Projective models of Picard modular varieties

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1 Introduction

In this paper we study moduli spaces of principally polarized abelian varieties (ppav's) with a polarizaton preserving automorphism. In fact we concentrate on the case in which the automorphism has order 3 or 4. These moduli spaces are quotients of the symmetric domain

$$H_{p,q} := U(p,q)/(U(p) \times U(q)) \hookrightarrow S_g,$$

(with S_g, $g = p+q$, the Siegel upper half space) by a discrete subgroup. These 'modular' embeddings of $H_{p,q}$ in S_g, and generalizations, were studied by Shimura and Satake, [Sat]. In the case of a level-2 subgroup and small genus, we find explicit projective varieties which are isomorphic to the satake compactification of these quotients. The spaces $H_{p,1}$ are in fact complex balls of dimension p and so we obtain projective models for (non-compact) ball quotients.

In most of the cases we consider, the abelian varieties are in fact Jacobians of curves (with an automorphism). A nice example is the case of the 2-dimensional family of genus 3 curves $y^3 = f_4(x)$ which have an isomorphism of order 3. It was already observed by Picard that these curves are parametrized by (a quotient) of $H_{2,1}$. The name Picard modular variety in the title thus refers to the obvious generalization of this quotient of $H_{2,1}$. This family of curves was also investigated by Holzapfel [Ho].

The satake compactification of $H_{2,1}/\Gamma_M(2)$ (for notations see the text) was already investigated by Hunt and Weintraub [HW] who proved that it is isomorphic to the complement of a set of 9 points in P^2. These 9 points are in fact the base points of the Hesse pencil

$$X^3 + Y^3 + Z^3 + \lambda XYZ.$$

The singular curves of this pencil also have a natural moduli interpretation (see thm 8.5). Their method of proof involves the Chern-inequalities for Ball-quotients. Our method instead uses second order theta functions, and can also be applied to Picard modular varieties which are not Ball-quotients (see prop. 10.11 for example).

Also in the case of the 3-dimensional family of genus 4 curves $y^3 = f_6(x)$ we recover a result from [HW]: the satake compactification is the Burkhardt quartic threefold $\mathcal{B}$ in P^4 (see thm 8.6). It is the unique quartic threefold whose singular locus consists of 45 nodes [JVSB]. The projective dual of this threefold is isomorphic to the satake compactification of $S_2/\Gamma_2(3)$, the moduli space of 2-dimensional abelian varieties with a level-3 structure. In section 9 we give a moduli-theoretic description of this birational isomorphism of moduli spaces.

Most of the varieties we find are actually rational. Using the theory of theta functions, it is however not hard to obtain projective models for finite covers, which are also Picard modular varieties, but which in general will not be rational. In a later paper we hope to study these covers, both geometrically and arithmetically.

The results of Hunt and Weintraub [HW] were the main motivation for studying these varieties. I am indebted to B.Hunt for several stimulating discussions. I would like to thank D. van Straten for his help with the computer program 'macaulay'.

2 Characteristics

References for this section are [M1], §2 and [I], V.6.

Let X be an abelian variety and let L be an ample line bundle on X with $\dim H^0(X,L) = 1$. The pair (X,L) is called a principally polarized abelian variety (ppav). One may replace L by any of its translates, that is, only the algebraic equivalence class of L is important for the definition of ppav.

Using the inversion

$$\iota : X \longrightarrow X, \qquad x \mapsto -x$$

one can pick out special bundles among the translates of L. A symmetric line bundle L is a line bundle satisfying:

$$\iota^* L \cong L.$$

Any line bundle is algebraically equivalent to 2^{2g} symmetric line bundles. In case L is a symmetric line bundle defining a principal polarization, the other symmetric line bundles algebraically equivalent to L are obtained by translating L by a point of order two.

2.1 Let L be a symmetric line bundle defining a principal polarization and choose an isomorphism:

$$\phi : L \longrightarrow \iota^* L, \qquad \text{with} \quad \phi(0) = 1 : L(0) \to \iota^* L(0) = L(0),$$

here $0 \in X$ is the origin of X, $L(x)$ is the fiber of L over $x \in X$ and $\phi(x) \in C^*$ is the restriction of ϕ to the fiber $L(x)$. Since the two-torsion points are fixed by ι and since $\iota^*\phi \circ \phi = id_L$, we can define a map:

$$e^L : X[2] \longrightarrow \{\pm 1\}, \qquad e^L(x) := \phi(x).$$

This map is not a homomorphism, instead it satisfies ([M1], § 2, Cor.1):

$$e^L(x+y) = e^L(x) \cdot e^L(y) \cdot e_2(x,y), \tag{2.1.1}$$

where $e_2(x,y)$ is the weil-pairing on the two-torsion points, in particular e_2 is a non-degenerate, bilinear, alternating (i.e. $e_2(x,x) = +1$) form with values in $\{\pm 1\}$. One should thus think of e^L as a quadratic form on $X[2]$, with associated bilinear form e_2.

With respect to a suitable (=symplectic) basis of $X[2]$, the weil-pairing is given by:

$$e_2(x,y) = (-1)^{\sum_{i=1}^g x_i y_{g+i} + x_{g+i} y_i}.$$

A straight forward computation shows that any map $q : X[2] \longrightarrow \{\pm 1\}$ satisfying relation 2.1.1, with $q := e^L$, is given by:

$$q(x) = (-1)^{\sum_{i=1}^g \epsilon_i x_i + \epsilon'_i x_{i+g} + x_i x_{i+g}}, \qquad \epsilon,\ \epsilon' \in \{0,1\}^g.$$

We will call ϵ, ϵ' the characteristics of q, or of L if $q = e^L$ (w.r.t. this basis of $X[2]$).

In particular there are exactly 2^{2g} q's associated with e_2. Since

$$e^{T_x^* L}(y) = e^L(y) \cdot e_2(x,y) \qquad (x, y \in X[2]),$$

with $T_x^* L$ the pull-back of L by $T_x : X \to X,\ y \mapsto x + y$, it follows easily that the map $M \mapsto e^M$, from symmetric line bundles algebraically equivalent to L to quadratic forms associated with e_2 is a bijection.

From the classification theory of quadratic forms one knows that there are two classes of q's. We call q even if $\sum_i \epsilon_i \epsilon'_i = 0 \bmod 2$ and odd otherwise. The even q's are characterised by the fact that they are trivial on a subspace of dimension g of the F_2-vector space $X[2]$. Another way to distinguish the even and odd forms is by counting the number of $x \in X[2]$ with $q(x) = 1$. When q is even there are $e(g)$ and for q odd there are $o(g)$ such points, with

$$e(g) := 2^{g-1}(2^g + 1), \qquad o(g) := 2^{g-1}(2^g - 1).$$

The number of even/odd $q's$ is also equal to $e(g)$ and $o(g)$ resp., in fact if for $x \in X[2]$ one easily verifies:

$$e^{T_x^* L} \text{ has the same parity as } e^L \quad \text{iff} \quad e^L(x) = +1.$$

The group $Sp(2g, F_2)$ of linear maps on the F_2 -vector space $X[2]$ which preserve the weil-pairing e_2, acts transitively on the even and on the odd quadratic forms associated with e_2.

Since $\dim H^0(X, L) = 1$, we can write $L = \mathcal{O}_X(\Theta_L)$, with Θ_L an effective symmetric divisor. Writing $m(x)$ for the multiplicity of Θ_L in $x \in X$ we have the relation ([M1], §2, prop.2):

$$e^L(x) = (-1)^{m(x) - m(0)} \qquad (x \in X[2]).$$

2.2 We relate the general theory above to the case of Jacobians of curves, which have a natural principal polarization. For a curve C, let $Pic^d(C)$ be the algebraic variety parametrizing divisor classes of degree d on C. Inside $Pic^{g-1}(C)$ there is a natural divisor:

$$\Theta := \{x \in Pic^{g-1}(C) \ : \quad h^0(x) := \dim H^0(C, \mathcal{O}_C(x)) > 0\}.$$

For any $\alpha \in Pic^{g-1}$ one obtains a divisor Θ_α in the abelian variety $Pic^0(C) = J(C)$ by:

$$\Theta_\alpha := \{x \in Pic^0(C) \;:\; \quad x + \alpha \in \Theta\}.$$

The corresponding line bundle

$$L_\alpha := \mathcal{O}_{J(C)}(\Theta_\alpha)$$

defines a principal polarization on $J(C)$.

Using Riemann-Roch it is easy to see that $i^*\Theta_\alpha = \Theta_{K-\alpha}$, with K the canonical class of C (indeed, $x \in i^*\Theta_\alpha$ iff $h^0(-x+\alpha) > 0$, but since $deg(-x+\alpha) = g-1$, this is equivalent with $h^0(x + (K-\alpha)) > 0$). In particular, Θ_α is symmetric iff $2\alpha = K$, divisor classes satisfying this condition are called theta characteristics (cf [M2]). A theta characteristic is called even/odd if e^{L_α} is even/odd.

From Riemann's theorem: $m_x(\Theta) = h^0(x)$ (with $x \in Pic^{g-1}(C)$), it follows that for a theta characteristic α we have:

$$e^{L_\alpha}(x) = (-1)^{m(x)-m(0)} = (-1)^{h^0(\alpha+x)-h^0(\alpha)} \qquad\qquad (x \in J(C)[2]).$$

Moreover, one has that e^{L_α} is an even quadratic form iff $h^0(\alpha)$ is an even integer.

2.3 We recall the main results on even theta characteristics on curves of genus ≤ 4. Note that by the Riemann-Kempf theorem, the effective divisors in a linear system α, with $deg(\alpha) = g-1$ and $h^0(\alpha) > 1$ are cut out on the canonical curve by linear subspaces in the tangent cone to Θ at α.

In case $g \leq 2$ we have $h^0(L) = 0$ for all even theta characteristics. In case $g = 3$, we also have $h^0(L) = 0$ for all even theta charateristics on C, except when C is hyperelliptic. On a hyperelliptic curve there is one (even) theta characteristic h with $h^0(h) > 0$. One has $h^0(h) = 2$ and h defines the $2:1$ map $C \to P^1$.

In case $g = 4$, there are either 0, 1 or 10 even theta characteristics with $h^0 \neq 0$, and $h^0 = 2$ for these. In case there are 10, the curve is hyperelliptic and the 10 theta characteristics are $3P_i$, where the 10 P_i are the 10 weierstrass points. In case C is not hyperelliptic, its canonical image lies on a unique quadric. If this quadric is a cone, then the curve has one theta characteristic α with $h^0(\alpha) = 2$. The linear system $|\alpha|$ is cut out by the lines on the cone. In case the quadric is smooth, C doesn't have an even theta characteristic α with $h^0(\alpha) > 0$.

3 Theta functions

3.1 We now study the ppav $X = X_\tau$, with $\tau \in S_g$, the Siegel upper half plane:

$$X_\tau := C^g/\Lambda_\tau, \qquad \Lambda_\tau := Z^g + \tau Z^g,$$

$$S_g := \{\tau \in M_g(C) : \quad {}^t\tau = \tau, \quad Im\,\tau > 0\}.$$

A symmetric line bundle $L := L_\tau$ defining the principal polarization on X_τ is:

$$L_\tau := (C^g \times C)/\Lambda_\tau, \qquad \text{with} \quad (m + \tau n)\cdot(z,t) := (z + m + \tau n, e^{-\pi i({}^tn\tau n + 2{}^tnz)}t)$$

the action of Λ_τ on $C^g \times C$.

3.2 The global sections of the bundles T_x^*L, with $x=(1/2)(\tau\epsilon+\epsilon')$, are given by the classical theta functions:

$$\theta[^{\epsilon}_{\epsilon'}](\tau,z):=\sum_{k\in Z^g} exp(\pi i^t(m+\epsilon/2)\tau(m+\epsilon/2)+2^t(m+\epsilon/2)(z+\epsilon'/2)),$$

here $\epsilon=(\epsilon_1,\ldots,\epsilon_g)$, $\epsilon'=(\epsilon',\ldots,\epsilon'_g)$ and $\epsilon_i,\ \epsilon'_i\in\{0,1\}$ are the characteristics of the bundle T_x^*L. In fact, one has $\theta[^{\epsilon}_{\epsilon'}](\tau,-z)=(-1)^{^t\epsilon\epsilon'}\theta[^{\epsilon}_{\epsilon'}](\tau,z)$ and the functions $\theta[^{\epsilon}_{\epsilon'}](\tau,z+(1/2)(m+\tau n)$ and $\theta[^{\epsilon+n}_{\epsilon'+m}](\tau,z)$ define sections of isomorphic line bundles, so:

$$e^{T_x^*L}(y)=(-1)^{^t\epsilon\epsilon'}(-1)^{^t(\epsilon+n)(\epsilon'+m)}\;=\;(-1)^{^tnm+^t\epsilon m+^t\epsilon' n}\qquad(y=(1/2)(m+\tau n)\in X_\tau[2]).$$

A basis of $H^0(X,L^{\otimes 2})$ is given by the 2^g second order thetafunctions:

$$\theta[^{\sigma}_{0}](2\tau,2z),\qquad\qquad(\sigma\in\{0,1\}^g).$$

3.3 For $x\in X[2]$, the line bundles T_x^*L are also symmetric and one has $(T_x^*L)\otimes(T_x^*L)\cong L^{\otimes 2}$ (by the theorem of the square: $(T_p^*L)\otimes(T_q^*L)\cong(T_{p+q}^*L)\otimes L$ for any line bundle L and $p,q\in X$). For each $x\in X[2]$ we thus have a multiplication map:

(3.3.1) $$H^0(X,T_x^*L)\otimes H^0(X,T_x^*L)\longrightarrow H^0(X,L^{\otimes 2}).$$

The multiplication map 3.3.1 is given by the theta relation ([I], IV.1):

(3.3.2) $$\theta[^{\epsilon}_{\epsilon'}](\tau,z)^2=\sum_\sigma(-1)^{^t(\epsilon+\sigma)\epsilon'}\theta[^{\sigma}_{0}](2\tau,0)\theta[^{\epsilon+\sigma}_{0}](2\tau,2z),$$

here σ runs over $\{0,1\}^g$.

