Math. Ann. 294, 741–754 (1992)

Mathematische Annalen

Prym varieties and the Verlinde formula

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Received June 14, 1991

Mathematics Subject Classification (1991): 14H60, 14H42

This note offers an interpretation of the Verlinde formula that brings into play the projective configuration of the Kummer variety of the Jacobian and its Pryms; we derive a novel proof for the formula in degree 4 (actually, a lower bound, the upper bound was proved by Bertram [Be]).

Let's briefly recall the meaning of the words; for more detail and references see below. What is referred to as the Verlinde formula is a Quantum-Field-Theoretic derivation of the dimension of certain cohomology groups; we shall only consider the following: let C be a Riemann surface of genus $g \geqq 2$, $\mathcal{S}M_C(2)$ the moduli space of semistable bundles of rank 2 and trivial determinant over C and $\mathcal{L}$ the ample line bundle over $\mathcal{S}M_C(2)$ that generates $\mathrm{Pic}\,(\mathcal{S}M_C(2)) \cong \mathbb{Z}$. Let

$$N_k := \left(\frac{k+2}{2}\right)^{g-1} \sum_{j=0}^{k} \frac{1}{\left(\sin \dfrac{\pi(j+1)}{k+2}\right)^{2g-2}}$$

and let

$$D_k := \dim H^0(\mathcal{S}M_C(2), \mathcal{L}^{\otimes k}).$$

Then the Verlinde formula asserts:

$$(*) \qquad D_k = N_k .$$

Recently Bertram [Be] proved an upper bound for D_k:

$$D_k \leqq N_k .$$

⋆ Research partially supported by the CNR Foundation while the autor was visiting the University of Pisa, Italy

⋆⋆ A visiting professorship at MSRI supported by NSF grant # DMS 8505550 is gratefully acknowledged

The equality was proven by Beauville in the case $k = 1$ [B1] and in [B2] he proved that N_2 gives a lower bound, so the equality $(*)$ is now proven for $k = 2$ as well.

Moreover, [B1] gives an identification of $H^0(\mathscr{S}M_C(2), \mathscr{L})$ with $V = H^0(\operatorname{Jac} C, L)$, where $L = \mathscr{O}_{\operatorname{Jac} C}(2\Theta)$ and Θ is a symmetric divisor defining the principal polarization. This gives us the link with Prym geometry. The map to the projective space $\mathbb{P}V$ determined by L sends $X := \operatorname{Jac} C$ to the Kummer variety $K := X/\pm$; for any point of order two $x \in X[2]$, the associated Prym of X, P_x an abelian variety of dimension $g - 1$, maps similarly to $\mathbb{P}V$:

$$\begin{array}{ccccc} X & \xrightarrow{2:1} & K & \subset & \mathbb{P}V \\ & & & & \cup \\ P_x & \longrightarrow & K(P_x) & \subset & \mathbb{P}V_x \end{array}$$

where the dimensions of $\mathbb{P}V$, $\mathbb{P}V_x$ are $2^g - 1$, $2^{g-1} - 1$, respectively. As reviewed in Sect. 1, the Kummer variety and the Kummers of all these Pryms (we call the Kummers of Pryms simply Pryms in the sequel) are in the image of

$$\varphi : \mathscr{S}M_C(2) \longrightarrow \mathbb{P}V \cong \mathbb{P}H^0(\mathscr{S}M_C(2), \mathscr{L}).$$

In particular, a polynomial which vanishes on $\varphi(\mathscr{S}M_C(2))$ must vanish on all the Pryms (as well on K).

Let $F \in S^kV$, so F is a homogeneous polynomial of degree k on $\mathbb{P}V$. The restriction of F to $\varphi(\mathscr{S}M_C(2))$ defines a map:

$$m_k : S^kV \longrightarrow H^0(\mathscr{S}M_C(2), \mathscr{L}^{\otimes k}).$$

Since $S^kV/\ker(m_k)$ can be identified with a subspace of $H^0(\mathscr{S}M_C(2), \mathscr{L}^{\otimes k})$, an upper bound for $\dim\ker(m_k)$ thus gives a lower bound for D_k. Since a polynomial which vanishes on $\varphi(\mathscr{S}M_C(2))$ must vanish on all the Pryms, we find that

