A compactification of a fine moduli space of curves

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Introduction

In [4] Deligne and Mumford define stable curves and they prove that the moduli space $\overline{\mathcal{M}_g}$ of stable curves of genus g is a 'compactification' of the moduli space $\mathcal{M}_g$ of smooth curves. For any given integer $m \geq 3$ (invertible on some base scheme) Mumford has constructed a fine moduli scheme $\mathcal{M}_{g,m}$ of curves of genus g with level-m-structure; moreover $\mathcal{M}_{g,m} \to \mathcal{M}_g$ is a Galois covering. It is useful to have a compactification of $\mathcal{M}_{g,m}$,

$$\begin{array}{ccc} \mathcal{M}_{g,m} & \hookrightarrow & ? \\ \big\downarrow & & \big\downarrow \\ \mathcal{M}_g & \hookrightarrow & \overline{\mathcal{M}_g}. \end{array}$$

We find definitions and properties of such a compactification in [4], page 106, in [12], Lecture 10, and in [1], § 2. With the convenient definitions, the results are not so difficult to find, and in this note we put these properties together. The main results are:

- *A compactification $\overline{\mathcal{M}_{g,m}}$, with a tautological family $\mathcal{D} \to \overline{\mathcal{M}_{g,m}}$ exists,* see Theorem (2.1).
- *However this space is not constructed as a coarse or a fine moduli scheme associated with a moduli functor.*
- *The compactification $\overline{\mathcal{M}_{g,m}}$ is a normal space, we describe the local structure of it, in particular for $g \geq 3$ this space is singular,* see Theorem (3.2).

The results of this note were written up as Dept. Math., Univ. Utrecht Preprint 301, August 1983. Our results were partly contained in [9], and we never published this preprint. However as there still seems to be some need for our point of view we publish these results now.

We thank J-L. Brylinski and J. Steenbrink for stimulating discussions.

1 Notations.

(1.1) Level structures. We fix an integer $m \in \mathbb{Z}_{\geq 1}$ (and soon we shall suppose $m \geq 3$). In [10], page 129 a level structure on an abelian variety X of dimension g is defined as an isomorphism $X[m] \cong (\mathbb{Z}/m)^{2g}$. Here we adopt a slightly different notation.

(1.2) Definition. Let S be a base scheme, and $m \in \mathbb{Z}_{\geq 1}$. Note that $((\mathbb{Z}/m)_S)^D \cong \mu_{m,S}$ (here superscript D refers to Cartier duality of finite group schemes). The natural bi-homomorphism

$$e : ((\mathbb{Z}/m)_S)^g \times (\mu_{m,S})^g) \times ((\mathbb{Z}/m)_S)^g \times (\mu_{m,S})^g) \longrightarrow \mu_{m,S}$$

defined by Cartier duality is called the symplectic pairing.

Let $X \to S$ be an abelian scheme of relative dimension g. Suppose m is invertible on S; i.e. there is a canonical morphism $S \to \mathrm{Spec}\ \mathbb{Z}[\frac{1}{m}]$. A *symplectic level m-structure on X/S* is an isomorphism

$$\phi : X[m] \xrightarrow{\sim} ((\mathbb{Z}/m)_S)^g \times (\mu_{m,S})^g$$

which identifies the Weil pairing $e_X : X[m] \times X[m] \to \mu_{m,S}$ with the symplectic pairing.

Let $C \to S$ be a smooth and proper curve over S. A *symplectic level m-structure on C/S* is a symplectic level m-structure on $J := \mathrm{Pic}^0_C$.

(1.3) Remark. Suppose m is invertible on S, and suppose a choice of a primitive m-th root of unity $\zeta_m \in \Gamma(S, \mathcal{O}_S)$ is possible, and has been made. Then we obtain an identification $(\mathbb{Z}/m)_S \cong \mu_{m,S}$, and the notion of a symplectic level-m-structure just given is the same as the one given in [10], page 129.

We could work over schemes over $T = \mathrm{Spec}\ \mathbb{Z}[\zeta_m, \frac{1}{m}]$, and define a level-$m$-structure using the identification of $(\mathbb{Z}/n)_T \cong \mu_{m,T}$.

(1.4) Remark. The definition given above can be generalized as follows. Let m be invertible on S and suppose given a finite flat group scheme $\mathcal{H} \to S$ such that every geometric fiber is isomorphic to $(\mathbb{Z}/m)^{2g}$ with a skew pairing $e : \mathcal{H} \times \mathcal{H} \to \mu_{m,S}$. Use this to define an e-symplectic pairing. The advantage of this is shown in the following example. Choose an elliptic curve E, say over $\mathbb{Q}$, and let $\mathcal{H} := E[m]$ plus its Weil-pairing, call it e. The modular curve representing full level m-structure with this e-symplectic pairing is representable (say $m \geq 3$), and it has a $\mathbb{Q}$-rational point, given by the existence of E.

If you feel that all these fine points are too fancy, just stick to a symplectic structure on $(\mathbb{Z}/m)^{2g}$, and working with base schemes over $T = \mathrm{Spec}\ \mathbb{Z}[\zeta_m, \frac{1}{m}]$, then there is no difference.

(1.5) We fix:

- an integer $g \geq 2$ (the genus),
- an integer $m \geq 3$ (the level),
- and an integer $\nu \geq 5$ (used in multi-canonical embeddings),
- and $N := (2\nu - 1)(g-1) - 1$; note that the ν-multi-canonical map gives $\Phi_{\nu \cdot K} : C \hookrightarrow \mathbb{P}$ with $\mathbb{P}$ a projective space of dimension N. We write PGL for $PGL(N)$ $(= Aut(\mathbb{P}))$.