3.4 Putting $z=0$ in the theta functions we obtain the theta constants which are used to map quotients of S_g to a projective space. The map

$$\Theta: S_g\longrightarrow P^{2^g-1},\qquad \tau\mapsto(\ldots:\theta[^{\sigma}_{0}](2\tau):\ldots)$$

factors over the congruence subgroup $\Gamma(2,4):=$

$$\left\{\begin{pmatrix} I+2A & 2B\\ 2C & I+2D\end{pmatrix}\in Sp(2g,Z)\ :\ diag(A^tB)\equiv diag(C^tD)\equiv 0 \bmod 2\right\}.$$

The induced map $\Theta: A_g(2,4):=S_g/\Gamma(2,4)\to P^{2^g-1}$ is known to be injective on tangent spaces, but for $g\geq 4$ it is not known if it is an injection

3.5 For even characteristics $m=[^{\epsilon}_{\epsilon'}]$ we define quadrics $Q_m\subset P^{2^g-1}$ by:

(3.5.1) $$Q_m:=\left\{x=(\ldots:x_\sigma:\ldots)\in P^{2^g-1}:\quad\sum_\sigma(-1)^{^t\sigma\epsilon}x_\sigma x_{\sigma+\epsilon}=0\right\}.$$

The $e(g)=2^{g-1}(2^g+1)$ quadrics obtained in this way are a basis of $H^0(P^{2^g-1},\mathcal{O}(2))$. Comparing the defining equation of Q_m with the formula for the multiplication map 3.3.2 (and using $^t\epsilon\epsilon'=0\bmod 2$) we find:

$$\Theta(\tau)\in Q_m\Longleftrightarrow\theta_m(\tau)=0\Longleftrightarrow m_0(\Theta_m)\geq 2,$$

i.e. iff the divisor Θ_m of the theta function θ_m vanishes with (even) multiplicity at $O\in X_\tau$.

3.6 In case X_τ is the product of two ppav's, the period matrix for X_τ can be chosen as

$$\tau = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}, \qquad \text{and then} \qquad \theta[{\epsilon \atop \epsilon'}](\tau, z) = \theta[{\epsilon_1 \atop \epsilon'_1}](\tau_1, z_1)\theta[{\epsilon_2 \atop \epsilon'_2}](\tau_2, z_2),$$

where $\tau_1 \in S_k$, $\tau_2 \in S_{g-k}$ and $\epsilon = (\epsilon_1, \epsilon_2) \in \{0,1\}^k \times \{0,1\}^{g-k}$ etc. Since $\theta[{\epsilon \atop \epsilon'}](\tau, -z) = (-1)^{{}^t\epsilon\epsilon'}\theta[{\epsilon \atop \epsilon'}](\tau, z)$, we see that if $m = (m_1, m_2)$ is even, but m_1 and m_2 are odd, then $\theta[{\epsilon \atop \epsilon'}](\tau, 0) = 0$. The image $\Theta(\tau)$ of such a period matrix thus lies on at least $2^{k-1}(2^k - 1) \cdot 2^{g-k-1}(2^{g-k} - 1)$ quadrics Q_m.

3.7 The map Θ can be extended to the satake compactification $A_g(2,4)^{sat}$ of $A_g(2,4)$ and we denote the extension by the same symbol. The boundary components are copies of $A_k(2,4)$ for $0 \le k \le g-1$. A point in $A_k(2,4)$ corresponds to a product of $(C^*)^{g-k}$ with an abelian variety of dimension k (since we are working with the satake compactification, extension data are 'forgotten'). Modulo the action of $Sp(2g, Z)$, any point in the boundary can be obtained as a limit:

$$\Theta(\tau_k) := \lim_{t\to\infty} \Theta(\tau(t)), \qquad \text{with} \quad \tau(t) := \begin{pmatrix} itI_{g-k} & 0 \\ 0 & \tau_k \end{pmatrix}.$$

Using the series defining the theta fuctions, one easily verifies:

$$\lim_{t\to\infty} \theta[{\epsilon_{g-k}\ \epsilon_k \atop \epsilon'_{g-k}\ \epsilon'_k}](\tau) = \begin{cases} \theta[{\epsilon_k \atop \epsilon'_k}](\tau_k) & \text{if } \epsilon_{g-k} = 0, \\ 0 & \text{if } \epsilon_{g-k} \neq 0. \end{cases}$$

Thus at a point $\Theta(\tau_{g-1})$ there vanish at least $2^{g-2}(2^{g-1}+1) + 2^{g-2}(2^{g-1}-1) = 2^{2g-2}$ characteristics (the two contributions come from the characteristics with $(\epsilon_1, \epsilon'_1) = (1,0)$ and $(1,1)$ respectively).

The next lemma collects the facts on the vanishing of the even theta nulls that we will need for our study of the Picard modular varieties.

3.8 Lemma. The following tables list the exact number of theta constants vanishing on the ppav's, or their limits, listed for $g = 2$ and $g = 3$ respectively.

#vanishing Q_m	moduli point
0	$J(C)$, C a smooth curve
1	$E_1 \times E_2$, E_i elliptic curves
4	$C^* \times E$, E elliptic curve
6	$(C^*)^2$

#vanishing Q_m	moduli point
0	$J(C),\ C$ smooth non-HE curve
1	$J(C),\ C$ smooth HE curve
6	$E \times J(C'),\ E$ an elliptic curve, C' smooth $g=2$ curve
9	$E_1 \times E_2 \times E_3,\ \ E_i$ elliptic curves
24	$(C^*)^2 \times E,\ E$ an elliptic curve

Proof. In case g=2, a ppav is either the Jacobian of a smooth genus 2 curve or the product of two elliptic curves (with the product polarization). Since a genus 2 curve cannot have an even theta characteristics with $h^0 > 1$, none of the theta constants vanishes at such a point. On a product of two elliptic curves exactly one theta constant is zero, if the period matrix is in the standard form it is $\theta[^{11}_{11}]$. In the boundary one finds the points corresponding to the other two varieties listed. Using the results stated above, the number of vanishing theta constants is easily found.

In case g=3, a ppav is either the Jacobian of a curve or a product of these. The number of vanishing theta constants for these and for the boundary points is then easily deduced from the results stated above. $\square$

4 Theta transformations and relations

4.1 Since $\Gamma(2,4)$ is a normal subgroup of $\Gamma_g := Sp(2g,Z)$, the finite group $\Gamma_g/\Gamma_g(2,4)$ acts on $A_g(2,4) = S_g/\Gamma_g(2,4)$ and also on its satake compactification $A_g(2,4)^{sat}$. The transformation theory of theta function (cf. [I],) shows that there is a (projective) representation

$$R : \Gamma_g/\Gamma_g(2,4) \longrightarrow Aut(P^{2^g-1})$$

such that the map $\Theta : A_g(2,4) \to P^{2^g-1}$ is $\Gamma_g/\Gamma_g(2,4)$-equivariant:

$$\begin{array}{ccc} A_g(2,4) & \xrightarrow{\Theta} & P^{2^g-1} \\ M \downarrow & & \downarrow R(M) \\ A_g(2,4) & \xrightarrow{\Theta} & P^{2^g-1} \end{array} \qquad\qquad (M \in \Gamma_g).$$

(The group $R(\Gamma_g) \subset Aut(P^{2^g-1})$ is the normalizer of the Heisenberg group H acting on P^{2^g-1}, in fact the action of H coincides with the action of $\Gamma_g(2)/\Gamma_g(2,4)$; [G] §3.)

4.2 We explain how compute $R(M)$ explicitly for some particular M's. Let $e_1, \ldots, e_{2g}$ be a basis of Z^{2g} for which the symplectic form E is given by:

$$E = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

We define a homomorphism $SL(2,Z)^g \to Sp(2g,Z)$ by:

$$(M_1, M_2, \ldots, M_g) \mapsto M := M_1 \oplus M_2 \oplus \ldots \oplus M_g,$$

with $M \in Sp(2g,Z)$ the matrix:

$$M_{kk} := (M_k)_{11}, \quad M_{k,g+k} := (M_k)_{12}, \quad M_{g+k,k} := (M_k)_{21}, \quad M_{g+k,g+k} := (M_k)_{22},$$

for $1 \le k \le g$ and with $M_{ij} = 0$ else.

Note that the map $(C^2)^{\otimes g} \to C^{2^g}$:

$$(x_0^{(1)}, x_1^{(1)}) \otimes \ldots \otimes (x_0^{(g)}, x_1^{(g)}) \mapsto (\ldots, x_\sigma, \ldots) \qquad \text{with} \quad x_\sigma := x_{\sigma_1}^{(1)} \cdot \ldots \cdot x_{\sigma_g}^{(g)}$$

($\sigma \in \{0,1\}^g$) induces an isomorphism: $P((C^2)^{\otimes g}) \cong P^{2^g-1}$.

For $U_i \in Aut(P^1)$ $(1 \le i \le g)$ we denote by

$$U_1 \otimes U_2 \ldots \otimes U_g \quad \in Aut(P^{2^g-1})$$

the map obtained from the U_i's via the isomorphism of projective spaces.

4.3 Proposition. 1. For the generators S, T of $SL(2,Z)$:

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

the projective transformations $R(S),\ R(T) \in Aut(P^1)$ are given by:

$$R(S) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad R(T) = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}.$$

2. For $M = M_1 \oplus M_2 \oplus \ldots \oplus M_g \in Sp(2g,Z)$, with $M_i \in SL(2,Z)$, we have:

$$R(M) = R(M_1) \otimes R(M_2) \otimes \ldots \otimes R(M_g).$$

Proof. A direct easy computation shows that ($\sigma \in \{0,1\},\ \tau \in S_1$):

$$\theta[{}^{\sigma}_{0}](2(\tau+1)) = i^\sigma \theta[{}^{\sigma}_{0}](2\tau), \qquad \text{thus} \quad R(T) = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}.$$

To find the matrix for S, we use the transformation formula (essentially the poisson summation formula):

$$\theta[{}^{\sigma}_{0}](-2/\tau) = \sqrt{\tau/2i} \cdot \theta[{}^{0}_{\sigma}](\tau/2).$$

To get back to the 2τ's, we use the identities:

$$\theta[{}^{0}_{\sigma}](\tau/2) = \theta[{}^{0}_{0}](2\tau) + (-1)^\sigma \theta[{}^{1}_{0}](2\tau).$$

Therefore:

$$R(S) = \sqrt{\tau/2i} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Since we deal with projective transformations, we can omit the factor $\sqrt{\tau/2i}$.

For the second point, we observe that $R(M) \in Aut(P^{2^g-1})$ does not depend on τ. Specializing the period matrix to $\tau = diag(\tau_1, \tau_2, \ldots, \tau_g)$ with $\tau_i \in S_1$, we get: $\theta[{}^{\sigma}_{0}](\tau) = \prod \theta[{}^{\sigma_i}_{0}](\tau_i)$ and $\theta[{}^{\sigma}_{0}](M\tau) = \prod \theta[{}^{\sigma_i}_{0}](M_i\tau_i)$. $\square$

4.4 To study the Picard modular varieties in case $g = 4$ we will need some equations for the (closure of the) image $\Theta(A_4(2,4))$ in P^{15}. We recall some of the facts (see [RF] and [G], §4 for proofs).

Three even characteristics m_1, m_2, m_3 are called asyzygeous if $m_1 + m_2 + m_3$ is an odd characteristic. A 4-tuple of characteristics is called asyzygeous if any three of the four are asyzygeous. For example,

$$\left[\begin{smallmatrix}11\\00\end{smallmatrix}\right], \quad \left[\begin{smallmatrix}01\\10\end{smallmatrix}\right], \quad \left[\begin{smallmatrix}01\\00\end{smallmatrix}\right], \quad \left[\begin{smallmatrix}11\\11\end{smallmatrix}\right] \qquad \text{and} \quad \left[\begin{smallmatrix}00\\00\end{smallmatrix}\right], \quad \left[\begin{smallmatrix}00\\01\end{smallmatrix}\right], \quad \left[\begin{smallmatrix}01\\00\end{smallmatrix}\right], \quad \left[\begin{smallmatrix}11\\11\end{smallmatrix}\right]$$

are asyzygeous 4-tuples. For an asyzygeous 4-tuple $m_1, \dots m_4$ of $g = 2$ characteristics, one has the relation:

$$\theta^4_{m_1}(\tau_2) \pm \theta^4_{m_2}(\tau_2) \pm \theta^4_{m_3}(\tau_2) \pm \theta^4_{m_4}(\tau_2) = 0 \qquad (\forall \tau_2 \in S_2).$$

The signs won't be of importance for us, the relation can in fact be checked by substituting the quadratic relations above for $\theta^2_{m_i}$, the result should be identically zero as polynomial in the four $\theta[^{\sigma}_{0}](2\tau_2)$'s.

To get relation for $g = 4$, we take two non-zero, even, $g = 2$ characteristics $n_1 = (x_1, y_1)$ and $n_2 = (x_2, y_2)$ with ${}^t x_1 y_2 + {}^t x_2 y_1 = 0 \bmod 2$. For an even genus 2 characteristic $m_i = (\epsilon, \epsilon')$ we define four even $g = 4$ characteristics by:

$$m_{i1} := \left[\begin{smallmatrix}00\,\epsilon\\00\,\epsilon'\end{smallmatrix}\right], \quad m_{i2} := \left[\begin{smallmatrix}00\,\epsilon\\1\,1\,\epsilon'\end{smallmatrix}\right], \quad m_{i3} := \left[\begin{smallmatrix}1\,1\,\epsilon\\00\,\epsilon'\end{smallmatrix}\right], \quad m_{i4} := \left[\begin{smallmatrix}11\,\epsilon\\11\,\epsilon'\end{smallmatrix}\right].$$

Then, for an asyzygeous 4-tuple m_i of even $g = 2$ characteristics, we have the following relation (obtained by applying a suitable $M \in \Gamma_g$ to the relations in [RF] and [G]):

$$\prod_{i=0}^{3} \theta_{m_{1i}}(\tau_4) \pm \prod_{i=0}^{3} \theta_{m_{2i}}(\tau_4) \pm \prod_{i=0}^{3} \theta_{m_{3i}}(\tau_4) \pm \prod_{i=0}^{3} \theta_{m_{4i}}(\tau_4) = 0 \qquad (\forall \tau_4 \in S_4).$$

To get a relation between the $\theta[^{\sigma}_{0}](2\tau_4)$'s, one multiplies eight expressions as above, with distinct signs. The relation obtained is a polynomial in the $\theta^2_{m_{ij}}(\tau_4)$. One can thus substitute the quadratic relations to obtain a polynomial of degree 32 in the $\theta[^{\sigma}_{0}](2\tau_4)$'s (which in general is not identically zero). Note that if $PV \subset P^{15}$ is a subspace contained in $Q_{m_{44}}$, then the restriction of this polynomial to PV is the square of a polynomial of degree 16.