$$\dim\ker m_k \leqq \dim\{F \in S^kV : F \text{ vanishes on all Pryms}\}.$$

By expressing geometric relations between K and the $K(P_x)$ in a suitable coordinate system for $\mathbb{P}V$, we obtain information on the quartics that vanish on all Pryms, see [vG, Theorem 1] and Theorem 1 below. For a generic C, Theorem 2(b) gives a lower bound for D_4, which turns out to be N_4. Thus, combining our results with those of Bertram, we have as a corollary a proof of the Verlinde formula for generic C (cf. Sect. 3 below) but since D_k is the same for all Riemann surfaces of the same genus (cf. [L]) we have $D_4 = N_4$ for all C.

We also obtain some information on the ideal of the variety $\varphi(\mathscr{S}M_C(2))$ (cf. Corollary 2):

Let C be a Riemann surface without vanishing thetanulls. Then a quartic F vanishes on $\varphi(\mathscr{S}M_C(2))$ iff *F vanishes on all the Pryms of C.*

Remarks. 1. [B2] shows that K and $\mathscr{S}M_C(2)$ are projectively normal in degree 2 iff C has no vanishing thetanulls. Our Corollary 2 implies that $\mathscr{S}M_C(2)$ is projectively normal in degree 4 if the same condition is satisfied.

2. Almost a footnote: an interpretation of the Verlinde numbers N_k in terms of the representation ring $R(SU(2))$ is offered by Bott and Szenes [Bott]; their algorithm gives indeed N_4, cf. Sect. 3.

1 The geometry of $\mathcal{S}M_C(2)$

In this section we derive from the theory of rank 2 bundles on C the Schottky-Jung relations (an identity between theta constants on $X = \operatorname{Jac} C$ and those of a Prym) and the Donagi relations (an identity between theta constants of two Pryms).

For any rank 2 bundle E on C and $z \in X$ we have $\det(E \otimes z) \cong \det(E) \otimes z^{\otimes 2}$. Thus the group $X[2]$ acts on $\mathcal{S}M_C(2)$ via tensorization. Since $\mathcal{L}$ is the unique ample generator of $\operatorname{Pic}(\mathcal{S}M_C(2))$, the pull-back of $\mathcal{L}$ by the automorphism of $\mathcal{S}M_C(2)$ defined by $x \in X[2]$ is isomorphic to $\mathcal{L}$. Therefore there is a projective representation:

$$U : X[2] \longrightarrow \operatorname{Aut}(\mathbb{P}H^0(\mathcal{S}M_C(2), \mathcal{L})), \quad \text{and} \quad U(x)\varphi(E) = \varphi(E \otimes x),$$

so φ is equivariant for the action of $X[2]$.

For $z \in X = \operatorname{Pic}^0(C)$, the rank 2 bundle $z \oplus z^{-1}$ has trivial determinant and is semi-stable (but not stable). Thus we have a map $X \longrightarrow \mathcal{S}M_C(2)$ which factors over K (since $z \oplus z^{-1} \cong z^{-1} \oplus z$):

$$\psi_K : K \longrightarrow \mathcal{S}M_C(2), \quad z \mapsto z \oplus z^{-1}.$$

The image of ψ_K is stable under the action of $X[2]$ on $\mathcal{S}M_C(2)$ and we have $U(x)\psi_K(z) = \psi_K(z \otimes x)$. Using the identification $\mathbb{P}V \cong \mathbb{P}H^0(\mathcal{S}M_C(2), \mathcal{L})$ and Mumford's theory of theta groups, we see that the projective representation U of $X[2]$ is isomorphic to the projectivization of the irreducible (Schrödinger) representation of the Heisenberg group H:

$$\begin{aligned} H = H_g &:= \{(t, \alpha, \alpha^*) \in \mathbb{C}^* \times (\mathbb{Z}/2)^g \times \operatorname{Hom}((\mathbb{Z}/2)^g, \mathbb{C}^*)\}, \quad \text{with} \\ &(t, \alpha, \alpha^*)(s, \beta, \beta^*) = (ts\beta^*(\alpha), \alpha + \beta, \alpha^*\beta^*) \end{aligned}$$

on a space V, with basis δ_σ, $\sigma \in (\mathbb{Z}/2)^g$ via:

$$(t, \alpha, \alpha^*) \cdot \delta_\sigma := t\alpha^*(\sigma + \alpha)\delta_{\sigma+\alpha}.$$