We write $S_n = \text{Spec } (\mathbb{Z}[\frac{1}{n}])$.

(1.6) By

$$\mathcal{M}_g \to S_1 := \text{Spec } (\mathbb{Z})$$

we denote the (coarse) moduli scheme of curves of genus g as defined and constructed in [4].

We write

$$\begin{array}{ccc} \mathcal{H}_g^0 & \hookrightarrow & \mathcal{H}_g \\ & \searrow & \downarrow \\ & & S \end{array}$$

for the Hilbert schemes of smooth (respectively stable) ν-canonically embedded curves of genus g. Note that

$$\mathcal{M}_g = \text{PGL} \backslash \mathcal{H}_g^0 \quad \text{and} \quad \overline{\mathcal{M}_g} = \text{PGL} \backslash \mathcal{H}_g.$$

The existence theorems for $\mathcal{M}_g$ and for $\mathcal{M}_{g,m}$ are contained in [10]. By [7] we conclude that PGL is geometrically reductive, and by [11], Th. 5.1 we know that its action on $\mathcal{H}_g$ is stable, hence the required geometric quotient exists.

(1.7) By

$$\mathcal{M}_{g,m} \to S_m$$

we denote the (fine) moduli scheme of curves with a simplectic level-m-structure. Note that this is a fine moduli scheme, i.e. there exists a universal curve

$$\mathcal{D}^0 \to \mathcal{M}_{g,m}$$

with a symplectic level-m-structure representing the functor of smooth curves of genus g with such levels. The level-m-structures will be symplectic, thus the covering $\tau : \mathcal{M}_{g,m} \to \mathcal{M}_g$ is Galois with group $\Gamma = \text{Sp}(2g, \mathbb{Z}/m)$ (cf. [10], 7.3). Note that $\mathcal{M}_{g,m} \to S_m$ is smooth (because $m \geq 3$), in particular $\mathcal{M}_{g,m}$ is a normal space.

Note that for every field k of characteristic not dividing m the fiber $\mathcal{M}_{g,m} \otimes$ Spec (K) is a regular variety.

(1.8) Definition. We define

$$\overline{\mathcal{M}_{g,m}} \longrightarrow S_m := \text{Spec } [\frac{1}{m}]$$

to be the normalization of $\overline{\mathcal{M}_g}$ in the function field of $\mathcal{M}_{g,m}$; thus $\mathcal{M}_{g,m} \hookrightarrow \overline{\mathcal{M}_{g,m}}$. Compare: [1], p. 307; [4], p. 106.

Note that for every field k of characteristic not dividing m the fiber $\overline{\mathcal{M}_{g,m}} \otimes$ Spec (K) is a normal variety, see [4], page 106, Th. (5.9)

(1.9) Remark. It would be much more natural to define a moduli functor of "stable curves with a level structures" first, and then try to have a coarse or fine moduli scheme, thus arriving at a definition of $\overline{\mathcal{M}_{g,m}}$. However we do not know such a representable moduli functor defining $\overline{\mathcal{M}_{g,m}}$ (and once the local structure is studied, see Section 3, it will be clear that no "easily defined" functor will do).

(1.10) Tautological curves. Let T be a scheme, and $f : T \to M$ a morphism, where M is a moduli space of curves, and consider a curve $\mathcal{C} \to T$ (plus extra structure ...). We say this is a *tautological curve* if it defines f. In particular, in such a case, for a geometric point $t \in T$ the moduli point of the fiber $\mathcal{C}_t$ equals $f(t)$:

$$[\mathcal{C}_t] \quad = f(t) \in M.$$

Sometimes this is also called a "universal curve", this terminology can be misleading! However if we have a fine moduli scheme, the universal curve is tautological.

2 Construction of a tautological family.

(2.1) Theorem: *Let $g \in \mathbb{Z}_{\geq 2}$ and $m \in \mathbb{Z}_{\geq 3}$. There is a unique*

$$\mathcal{D} \to \overline{\mathcal{M}_{g,m}}$$

which is tautological, and a symplectic level-m-structure on

$$\mathcal{D}|_{\mathcal{M}_{g,m}} =: \mathcal{D}^0 \to \mathcal{M}_{g,m}$$

representing the moduli functor of smooth curves with level structure.

One could also give the theorem for $g = 0$, which would be pedantic, but useful for later use in the case of moduli spaces of pointed curves; in that case the first claim of the theorem is still valid. For $g = 1$, considering curves of $g = 1$ with one base point (called elliptic curves) the theorem is not so difficult and well-known. From now on we suppose $g \geq 2$.

(2.2) Suppose given an open set $U \subset T$ in a scheme, and suppose given a stable curve over U. If this curve extends to a stable curve over T, if U is dense in T, and if T is normal, this extension to a stable curve is unique once this is possible (this follows using [4], Th. (1.11), and Zariski's Main Theorem). The uniqueness in the theorem is not the surprising part, but existence will require some work.

(2.3) Definition. We write

$$\mathcal{H}^0_{g,m} \longrightarrow S$$

for the Hilbert scheme of ν-canonically embedded smooth curves of genus g with symplectic level-m-structure. Note that $\mathrm{PGL}\backslash\mathcal{H}^0_{g,m} = \mathcal{M}_{g,m}$.