5 Abelian varieties with an automorphism

Let ϕ be an automorphism of the principally polarized abelian variety (X, L), that is: $\phi : X \to X$ is an automorphism (with $\phi(O) = O$) and $\phi^* L$ is algebraically equivalent to L. For completeness sake, we recall the following well known lemmas and proposition.

5.1 Lemma. Let $Aut(X_\tau)$, $\tau \in S_g$, be the automorphism group of the ppav X_τ. Then:

$$Aut(X_\tau) \;\cong\; \{M \in Sp(2g, Z) : \quad M \cdot \tau = \tau\}.$$

Proof. We use that $S_g \cong Sp(2g,R)/U(g)$ also parametrizes the complex structures J on $V_R := Z^{2g} \otimes R$ which are symplectic w.r.t. to a (fixed) form E and which are positive, that is $E(x, Jx) > 0$ for all non-zero $x \in V_R$. The action of $Sp(2g,R)$ on these complex structures is given by conjugation $M : J \mapsto MJM^{-1}$. The period matrix τ determines in fact an isomorphism $H_1(X_\tau, Z)$ with Z^{2g}, the complex stucture is the one on the tangent space at $O \in X$ and E is the polarization.

For $\phi \in Aut(X_\tau)$, let ϕ_* be the map induced by ϕ on $H_1(X_\tau, Z)$. Then $\phi_* \in Sp(E) = Sp(2g,Z)$ since ϕ preserves the polarization. Moreover, since ϕ is a holomorphic map, ϕ_* commutes with J_τ and thus ϕ_* fixes τ. The other inclusion is easy. $\square$

5.2 Lemma. Let τ_0 be a fixed point of $M \in Sp(2g,Z)$. Then the fixed point locus of M,

$$S_g^M := \{\tau \in S_g : \ M \cdot \tau = \tau\}$$

is a connected, smooth, complex submanifold of S_g.

Proof. We follow [F], Hilfssatz III, 5.14, p.196. There is an isomorphism of complex manifolds

$$S_g \xrightarrow{\cong} E_g := \{W \in M_n(C) : \ {}^tW = W, \ I -^t W\overline{W} > 0\},$$

which maps τ to 0 and the image of S_g^M is defined by the equations, linear in w_{ij}: $w_{ij}\zeta_i\zeta_j = w_{ji}$, where the ζ_i are eigenvalues of M. For W in the image of S_g^M, the matrices tW, with $0 \leq t \leq 1$, then also lie in the image and connect 0 to W. $\square$

The following easy lemma will be used to find the projective models of the moduli spaces of the abelian varieties with automorphisms.

5.3 Lemma. Let $M \in Sp(2g,Z)$ be an element of finite order. Let $H \subset S_g$ be the fixed point set of M.

Then $\Theta(H)$ is contained in an eigenspace of $R(M)$.

Proof. For $\tau \in H$ we have $M\tau = \tau$ and thus $\Theta_g(\tau) = \Theta_g(M\tau) \ \in P^{2^g-1}$. Since Θ is equivariant for the action of $Sp(2g,Z)$, we must have $R(M)\Theta(\tau) = \lambda(\tau)\Theta(\tau)$ in C^{2^g} for some $\lambda(\tau) \in C$. Since H is connected and $R(M)$ has only a finite number of eigenvalues, $\lambda(\tau)$ is constant. $\square$

5.4 We will be particularly interested in the case that ϕ has order three or four and that $d\phi \in End(T_0X)$ has no eigenvalues equal to ± 1. Then $d\phi$ has p eigenvalues λ and q eigenvalues $\bar{\lambda}$ with $p + q = g$:

$$d\phi \sim diag(\underbrace{\lambda, \ldots, \lambda}_{p}, \underbrace{\bar{\lambda}, \ldots, \bar{\lambda}}_{q}) \quad \in End(T_0X),$$

and we will call ϕ an isomorphism of type (p,q).

For such a pair (X_{τ_0}, ϕ) we define $H(\phi_*) \subset S_g$ to be the set of period matrices τ for which ϕ_* induces an automorphism of type (p,q) on X_τ. The following (well-known) proposition shows that $H(\phi_*)$ depends indeed only on ϕ_* and that it is an Hermitian symmetric domain.

5.5 Proposition. Let ϕ be an automorphism of type (p, q) of the ppav (X_{τ_0}, L) and let $M := \phi_* \in Sp(2g, Z)$. Then:

1. $$H(M) = S_g^M, \qquad \text{with} \quad S_g^M := \{\tau \in S_g : \quad M \cdot \tau = \tau\},$$

   the fixed point set of M in S_g.

2. The centralizer of M in the group $Sp(2g, R)$ is isomorphic to $U(p, q)$, the unitary group of a hermitian form of signature (p, q).

3. There is an isomorphism of complex manifolds:

   $$H(M) \cong H_{p,q} := U(p, q)/(U(p) \times U(q)),$$

   where $U(p) \times U(q)$ is a maximal compact subgroup of $U(p, q)$.

Proof. Lemma 5.2 shows that $H(M) \subset S_g^M$. Conversely, if $\tau \in S_g^M$ then M defines an automorphism ϕ_τ on X_τ. Since S_g^M is connected and the type of $M = \phi_{\tau_0}$ on X_{τ_0} is (p, q), the type of ϕ_τ is also (p, q).

We now prove the last point. Let X_τ be a ppav on which M induces and automorphism of type (p, q). We will denote the complex vector space $T_0 X_\tau$ by (V_R, J_τ), so $V_R = H_1(X_\tau, R)$ and J_τ is the complex structure ($J_\tau^2 = -I$).

The map M defines also a complex structure (V_R, J_M) on V_R by:

$$J_M : V_R \longrightarrow V_R, \quad \text{with} \quad J_M := \begin{cases} M & \text{if } M^4 = I \\ \frac{1}{\sqrt{3}}(I + 2M) & \text{if } M^3 = I. \end{cases}$$

Since $M J_\tau = J_\tau M$ (the automorphism defined by M is holomorphic) and M is symplectic, we find:

$$J_\tau J_M = J_M J_\tau \qquad \text{and} \quad E(J_M x, J_M y) = E(x, y) \qquad (\forall x, y \in V_R).$$

The map J_M is thus C-linear on (V_R, J_τ). On the complex vector space (V_R, J_τ), J_τ acts by definition as scalar multiplication by i. Since this complex space is $T_0 X$ and M has type (p, q), J_M has two eigenspaces $V_\pm(\tau)$ in (V_R, J_τ) of dimension p and q and with eigenvalue i and $-i$ respectively:

$$V_R := V_+(\tau) \oplus V_-(\tau).$$

Since J_M and J_τ commute, these spaces can also be considered as complex subspaces of (V_R, J_M). Moreover J_τ can be recovered from the $V_\pm(\tau) \subset V_R$ by defining:

(5.5.1) $$J_\tau := J_M \quad \text{on} \quad V_+(\tau), \qquad J_\tau := -J_M \quad \text{on} \quad V_-(\tau).$$

We will now determine which decompositions of (V_R, J_M) correspond to $\tau \in H(M)$. Since the polarization on X_τ is given by E, the hermitian form

$$H_\tau(x, y) := E(x, J_\tau y) + iE(x, y)$$

on the complex vector space (V_R, J_τ) is positive definite (since $E(x, J_\tau x) > 0$ for all non-zero $x \in V_R$). Using that J_M is symplectic and that $MJ_M = J_M M$, one easily verifies that:

$$H_M(x, y) := E(x, J_M y) + iE(x, y)$$

is a hermitian form on the complex vector space (V_R, J_M) (in particular: $H_M(x, y) = \overline{H_M(y, x)}$ and $H_M(x, J_M y) = iH_M(x, y)$). From the equations 5.5.1 and the positive definiteness of H_τ it follows that:

$$\begin{cases} H_M(x, x) = H_\tau(x, x) & > 0 \qquad \forall x \in V_+(\tau) - \{0\} \\ H_M(y, y) = -H_\tau(y, y) & < 0 \qquad \forall y \in V_-(\tau) - \{0\} \end{cases}$$

Moreover, if $x \in V_+(\tau)$ and $y \in V_-\tau$, then, since J_M and J_τ commute and J_τ, J_M are symplectic, we get:

$$H_M(x, y) = H_M(J_\tau x, J_\tau y) = H_M(J_M x, -J_M y) = H_M(x, -y) = -H_M(x, y),$$

so $V_+(\tau)$ and $V_-(\tau)$ are perpendicular in V_R w.r.t. H_M. In particular, H_M is a hermitian form of signature (p, q).

Conversely, let $V_R = V_+ \oplus V_-$ be a decomposition into two J_M-complex subspaces of dimension p and q respectively, which are perpendicular for H_M and on which $H_M|V_\pm$ is $\pm$-definite. Define J by $J = \pm J_M$ on $V_\pm$, then we obtain a complex structure on V_R, with a symplectic J, a positive definite $E(\cdot, J\cdot)$, $JM = MJ$ and M defines an automorphism of type (p, q).

Therefore we can identify $H(M)$ with the set of p-dimensional complex subspaces V_+ of (V_R, J_M) on which H_M is positive definite (then $V_- = V_+^\perp$ w.r.t. H_M). The group $U(H_M) \cong U(p, q)$ acts transitively on the V_+'s, and stabilizer of a given V_+ is $U(p) \times U(q)$ (stabilizing V means also stabilizing $V_+^\perp$ and H_M is definite on V_+ and $V_+^\perp$). Thus we get $H(M) \cong U(p, q)/(U(p) \times U(q))$.

For the second point we observe that, since $Im(H_M) = E$, we have $U(H_M) \subset Sp(2g, R)$. Since $A \in U(H_M)$ is C-linear on (V_R, J_M) it commutes with J_M and thus with M, so $U(p, q) \subset C_{Sp}(M)$. Conversely, if A commutes with M it is C-linear on (V_R, J_M), and $E(Ax, Ay) = E(x, y)$, $MA = AM$ imply $H_M(Ax, Ay) = H_M(x, y)$. $\square$

5.6 Remark. Note that a hermitian form of signature $(p, 1)$ is given, w.r.t. a suitable basis, by: $|z_1|^2 + \ldots + |z_p|^2 - |z_{p+1}|^2$. Each subspace of C^{p+1} on which it is negative definite, is spanned by a (unique) $(z_1, \ldots, z_p, 1)$ with $\sum_{i=1}^p |z_i|^2 < 1$. The domains $H_{p,1}$ are thus isomorphic to p-dimensional complex balls:

$$H_{p,1} \cong \{z \in C^p : \sum_{i=1}^p |z_i|^2 < 1\}.$$

In general we have:

$$\dim H_{p,q} = pq.$$

6 Discrete subgroups of $SU(p,q)$

6.1 On an eigenspace $PV \subset P^{2^g-1}$ of $R(M)$ there acts a subgroup of the finite group $\Gamma_g/\Gamma_g(2,4)$. We will use this group to study the geometry of the Picard modular varieties.

In this section we will only consider the case that M corresponds to an automorphism of order 3 of type (p,q) (the results in this section are in fact independent of p, q). Then M satisfies $M^2+M+I=0$. For any $\tau \in S_g^M$, the lattice Λ_τ becomes a $Z[\omega]$-module by defining:

$$\omega \cdot \lambda := M\lambda \qquad (\lambda \in \Lambda_\tau).$$

Since the class number of $Z[\omega]$ is one and S_g is simply connected, we can identify the Λ_τ's with a fixed Λ and $\Lambda \cong Z[\omega]^g$.

Similarly, the action of M on the group $X_\tau[2]$ defines the structure of an $F_4 = F_2(\rho)$ vector space on $X_\tau[2]$ by defining $\rho \cdot v := Mv$, since both satisfy $x^2+x+1=0$.

6.2 Lemma. Let $M \in Sp(2g,Z)$ satisfy $M^2+M+I=0$. Then:

1. The map $H_M : \Lambda \times \Lambda \to Z[\omega]$ defined by:

   $$H_M(x,y) := E(x,My) - \omega E(x,y)$$

   defines a non-degerate hermitian form on $Z[\omega]^g$.

2. For any commutative Z-algebra A we define a group by:

   $$U(H_M)(A) := \{N \in Sp(2g,A) : \; H_M(Nx,Ny) = H_M(x,y) \quad \forall x,y \in \Lambda \otimes_Z A\}.$$

   Then:

   $$U(H_M)(Z) = Sp(2g,Z) \cap U(H_M)(R) \quad \text{and} \quad U(H_M)(R) = C_{Sp(2g,R)}(M).$$

3. For all primes $p \equiv 2 \bmod 3$ we have $Z[\omega] \otimes F_p \cong F_{p^2}$. The form H_M defines a non-degenerate hermitian form on $F_{p^2}^g$, the group $U(p,q)(F_p)$ is isomorphic to $U(g,F_{p^2})$, and there is a surjective reduction map

   $$\Gamma_M := U(H_M)(Z) \longrightarrow U(g,F_{p^2}).$$

Proof. We refer to the proof of prop. 5.5 for most these statements, since H_M is equivalent to the form considered there. For the notation we observe that for a quadratic extension $F_q \subset F_{q^2}$ any two non-degenerate hermitian forms are equivalent (there is no such thing as signature there), and the unitairy group of such a form is denoted here by $U(g,F_{q^2})$. The non-degeneracy of the reduction of H_M follows from $Im(H_M)=E$. The surjectivity of the reduction map is known as the strong approximation theorem. $\square$

6.3 We apply the lemma to study the restriction of the projective representation R to $\Gamma_M := U(H_M)(Z) \subset Sp(2g, Z) = \Gamma_g$. Somewhat surprisingly, the representation factors in fact over

$$\Gamma_M(2) := \Gamma_M \cap \Gamma_g(2), \qquad \text{with} \quad \Gamma_M := U(H_M)(Z).$$

6.4 Proposition. Let $M \in Sp(2g, Z)$ satisfy $M^2 + M + I = 0$. Let $V \subset C^{2^g}$ be an eigenspace of (a lift to $GL(2^g, C)$ of) $R(M)$.