Therefore each $U(x)$ has two eigenspaces in $\mathbb{P}V$, each of dimension $2^{g-1} - 1$, and, since the Weil-pairing $E : X[2] \times X[2] \to \mathbb{Z}/2$ corresponds to the commutator in H, we have:

$$U(x)U(y) = (-1)^{E(x,y)}U(y)U(x).$$

Thus if $E(x, y) = 1$, the eigenspaces of $U(x)$ and $U(y)$ do not intersect. On an eigenspace $\mathbb{P}V_x$ of $U(x)$ the group

$$x^\perp := \{y \in X[2] : E(x, y) = 0\}$$

acts, and since $U(x)$ acts trivially on an eigenspace, the action factors over $x^\perp/\langle x\rangle \cong (\mathbb{Z}/2)^{2(g-1)}$. This action is the projectivization of the Schrödinger representation of $H' = H_{g-1}$ on V_x.

When $E(x, y) = 0$, the space $\mathbb{P}V_{x,y} := \mathbb{P}V_x \cap \mathbb{P}V_y$, is an eigenspace of $U(\bar{y})$,

$$\bar{y} := y + \langle x\rangle \in x^\perp/\langle x\rangle$$

so $\dim \mathbb{P}V_{x,y} = 2^{g-2} - 1$, the group $(x^\perp \cap y^\perp)/\langle x, y\rangle \cong (\mathbb{Z}/2)^{2(g-2)}$ acts on it and the action is the projectivization of the Schrödinger representation of $H'' = H_{g-2}$ on $V_{x,y}$.

For any ppav (A, Θ_A), with symmetric Θ_A, the map $\varphi_A : A \to \mathbb{P}H^0(A, L_A)$, with $L_A := \mathcal{O}_A(2\Theta_A)$ is equivariant under the action of $A[2]$ and factors over $A/\pm$. If the decomposition of this ppav into indecomposable ppav's is given by:

$$(A, \Theta_A) \cong \prod_i (A_i, \Theta_{A_i}), \quad \text{then} \quad A \xrightarrow{\varphi} K(A) := \prod (A_i/\pm) \subset \mathbb{P}H^0(A, L_A)$$

and we call $K(A)$ the Kummer variety of A. In case A is a Jacobian, then $K(A) \cong A/\pm$.

For non-zero $x \in A[2]$, let:

$$S_x := \{z \in A : z^{\otimes 2} \cong x\}.$$

The set S_x is a principal homogeneous space for $A[2]$, and since for $z \in S_x$, $\varphi(z) = \varphi(z^{-1}) = \varphi(z \otimes x)$, the image of S_x lies in the union of the two eigenspaces of $U(x)$ in $\mathbb{P}H^0(A, L_A)$, and each eigenspace gets half the points. If A is indecomposable, then $K(A) \cap \mathbb{P}V_x$ consists exactly of the image of one half of S_x (since then $\varphi(z) = \varphi(z \otimes x)$ is equivalent to $z \in S_x$) and since φ is 2:1, that is a set of 2^{2g-2} points.

A point of order two $x \in X[2]$ defines an unramified 2:1 cover C_x of C and a (principally polarized) abelian variety (P_x, Θ_{P_x}):

$$\pi_x : C_x \longrightarrow C, \quad P_x := \ker(Nm_x : \operatorname{Jac} C_x \longrightarrow X)^0,$$

where ^{0} stands for the connected component containing $0 \in \operatorname{Jac} C_x$. Here Nm_x is the norm map, it maps $D \mapsto \pi_{x*}D$ for any divisor D on C_x, and it is a surjective homomorphism.