(2.4) Definition ([12], p. 137; this definition differs from the one given in [1], p. 307). We define

$$\mathcal{H}_{g,m} \longrightarrow S$$

to be the normalization of $\mathcal{H}_g$ in the function field of $\mathcal{H}^0_{g,m}$,

$$\begin{array}{ccc} \mathcal{H}^0_{g,m} & \hookrightarrow & \mathcal{H}_{g,m} \\ \big\downarrow & & \big\downarrow \\ \mathcal{H}^0_g & \hookrightarrow & \mathcal{H}_g. \end{array}$$

Note that $\mathcal{H}_g$ and $\mathcal{H}_{g,m}$ are non-complete varieties. This is the reason we write $\mathcal{H}_{g,m}$ in stead of a notation like $\overline{\mathcal{H}^0_{g,m}}$.

(2.5) Remark. A point $y \in \mathcal{H}_g$ corresponds to a ν-canonically embedded curve $C_y \subset \mathbb{P}$. A point $x \in \mathcal{H}^0_{g,m}$ corresponds to a pair $x = (C_y \subset \mathbb{P},\, \phi)$ where

$$\phi : J(C_y)[m] \xrightarrow{\sim} (\mathbb{Z}/m)^{2g}$$

is a symplectic isomorphism. For $a \in \mathrm{PGL}$ we define

$$ax = (C_{ay} \subset \mathbb{P},\, \phi \circ (a_{|C_y})^*)$$

where

$$\begin{array}{rcc} C_y & \longrightarrow & \mathbb{P} \\ a_{|C_y}\big\downarrow & & \big\downarrow a \\ a(C_y) = C_{ay} & \longrightarrow & \mathbb{P} \end{array}$$

and the Picard aspect of the Jacobian variety gives isomorphisms:

$$J(C_{ay})[m] \xrightarrow{(a_{|C_y})^*} J(C_y)[m] \xrightarrow{\phi} (\mathbb{Z}/m)^{2g}.$$

This gives an action

$$\mathrm{PGL} \times \mathcal{H}^0_{g,m} \longrightarrow \mathcal{H}^0_{g,m}$$

which extends uniquely to an action of PGL on $\mathcal{H}_{g,m}$ (by [15], Lemma 6.1).

For $h \in \Gamma = \mathrm{Sp}(2g, \mathbb{Z}/m)$ and $x \in \mathcal{H}_{g,m}$ we define $h \cdot x$ in the natural way:

$$h \cdot x = h \cdot (C_y \subset \mathbb{P},\, \phi) := (C_y \subset \mathbb{P},\, h \circ \phi).$$

This action commutes with the action of PGL:

$$h \cdot ax = (C_{ay} \subset \mathbb{P},\, h \circ \phi \circ (a_{|C_y})^*) = a(h \cdot x).$$

The action of Γ on $\mathcal{H}^0_{g,m}$ extends to an action on $\mathcal{H}_{g,m}$ and we obtain an action:

$$(\mathrm{PGL} \times \Gamma) \times \mathcal{H}_{g,m} \longrightarrow \mathcal{H}_{g,m}.$$

Note that PGL also acts on the universal families $\mathcal{C} \to \mathcal{H}_{g,m}$ and $\mathcal{E} \to \mathcal{H}_g$. we conclude that in the diagram

$$\begin{array}{ccccccccc}
\mathcal{H}_{g,m} & \supset & \mathcal{H}^0_{g,m} & \longrightarrow & \mathrm{PGL}\backslash\mathcal{H}^0_{g,m} = & \mathcal{M}_{g,m} & \subset & \overline{\mathcal{M}_{g,m}} & \\
\pi\big\downarrow & & \big\downarrow & & & \big\downarrow & & \big\downarrow\tau & \\
\mathcal{H}_g & \supset & \mathcal{H}^0_g & \longrightarrow & \mathrm{PGL}\backslash\mathcal{H}^0_g = & \mathcal{M}_g & \subset & \overline{\mathcal{M}_g} = \mathrm{PGL}\backslash\mathcal{H}_g &
\end{array}$$

the vertical arrows are Galois coverings, all with Galois group $\Gamma = \mathrm{Sp}(2g, \mathbb{Z}/m)$.

(2.6) Claim. *The action of* PGL *on $\mathcal{H}_{g,m}$ has no fixed points.*

(2.7) We will show that this claim proves the theorem. In fact, we will have:

$$\mathrm{PGL}\backslash\mathcal{H}_{g,m} = \overline{\mathcal{M}_{g,m}},$$

and the universal family of curves $\mathcal{C} \to \mathcal{H}_{g,m}$ descends to a family of stable curves:

$$\begin{array}{ccc}
\mathcal{C} & \longrightarrow & \mathrm{PGL}\backslash\mathcal{C} = \mathcal{D} \\
\big\downarrow & & \big\downarrow \\
\mathcal{H}_{g,m} & \longrightarrow & \mathrm{PGL}\backslash\mathcal{H}_{g,m} = \overline{\mathcal{M}_{g,m}}.
\end{array}$$

(2.8) Notations. Before we give proofs we introduce some further notations. Let $\mathcal{E} \to \mathcal{H}_g$ be the universal family; by results of Raynaud, cf. [13], it can be proved that

$$J := \mathrm{Jac}(\mathcal{E}/\mathcal{H}_g) := \mathbf{Pic}^0(\mathcal{E}/\mathcal{H}_g)$$

exists and that this formation commutes with base change (cf. [4], Th. (2.5) for precise arguments). Denote by $J[m] \subset J \to S$ the scheme of m-torsion points.