1. Then PV is stable under action of Γ_M and the action factors to give a projective representation:

   $$R : \Gamma_M/\Gamma_M(2) \cong U(g, F_4) \longrightarrow PV.$$

   This representation factors over the center $< M > \subset U(g, F_4)$ to give a projective representation of $PU(g, F_4^g)$.

2. The map $\Theta : S_g/\Gamma_g(2, 4) \to P^{2^g-1}$ restricts to give a map:

   $$\Theta : H(M)/\Gamma_M(2) \longrightarrow PV$$

   with V a certain eigenspace of $R(M)$ and Θ is equivariant for the action of $\Gamma_M/\Gamma_M(2) \cong U(g, F_4)$.

Proof. Since Γ_M is contained in the centralizer of M, we get $R(M)R(g) = \lambda_g R(g)R(M)$ in $GL(2^g, C)$ and we may assume that the eigenvalues of $R(M)$ are cube roots of unity. Then also λ_g must be a cube root of unity. If $\lambda_g \neq 1$, then $R(g)$ would permute the 3 eigenspaces of $R(M)$ cyclically. These spaces would thus have the same dimension, but 2^g is not divisible by 3. Therefore $\lambda_g = 1$ for all $g \in \Gamma_M$ and Γ_M stabilizes each PV.

The center of $U(g, F_4)$ consists of the scalar multiples of the identity. Since M acts by the scalar $\rho \in F_4$, the center is just $< M >$ which indeed acts trivially on each projectivized eigenspace of M.

The representation R factors over the subgroup $\Gamma_M \cap \Gamma_g(2, 4)$. Since $\Gamma_M = C_{Sp(2g,Z)}(M)$ and $U(g, F_4) \cong \Gamma_M/\Gamma_M(2) \cong C_{Sp(2g,F_2)}(M)$ (the last iso is proven as in (2) of prop.5.5), it suffices to show that $C_G(M) \cong C_{Sp(2g,F_2)}(M)$, with $C_H(M)$ the centralizer of M in the group H.

The exact sequence, with $F_2^{2g} = \Gamma_g(2)/\Gamma_g(2, 4)$:

$$0 \longrightarrow F_2^{2g} \longrightarrow G \longrightarrow Sp(2g, F_2) \longrightarrow 1$$

defines, by conjugation, an action of $Sp(2g, F_2)$ on F_2^{2g} which is just the standard action. So if $x \in F_2^{2g}$ is represented by $A_x \in \Gamma_g(2)$, we have:

$$BA_x = A_{Bx}B \quad \in \Gamma_g(2)/\Gamma_g(2, 4) \qquad (\forall B \in \Gamma_g).$$

Let $B \in \Gamma_g$ and suppose that $BM = MB$ in $\Gamma_g/\Gamma_g(2)$. Then $BM = A_xMB$ in $\Gamma_g/\Gamma_g(2, 4)$, for an $x \in F_2^{2g}$. For $y \in F_2^{2g}$ define $B_y := A_yB \in \Gamma_g$. Then, in G, we get:

$$\begin{aligned} B_yM &= A_yBM \\ &= A_yA_xMB \\ &= A_{x+y}MA_y(A_yB) \\ &= A_{x+y+My}MB_y. \end{aligned}$$

Since M satisfies the equation $M^2 + M + I = 0$, we see that there is a unique y such that B_y commutes with M in G: $y = Mx$.

We conclude that the canonical homomorphism $C_G(M) \to C_{Sp(2g,F_2)}(M)$ (induced by $G = \Gamma/\Gamma(2,4) \to \Gamma/\Gamma(2)$) is indeed an isomorphism.

The last statement follows from the previous results. $\square$

6.5 We study the action of the group $U(g, F_4) = U(H_M)(F_2) \subset Sp(2g, F_2) = Sp(X[2], e_2)$ on the quadratic forms associated with the weil-pairing. In the next section we will give a geometrical interpretation for some of the results.

6.6 Proposition. The map:

$$q_M : X[2] \longrightarrow \{\pm 1\}, \qquad q_M(x) := e(x, Mx) = (-1)^{H_M(x,x)},$$

is a quadratic form associated with e_2. It is even iff g is even.

The group $U(g, F_4)$ acts transitively on the even quadratic forms if g is odd. If g is even, it has two orbits on this set and one orbit consists of $\{q_M\}$. The same is true for the odd quadratic forms after changing the parity of g.

The unique $U(g, F_4)$-invariant quadric is also the unique M-invariant quadric.

Proof. Since M is symplectic and $E(x, y) = E(y, x)(\in F_2)$ we have: $E(x, My) = E(Mx, M^2y) = E(Mx, y) + E(Mx, My) = E(y, Mx) + E(x, y)$. Therefore:

$$\begin{aligned} E(x + y, M(x + y)) &= E(x, Mx) + E(y, My) + E(x, My) + E(y, Mx) \\ &= E(x, Mx) + E(y, My) + E(x, y), \end{aligned}$$

so q_M is associated with e_2. Since $q_M(x) = E(x, Mx) = E(Mx, M^2x) = q_M(Mx)$, and 0 is the only point invariant under M, the number of $x \in X[2]$ with $q_M(x) = +1$ is congruent to 1 mod. 3 and q_M is the only M-invariant quadric. Since $e(g) \equiv 1$ mod. 3 iff g is even, and $o(g) \equiv 1$ mod. 3 iff g is odd, we get that q_M is even iff g is even.

The form q^M is obviously fixed by $U(g, F_4) = U(H_M)(F_2)$. In case g is even, the other even forms are $x \mapsto q_M(x)e_2(x, y)$ with $y \in X[2]$ satisfying $q_M(y) = (-1)^{H_M(y,y)} = +1$. Similarly, the odd forms are given by the y with $q_M(y) = (-1)^{H_M(y,y)} = -1$. Since the unitary group acts transitively on the set of non-zero vectors with a fixed length, it also acts transitively on the set of odd quadratic forms and on the complement of $\{q_M\}$ in the set even forms. The proof for g is odd is similar. $\square$

7 Curves with automorphism of type (p, q)

7.1 In this section we study the curves with an automorphism ϕ such that $\phi^* : JC \to JC$ is of type (p, q), the results are in the lemmas 7.2 and 7.6 respectively. In lemma 7.9 we determine the map ϕ_* on $H_1(JC, Z)$.

7.2 Lemma. Let $\phi : C \to C$ be an automorphism of order 3 of a smooth genus g-curve such that $\phi^* : JC \to JC$ has type (p, q).

Then C is a $3:1$ cyclic cover of P^1 and can be defined by an equation:

$$y^3 = f_k(x)g_l^2(x), \qquad \text{and} \quad k + 2l \equiv 0 \bmod 3,$$

with f_k and g_l polynomials of degree k and l respectively, relatively prime and without multiple roots, where

$$p = (1/3)(k + 2l) - 1, \qquad q = (1/3)(2k + l) - 1$$

and ϕ is defined by $(x, y) \mapsto (x, \omega y)$, with $\omega^3 = 1$, $\omega \neq 1$.

The only cases in which pq, the dimension of the space $S_g^{\phi_*}$, is equal to $k+l-3$, the dimension of the family of covers, is (up to permutation) for:

$$(k, l) \in \{(3, 0),\ (2, 2),\ (1, 4),\ (0, 6)\}.$$

These correspond to curves with genus 1, 2, 3, 4 resp. with types (1,0), (1,1), (2,1), (3,1) respectively.

Proof. Since there are no holomorphic differentials on C which are invariant under ϕ, we have $3:1$ map $C \to C/ < \phi > \cong P^1$. Assuming that ∞ is not a ramification point, we get the desired equation with the condition on $k + 2l$. Since there are $k + l$ branch points, the genus of C is $k + l - 2$.

To find the type of the map induced by a covering automorphism, we use the holomorphic Lefschetz trace formula. The local contrubitions from a ramification point over a zero of f_k is $\frac{1}{1-\omega} = \frac{1}{3}(2+\omega)$ and over a zero of g_l it is $\frac{1}{1-\omega^2} = \frac{1}{3}(1-\omega)$. The trace formula [GH] then gives:

$$1 - (p\omega + q\omega^2) = \frac{1}{3}\left(k(2 + \omega) + l(1 - \omega)\right),$$

and since $\omega^2 = -1 - \omega$ this gives the stated formula.

Finally we observe that $k+l-3 = p+q-1$ is equal to pq iff $(p-1)(q-1) = 0$. The values for k and l are then easily determined. $\square$

7.3 In case $g = 2$ these curves can be defined by:

$$Y^3 = (X - a)(X - b)(X - c)^2(X - d)^2, \qquad \text{or by} \quad V^2 = (U^3 + 1)(U^3 + \lambda),$$

where the first equation emphasizes the $3:1$ covering and the second emphasizes the fact that the curves are hyperelliptic. A basis for the holomorphic one forms for the first curve is given by $\frac{dX}{Y}$, $(X - c)(X - d)\frac{dX}{Y}$. The ϕ-invariant even theta characteristic is given by $D_3 - K$, where D_3 is the sum of the three ramification points over $U^3 + 1 = 0$ and K is the canonical class.

7.4 In case $g = 3$, so $(k, l) = (4, 1)$, we can move the branch point which is the zero of g_1 to infinity. The equation for C can then be homogenized to give $y^3 z = f_4(x, z)$, which defines a smooth quartic curve in P^2, the canonical curve. Note C has a hyperflex l defined by $z = 0$, so a canonical divisor is $l \cdot C = 4P$, $P = (0 : 1 : 0)$. The ϕ-invariant odd theta characteristic is given by $2P$.

7.5 In the $g = 4$, $(3,1)$ case, the curve is given by $y^3 = f_6(x)$. A basis for $H^0(C_4, \Omega^1_{C_4})$ is given by:

$$\omega, \quad x\omega, \quad x^2\omega, \quad y\omega, \qquad \text{with} \quad \omega = y^{-2}dx,$$

and thus the canonical embedding $C_4 \hookrightarrow PH^0(C_4, \Omega^1_{C_4})$ is given by: $(x, y) \mapsto (x_0 : x_1 : x_2 : x_3) = (1 : x : x^2 : y)$. Note that C lies on the cone defined by $x_0x_2 = x_1^2$ and that the rulings of the cone correspond to the global sections of an effective, even theta characteristic. This is the even theta characteristic fixed by ϕ.

7.6 Lemma. Let $\phi : C \to C$ be an automorphism of order 4 of a smooth genus g-curve such that $\phi^* : JC \to JC$ has type (p, q) $(p + q = g)$.

Then C is a hyperelliptic curve and can be defined by an equation:

$$y^2 = xf_g(x^2); \qquad \phi : (x, y) \mapsto (-x, iy),$$

(with f_g a polynomial of degree g) and

$$(p, q) = (g/2,\, g/2) \quad \text{is } g \text{ is even}, \quad (p, q) = ((g + 1)/2, (g - 1)/2) \quad \text{if } g \text{ is odd}.$$

The only cases in which pq, the dimension of the space $S_g^{\phi_*}$, is equal to $g - 1$, the dimension of the family of covers, is (up to permutation) for:

$$g = 1,\, 2,\, 3 \qquad \text{and} \qquad (p, q) = (1, 0),\ (1, 1),\ (2, 1) \quad \text{respectively.}$$

Proof. In case C is a curve with an automorphism ϕ of order 4 of type (p, q), then ϕ^2 acts as -1 on $H^0(C, \Omega^1_C)$ and thus $C/ < \phi^2 > \cong P^1$. Therefore C is a hyperelliptic curve with HE-involution ϕ^2.

Since ϕ and ϕ^2 commute, the map ϕ induces an involution on P^1 which can be put in the normal form $x \mapsto -x$. The set of branch points is thus invariant under this map. Since the eigenvalues of ϕ^* on $H^{1,0}$ are i and $-i$, the trace of ϕ^* on $H^1(C, Q)$ is $g(i - i) = 0$. The Lefschetz trace formula (for ϕ^* on $H^i(C, Q)$) then shows that case ϕ has two fixed points, which thus map to the fixed points $0, \infty$ of ϕ^2 on P^1 and these are branch points. The equation for C then has the desired form, and the automorphism ϕ is a lift of the map $x \mapsto -x$ on P^1.

A basis for the holomorphic one-forms is given by the $\frac{X^l dX}{Y}$ with $0 \leq l \leq g-1$, thus the endomorphism is of the type stated. $\square$

7.7 Remark. In case $g = 3$ these curves were also studied by Shimura ([Sh]) and K. Matsumoto ([Ma]), in fact they consider the genus 3 curves C defined by:

$$w^4 := z^2(z - 1)^2(z - \lambda)(z - \mu),$$

the automorphism of order 4 on these curves is given by $(v, w) \mapsto (v, iw)$.

The 2:1 map of C to P^1 is given by $(z, w) \mapsto u := \frac{w^2}{z(z-1)(z-\mu)}$ ([Ma], prop.1.1). Therefore we get $(*)$ $\quad w^2 = uz(z - 1)(z - \mu)$. Using the equation for C one finds $u^2 = \frac{z-\lambda}{z-\mu}$ and thus $z = \frac{u^2-\lambda}{u^2-\mu}$. Substituting this in $(*)$ and normalizing the result one obtains an equation as in lemma 7.6.

7.8 The following proposition shows that the map $\phi_* \in Sp(2g, Z)$ induced by an automorphism of type (p, q) on a curve C is completely determined (up to conjugation and inversion) by (p, q) and the order of ϕ.