One has $\ker(\pi_x^* : X \to \operatorname{Jac}(C_x)) \cong \langle x\rangle$, π_x^* induces an isomorphism:

$$\pi_x^* : x^\perp/\langle x\rangle \xrightarrow{\cong} P_x[2] \quad \text{and} \quad \ker(Nm_x) = P_x + (P_x + \pi_x^* y),$$

for (any) $y \in X[2]$ with $E(x, y) = 1$. Since $Nm_x(p \otimes \pi_x^* z) = Nm_x(p) \otimes z^{\otimes 2}$, the group $X[2]$ acts on $\ker(Nm_x)$ via $p \mapsto p \otimes \pi_x^* y$, $y \in X[2]$ and this action factors over $X[2]/\langle x\rangle \cong (\mathbb{Z}/2)^{2g-1}$.

Since $\pi_{x*}\mathcal{O}_{C_x} \cong \mathcal{O}_C \oplus x$ we have $\det \pi_* \mathcal{O}_{C_x} \cong x$. For any $p \in \ker(Nm_x)$, the rank two bundle $\pi_{x*}p$ on C then also has determinant x. Fixing a $z_x \in X$ with $z_x^{\otimes 2} \cong x$, we get a map:

$$\psi_x : \ker(Nm_x) \cong P_x \cup P_x \longrightarrow \mathcal{SM}_C(2), \quad p \mapsto (\pi_{x*}p) \otimes z_x.$$

Note that the actual map depends on the choice of z_x, but that if also $z_x'^{\otimes 2} \cong x$, then $z_x' \cong z_x \otimes y$ for some $y \in X[2]$. Thus if ψ_x' is the map defined by z_x', then we have $\psi_x'(p) = U(y)\psi_x(p)$ for all $p \in \ker(Nm_x)$.

By the projection formula we have, for any $p \in \ker(Nm_x)$ and $z \in X$:

$$\pi_{x*}(p) \otimes z \cong \pi_{x*}(p \otimes \pi_x^* z) \quad \text{thus} \quad \psi_x(p) \otimes y = \psi_x(p \otimes \pi_x^* y) \quad (\forall y \in X[2])$$

and so the map ψ_x is equivariant for the actions of $X[2]$ on $\ker(Nm_x)$ and $\mathcal{S}M_C(2)$. Moreover, the choice of z_x in the definition of ψ_x does not affect the image of ψ_x (i.e. $\mathrm{Im}(\psi_x) = \mathrm{Im}(\psi'_x)$).

Since both ψ_x and φ are equivariant for the $X[2]$-action, and since $\langle x\rangle$ acts trivially on $\ker(Nm_x)$, we have, for all $p \in \ker(Nm_x)$:

$$U(x)\varphi(\psi_x(p)) = \varphi(\psi_x(p)), \quad \text{thus} \quad \psi_x(\ker(Nm_x)) \subset \mathbb{P}V_x^+ \cup \mathbb{P}V_x^-,$$

the union of the two eigenspaces of $U(x)$ in $\mathbb{P}V$. Since $U(y)$, with $E(x,y) = 1$, interchanges the two components of $\ker(Nm_x)$ and the two eigenspaces, there is one component of $\ker(Nm_x)$ in each eigenspace. We can restrict $\varphi \circ \psi_x$ to get a map:

$$\varphi_x : P_x \longrightarrow \mathbb{P}V_x$$

where $\mathbb{P}V_x$ is the appropriate eigenspace of $U(x)$. The description of φ by Narasimhan and Ramanan [NR] and Beauville [B1], combined with results of Mumford's [M] allows us to identify φ_x.

Proposition 1. *The map $\varphi_x : P_x \longrightarrow \mathbb{P}V_x$ is the natural map*

$$P_x \longrightarrow K(P_x) \subset \mathbb{P}H^0(P_x, L_{P_x}) \cong \mathbb{P}V_x \quad \textit{with} \quad L_{P_x} := \mathcal{O}(2\Theta_{P_x}).$$

Proof. Let $\Theta_{g-1} := \{z \in \mathrm{Pic}^{g-1}(C) : H^0(C,z) \neq 0\}$ be the theta divisor of C. Then there is a unique $w \in \mathrm{Pic}^{g-1}(C)$, in fact a theta characteristic, such that $\Theta = \{z \otimes w^{-1} : z \in \Theta_{g-1}\}$. The map $\varphi : \mathcal{S}M_C(2) \longrightarrow \mathbb{P}V \cong |2\Theta| \cong |2\Theta_{g-1}|$ is given by:

$$\varphi : E \mapsto D_E \in |2\Theta_{g-1}|, \quad D_E := \{z \in \mathrm{Pic}^{g-1}(C) : H^0(C, E \otimes z) \neq 0\}.$$