Note that the morphism $J[m] \to S$ is étale and quasi-finite. Let $x \in \mathcal{H}_{g,m}$, $y = \pi(x) \in \mathcal{H}_g$, let $U_y \subset \mathcal{H}_g$ be the formal neighborhood of y in $\mathcal{H}_g$, and let $\Delta = \Delta_y \subset U_y$ denote the locus of points over which $\mathcal{E} \to \mathcal{H}_g$ is not smooth. By [4], 1.9, we know that Δ is a divisor with normal crossings. We denote by v the generic point of U_y, and by $\pi_1(U_y - \Delta, v)$ the algebraic fundamental group prime to the characteristic of $k(y)$. Note that

$$J[m]_{|(U_y - \Delta)} \longrightarrow U_y - \Delta$$

is an étale covering, thus we obtain a representation

$$R : \pi_1(U_y - \Delta, v) \longrightarrow \mathrm{Sp}(J_v[m]).$$

The image of this 'monodromy' R is denoted by

$$I_y := R(\pi_1(U_y - \Delta, v)) \subset \mathrm{Sp}(J_v[m])$$

(here $J_v[m]$ is considered as the abstract group $J_v[m](\overline{k(v)})$).

(2.9) Lemma. *The fiber $\pi^{-1}(y) \subset \mathcal{H}_{g,m}$ is given by*

$$\pi^{-1}(y) = I_y \backslash \{\phi |\, \phi : J_v[m] \xrightarrow{\sim} (\mathbb{Z}/m)^{2g},\ \phi \text{ symplectic}\}.$$

Proof. Consider:

$$\begin{array}{ccccccc} X = & \pi^{-1}(Y) & \hookrightarrow & \pi^{-1}(U_y) & \supset & \pi^{-1}(y) \\ \big\downarrow \pi^0 & & & \pi \big\downarrow & & \\ Y = & U_y - \Delta & \hookrightarrow & U_y & \supset & \{y\}. \end{array}$$

Because we work with stable curves, the monodromy around each component of Δ is unipotent, so of order dividing n, so not divisible by $\mathrm{char}(k(y))$ (we work over $S_n = \mathrm{Spec}\ (\mathbb{Z}[n^{-1}])$). The covering $\pi^0 : X \to Y$ is étale so each component is determined by the monodromy R. Because R is tame, because Δ is a divisor with normal crossings, and because $\mathcal{H}_{g,m}$ is normal we see that π is a generalized Kummer covering in the sense of [6], page 12 (use [6], page 39, Corollary 2.3.4). For Kummer coverings one easily proves that the fiber over a point (as a set) is canonically isomorphic with he orbits of the inertia group in the Galois group. Hence the same result follows for generalized Kummer coverings. By definition of $\mathcal{H}_{g,m}$ the fiber $\pi^{-1}(v)$ corresponds to the set of all ϕ as indicated. Thus the lemma follows. □

(2.10) Lemma. *We have:*

$$J_y[m] = (J_v[m])^{I_y}.$$

Proof. First we note that $J[m] \to \mathcal{H}_g$ is an étale morphism, hence a point of $J_y[m]$ extends to a section of $J[m]_{|U_y} \to U_y$, cf. [5]=EGA, IV^4.18.5.17; thus

$$J_y[m] \hookrightarrow (J_v[m])^{I_y}.$$

Let k be the number of singular points of the curve $C = \mathcal{E}_y$. By [4], Theorem (1.6) we can choose local coordinates $\{t_i\}$ $(1 \le i \le N)$ in $y \in U_y \subset \mathcal{H}_g$ so that $\Delta \subset U_y$ is given by the union of the divisors defined by $t_j = 0$, $1 \le j \le k$; moreover the singularities of $\mathcal{E}_y$ are locally given inside $\mathcal{E}$ by equations of the form $a_j b_j - t_j = 0$. Define $V_i \subset U_y$ by the equations $t_i = t_{i+1} = \ldots = t_N = 0$, thus $V_1 = \{y\}$, $V_i \subset V_{i+1}$ and $V_{N+1} = U_y$; let $v_i \in V_i$ be the generic point. In each step $V_i \subset V_{i+1}$, $1 \le i \le N$, we can apply local monodromy on one parameter, cf. [16], page 495, Lemma 2: let $\mathcal{A}$ be the Néron minimal model of $J_{v_{i+1}}$over $\mathrm{Spf}(k(v_i)[[t_i]])$ with special fiber $A := \mathcal{A}(t_i \mapsto 0)$, then the invariants under the monodromy group in $J_{v_{i+1}}[m]$ are precisely $A[m]$. Because the singularities are ordinary quadratic singularities, by [2], page 192, Proposition (3.3.5), we conclude that the relevant part of the monodromy equals

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

or is trivial thus $A[m] \cong (\mathbb{Z}/m)^{m-1}$ when $J_{v_{i+1}}[m] \cong (\mathbb{Z}/m)^m$ in the first case and $A[m] \cong J_{v_{i+1}}[m]$ in the second case. We conclude that $J_y[m] \supset (J_v[m])^{I_y}$.