7.9 Proposition. Let $\mathcal{A}_g = S_g/\Gamma_g$ be the moduli space of ppav's of dimension g. Let $\mathcal{N}_{k,(p,q)} \subset \mathcal{A}_g$ be the closure in $\mathcal{A}_g$ of the (irreducible) set of jacobians of curves with an automorphism ϕ_k of order k ($k = 3$, 4) and type (p, q). Then:

1. there is a point in $\mathcal{N}_{k,(p,q)}$ corresponding to the ppav (with product polarization) E_k^g, with

   $$E_3 := C/(Z + \omega Z), \quad E_4 := C/(Z + iZ), \quad \text{and} \quad \phi_{k*} = M^{\oplus p} \oplus (M^{k-1})^{\oplus q},$$

   where $M \in SL(2, Z)$ induces the automorphism of order k on E_k.

2. In case $g = 2$, 3 a point in $\overline{\mathcal{N}_{k,(p,q)}} \subset \overline{\mathcal{A}_g}$, which is in the boundary of $\mathcal{A}_g$, corresponds to $(C^*)^2$ and $E_k \times (C^*)^2$ respectively.

Proof. The irreducibility of $\mathcal{N}_{k,(p,q)}$ follows from the previous lemmas. One can degenerate the curves till they become trees of the elliptic curves E_k and ϕ_k will map each of the E_k to itself, so $\phi_{k*} = M^{\oplus p} \oplus (M^{k-1})^{\oplus q}$.

More explicitly, let first $k = 3$. Consider the one parameter family of genus $g \geq 2$ curves defined by $Y^3 = (X^2 - t^6)g(X)$, with X, g, $X^2 - t^6$ relatively prime. Letting $t \to 0$ and normalizing the singular curve obtained, one finds a curve C of genus $g - 1$ whith an automorphism of order three. The other component appears after blowing up the point $(X, Y, t) = (0, 0, 0)$. Substituting $Y := t^2 Y$ and $X = t^3 X$ one finds, upon $t \to 0$, the curve $Y^3 = X^2 - 1$, i. e. the elliptic curve E_3. The Jacobian of the special fiber is thus the product of this elliptic curve and $J(C)$, and the automorphism on E_3 is induced by $(X, Y) \mapsto (X, \omega Y)$ on $J(C)$. Proceeding in this way, one will obtain E_3^g and the automorphism as stated.

In case $k = 4$, write $f_g(x^2) = (x^2 + a_1) \dots (x^2 + a_g)$ and let $a_g \to 0$, then the stable reduction gives a curve with two components, one isomorphic to E_4, the other isomorphic to a similar curve of genus $g - 1$. Therefore the curve E_4^g is in the (closure of) the locus defined by these curves.

Since the ring $Z[\omega]$ resp. $Z[i]$ will act on the character group of the torus part of a semi-stable abelian variety in the limit, this torus part must have an even dimension. Since this ring also acts on the abelian part, we get the result for $g = 2, 3$.

More explicitly, if $k = 4$, let $a_g \to a_{g-1}$ (for $g \geq 2$). The corresponding stable curve is a curve with two nodes (permuted by the automorphism), whose normalization is a curve of the same type of genus $g-2$. In particular, if $g = 2$, 3 the normalization is a P^1 or the curve E_4 respectively. The boundary of (the image of) $H_{1,g-1}$ in the Satake compactification of A_g then consists of one point in the A_{g-2}-stratum. □

8 Automorphism of order three

8.1 In this section we determine the image of S_g^M under the Θ-map. The matrix M corresponds to an automorphism of type $(g-1,1)$, g =2, 3, 4 and moreover M is obtained from an automorphism of a curve of genus g.

8.2 Let $M_0 := TS \in Sp(2,Z) = SL(2,Z)$, then M_0 has order 3,

$$M_0 := \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix},$$

and any element of order 3 in $SL(2,Z)$ is conjugate with M_0 or M_0^2. The point $\tau_0 := \frac{1}{2}(1+i\sqrt{3}) \in S_1$ is the unique fixed point of M and M_0 induces an automorphism, also denoted by M_0, of order three on the elliptic curve $E_3 := C/(Z+\tau_0 Z)$. On the tangent space at $O \in E_3$, M_0 will act via a primitive 3-rd root of unity ω.

Let $M_{p,g-p}$ be the matrix with p blocks equal to M_0 and $g-p$ diagonal equal to M_0^2 (in particular, $M_0 = M_{1,0}$):

$$M_{p,g-p} := M_0^{\oplus p} \oplus (M_0^2)^{\oplus g-p} \quad \in Sp(2g,Z).$$

Then $M_{p,g-p}$ induces an automorphism of order three of type (p,q) on the principally polarized abelian variety E_3^g.

If $M = M_{p,q}$, the hermitian form H_M as defined in lemma 6.2, is given by: $H_M(z,z) = \sum_{i=1}^p |z_i|^2 - \sum_{j=1}^q |z_{p+j}|^2$.

8.3 Lemma. 1. The projective transformation $R(M_0)$ is given by:

$$R(M_0) = (-1+i)^{-1} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}.$$

Eigenvectors of $R(M_0)$ are:

$$v_\pm := \begin{pmatrix} 1 \\ \mu_\pm \end{pmatrix}, \quad \mu_\pm := \frac{(1+i)(-1\pm\sqrt{3})}{2},$$

with eigenvalues $\frac{1}{2}(-1 \pm \sqrt{-3})$.

2. For $g \geq 2$, $R(M_{p,g-p})$ has three eigenvalues $\lambda = 1, \omega$ and ω^2 resp. and corresponding eigenspaces $V_\lambda \subset C^{2^g}$. We have:

$$\dim V_1 = \tfrac{1}{3}(2^g - (-1)^{g-1}2), \qquad \dim V_\omega = \dim V_{\omega^2} = \tfrac{1}{3}(2^g + (-1)^{g-1}).$$

3. The only characteristic fixed by $M_{p,g-p}$ is $\left[\begin{smallmatrix} 11\dots1 \\ 11\dots1 \end{smallmatrix}\right]$, it is even iff g is even.

Proof. Since $M_0 = TS$, we have $R(M_0) = R(T)R(S)$, the factor in front is chosen so that $R(M_0)$, as element of $GL(2,C)$, has order 3.

Write $M_{p,g-p} = M_{1,0} \oplus M_{p-1,g-p}$, the case $p = 0$ can be handled analogously. Let $V'_{\omega^k} \subset C^{2^{g-1}}$ be the eigenspace of $R(M_{p-1,g-p})$ with eigenvalue ω^k, $k = 0,1,2$. Then

$$V_1 = v_- \otimes V'_\omega \oplus v_+ \otimes V'_{\omega^2}, \qquad V_\omega = v_+ \otimes V'_1 \oplus v_- \otimes V'_{\omega^2}.$$

Thus: $m_g := dim\, V_1 = 2n_{g-1}$, $n_g := dim\, V_\omega = m_{g-1} + n_{g-1}$, with $n_{g-1} = dim\, V'_\omega = dim\, V'_{\omega^2}$. From these relations the formulas easily follow. The block-form of $M_{p,q}$ and the fact for $g = 1$ the only characteristic fixed by M_0 is $[{1 \atop 1}]$ imply the last statement. $\square$

8.4 Theorem. Let $PV \subset P^3$ be the eigenspace of $R(M_{1,1})$ which contains $\Theta(E_3^2)$. Let $M := M_{1,1}$ and let $B_1 := H(M) \subset S_2$. Then:

1. The satake compactification of $B_1/\Gamma_M(2)$ is isomorphic to P^1 and
$$(B_1/\Gamma_M(2))^{sat} \cong \overline{\Theta(B_1)} \cong PV \cong P^1.$$
2. the general point of B_1 corresponds to the g=2 curve $y^2 = f_2(x^3)$.
3. $B_1/\Gamma_M(2) \cong \Theta(B_1)$ is the complement in P^1 of a set of 3 points.
4. There are precisely two points in $\Theta(B_1)$ which correspond to a product of two elliptic curves, each of these points in fact corresponds to E_3^2 (with a certain level-two structure).

Proof. Since $R(M_{1,1})(v_+ \otimes v_+) = \omega \cdot \omega^2 (v_+ \otimes v_+)$, the eigenspace V of $R(M)$ which contains $v_+ \otimes v_+$ is V_1 and thus has dimension 2. (It also contains $v_- \otimes v_-$, the other two eigenspaces of $R(M)$ are one dimensional and are spanned by $v_+ \otimes v_-$ and $v_- \otimes v_+$ respectively.) Since $\dim B_1 = 1$ we get: $\overline{\Theta(B_1)} = PV$, this projective line in P^3 will be denoted by L.

The map $\Theta : S_2/\Gamma_2(2,4) \to P^3$ induces an isomorphism $(S_2/\Gamma_2(2,4))^{sat} \cong P^3$ (cf. [GN]). Thus $\overline{\Theta(B_1/\Gamma_M(2))} \cong (B_1/\Gamma_M(2))^{sat}$ and $\Theta(B_1/\Gamma_M(2)) \cong B_1/\Gamma_M(2)$.

The quadric Q_m with $m = [{11 \atop 11}]$ is fixed under the $U(2, F_4)$-action, and cuts L in the points $(1:0)$ and $(0:1)$ (where $(x:y)$ corresponds to $xv_+ \otimes v_+ + yv_- \otimes v_-$). These two points correspond to $E_3 \times E_3$. (This can be shown by explicit computation, but one can also use that the points of L parametrize the Jacobians of cyclic $3:1$ covers of P^1, (cf. prop. 7.9), and thus there is no theta constant which vanishes identically on B_1. Since Q_m contains all points of the form $v \otimes w$ (which correspond to products of elliptic curves) it contains the points $v_+ \otimes v_+$ and $v_- \otimes v_-$. Thus the quadric Q_m doesn't contain L, and so meets L in at most two points.)

The nine remaining quadrics come in three trios (=orbits) under the action of $R(M)$, they are:

$$\{[{00 \atop 00}],\ [{10 \atop 01}],\ [{00 \atop 11}]\}, \qquad \{[{11 \atop 00}],\ [{00 \atop 10}],\ [{00 \atop 01}]\}, \qquad \{[{00 \atop 11}],\ [{01 \atop 00}],\ [{10 \atop 00}]\}.$$

Since L is an eigenspace of M, each tquadric from a trio intersects L in the same set. Since at least three quadrics vanish in a point of this set, the point must be a cusp, and thus either 4 or 6 Q_m's vanish there. As the number of vanishing Q_m's is a multiple of 3, there are 6 vanishing Q_m's and the point corresponds to $(C^*)^2$. Using that $U(2, F_4)$ acts transitively on the 9 non-invariant Q_m's, we conclude that there are three cusps, and that each Q_m intersects L in two distinct points. $\square$

8.5 Theorem. Let $PV \subset P^7$ be the eigenspace of $R(M)$, with $M := M_{2,1}$, which contains $\Theta(E_3^3)$. Let $B_2 := S_3^M \subset S_3$, so B_2 is a complex 2-ball. Then:

$$(B_2/\Gamma_M(2))^{sat} \cong \overline{\Theta(B_2)} \cong PV \cong P^2.$$

Moreover,

1. the general point of B_2 corresponds to the jacobian of the g=3 curve $y^3 = f_4(x)$.

2. $B_2/\Gamma_M(2) \cong \Theta(B_2)$ is the complement in P^2 of 9 points, the cusps, which are the base locus of the Hesse pencil:

$$(X^3 + Y^3 + Z^3) + \lambda XYZ$$

   (so the 9 cusps are: $(-1:\epsilon:0)$, $(-1:0:\epsilon)$ and $(0:-1:\epsilon)$ with $\epsilon^3 = 1$).

3. The 4 singular fibers of the Hesse pencil (for $\lambda = \infty$ and $\lambda^3 = -27$) consist of 3 lines each. The twelve lines parametrize ppav's $E_3 \times A$, where A is an abelian surfaces.

4. Each Q_m intersects PV in two lines from a singular fiber, the intersection point of these two lines corresponds to E_3^3.

Proof. Let V be the eigenspace containing $v_+ \otimes v_+ \otimes v_+$. It has dimension three, $v_+ \otimes v_- \otimes v_-$, $v_- \otimes v_+ \otimes v_-$ are also in V. Since $\dim B_2$=2, we conclude that $\overline{\Theta(B_2)} = PV \subset \overline{\Theta(A_3(2,4))}$.

For (1) see prop. 7.9. We denote by $B_1,\ B_1' \subset B_2$ the following two copies of $S_2^{M_{1,1}} \subset S_2$:

$$B_1 = \left\{ \begin{pmatrix} \tau_0 & 0 & 0 \\ 0 & \tau_{11} & \tau_{12} \\ 0 & \tau_{21} & \tau_{22} \end{pmatrix} \right\}, \quad B_1' = \left\{ \begin{pmatrix} \tau_{11} & 0 & \tau_{12} \\ 0 & \tau_0 & 0 \\ \tau_{21} & 0 & \tau_{22} \end{pmatrix} \right\}, \quad \tau_2 := \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix} \in S_2^{M_{1,1}}.$$

For $\tau \in B_1$ we have $\theta[{\sigma \atop 0}](2\tau) = \theta[{\sigma_1 \atop 0}](2\tau_0)\theta[{\sigma_2 \sigma_3 \atop 0\ 0}](2\tau_2)$ and a similar decomposition holds for $\tau \in B_1'$. Therefore

$$l := \Theta(B_1) \qquad \text{and} \qquad l' := \Theta(B_1')$$

are two lines in PV, each of which can be identified with $L := PV$ from thm 8.4. In particular, on each there are two points corresponding to E_3^3 and there are three cusps, corresponding to $E_3 \times (C^*)^2$ while all the other points correspond to $E_3 \times A_2$, where A_2 is an abelian surface which is not a product of elliptic curves.

Note that l is the line connecting the points $v_{+++} := v_+ \otimes (v_+ \otimes v_+) = \Theta(diag(\tau_0, \tau_0, \tau_0))$ and $v_{+--} := v_+ \otimes (v_- \otimes v_-)$, whereas l' is the line on v_{+++} and $v_{-+-} := v_- \otimes (v_+) \otimes v_-$. Using the $g = 2$ result, we see that both v_{+--} and v_{-++} correspond to E_3^3.