For $p \in \ker(Nm_x)$ we have:

$$H^0(C, (\pi_{x*}p) \otimes z_x \otimes z) = H^0(C, \pi_{x*}(p \otimes \pi_x^+(z_x \otimes z))) = H^0(C_x, p \otimes \pi_x^*(z_x \otimes z)).$$

The degree of $\pi_x^* z$ is $2(g-1) = g_x - 1$ with g_x the genus of C_x, therefore

$$D_{\pi_{x*}(p)\otimes z_x} = D_p := \{z \in \mathrm{Pic}^{g-1}(C) : \pi_x^*(z) \in p^{-1} \otimes (\pi_x^* z_x)^{-1} \otimes \widetilde{\Theta}_{g_x-1}\},$$

with $\widetilde{\Theta}_{g_x-1} \subset \mathrm{Pic}^{g_x-1}(C)$ the theta divisor of C_x, and Mumford [M, p. 334] proved that $p \mapsto D_p$ gives the natural map $P_x \to \mathbb{P}H^0(P_x, L_{P_x})$. $\square$

From now on, we simply write K and $K(P_x)$ for the images of the maps $\varphi_K := \varphi \circ \psi_K$ and φ_x respectively.

Proposition 2. *For any curve C and any $x, y \in X[2] - \{0\}$ with $E(x,y) = 0$ we have:*

(i) *The SJ (Schottky-Jung) relations:*

$$K \cap \mathbb{P}V_x = K(P_x[2])\,. \tag{SJ}$$

(ii) *The Donagi relations:*

$$\varphi_y(S_{\bar{x}}) \cap \mathbb{P}V_{x,y} = \varphi_x(S_{\bar{y}}) \cap \mathbb{P}V_{x,y} \tag{D}$$

with $\bar{x} := x + \langle y\rangle \in y^\perp/\langle y\rangle \cong P_y[2]$ and $\bar{y} := y + \langle x\rangle \in x^\perp/\langle x\rangle \cong P_x[2]$.

Proof. The Schottky-Jung relations follow from $\pi_*(\mathcal{O}_{C_x}) \cong \mathcal{O}_C \oplus x$ which, after a translation to get trivial determinant, implies that $\varphi_K(y) = \varphi_x(0)$ for some $y \in X[2]$. Since the maps are equivariant under the action of $x^\perp$, which stabilizes $\mathbb{P}V_x$ and acts there via $x^\perp/\langle x\rangle \cong P_x[2]$, the first point follows.

More precisely, let $\varphi_K(z_x) \in K\cap\mathbb{P}V_x$, then$z_x^{\otimes 2} \cong x$ and $K\cap\mathbb{P}V_x = \{\varphi_K(z_x\otimes y) : y \in x^\perp\}$. Let z_x define $\varphi_x : P_x \to \mathbb{P}V_x$ then:

$$\varphi_K(z_x) := \varphi(z_x \oplus z_x^{-1}) = \varphi((\mathcal{O}_C \oplus x) \otimes z_x) = \varphi_x(\mathcal{O}_{C_x})\,.$$

Using the action of $x^\perp$ and the identification $\pi_x^* : x^\perp/\langle x\rangle \longrightarrow P_x[2]$ we get, with y running over $x^\perp$:

$$\begin{aligned} K \cap \mathbb{P}V_x &= \{\varphi_K(z_x \otimes y)\} = \{U(y)\varphi_K(z_x)\} \\ &= \{U(y)\varphi_x(\mathcal{O}_{C_x})\} = \{\varphi_x(\pi_x^* y)\} = K(P_x[2])\,. \end{aligned}$$

