K| & \subset & S^4C_3 \end{array}$$

Since C_7 is an unramified cyclic 3:1 covering of C_3, the map $\pi^{(4)} : S^4C_7 \to S^4C_3$ (with S^kC the k-fold symmetric product of the curve C) can be shown to factor over a fourfold T such that $\tau : T \to S^4C_3$ is an unramified cyclic 3:1 cover. In S^4C_3 lies a $P^2 = |K|$, the linear system of effective, canonical divisors on C_3. Since P^2 is simply connected, $\tau^{-1}(P^2)$ must be reducible, say

$$\tau^*(P^2) = S_1 + S_2 + S_3 \quad (\in Z^2(T)).$$

Thus $\pi^{(4)*}|K|$ has at least three irreducible components $\tilde{S}_i$ in S^4C_7. Using the composition

$$\phi : S^4C_7 \longrightarrow J(C_7) \approx_{isog} J(C_3) \times P \longrightarrow P,$$

one obtains cycles $\phi_*\tilde{S}_i$ in P. Schoen proves that linear combinations of the cycle classes of the $\phi_*\tilde{S}_i$ span the space of Weil-Hodge cycles for the general P, $[P] \in \mathcal{H}$. By specialization (over a one parameter family) one finds that the space of Weil-Hodge cycles on any P, $[P] \in \mathcal{H}$ is spanned by cycle classes.

7.3 Similarly, the general 6 dimensional abelian variety of Weil-type with $K = Q(\sqrt{-3})$ and $det\, H = 1$ is obtained as the Prym of an unramified 3:1 cover $C_{10} \to C_4$ (cf. [F]). Schoen's results imply that the cycles obtained from $|K| \cong P^3 \subset S^6C_4$ will again span the space of Weil-Hodge cycles, proving the Hodge $(3,3)$ conjecture for such an abelian variety.

7.4 Schoen observed that one can also obtain abelian varieties of Weil-type with field $Q(i)$ and $det\,H = 1$ via a Prym construction. Starting from a curve C_{h+1} of genus $h+1$ and a cyclic subgroup of order 4 of $J(C_{h+1})$, one has a tower of unramified 2:1 coverings:

$$C_{4h+1} \longrightarrow C_{2h+1} \longrightarrow C_{h+1}.$$

The Prym variety P of the 2:1 covering $C_{4h+1} \to C_{2h+1}$ is a (principally polarized) abelian variety of dimension $2h$, of Weil-type with field $Q(i)$ (note that C_{4h+1} has an automorphism of order 4) and with $det\,H = 1$. Each P is in fact a member of the family constructed in Example 5.12 (see [vG]).

A variation on a proof in [S] shows that the general abelian fourfold of Weil-type with field $Q(i)$ and $det\,H = 1$ is isogeneous to such a P (cf. [vG]) with $h = 2$. The cycle construction as before gives the proof of the Hodge conjecture for the general member of this family of abelian fourfolds.

7.5 We sketch the method used in [vG] to prove the Hodge conjecture for the general abelian varieties of Weil-type with $K = Q(i)$ and $det\,H = 1$.

In Example 5.12 ('universal') families of such (principally polarized) abelian varieties were constructed. Using the easy description of any member X, one can actually obtain useful information on the multiplication maps $S^nH^0(X,L) \to H^0(X, L^{\otimes n})$ where L is an ample line bundle on the abelian variety X.

We will now restrict ourselves to the case $\dim X = 4$ and $L = \mathcal{O}(2\Theta)$, with Θ a symmetric divisor defining the principal polarization. The automorphism of order 4 of X acts as an automorphism of order 2 on $H^0(X,L)$, splitting it in a direct sum of a 10 dimensional even part H^0_+ and a 6 dimensional odd part H^0_-. Thus, by composing the natural map with the projection, we have a rational map

$$\Phi^-_L : X \longrightarrow P^{15} = PH^0(X,L) \longrightarrow P^5 \cong PH^0_-.$$

The (closure of) the image of X turns out to be a smooth quadric Q (for this one computes the kernel of the map $S^2H^0(X,L) \to H^0(X, L^{\otimes 2})$ and shows that it contains a quadric which lies in the subspace $S^2H^0_- \subset S^2H^0(X,L)$).

Pulling back the rulings of Q to X along Φ^-_L produces cycles whose classes do not lie in $\wedge^2 B^1$. Using the action of $Q(i)^*$ on $H^4(X,Q)$, one finds cycles which span B^2 for the general X in the family.

References

[DMOS] P. Deligne, *Hodge cycles on abelian varieties*, in: Hodge Cycles, Motives, and Shimura Varieties. LNM 900, Springer Verlag, pp. 9-100, (1982).

[F] C. Faber, *Prym varieties of triple cyclic covers*, Math. Z. **199**, 61-79 (1988).

[Fay] J. D. Fay, *Theta Functions on Riemann Surfaces*, LNM 352, Springer Verlag (1973).

[Fu] W. Fulton and J. Harris, *Representation Theory*, GTM 129, Springer-Verlag New York Inc. 1991.

[G] M. Green, lectures in this volume.

[vG] B. van Geemen, *Theta functions and cycles on some abelian fourfolds*, To appear in: Mathematische Zeitschrift.

[H] F. Hazama, *Algebraic cycles on certain abelian varieties and powers of special surfaces*, J. Fac. sci. Univ. Tokyo, Sect. IA, Math. **31**, 487-520 (1984).

[L] W. Landherr, *Äquivalenz Hermitescher Formen über einem beliebigen algebraischen Zahlkörper*, Abh. Math. Semin. Hamburg Univ. **11** 245-248 (1936).

[Ma] A. Mattuck, *Cycles on abelian varieties*, Proc. A.M.S. **9**, 88-98 (1958).

[MoZ] B. Moonen and Yu. Zarhin, *Hodge classes and Tate classes on simple abelian fourfolds*, Preprint Utrecht University (1993).

[Mum] D. Mumford, *Families of abelian varieties*, in: Algebraic Groups an Discontinuous Subgroups, Proc. Symp. Pure Math. **9**, A.M.S. Providence, R.I. 347-351 (1966).

[Mu] J. P. Murre, lectures in this volume.

[Mur] V. K. Murty, *Exceptional Hodge classes on certain abelian varities*, Math. Ann. **268**, 197-206 (1984).

[Po] H. Pohlman, *Algebraic cycles on abelian varieties of complex multiplication type*, Ann. of Math. **88**, 161-180 (1968).

[R] K. A. Ribet, *Hodge classes on certain types of abelian varieties*, Amer. J. Math. **105**, 523-538 (1983).

[S] C. Schoen, *Hodge classes on self-products of a variety with an automorphism*, Comp. Math. **65**, 3-32 (1988).

[Sh] G. Shimura, *On analytic families of polarized abelian varieties and automorphic functions*, Ann. Math. **78**, 149-192 (1963).

[Tat] J. Tate, *Algebraic cycles and poles of zeta functions*, in: Arithmetical Algebraic Geometry. Harper and Row, New York pp. 93-110 (1965).

[Tan] S. G. Tankeev, *On algebraic cycles on surfaces and abelian varieties*, Math. USSR Izv. **18**, 349-380 (1982).

[V] C. Voisin, lectures in this volume.

[W] A. Weil, *Abelian varieties and the Hodge ring*, in: Collected Papers, Vol. III, 421-429. Springer Verlag (1980).

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