Since B_2 parametrizes Jacobians of $y^3 = f_4(x)$, which are non-HE genus 3 curves, none of the Q_m's vanishes identically on PV. As $\theta_n(\tau)$, with $n = [{110 \atop 110}]$, vanishes on both B_1 and B_1', we conclude that the quadric Q_n intersects PV in the lines l and l'. Note that the intersection point v_{+++} of l and l' corresponds to E_3^3. Since $U(3, F_4)$ acts transitively on the quadrics, every quadric intersects $P(V_+)$ in two (distinct) lines, and the intersection point of these two lines corresponds to E_3^3. Since 6 quadrics vanish on a line, we find 12 such lines ($36 \cdot 2 = 6n$).

In a general point of l there vanish exactly $6 = 2 \cdot 3$ quadrics Q_m (those with (even) $m = \left[\begin{smallmatrix}1ab\\1cd\end{smallmatrix}\right]$), whereas on l' the 6 quadrics with $m = \left[\begin{smallmatrix}1a1\\1b1\end{smallmatrix}\right]$ vanish. The trio $\left[\begin{smallmatrix}11a\\11b\end{smallmatrix}\right]$ vanishes on both l and l'. Thus we found 9 quadrics vanishing in v_{+++}, and since that point corresponds to E_3^3 there are no more quadrics vanishing there. The set of these characteristics we denote by

$$S := \{\left[\begin{smallmatrix}110\\110\end{smallmatrix}\right], \left[\begin{smallmatrix}111\\110\end{smallmatrix}\right], \left[\begin{smallmatrix}110\\111\end{smallmatrix}\right], \left[\begin{smallmatrix}101\\101\end{smallmatrix}\right], \left[\begin{smallmatrix}111\\101\end{smallmatrix}\right], \left[\begin{smallmatrix}101\\111\end{smallmatrix}\right], \left[\begin{smallmatrix}011\\011\end{smallmatrix}\right], \left[\begin{smallmatrix}111\\011\end{smallmatrix}\right], \left[\begin{smallmatrix}011\\111\end{smallmatrix}\right]\}.$$

Next we consider the quadrics vanishing in v_{+--} and v_{-+-}. Since each point corresponds to E_3^3, there are 9 quadrics vanishing in each point. In the point $v_- \otimes v_- \in L$ the quadric with characteristic $\left[\begin{smallmatrix}11\\11\end{smallmatrix}\right]$ vanishes, so the quadrics with characteristics $m = \left[\begin{smallmatrix}a11\\b11\end{smallmatrix}\right]$ or $m = \left[\begin{smallmatrix}1ab\\1cd\end{smallmatrix}\right]$ vanish in v_{+--}. Therefore the 9 quadrics vanishing in v_{+--} are the same as those that vanish in v_{+++}. A similar argument on l' shows that these 9 also vanish in v_{-+-}.

A quadric Q_m with $m = \left[\begin{smallmatrix}a11\\b11\end{smallmatrix}\right]$ intersects PV in two lines, one of wich is l', the other will be denoted by l''. Since $l' \cap l''$ must correspond to E_3^3, and there only two such points on l', $l' \cap l''$ must be either v_{+++} or v_{-+-}. Since Q_m also vanishes in $v_{+--} \in l$, but Q_m doesn't vanish on l, we conclude that $l \cap l'' = v_{-+-}$. Since Q_m also vanishes in $v_{+--} \in l$, the l'' is the line connecting the points v_{+--} and v_{-+-}.

Thus on each line of the triangle $T_S := \{l,\ l',\ l''\}$ there vanish 6 of the 9 Q_m with $m \in S$, and in each vertex there all 9 vanish. Since T_S is completely determined by any one of the 9 $m \in S$ (intersect Q_m with PV, that gives two lines from T_S, the third connects the points corresponding to E_3^3 on each) and since $U(3, F_4)$ acts transitively on the m's, we find that the 12 lines make up 4 triangles.

Consider again T_S. The remaing 27 Q_m's (i.e. $m \notin S$) thus intersect PV in 9 lines (making 3 triangles like T_S). Let m be such a line and let $P = m \cap l$. Since in $P \in l$ there vanish at least $6 + 6 = 12$ thetanull's, and thus P is a cusp. Therefore each of the 9 remaining lines intersects l in a cusp, which corresponds to $E_3 \times (C^*)^2$. In P there must then vanish 24 thetanull's, so P lies on 4 lines and we see that through each of the 3 cusps of l there pass 3 of the 9 remaining lines. The same is of course true for any of the lines.

In particular, we find a configuration of 12 lines, meeting in 12 points (corresponding to E_3^3) in pairs and in the 9 points, the cusps, four of the lines meet. This configuration is in fact uniquely determined up to projective equivalence and is known as the Hesse-Configuration (see [BHH], 2.3A, p.71-75). This configuration is formed by the 12 lines from the degenerate fibers of the Hesse pencil.

Using the action of $\Gamma_M/\Gamma_M(2)$ on PV, we see that $\Theta : B_2/\Gamma_M(2) \to PV$ has degree one. Thus Θ gives a birational isomorphism of the normal varieties $(B_2/\Gamma_M(2))^{sat}$ and P^2. Since Θ induces a bijection (use the description of the Satake compactification in [HW]), it is in fact an isomorphism. $\square$

8.6 Theorem. Let $PV \subset P^{15}$ be the eigenspace of $R(M_{3,1})$ which contains $\Theta(E_3^4)$. We write $B_3 := H(M_{3,1})$, the complex 3-ball. Then:

$$(B_3/\Gamma_M(2))^{sat} \cong \overline{\Theta(B_3)} \cong \mathcal{B} \subset PV \cong P^4,$$

where B is the Burkhardt quartic threefold, defined by the equation:

$$Y_0^4 - Y_0(Y_1^3 + Y_2^3 + Y_3^2 + Y_4^3) + 3Y_1Y_2Y_3Y_4.$$

Moreover:

1. the general point of B_3 corresponds to the jacobian of a genus 4 curve $y^3 = f_6(x)$.
2. The singular locus of $\mathcal{B}$ consists of 45 nodes, these points correspond to the cusps, thus $B_3/\Gamma_M(2) \cong \Theta(B_3) = \mathcal{B}_{smooth}$.
3. There are precisely 40 (linear) P^2's inside $\mathcal{B}$, these parametrize products of abelian threefolds with the elliptic curve E_3.
4. The space PV is contained in the (invariant) quadric Q_m, with $m = \left[\begin{smallmatrix}1111\\1111\end{smallmatrix}\right]$. The other 145 Q_m form 45 orbits of 3 under the action of $R(M_{3,1})$. Each of these Q_m's intersects PV in a cone, i.e. a quadric with one singular point. Each cone is the tangent cone to $\mathcal{B}$ at some cusp.
5. There is a natural bijection between the nodes of $\mathcal{B}$ and the 45 $R(M_{3,1})$-orbits of quadrics given by associating to a node its tangent cone, and to an obit of Q_m's the singular point of $Q_m \cap PV$.

Proof. By a direct computation, or by observing that there are no invariant quadrics under the action of $PU(4, F_4)$ on PV, one finds that the equation for Q_n, with $n = \left[\begin{smallmatrix}1111\\1111\end{smallmatrix}\right]$, vanishes identically on PV. (Note that since B_3 parametrizes curves with a (unique) vanishing even theta null (see 7.5), it is clear that Q_n vanishes on the image of B_3.)

To find the equation of the threefold $\Theta(B_3)$ in P^4, we use the two equations, of degree 32 in the coordinates of the P^{15}, for $\Theta(S_4)$. Since the θ_n vanishes on B_3, the equations become squares when restricted to PV, and thus we have to investigate two equations of degree 16 in the 5 variables of PV. Using the computer program 'macaulay', we found that, over F_{37}, the common factor of the two polynomials has degree 4 and is irreducible. From this we conclude that, over C, the common factor F also has degree 4 and that $\overline{\Theta(B_3)}$ is defined by F.

The action of $PU(4, F_4)$ on PV can be lifted to a linear representation of its Schur multiplier (a 2:1 cover, see [A]) on V. This representation is irreducible (use the restrictions of subgroups to the P^2's below) and from the character table in [A] one finds that the representation on V factors in fact over $PU(4, F_4)$ and thus coincides (upto conjugation) with the representation of $PSp(4, F_3) \cong PU(4, F_4)$ studied by Burkhardt [Bu]. He proved that there is a unique invariant of degree 4 on PV whose zero locus is the Burkhardt quartic. Since $\overline{\Theta(B_3)}$ is invariant under the action of the group $PU(4, F_4)$ and is defined by a polynomial of degree 4, we conclude that $\overline{\Theta(B_3)} \cong \mathcal{B}$.

Inside of $B_3 = H(M_{3,1})$ one finds a copy of $B_2 = H(M_{2,1})$, by considering only period matrices of the form:

$$\begin{pmatrix} \tau_0 & 0 \\ 0 & \tau_3 \end{pmatrix}, \qquad \tau_3 \in B_2 = H(M_{2,1}).$$

The restriction of the Θ-map for $g = 4$ to this copy of B_2 is just the Θ-map for $g = 3$ (use that the theta's become products on this B_2). The closure of the image of this copy of B_2 is thus isomorphic to $\overline{\Theta(B_2)} = P^2$, and it lies in H. Since there are exactly 40 P^2's in $\mathcal{B}$ and $U(4, F_4)$ acts transitively on them (cf. [Ba]), we find (2).

A direct computation shows that that a Q_m intersects PV in a cone (with a unique singular point) and that this point is a cusp of $\overline{\Theta(B_3)}$. Since $U(4, F_4)$ acts transitively on the Q_m's and on the nodes of $\mathcal{B}$, (3) and (4) follow.

The proof of the isomorphism $(B_3/\Gamma_M(2))^{sat} \cong \mathcal{B}$ is similar to the one in theorem 8.5.

(It is not hard to show that $Q_m \cap \overline{\Theta(B_3)}$ must consist of 8 P^2's, each $P^2 \cong \overline{\Theta(B_2)}$. So if one could prove directly that this intersection were transversal, then it would follow that $deg(\overline{\Theta(B_3)}) = 4$ and the argument with invariants would show it to be isomorphic to $\mathcal{B}$. In particular, the computer computations could then be avoided.) $\square$

8.7 Remark. The fourfold $\overline{\Theta(H(M_{2,2}))} \subset P^5 \cong PW$, an eigenspace of $R(M_{2,2})$, is related to the invarian theory of $W(E_6)$, the Weyl group of the rootsystem E_6. In fact, the group $PU(4, F_4)$ is a subgroup of index 2 in $W(E_6)$, and W can be identified with $R(E_6) \otimes_Z C$. We hope to discuss this fourfold and its relation with E_6 in a later article.

9 An isomorphism of moduli spaces

9.1 The projective dual of the Burkhardt quartic $\mathcal{B}$ in P^4 is isomorphic to the satake compactification of $S_2/\Gamma_2(3)$ (see [SB], [HW]). We will show that the Burkhardt is a compactification of the moduli space of of curves defined by $y^3 = f_6(x)$ with a certain type of level-2 structure. We then give a moduli interpretation of the birational isomorphism of this moduli space with $S_2/\Gamma_2(3)$.

9.2 Let $J_4 := J(C_4)$ be the jacobian of a (smooth, projective) genus 4 curve C_4 defined by an equation $y^3 = f_6(x)$. On the group $J_4[2]$ of 2-torsion points there is a natural structure of hermitian F_4-vector space, using the automorphism of order 3 and H_M. We define a hermitian level-2 structure to be an isomorphism α of F_4-vector spaces:

$$\alpha : J_4[2] \xrightarrow{\cong} F_4^4, \qquad \text{such that} \quad H_M(x, y) = H^0(\alpha(x), \alpha(y)),$$

where the hermitian form H^0 on F_4^4 is defined by:

$$H^0(u, v) := {}^t u H^0 \bar{v}, \qquad H^0 := \begin{pmatrix} 1 & 0 & \rho & \rho \\ 0 & 1 & \rho & 0 \\ \rho^2 & \rho^2 & 1 & 0 \\ \rho^2 & 0 & 0 & 1 \end{pmatrix},$$

so H^0 also denotes the matrix defining the hermitian form H^0, and where $F_4 = F_2(\rho)$. (Since both H_M and H^0 are non-degerate hermitan forms on a 4- dimensional F_4-vector space, such isomorphisms exist, and they form a principally homogeneous space under the action of $U(H^0)$ by $A \cdot \alpha := A \circ \alpha$ $(A \in U(H^0))$. Note that α is determined by the four-tuple (also denoted by α)

$$\alpha := (x_1, \ldots, x_4) \in J_4[2]^4 \qquad \text{with} \quad x_i := \alpha^{-1}(e_i),$$

where e_i is the i-th basis vector of F_4^4. Fixing an isomorphism (of abelian groups) $F_4^4 \cong (Z/2Z)^8$ and noting that $Im\, H_M$ is the weil-pairing, one sees that a hermitian level-2 structure α gives a level-2 structure, also denoted by α.

Since each curve C_4 has an automorphism ϕ of order 3, the (hermitian) level-2 structures α and $\alpha \circ \phi_*$ give rise to the same moduli point. In the projective space $P(J_4[2]) \cong P^3(F_4) = (F_4^4 - \{0\})/ < \rho >$ (so we consider $J_4[2]$ again as a F_4 vector space) we consider the set:

$$S := \{\bar{x} \in P(J_4[2]) : \; x \in J_4[2] - \{0\}, \quad H_M(x, x) = 1\}$$

of anisotropic points. Since the points x with $H_M(x, x) = 1$ correspond canonically to the odd theta characteristics, the cardinality of S is $120/3 = 40$. A projective hermitian level-2 structure on J_4 is defined to be an ordered four-tuple

$$\alpha_S(\bar{x}_1, \ldots, \bar{x}_4) \in S^4, \qquad \text{with :} \quad H_M(x_i, x_j) \neq 0 \text{ iff } H^0(e_i, e_j) \neq 0,$$

here $x_i \in J_4[2]$ are lifts of the $\bar{x}_i \in S$.