For the second point, we consider the unitary rank 2 vector bundle E on C defined by the representation:

$$\varrho_E : \pi_1(C) \longrightarrow SU(2)\,, \quad \begin{cases} \alpha_k \mapsto I\,, \quad 1 \leqq k \leqq g\,, \quad \beta_l \mapsto I\,, \quad 3 \leqq l \leqq g\,, \\ \beta_1 \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \beta_2 \mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \end{cases}$$

here the α_i, β_j are standard generators of $\pi_1(C)$. To prove the Donagi relation, we will first show that $E \cong \pi_{x*}q_x \cong \pi_{y*}q_y$, for $x, y \in X[2]$, $q_x \in \mathrm{Jac}\,(C_x)$, $q_y \in \mathrm{Jac}\,(C_y)$, which is equivalent to proving that

$$\varrho_E \sim \mathrm{Ind}_{\pi_1(C_x)}^{\pi_1(C)}(\varrho_x) \sim \mathrm{Ind}_{\pi_1(C_y)}^{\pi_1(C)}(\varrho_y)\,,$$

with $\varrho_x : \pi_1(C_x) \to U(1)$ the representation corresponding to q_x etcetera.

Let C_x be the unramified 2:1 cover of C defined by the subgroup $\ker(\varepsilon_x)$ of $\pi_1(C)$ with

$$\varepsilon_x : \pi_1(C) \longrightarrow U(1)\,, \quad \varepsilon_x(\beta_1) = -1\,, \ \varepsilon_x(\alpha_i) = \varepsilon_x(\beta_j) = 1\,,$$

with $1 \leqq i \leqq g$, $2 \leqq j \leqq g$. Similarly, we define C_y by a character ε_y with $\varepsilon_y(\beta_2) = -1$, and ε_y trivial on the others generators.

Note that $\pi_{x*} : \pi_1(C_x) \xrightarrow{\cong} \ker(\varepsilon_x) \subset \pi_1(C)$, and that $\pi_1(C_x)$ is the fundamental group of a Riemann surface of genus $2g-1$. For suitable generators γ_i, δ_i of $\pi_1(C_x)$ the homomorphism π_{x*} is given by:

$$\begin{array}{cccccccccccccccc} & \gamma_1, & \gamma_2, & \ldots, & \gamma_g, & \gamma_{g+1}, & \ldots, & \gamma_{2g-1}, & \delta_1, & \delta_2, & \ldots, & \delta_g, & \delta_{g+1}, & \ldots, & \delta_{2g-1} \\ \pi_{x*} & \downarrow & \downarrow & & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & & \downarrow \\ & \alpha_1, & \alpha_2', & \ldots, & \alpha_g', & \alpha_2, & \ldots, & \alpha_g, & \beta_1^2, & \beta_2', & \ldots, & \beta_g', & \beta_2, & \ldots, & \beta_g \end{array}$$

with $\lambda' := \beta_1 \lambda \beta_1^{-1}$ and π_{y*} is given by a similar prescription, with the roles of 1 and 2 interchanged.

Then it is easy to check that the restriction of ϱ_E to $\pi_1(C_x) \subset \pi_1(C)$ is reducible; invariant subspaces are $W_x := \langle \binom{1}{1} \rangle$ and $W_x^\perp$. Since

$$\pi_1(C) = \pi_1(C_x) \cup \beta_1 \pi_1(C_x) \quad \text{and} \quad \mathbb{C}^2 = W_x \oplus W_x^\perp = W_x \oplus \varrho_E(\beta_1) W_x \, ,$$

it follows that $\varrho_E = \mathrm{Ind}_{\pi_1(C_x)}^{\pi_1(C)}(\varrho_x)$, with $\varrho_x : \pi_1(C_x) \to U(W_x) \cong U(1)$ the restriction of ϱ_E, see [S]. Similarly, the restriction to $\pi_1(C_y)$ has invariant subspaces $W_y := \langle \binom{1}{0} \rangle$ and $W_y^\perp = \varrho_E(\beta_2) W_y$, and $\varrho_E = \mathrm{Ind}_{\pi_1(C_y)}^{\pi_1(C)}(\varrho_y)$.