Note that the term "relevant part of the monodromy" has the following meaning. Consider a stable curve C_0 with singularities $P_1, \cdots, P_d$. The universal deformation space D of C_0 contains $\Delta \subset D$, the discriminant locus, the closed part over which the universal curve is non-smooth. This is a union of divisors $\Delta = \cup_{1 \le i \le d}\ H_i$, such that "$P_i$ stays singular" over H_i. The monodromy around H_i is trivial iff deleting P_i gives a disconnected curve $C_0 - \{P_i\}$. In this case the generic point of H_i parameterizes a "curve of compact type", i.e. a curve where the Jacobian variety is an abelian variety. The monodromy around H_i is non-trivial, and in fact is as indicated as above, iff the curve $C_0 - \{P_i\}$ is connected; this is the case if the generic point of H_i parameterizes an irreducible (singular) curve, and in this case its Jacobian variety is not an abelian variety. $\square$

(2.11) Proposition. *In the diagram*

$$\begin{array}{ccc} F := \{\phi |\, \phi : J_v[m] \xrightarrow{\sim} (\mathbb{Z}/m)^{2g},\ \phi \textit{ symplectic}\} & \longrightarrow & I_y\backslash F \\ \downarrow & \swarrow \eta & \\ F' := \{\rho |\, \rho : J_y[m] \hookrightarrow (\mathbb{Z}/m)^{2g},\ \rho \textit{ symplectic}\} & & \end{array}$$

the natural map η is surjective, but in general η is not injective.

Proof. From the preceding lemma we have $J_y[m] \hookrightarrow (J_v[m])^{I_y}$, and this shows the existence of η. The symplectic structure on $J_y[m]$ (which is degenerate if and only if J_y is not an abelian variety) is induced by the symplectic structure on $J_v[m]$, and it is not difficult to see that a given ρ extends:

$$\begin{array}{ccc} J_v[m] & \stackrel{\exists}{\longrightarrow} & (\mathbb{Z}/m)^{2g} \\ \cup & & \rho\uparrow \\ (J_v[m])^{I_y} & \supset & J_y[m]. \end{array}$$

We choose an irreducible curve C with two ordinary double points. We can choose a symplectic base (intersection form in standard form)

$$\{\alpha_1,\dots,\alpha_g,\beta_1,\dots,\beta_g\} \quad \text{for} \quad J_v[m]$$

such that I_y acts on $J_v[m]$ by

$$\left\{ \begin{array}{ccc} \multicolumn{3}{l}{\{\alpha_1,\dots,\alpha_{g-2},\beta_1,\dots,\beta_g\} \subset (J_v[m])^{I_y} = J_y[m],} \\ \alpha_{g-1} & \longmapsto & \alpha_{g-1}+k_1\beta_{g-1}, \\ \alpha_g & \longmapsto & \alpha_g+k_2\beta_g, \end{array} \right.$$

with k_1, $k_2 \in \mathbb{Z}/m$ depending on which element of I_y is acting, i.e. I_y acts via matrices of the form

$$\left(\begin{array}{c|c} 1 & \begin{array}{ccc} 0 & \vdots & 0 \\ \cdots & & \cdots \\ 0 & \vdots & B \end{array} \\ \hline 0 & 1 \end{array} \right) \qquad B = \left(\begin{array}{cc} k_1 & 0 \\ 0 & k_2 \end{array} \right)$$

where 1 stands for a diagonal-1-matrix and 0 for a zero matrix; this determines the action of I_y, and

$$\{\alpha_1,\dots,\alpha_{g-2},\beta_1,\dots,\beta_g\}$$

is a base for $J_y[m]$. Take a ρ and extend it to some ϕ. Note that the matrix

$$N = \left(\begin{array}{c|c} 1 & \begin{array}{ccc} 0 & \vdots & 0 \\ \cdots & & \cdots \\ 0 & \vdots & A \end{array} \\ \hline 0 & 1 \end{array} \right) \qquad A = \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)$$

is symplectic, hence $\psi := \phi \circ N$ is symplectic. Note that $N \notin I_y$, thus ϕ and ψ define different elements of $I_y\backslash F$, but they restrict to the same $\rho \in \eta(I_y\backslash F)$. This proves Proposition (2.10).

(2.12) Remark. Note that this shows that [12], p. 137, Theorem (10.6) is incorrect.

One should like to have an a-priori definition of the functor which is represented by the scheme $\overline{\mathcal{M}_{g,m}}$ (and the same for $\mathcal{H}_{g,m}$); we were unable to do so, and the previous proposition indicates why we have taken Definition (1.8).

(2.13) Proof of the claim (2.6). We denote by

$$\mathbb{P} \supset \mathcal{C} \longrightarrow \mathcal{H}_{g,m}$$

the universal family, by $J = \mathrm{Jac}(\mathcal{C}/\mathcal{H}_{g,m})$ its relative Jacobi scheme (which is the pull-back by $\mathcal{H}_{g,m} \to \mathcal{H}_g$ of $\mathrm{Jac}(\mathcal{E}/\mathcal{H}_g)$), and by

$$\mathcal{N} = J[m] \ \supset \mathcal{N}^0 = \mathcal{N}_{|\mathcal{H}^0_{g,m}}$$

endowed with universal symplectic level-m-structure

$$\Phi : \mathcal{N}^0 \xrightarrow{\sim} (\mathbb{Z}/n\mathbb{Z})^{2g} \times \mathcal{H}^0_{g,m}.$$

Let $a \in \mathrm{PGL}$, $x \in \mathcal{H}_{g,m}$ and $a(x) = x$. By the universal property of Hilbert schemes there exists a unique morphism A such that