9.3 Lemma. 1. The map: $\alpha = (x_1, \ldots, x_4) \mapsto \alpha_S = (\bar{x}_1, \ldots, \bar{x}_4)$, induces a bijection:

$$\left\{ \begin{array}{c} \text{hermitian level-2} \\ \text{structures on } J_4[2] \end{array} \right\} \Big/ < \phi_* > \quad \longrightarrow \left\{ \begin{array}{c} \text{projective hermitian} \\ \text{level-2 structures on } J_4[2] \end{array} \right\}.$$

2. The moduli space of the jacobians of the curves $y^3 = f_6(x)$ with a hermitian level-2 structure is isomorphic to a Zariski open subset of $(H_{3,1}/\Gamma_{M_{3,1}}(2))^{sat} = \mathcal{B}$, the Burkhardt quartic.

Proof. Let $\alpha_S = (\bar{x}_1, \ldots, \bar{x}_4)$ be a projective hermitian level-2 structure and let $x_1 \in J_4[2]$ be a lift of $\bar{x}_1$. Since for a hermitian level-2 structure we demand that $H^0(\alpha(x_1), \alpha(x_3)) = H^0(\alpha(x_1), \alpha(x_4)) = \rho$, the lifts x_3, x_4 of $\bar{x}_3$ and $\bar{x}_4$ are uniquely determined. Also x_2 is now determined by $H^0(x_2, x_3) = \rho$. It is straightforward to check that $\alpha := (x_1, \ldots, x_4)$ is indeed a hermitian level-2 structure.

For the last point we observe that $H_{3,1} \cong S_4^M$ parametrizes the jacobians of these curves with a symplectic basis of the period lattice and for which ϕ_* corresponds to the (fixed) element $M = M_{3,1} \in Sp(8, Z)$. Fixing M mod $\Gamma_M(2)$ is the same as fixing the hermitian form H_M on $J_4[2]$, whence the result. The $\cong$ was proved in thm. 8.6 $\square$

9.4 We now consider the jacobian $J_2 := J(C_2)$ of a genus 2 curve defined by an equation $y^2 = f_6(x)$. Recall that a level-3 structure on J_2 is a symplectic isomorphism:

$$\beta : (J_2[3], e_3) \xrightarrow{\cong} (F_3^4, E_3), \qquad \text{with} \quad e_3(x,y) = \rho^{E_3(\beta(x),\beta(y))},$$

where e_3 is the μ_3-valued weil-pairing and $E_3 : F_3^4 \times F_3^4 \to F_3$ is a (fixed) symplectic form. We will take:

$$E_3(u,v) = {}^t u E_3 v, \qquad \text{with} \quad E_3 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

We will identify the level-3 structure β with the 4-tuple $\beta = (x_1, \dots, x_4) \in X_2[3]^4$ with $x_i := \beta^{-1}(f_i)$, here f_i is the i-th standard basis vector of F_3^4.

Since $-1 \in Aut(J_2)$, we define:

$$T = T(J_2) := P(J_2[3]) \cong P(F_3^4) = P^3(F_3)$$

and we define a projective level-3 structure to be a 4-tuple

$$\beta_T = (\bar{x}_1, \dots, \bar{x}_4) \in T^4 \qquad \text{with} \quad e_3(x_i, x_j) \neq 1 \quad \text{iff} \quad (E_3)_{ij} \neq 0.$$

As in lemma 9.3, the map:

$$\beta = (x_1, \dots, x_4) \mapsto \beta_T = (\bar{x}_1, \dots, \bar{x}_4),$$

induces a bijection:

$$\left\{ \begin{array}{c} \text{level-3 structures} \\ \text{on } J_2 \end{array} \right\} \Big/ \{\pm 1\} \quad \longrightarrow \left\{ \begin{array}{c} \text{projective level-3} \\ \text{structures on } J_2 \end{array} \right\}.$$

9.5 The finite (simple) groups $PU(4, F_4)$ and $PSp(4, F_3)$ are isomorphic (see [A]) and the set of projective hermitian level-2 structures on J_4 and the set of projective level-2 structures respectively are principal homogeneous spaces on these groups. To get an explicit isomorphism of these principal homogeneous spaces, it suffices to give an explicit isomorphism (of homogeneous spaces) between S and T, since an isomorphism $\Phi : S \to T$ will preserve H and e_3, in the sense that (for $x, y \in J_4[2],\ x \neq y,\ \bar{x},\ \bar{y} \in S$):

$$H(x,y) = 0 \quad \text{iff} \quad e_3(u,v) = 1, \qquad \text{when} \quad \bar{u} = \Phi(\bar{x}),\ \ \bar{v} = \Phi(\bar{y}).$$

(The existence of an isomorphism $\Phi : S \to T$ is stated in [A], p.26, to get that $H(x,y) = 0$ iff $e_3(u,v) = 1$, it suffices to observe, since the forms are 'preserved', that for a non-degerate hermitian form H on F_4^4 and $x \in S \subset F_4^4 - \{0\}$ (so $H(x,x) = 1$), the subspace $x^\perp$ has $4^3 = 64 = 1 + 3 \cdot 21$ elements and that $3 \cdot 12$ of these have $H(y,y) = 1$. Similarly the subspace $< u >^\perp$ of $u \in F_3^4 - \{0\}$ w.r.t. a symplectic form has $3^3 = 27 = 3 + 2 \times 12$ elements.)

The desired birational isomorphism of moduli spaces now follows from the following proposition.

9.6 Proposition. Let $C_4,\ C_2$ be the (smooth, projective) curve of genus 4, genus 2 respectively, defined by:

$$y^3 = f_6(x) \qquad y^2 = f_6(x).$$

Then there are natural bijections between the three sets:

$$S(J(C_4)), \qquad T(J(C_2)) \qquad P := \{(f_2(x), f_3(x)) :\ f_6 = f_3^2 - f_2^3\}/\sim,$$

where $(f_2, f_3) \sim (g_2, g_3)$ if $f_2^3 = g_2^3$ and $f_3^2 = g_3^2$.

Hence the varieties $H_{3,1}/\Gamma_{M_{3,1}}(2)$ and $S_2/\Gamma_2(3)$ are birationally isomorphic.

Proof. The set $S(J_4)$ is canonically isomorphic to the set of odd theta characteristics of the curve C_4 modulo the action of the automorphism ϕ of order three. Since C_4 is non-hyperelliptic, the effective divisors D with $2D = K_{C_4}$ correspond to planes $H_D \subset P^3$ which are tangent to the canonical curve at each intersection point. Recall that the canonical embedding $C_4 \hookrightarrow P(H^0(C_4, \Omega^1_{C_4})$ is given by (cf. 7.5): $(x, y) \mapsto (1 : x : x^2 : y)$ and that C_4 lies on the cone defined by $x_0x_2 = x_1^2$.

The planes H_D defining odd theta characteristics don't pass through the vertex $(0 : 0 : 0 : 1)$ of the cone. Their equation may thus be written as: $x_3 = ax_0 + bx_1 + cx_2$. Then $H_D \cdot C_4$ is defined by the equations:

$$y^3 = f_6(x), \qquad y = -f_2(x), \quad \text{with} \quad f_2(x) = a + bx + cx^2.$$

Therefore H_D defines an odd theta characteristic iff $f_6(x) + f_2(x)^3 = f_3(x)^2$ for some f_3. Since $g(C_4) = 4$, we have $h^0(C_4, \alpha) = 1$ for all odd theta characteristics α, so for each α there is a unique H_D as above. Since $\phi(x, y) = (x, \omega y)$, the theta characteristic $\phi^* D$ is defined by $y = \omega^2 f_2$. This shows the natural bijection between $S(J(C_4))$ and P.

We recall that the map from $C^{(2)}$, the second symmetric product of C_2, to $J_2 = J(C_2) = Pic^0(C_2)$:

$$C^{(2)} \longrightarrow J_2, \qquad D \mapsto D - h,$$

(where h is the divisor (class) with $deg\, h = 2,\ h^0(h) = 2$) is surjective, and is an isomorphism outside $|h| \cong P^1 \subset C^{(2)}$ which is mapped to $0 \in J_2$. The points of order three on J_2 thus correspond to effective divisors of degree two, $D_2 \in C^{(2)}$, with $h^0(D_2) = 1$ and with $3D_2 \equiv 3h$. Since $1, x, x^2, x^3, y$ are a basis of $H^0(C_2, h^{\otimes 3})$, the zero locus of the section $s := f_3(x) - y$ on C_2 is given by:

$$y^2 = f_6(x), \qquad y = f_3(x).$$

The divisors D_2 corresponding to the points of order 3 thus correspond to the polynomials f_3 which satisfy $-f_6 + f_3^2 = f_2^3$ for some f_2. Since $D_2 + i^* D_2 = 2h$ (with $i : C_2 \to C_2$ the HE involution, we see that $-(D_2 - h) = i^*(D_2) - h$ is cut out by the section $i^* s = f_3 + y$. This gives the natural bijection between $T(J(C_2))$ and P. $\square$

10 Automorphism of order 4

10.1 In the first part of this section we investigate $\Theta(S_g^N)$ with

$$N = N_{p,q} = S^{\oplus p} \oplus (S^3)^{\oplus q}, \qquad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL_2(Z).$$

In the second part we study the case of a matrix M inducing an automorphism of type (n,n) and we determine the image of S_4^M under the Θ-map.

10.2 Note that S defines an automorphism of order 4 on the elliptic curve

$$E_4 := C/(Z + iZ).$$

In particular, $diag(i,\dots,i) \in S_g$ lies in $\Theta(S_g^{N_{p,q}})$, for all p,q.

10.3 Lemma. 1. The element $S \in SL(2,Z)$ of order four acts like:

$$R(S) = \sqrt{2}^{-1} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad \text{and} \quad v_\pm := \begin{pmatrix} 1 \\ \mu_\pm \end{pmatrix}, \quad \mu_\pm := -1 \pm \sqrt{2},$$

are two eigenvectors of $R(S)$. The eigenvalues of $U(N)$ are ± 1.

2. For all (p,q) we have $R(N_{p,q}) = R(N^{\oplus g})$. The map $R(N^{\oplus g})$ has two eigenvalues $\lambda = \pm 1$ and the corresponding eigenspaces are denoted by $V_\pm \subset C^{2^g}$. We have:
$$\dim V_+ = \dim V_- = 2^{g-1}.$$

3. Let $B := T^2 S T^2 S \in SL(2,Z) = \Gamma_1$. Then $B \in \Gamma_1(2)$ and
$$R(B) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \text{and} \quad R(N)R(B) = -R(B)R(N).$$
Moreover $R(B)v_+ = v_-$ and $R(B)v_- = v_+$.

Proof. Since $T^2 \equiv I$ mod. 2 and $S^2 = -I \equiv I$ mod. 2, we get $B \equiv I$ mod. 2 so $B \in \Gamma_1(2)$. Since $N_{p,q} \cdot (N^{\oplus g})^{-1}$ is a diagonal matrix with entries ± 1, and since all these matrices are in $\Gamma_g(2,4)$, we have $R(N_{p,q}) = R(N^{\oplus g})$.

As $R(S)$ and $R(T)$ have been determined and R is a projective representation, the matrix $R(B)$ is easy to compute and the other statements follow. □

10.4 Theorem. Let $N = N_{1,1}$ and let PV be the eigenspace of $R(N)$ which contains $\Theta(E_4^2)$. Let $B_1 := S_2^N$. Then:

1. the general point of B_1 corresponds to the jacobian of a genus 2 curve $y^2 = x f_2(x^2)$.
2. $$\overline{\Theta(B_1)} \cong PV = P^1.$$

3. The complement of $\Theta(B_1)$ in P^1 consists of two points, the cusps.

4. There are precisely 4 points in $\Theta(B_1)$ which correspond to a product of two elliptic curves, each of these points corresponds in fact to E_4^2.

Proof. The eigenspace $L := PV$ is spanned by $v_{++} := v_+ \otimes v_+ = \Theta(E_3^2)$ and $v_{--} := v_- \otimes v_-$. Since a general point of L corresponds to the Jacobian of a smooth genus two curve (prop. 7.9), none of the Q_m vanishes identically on L. Since Q_m with $m = \left[\begin{smallmatrix}11\\11\end{smallmatrix}\right]$ vanishes on all points of the form $v \otimes v$, we see that $Q_m \cap L$ consists of the two points v_{++} and v_{--} and since these correspond both to E_4^2, Q_m is the only quadric vanishing in these points.

The orbits of N on the even characteristics are:

$$\left[\begin{smallmatrix}11\\11\end{smallmatrix}\right], \quad \left[\begin{smallmatrix}00\\00\end{smallmatrix}\right], \quad \{\left[\begin{smallmatrix}10\\00\end{smallmatrix}\right], \left[\begin{smallmatrix}00\\10\end{smallmatrix}\right]\}, \quad \{\left[\begin{smallmatrix}01\\00\end{smallmatrix}\right], \left[\begin{smallmatrix}00\\01\end{smallmatrix}\right]\}, \quad \{\left[\begin{smallmatrix}11\\00\end{smallmatrix}\right], \left[\begin{smallmatrix}00\\11\end{smallmatrix}\right]\}, \quad \{\left[\begin{smallmatrix}10\\01\end{smallmatrix}\right], \left[\begin{smallmatrix}01\\10\end{smallmatrix}\right]\}.$$

Let $B_2 := B \oplus B$, then $R(B_2) = R(B) \otimes R(B)$ and since $B_2 \in \Gamma_2(2)$, $R(B_2)$ fixes the characteristics. One easily computes the action of B_2 on the equations of the quadrics:

$$R(B_2)Q\left[\begin{smallmatrix}ab\\cd\end{smallmatrix}\right] = (-1)^{a+b+c+d}Q\left[\begin{smallmatrix}ab\\cd\end{smallmatrix}\right].$$

Thus at both of the two fixed points of the involution $R(B_2)$ on L the four quadrics $\left[\begin{smallmatrix}10\\00\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}00\\10\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}01\\00\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}00\\01\end{smallmatrix}\right]$ vanish. These points are thus cusps, and there are 6 quadrics vanishing in each of the points. Since there are only 5 quadrics left, we conclude that there are precisely two cusps.