Let $z_x \in X$ define the map ψ_x, and let $p_x := q_x \otimes \pi_x^* z_x$, then $p_x \in \ker(Nm_x)$ and, after tensoring by some $y \in X[2]$ if necessary, we get $p_x \in P_x$. Then:

$$\psi_x(p_x) := \pi_{x*}(p_x) \otimes z_x \cong \pi_{x*}(q_x \otimes \pi_x^* z_x \otimes \pi_x^* z_x) \cong \pi_{x*}(q_x) \cong E$$

since $z_x^{\otimes 2} = x$ implies $(\pi_x^* z_x)^{\otimes 2} \cong \pi_x^* x \cong \mathcal{O}_{C_x}$. Since $p_x^{\otimes 2} \cong q_x^{\otimes 2}$ is the line bundle defined by the character ϱ_x^2 of $\pi_1(C_x)$ and since $\varrho_x^2 = \varepsilon_y$, the character defining C_y when restricted to $\pi_1(C_x)$ (easy verification), we have that $p_x \in S_{\bar{y}}$. Using a similar argument for y and using the equivariance of ψ_x, ψ_y for the action of $x^\perp \cap y^\perp$, the stabilizer of $\mathbb{P}V_{x,y}$ in $X[2]$, we get the desired result. $\square$

2 The multiplication maps

To find the polynomials which vanish on all Pryms, we study various multiplication maps. We use the translation of the SJ- and D-relations in coordinates to derive geometrical information on these polynomials. Before we launch into calculations, let us sketch the philosophy: since a basis of $H^0(X, L)$ is given by theta functions of order 2, both (SJ) and (D) turn out to have expressions in terms of thetanulls (for suitable choices of period matrices of X and P_x, τ and π_x, say). These thetanulls happen to give the entries of the matrices of the (H-equivariant) multiplication maps:

$$p_4 : S^4 V \longrightarrow H^0(X, L^{\otimes 4})_+ \quad p_4 : S^4 V_x \longrightarrow H^0(P_x, L_{P_x}^{\otimes 4})_+ \, .$$

Let k be even; then the action of H on $S^k V$ factors through an abelian group and since $\mathbb{C}^*$ acts via $t \mapsto t^k$, the essential part of the action is given by a character of $H/\mathbb{C}^* \cong (\mathbb{Z}/2)^g \times \mathrm{Hom}((\mathbb{Z}/2)^g, \mathbb{C}^*)$; let $\chi \in Ch(H)$, the group of characters of $H/\mathbb{C}^*$, with 0 denoting the trivial character. Then we have a decomposition in character spaces:

$$S^4V = \bigoplus_{\chi} S^4_\chi V\,.$$

Given a non-trivial $\chi \in Ch(H)$, we can choose an automorphism ϕ of H, which is the identity on $\mathbb{C}^*$, such that

$$\phi_\chi = \chi_1 := ((\alpha_1, \ldots, \alpha_g, \alpha_1^*, \ldots, \alpha_g^*) \mapsto (-1)^{\alpha_g})\,.$$

The element $\bar{h} \in Ch(H)/\mathbb{C}^*$ corresponding to χ_1, that is $[(t,h),(s,l)] = (\chi_1(l), 0)$, is then

$$\bar{h} = ((0,\ldots,0),(0,\ldots,0,1)) \in (\mathbb{Z}/2)^g \times \mathrm{Hom}((\mathbb{Z}/2)^g, \mathbb{C}^*)$$

since the commutator in H is: $[(t,\alpha,\alpha^*),(s,\beta,\beta^*)] = (\alpha^*(\beta)\beta^*(\alpha), 0, 1)$.

As in [vG], we choose bases as follows:

(i) in V: $X_\sigma = \theta_{\frac{\sigma}{2}0}(2\tau, 2z) \quad \sigma \in (\mathbb{Z}/2)^g$

(ii) in S^4_0V:

$$P_I = \sum_\sigma X_\sigma X_{\sigma+\varrho} X_{\sigma+\nu} X_{\sigma+\varrho+\nu}$$

for all $I = \{0, \varrho, \nu, \varrho+\nu\} \subset (\mathbb{Z}/2)^g$.