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "A"] \arrow[dr] & a^{-1}\mathcal{E} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\
& \mathcal{H}_g \arrow[r, "a"] & \mathcal{H}_g
\end{tikzcd}$$

commutes, because $a(x) = x$, for $y = \pi(x) \in \mathcal{H}_g$ we have $a(y) = y$, thus

$$\begin{tikzcd}
\mathcal{E}_y \arrow[r, hook] \arrow[d, "A_y"'] & \mathbb{P} \arrow[d, "a"] \\
(a^{-1}\mathcal{E}) = \mathcal{E}_y \arrow[r, hook] & \mathbb{P}
\end{tikzcd}$$

hence $A_y \in \mathrm{Aut}(\mathcal{E}_y)$. If we show that $A_y = \mathrm{id}$, it follows that $a = \mathrm{id}$ (because $\mathcal{E}_y \hookrightarrow \mathbb{P}$ is ν-canonical). Consider

$$F' := \{\rho |\, \rho : J_y[m] \hookrightarrow (\mathbb{Z}/m)^{2g},\ \rho \,\text{symplectic}\,\},$$

and let

$$a_* : F' \longrightarrow F', \qquad a_*(\rho) := \rho \circ (A_y)^*.$$

Note that $a \in \mathrm{PGL}$ acts on $\mathcal{N}^0 \to \mathcal{H}^0_{g,m}$; it follows that

$$\begin{tikzcd}
(\mathbb{Z}/m)^{2g} \times \mathcal{H}^0_{g,m} & & (\mathbb{Z}/m)^{2g} \times \mathcal{H}^0_{g,m} \\
\mathcal{N}^0 \arrow[u, "\Phi"] \arrow[dr] & a^{-1}(\mathcal{N}^0) \arrow[l, "a^*"'] \arrow[r] \arrow[ur, "a^{-1}(\Phi)"] \arrow[d] & \mathcal{N}^0 \arrow[u, "\Phi"'] \arrow[d] \\
& \mathcal{H}^0_{g,m} \arrow[r, "a"] & \mathcal{H}^0_{g,m}
\end{tikzcd}$$

has the property

$$\Phi \circ a^* = a^{-1}(\Phi).$$

If $ay = y$ then a maps $\pi^{-1}(y)$ to itself and we are going to prove

(2.14) *With notations as in Lemma (2.9), the diagram:*

$$\begin{array}{ccc} \pi^{-1}(y) & \xrightarrow{a} & \pi^{-1}(y) \\ \| & & \| \\ I_y\backslash F & \xrightarrow{a} & I_y\backslash F \end{array}$$

gives rise to a map a' such that

$$\begin{array}{ccc} I_y\backslash F & \xrightarrow{a} & I_y\backslash F \\ \eta\downarrow & & \downarrow\eta \\ F' & \xrightarrow{a'} & F' \end{array}$$

and

$$a' = a_* : F' \longrightarrow F'$$

as constructed above.

Indeed, we have seen that $\Phi \circ a^* = a^{-1}(\Phi)$ on $\mathcal{N}^0$, since $a(y) = y$ we get $\phi \circ a^* = a^{-1}(\phi)$ where $\phi = \Phi_{\mathcal{N}_y}$. $\square$

(2.15) **Serre's lemma.** *Let C be a stable curve and let $A \in \mathrm{Aut}(C)$ be such that*

$$(A^* : \mathrm{Jac}(C)[m] \longrightarrow \mathrm{Jac}(C)[m]) = \mathrm{id}$$

(where $n \geq 3$ is an integer, prime to the characteristic); then $A = \mathrm{id}$.
See [1], Lemma 4.

(2.16) **End of proof of the claim (2.6).** We have assumed that $ax = x$, so for $y = \pi x$ we can apply the claim, thus there exists $\rho \in F'$, with $a'(\rho) = \rho$; this implies

$$((A_y)^* : J_y[m] \longrightarrow J_y[m]) = \mathrm{id}_{J_y[m]}$$

(because ρ is injective). We see that the (2.14) and Serre's Lemma prove the Claim (2.6). $\square$

(2.17) We show that Claim (2.6) implies the theorem. We see that $\overline{\mathcal{M}_{g,m}}$ is a geometric quotient of a Hilbert scheme. For $\nu \geq 5$ the points of $\mathcal{H}_g$ are stable with respect to the action of of PGL (cf. [11], Theorem 5.1), so the same holds for points of $\mathcal{H}_{g,m}$. Hence the quotient $\mathrm{PGL}\backslash\mathcal{H}_{g,m}$ exists (same arguments as in [11]), it is a normal variety (cf. [10], p. 5 and use that $\mathcal{H}_{g,m}$ is integral and normal). We have:

$$\begin{array}{ccccc} & & \mathcal{M}_{g,m} = \mathrm{PGL}\backslash\mathcal{H}_{g,m} & & \\ & i \swarrow & \downarrow & \searrow i' & \\ \overline{\mathcal{M}_{g,m}} & & M & & \mathrm{PGL}\backslash\mathcal{H}_g. \\ & \tau \searrow & \downarrow & \swarrow \tau' & \\ & & \overline{\mathcal{M}_g} & & \end{array}$$

Here i, i' are inclusions and τ, τ' are finite maps from normal spaces which coincide on the set $\mathcal{M}_{g,m}$ which is dense in both; by uniqueness of the normalization we conclude (and may identify)

$$\overline{\mathcal{M}_{g,m}} = \mathrm{PGL}\backslash\mathcal{H}_{g,m}.$$

The action of PGL on $\mathcal{H}_{g,m}$ extends naturally to the universal curve $\mathcal{C} \to \mathcal{H}_{g,m}$, and by the Claim (2.6) we conclude that this action has no fixed points on $\mathcal{C}$. Thus

$$\mathrm{PGL}\backslash\mathcal{C} =: \mathcal{D} \longrightarrow \overline{\mathcal{M}_{g,m}} = \mathrm{PGL}\backslash\mathcal{H}_{g,m}$$

is a family of stable curves (which extends the universal family over $\mathcal{M}_{g,m}$). This proves Theorem (2.1). $\square$