Since L is an eigenspace for $R(N)$, in an intersection point P of L and Q_n also $Q_{R\cdot n}$ vanishes. Thus if $n = \left[\begin{smallmatrix}11\\00\end{smallmatrix}\right]$, then also $R \cdot n = \left[\begin{smallmatrix}00\\11\end{smallmatrix}\right]$ vanishes in P. Thus P must be one of the two cusps. The same holds for $m = \left[\begin{smallmatrix}10\\01\end{smallmatrix}\right]$. The only way in which this can work out is that both n and $R(N) \cdot n$ are tangent to L at one cusp and m and $R(N) \cdot m$ are tangent at the other cusp.

Since $m = \left[\begin{smallmatrix}00\\00\end{smallmatrix}\right]$ is fixed by B_2 but Q_m cannot intersect L in the fixed points of $R(B_2)$ on L (which are the cusps), we conclude that Q_m intersects L in two distinct points. $\square$

10.5 Remark. In the case $g = 3$ we have a surface $S := \overline{\Theta(B_2)} \subset PV = P^3$, here $B_2 := H(N_{2,1}) \subset S_3$. Since B_2 parametrizes hyperelliptic jacobians, see 7.9, there is one Q_m which vanishes identically on S. This Q_m is thus invariant under $R(N_{2,1})$ and a computation shows that none of the 4 $R(N_{2,1})$-invariant Q_m's vanishes identically on PV. Therefore $S = Q_m \cap PV$, for one of these m's and it is in fact a smooth quadric in P^3. We hope to describe the cusps etc. of this surface later.

10.6 We will now examine abelian varieties with an automorphism of order 4 of type (n, n), but where the automorphism is not given by $N_{n,n}$. Consider the following $4n \times 4n$ matrix M which is symplectic w.r.t to the standard form E:

$$M := \begin{pmatrix} 0 & I & & \\ -I & 0 & & \\ & & 0 & I \\ & & -I & 0 \end{pmatrix}, \qquad E = \begin{pmatrix} & & I & \\ & & & I \\ -I & & & \\ & -I & & \end{pmatrix}.$$

10.7 Proposition. 1. The fixed point set of M on S_{2n} is:

$$S_{2n}^M = \left\{ \tau = \begin{pmatrix} \tau_1 & \tau_{12} \\ -\tau_{12} & \tau_1 \end{pmatrix} \in S_{2n} \;:\; \tau_1 \in S_n,\ {}^t\tau_{12} = -\tau_{12} \right\}$$

and $\dim S_{2n}^M = n^2 = \frac{1}{2}n(n+1) + \frac{1}{2}n(n-1)$.

2. For $\tau \in S_{2n}^M = H(M)$ the abelian variety X_τ has an automorphism ϕ of order 4 and type (n,n). Thus $H(M) \cong U(n,n)/(U(n) \times U(n))$.

3. Let ϵ_1, ϵ_2, ϵ_1', $\epsilon_2' \in (Z/2Z)^n$ and let $\tau \in S_{2n}$. Then:

$$\theta[{}^{\epsilon_1\ \epsilon_2}_{\epsilon_1'\ \epsilon_2'}](M\cdot\tau) = (-1)^{\epsilon_2^t\epsilon_2'}\theta[{}^{\epsilon_2\ \epsilon_1}_{\epsilon_2'\ \epsilon_1'}](\tau) \qquad \text{and} \quad \theta[{}^{\epsilon_1\ \epsilon_2}_{0\ \ 0}](2M\cdot\tau) = \theta[{}^{\epsilon_2\ \epsilon_1}_{0\ \ 0}](2\tau).$$

4. The projective automorphism $R(M) \in Aut(P^{2^g-1})$ is given by:

$$R(M)(\ldots : x_\sigma : \ldots) = (\ldots : y_\sigma : \ldots), \qquad y_{(\epsilon_1,\epsilon_2)} := x_{(\epsilon_2,\epsilon_1)}.$$

5. The image of $H(M)$ under the map $\Theta : S_{2n} \to P^{2^{2n}-1}$ lies in the eigenspace PV of dimension $2^{2n-1}+2^{n-1}-1$ of $R(M)$ which is defined by the $\binom{2^n}{2} = 2^{2n-1}-2^{n-1}$ linear equations:

$$X_{\epsilon_1\ \epsilon_2} - X_{\epsilon_2\ \epsilon_1} = 0, \qquad \epsilon_1,\ \epsilon_2 \in (Z/2Z)^n.$$

6. The restriction of Θ to the submanifold $S_n \subset H(M)$ consisting of the matrices with $\tau_{12} = 0$, is the composition of the Θ-map for $g = n$, $\Theta_n : S_n \to P^{2^n-1}$, with the second Veronese map $P^{2^n-1} \to P^{2^{2n-1}+2^{n-1}-1} \cong PV$. In particular, $\Theta(H(M))$ spans the $P^{2^{2n-1}+2^{n-1}-1}$.

Proof. We have $X_\tau = C^{2n}/(I\ \tau)$ and to define ϕ we must give a C-linear map $d\phi : T_0A = C^{2n} \to T_0A$ which on the lattice Λ_τ induces $\phi_* := M$. The (easily verified) matrix equality

$$d\phi(I\ \tau) = (I\ \tau)M, \qquad \text{with} \quad d\phi := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

(where 0 and I are $n \times n$ matrices) thus in fact defines $\phi : X_\tau \to X_\tau$. Since the eigenvalues of $d\phi$ are i and $-i$, each with multiplicity n we have that ϕ is of type (n,n).

The formulas are easy consequences of Igusa's transformation formula, cf.[I]. In fact, denoting the two diagonal blocks of M by A, we get directly from the series defining the theta functions that

$$\theta[{}^{\epsilon}_{\epsilon'}](M\cdot\tau) = \theta[{}^{\epsilon A}_{\epsilon' A}](\tau)$$

and then one must use [I], $(\theta.2)$, p.39 to make ϵA and $\epsilon'^t A^{-1}$ have entries in $\{0,\ 1\}$. The second formula is a special case of the first one since $2(M\tau) = M(2\tau)$.

To find the eigenspace PV, note that for $\tau \in S_{2n}^M$ we have $\theta[{}^{\epsilon_1\epsilon_2}_{0\ 0}](2\tau) = \theta[{}^{\epsilon_2\epsilon_1}_{0\ 0}](2\tau)$.

If $\tau_{12} = 0$ then $\theta[\begin{smallmatrix}\epsilon_1\epsilon_2\\0\ 0\end{smallmatrix}](2\tau) = \theta[\begin{smallmatrix}\epsilon_1\\0\end{smallmatrix}](2\tau_1)\theta[\begin{smallmatrix}\epsilon_2\\0\end{smallmatrix}](2\tau_1)$ with $\tau_1 \in S_n$, which implies the last statement. □

The following corollary follows trivially from proposition 10.7 and will allow us to find the equations for the image of $H(M)$.

10.8 Corollary. Let $[\begin{smallmatrix}\epsilon\\\epsilon'\end{smallmatrix}]$ be an odd (i.e. $\epsilon^t\epsilon' \equiv 1 \mod 2$) characteristic. Then

$$\theta[\begin{smallmatrix}\epsilon & \epsilon\\\epsilon' & \epsilon'\end{smallmatrix}](\tau) = 0 \qquad (\forall \tau \in H(M)).$$

In particular, there are $2^{n-1}(2^n - 1)$ even theta constants which vanish identically on $H(M)$.

10.9 Remark. We observe that since the dimensions of the eigenspaces of $R(M)$ are not equal, while the eigenspaces of $R(N_{n,n})$ have the same dimension, M cannot be conjugated in $Sp(4n, Z)$ with $N_{n,n}$.

In case $g = 4$ we see that 6 even theta constants vanish on $\tau \in H(M)$. These points do not correspond to Jacobians of curves (see for example prop. 7.9). Since for general τ, the abelian variety X_τ has $NS(X_\tau) \cong Z$ (see [W]), X_τ is not isogeneous to a product of abelian varieties. Therefore we found a new 4 dimensional subvariety of the locus $\theta^{ind}_{null,6}$ from [Deb].

10.10 We now consider the case $n = 2$, so the 4-dimensional $H(M) \subset S_4$ is mapped to a P^9 by the second order theta constants. We will show that the image is the complete intersection of 5 quadrics.

10.11 Proposition. The closure of the image of the map

$$\Theta_4 : H(M) \cong H_{2,2} \longrightarrow PV \cong P^9$$

is the complete intersection of the following 5 quadrics (here $Z_0, \dots, Z_4$ and $W_0, \dots W_4$ are the coordinates on PV):

$$\begin{array}{ccccccccccc}
Z_0^2 & & & & & = & W_0^2 & +W_1^2 & +W_2^2 & +W_3^2 & -W_4^2 \\
& Z_1^2 & & & & = & W_0^2 & & +W_2^2 & & -W_4^2 \\
& & Z_2^2 & & & = & W_0^2 & +W_1^2 & & & -W_4^2 \\
& & & Z_3^2 & & = & & W_1^2 & & +W_3^2 & -W_4^2 \\
& & & & Z_4^2 & = & & & W_2^2 & +W_3^2 & -W_4^2
\end{array}$$

Proof. From corollary 10.8 we know that 6 even theta constants vanish on $H(M)$. The quadratic relations between the first and second order theta constants (see 3.3.2) thus imply that the image of $H_{2,2}$ lies in 6 quadrics. Since the image lies also in $PV \cong P^9$, we restrict the quadrics to this projective space. As coordinates on PV we choose:

$$X_{0000},\ X_{0101},\ X_{1010},\ X_{1111},\ X_{0001},\ X_{0010},\ X_{0011},\ X_{0110},\ ,X_{0111},\ X_{1011}.$$

In these coordinates, the restriction of $\frac{1}{2}\theta[\begin{smallmatrix}1010\\1010\end{smallmatrix}]^2$ is given by:

$$X_{0000}X_{1010} + X_{0101}X_{1111} - X_{0010}^2 - X_{0111}^2 + 2(X_{0001}X_{1011} - X_{0011}X_{0110}).$$

Proceeding in this way, one finds 6 quadrics, and it is easy to check that:

$$\theta\left[\begin{smallmatrix}1010\\1010\end{smallmatrix}\right]^2-\theta\left[\begin{smallmatrix}1010\\1111\end{smallmatrix}\right]^2-\theta\left[\begin{smallmatrix}1111\\1010\end{smallmatrix}\right]^2+\theta\left[\begin{smallmatrix}1111\\0101\end{smallmatrix}\right]^2-\theta\left[\begin{smallmatrix}0101\\0101\end{smallmatrix}\right]^2+\theta\left[\begin{smallmatrix}0101\\1111\end{smallmatrix}\right]^2$$

gives a quadric which is identically zero on PV. Note that the quadric given by the theta constant $\theta\left[\begin{smallmatrix}1010\\1010\end{smallmatrix}\right]^2-\theta\left[\begin{smallmatrix}1010\\1111\end{smallmatrix}\right]^2$ is $4(X_{0001}X_{1011}-X_{0011}X_{0110})$.

Define new coordinates X_i by:

$$\begin{aligned}X_{0000}&=X_0+X_1+X_2+X_3\\X_{0101}&=X_0-X_1+X_2-X_3\\X_{1010}&=X_0+X_1-X_2-X_3\\X_{1111}&=X_0-X_1-X_2+X_3.\end{aligned}$$

In particular, one has:

$$X_{0000}X_{1010}+X_{0101}X_{1111}=2(X_0^2+X_1^2-X_2^2-X_3^2).$$

Coordinates Y_i are defined by:

$$\begin{aligned}X_{0001}&=Y_0+Y_1 & X_{1011}&=Y_0-Y_1\\X_{0010}&=Y_2+Y_3 & X_{0111}&=Y_2-Y_3\\X_{0110}&=Y_4+Y_5 & X_{0011}&=Y_4-Y_5\end{aligned}$$

In particular, one has:

$$X_{0010}^2+X_{0111}^2=2(Y_2^2+Y_3^2),\quad 2(X_{0001}X_{1011}-X_{0011}X_{0110})=2(Y_0^2-Y_1^2-Y_4^2+Y_5^2).$$

In these new coordinates the equation of each of the 6 vanishing even theta constants is a sum of squares, for example, $\theta\left[\begin{smallmatrix}1010\\1010\end{smallmatrix}\right]^2$ corresponds to:

$$X_0^2+X_1^2-X_3^2-X_4^2+Y_0^2-Y_1^2-Y_2^2-Y_3^2-Y_4^2+Y_5^2,$$

and $\theta\left[\begin{smallmatrix}1010\\1111\end{smallmatrix}\right]^2$ corresponds to:

$$X_0^2+X_1^2-X_3^2-X_4^2-Y_0^2+Y_1^2-Y_2^2-Y_3^2+Y_4^2-Y_5^2.$$

Taking suitable lineair combinations one finds

$$\begin{array}{llll}X_0^2-X_1^2+X_2^2-X_3^2 & -Y_0^2-Y_1^2 & & \\ X_0^2+X_1^2-X_2^2-X_3^2 & & -Y_2^2-Y_3^2 & \\ X_0^2-X_1^2-X_2^2+X_3^2 & & & -Y_4^2-Y_5^2\\ & -Y_0^2+Y_1^2 & & +Y_4^2-Y_5^2\\ & & -Y_2^2+Y_3^2 & +Y_4^2-Y_5^2.\end{array}$$

Finally, by substracting the first equation from the second and the third, one can express the squares of $Z_0:=X_0,\ Z_1:=X_1,\ Z_2:=X_2,\ Z_3:=Y_0,\ Z_4:=Y_2$ as linear combinations of the squares of $W_0:=X_3,\ W_1:=Y_1,\ W_2:=Y_3,\ W_3:=Y_4,\ W_4:=Y_5$.

The equations in the statement of the proposition define a variety X which is a $2^5:1$-covering of the P^4 with coordinates W_i, from which the irreducibility of X is easily seen. Since the four dimensional $\Theta(H(M))$ lies in X we thus have $X=\overline{\Theta(H(M))}$. $\square$

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