3 Local structure of the moduli space

(3.1) We introduce some notation, needed below. We choose $x \in \mathcal{H}_{g,m}$ and denote by C the corresponding curve. The images of this point are denoted as follows:

$$\begin{array}{rcccl} x \in & \overline{\mathcal{H}^0_{g,m}} & \longrightarrow & \overline{\mathcal{M}_{g,m}} & \ni z \\ & \downarrow \pi & & \tau \downarrow & \\ \pi(x) = y \in & \mathcal{H}_g & \longrightarrow & \overline{\mathcal{M}_g} & \ni w = \tau(z). \end{array}$$

We denote by

$$I = I_x = \mathrm{Inertia}(x \in \overline{\mathcal{H}^0_{g,m}} \xrightarrow{\pi} \mathcal{H}_g) \ \subset \Gamma$$

the inertia group at x of the covering π, and analogously

$$G = G_z = \mathrm{Inertia}(z \in \overline{\mathcal{M}_{g,m}} \xrightarrow{\tau} \overline{\mathcal{M}_g}) \ \subset \Gamma;$$

note that we identified the Galois groups

$$\mathrm{Gal}(\overline{\mathcal{H}^0_{g,m}} \longrightarrow \mathcal{H}_g) = \Gamma = \mathrm{Gal}(\overline{\mathcal{M}_{g,m}} \longrightarrow \mathcal{M}_{g,m}).$$

A choice of an imbedding $k(v) \subset k(U_x) \subset \overline{k(v)}$ indices an isomorphism $I_y \cong I_x$ (which explains the notation).

Let $k = \overline{k(w)}$, an algebraically closed field. We write $W = k$ if $\mathrm{char}(k) = 0$, and $W = W_\infty(k)$ the ring of infinite Witt vectors with coordinates in k if $\mathrm{char}(k) = p > 0$. Note that the universal deformation space

$$\mathcal{X} \longrightarrow \mathrm{Spf}(W[[t_1, \ldots, t_N]]) = D$$

exists (cf. [4], pp. 81–83).

(3.2) Theorem.

(1) *There is an exact sequence of groups*

$$0 \to I_x \longrightarrow G_z \xrightarrow{\beta} \mathrm{Aut}(C) \to 0.$$

(2) *The fibers of π and τ are:*

$$\pi^{-1}(y) \cong F/I_y, \quad \text{and} \quad \tau^{-1}(w) \cong F/G_z$$

(with F as in Proposition (2.11)).

(3) *We have the following isomorphisms and a commutative diagram:*

$$\begin{array}{cccc}
U_z & & \hookrightarrow & \overline{\mathcal{M}_{g,m}} \\
\downarrow & & & \\
I\backslash U_z & \cong D & & \downarrow \\
\downarrow & \downarrow & & \\
G\backslash U_z & \cong \mathrm{Aut}(C)\backslash D & \cong U_w \subset & \overline{\mathcal{M}_g}.
\end{array}$$

(4) *If $g \geq 3$ the schemes $\mathcal{H}_{g,m}$ and $\overline{\mathcal{M}_{g,m}}$ have singularities in codimension 2.*

(3.3) Remark. Suppose that $w \in \mathcal{M}_g$, i.e. the curve C is nonsingular. Then $I_x = \{1\}$, we have $G_z = \mathrm{Aut}(C)$, and $D \cong U_z$ (note that $m \geq 3$), and locally on $\mathcal{M}_g$ we have: $U_w \cong \mathrm{Aut}(C)\backslash D \cong \mathrm{Aut}(C)\backslash U_z$. This is well-known.

(3.4) Proof. By (2.7) we have isomorphisms

$$C := \mathcal{E}_y \cong \mathcal{C}_x \cong \mathcal{D}_z.$$

Let $h \in G_z \subset \Gamma$. Then $h \in \Gamma \cong \mathrm{Gal}(\mathcal{H}_{g,m} \to \mathcal{H}_g)$ operates on $\mathcal{H}_{g,m}$, and from $h \in G_z$ it follows that

$$z = x \bmod \mathrm{PGL} = hx \bmod \mathrm{PGL}$$

hence there exists an element $a \in \mathrm{PGL}$, unique by (2.6), such that

$$hx = ax.$$

Then $ay = y$, thus

$$\begin{array}{ccc} C & \xrightarrow{y} & \mathbf{P} \\ \downarrow{\scriptstyle a_{|C}} & & \downarrow{\scriptstyle a} \\ C & \xrightarrow{y} & \mathbf{P} \end{array}$$

and we define

$$\beta_x(h) = a_{|C} \ \in \mathrm{Aut}(C).$$

Clearly β_x is a group homomorphism. If $\beta_x(h) = \mathrm{id}_C$, then $a = \mathrm{id}_C$, so $h \in I_x$; if $h \in I_x$ then $a = \mathrm{id}$, thus $\beta_x(h) = \mathrm{id}_C$. Equivalently

$$\mathrm{Ker}(\beta_x) = I_x.$$

By Lemma (2.9) we conclude:

$$\sharp\pi^{-1}(y) = \sharp\Gamma/\sharp I_x;$$

the group $\mathrm{Aut}(C)$ acts faithfully on the fiber $\pi^{-1}(y)$, it acts via PGL, so every $\mathrm{Aut}(C)$-orbit is mapped to a point in $\overline{\mathcal{M}_{g,m}}$, thus

$$\sharp\pi^{-1}(w) \leq \sharp\pi^{-1}(y)/\sharp\mathrm{Aut}(C).$$

Moreover

$$\sharp\pi^{-1}(w) = \sharp\Gamma/\sharp G,$$

thus we conclude that $\sharp G = \sharp\mathrm{Aut}(C) \cdot \sharp I_x$, and we have proved exactness of the sequence in (3.2).1.

Note that (3.2).2 has already been proved (cf. Lemma (2.9) plus the identification of I_x and I_y).

Using (2.7), the isomorphism $C \cong \mathcal{D}_z$ and the universality of the deformation space D of the curve C we obtain canonically a commutative diagram

$$\begin{array}{ccccccc} & U_x & \longrightarrow & U_z & & & \\ & \downarrow & & \downarrow & & & \\ I_x\backslash U_x = & U_y & \longrightarrow & D & \longrightarrow \mathrm{Aut}(C)\backslash D & \longrightarrow U_w, & \end{array} \qquad (C \cong \mathcal{D}_z).$$

Note that $U_x \to D$ is surjective (any point of D can be lifted to U_y by taking a base for the sections in the ν-canonical sheaf, and $U_x \to U_y$ is surjective), hence $U_z \to D$ is surjective, and we conclude that $U_z \to D$ factors as follows

$$U_z \longrightarrow I_x\backslash U_z \longrightarrow D.$$

Moreover, since $\overline{\mathcal{M}_{g,m}} \to \overline{\mathcal{M}_g}$ is a Galois covering and $\overline{\mathcal{M}_g}$ is normal

$$G_z \backslash U_z \xrightarrow{\sim} U_w.$$

Thus we obtain a commutative diagram

$$\begin{array}{ccc} & & U_z \\ & \swarrow & \downarrow \\ I_x \backslash U_z & \longrightarrow & D \\ \downarrow & & \downarrow \\ G_z \backslash U_z & \longrightarrow & \mathrm{Aut}(C) \backslash D \\ & \cong\searrow & \downarrow \\ & & U_w \end{array}$$

and we conclude

$$\mathrm{Aut}(C) \backslash D \xrightarrow{\sim} U_w$$

(this seemed to be known, cf. [8], §1). Using (1) we conclude the proof of (3.2).3.

To prove (3.2).4 let $g \geq 3$ and let C be a stable curve obtained by choosing regular curves of genus

$$g(C'') = i \geq 1, \qquad g(C') = g - i - 1 \geq 1,$$

which intersect transversally in 2 different points

$$\{P, Q\} = C' \cap C''.$$

The universal deformation family $\mathcal{X} \to D$ is smooth over $D - \Delta$, and $\Delta = \Delta_P \cup \Delta_Q$ consists of two divisors intersecting transversally. Let $d \in \Delta_P,\ d \notin \Delta_Q$ and $e \in \Delta_Q,\ e \notin \Delta_P$ and let $\mathcal{X}_d,\ \mathcal{X}_e$ denote the fibers of $\mathcal{X}$ over d and e respectively. Note that

$$\mathrm{Jac}(\mathcal{X}_d)[m] \cong \mathrm{Jac}(C)[m] \cong \mathrm{Jac}(\mathcal{X}_e)[m] \quad (\cong (\mathbb{Z}/m)^{2g-1}),$$

and one easily sees that

$$\mathbb{Z} \times \mathbb{Z} \cong \pi_1(D - \Delta, v) \xrightarrow{R} \mathrm{Aut}(\mathrm{Jac}(\mathcal{X}_v)[m]),$$

where the isomorphism can be taken such that

$$\mathrm{Ker}(R) = \langle (n, 0),\ (0, n),\ (1, -1) \rangle, \qquad R(\pi_1(D - \Delta), v) \cong \mathbb{Z}/m;$$

moreover the *unique* normal cover $U \to D$ which is étale outside Δ and which is given by this representation R has a local description

$$U = \mathrm{Spf}(W[[t_1, \ldots, t_N]][T]/(T^N - t_1 t_2))$$

(which is a singularity of ''type A_{n-1}'') (Proof: this U is normal, and outside $t_1 = 0 = t_2$ it has the correct structure, hence it is the one we are looking for.)

By what has been proved we know that $U \cong U_z$, thus for the choice of C we made, any point $z \in \tau^{-1}([C])$ is singular on $\overline{\mathcal{M}_{g,m}}$; clearly this gives a closed subset in codimension two as $\overline{\mathcal{M}_{g,m}}$ is normal, so is non-singular in codimension one. If we take

$$\mathcal{E}_{|U_y} \longrightarrow U_y$$

the same arguments apply, and we conclude that any point in $\mathcal{H}_{g,m}$ above the $[C] \in \overline{\mathcal{M}_g}$ chosen above is singular on $\mathcal{H}_{g,m}$. This concludes the proof of Theorem (3.2). □

(3.5) Remark. It is easy to check that $\overline{\mathcal{M}_{g,m}} \to S_m$ is smooth if $g = 2$ and $m \geq 3$. This can be proved by a direct (easy) verification. We can also use [9], page 91, Satz I.

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