(iii) in $S^4_{\chi_1}V$:

$$R_J = \sum_\delta X_{(\delta 0)} X_{(\delta+\beta 0)} X_{(\delta+\gamma 0)} X_{(\delta+\beta+\gamma 0)} - X_{(\delta 1)} X_{(\delta+\beta 1)} X_{(\delta+\gamma 1)} X_{(\delta+\beta+\gamma 1)}$$

for all $J = \{0, \beta, \gamma, \beta+\gamma\} \subset (\mathbb{Z}/2)^{g-1}$.

(iv) in the following subspace of $H^0(X, L^{\otimes 4})$, which contains the image $p_4(S^4_\chi V)$: $[2]^*H^0(X, L_\chi)_+$, where $L_\chi = T_y^*L$, $2y = x$, and $x \in X[2]$ corresponds to χ (the subscript $+$ means even thetas, note all elements of V are even!):

$$\theta_{\frac{\sigma}{2}0}(2\tau, 4z)\,, \qquad \sigma \in (\mathbb{Z}/2)^g\,, \qquad \text{for } \chi = 0$$

$$\theta_{\left(\frac{\alpha}{2}0\right)\left(0\frac{1}{2}\right)}(2\tau, 4z)\,, \qquad \alpha \in (\mathbb{Z}/2)^{g-1}\,, \qquad \text{for } \chi = \chi_1.$$

The matrices of the multiplication map p_4 are computed in [vG, Proposition 4]:

(i)′

$$p_4 : S^4_0(X, L) \to [2]^*H^0(X, L_x)_+\,, \quad P_I \xrightarrow{p_4} \sum_\lambda C_{I,\lambda} \theta_{\frac{\lambda}{2}0}(2\tau, 4z)$$

where $\lambda \in (\mathbb{Z}/2)^g$ and

$$C_{I,\lambda}(\tau) = (\theta_{\frac{\lambda+\varrho}{2}0}\theta_{\frac{\lambda+\nu}{2}0}\theta_{\frac{\lambda+\varrho+\nu}{2}0})(2\tau, 0)\,;$$

(ii)$'$

$$p_4 : S^4_{\chi_1} V \to [2]^* H^0(X, L_\chi)_+ , \quad R_J \xrightarrow{p_4} \sum_\sigma D_{J,\delta}\theta_{\left(\frac{\delta}{2}0\right)\left(0\frac{1}{2}\right)}(2\tau, 4z)$$

(where $\delta \in (\mathbb{Z}/2)^{g-1}$ and

$$D_{J,\delta}(\tau) = (\theta_{\left(\frac{\delta+\beta}{2}0\right)\left(0\frac{1}{2}\right)}\theta_{\left(\frac{\delta+\gamma}{2}0\right)\left(0\frac{1}{2}\right)}\theta_{\left(\frac{\delta+\beta+\gamma}{2}0\right)\left(0\frac{1}{2}\right)})(2\tau, 0) .$$

In [vG], the (SJ) relations are shown to be equivalent to $\theta_{\left(\frac{\delta}{2}0\right)\left(0\frac{1}{2}\right)}(2\tau, 0) = \theta_{\frac{\delta}{2}0}(2\pi_x, 0)$, for suitable period matrices τ of X and π_x of P_x thus, (ibid., Proposition 6), the matrices of the multiplication maps p_4 in the above bases (ii), (iii), (iv) differ by a non-zero multiplicative constant, where the maps are:

$$p_4 : S^4_{\chi_1} V \longrightarrow [2]^* H^0(X, L_\chi)_+ \quad \text{and} \quad p_4 : S^4_0 V_{P_x} \longrightarrow [2]^* H^0(P_x, L_{P_x})_+ .$$

Note that the identification $S^4_{\chi_1} V \cong S^4_0 V_{P_x}$ maps $R_J \in S^4_{\chi_1} V$, $J = \{0, \varrho, \nu, \varrho + \nu\} \subset (\mathbb{Z}/2)^{g-1}$ to $P_J \in S^4_0 V_{P_x}$, which is now a polynomial in the 2^{g-1} variables X_δ, $\delta \in (\mathbb{Z}/2)^{g-1}$. The identification is thus obtained by restricting R_J to the eigenspace V_x defined by $X_{(\delta 1)} = 0$ and then putting $X_{(\delta 0)} = X_\delta$. More intrinsically, we use the action of H_{g-1} to identify V_x and $H^0(P_x, L_{P_x})$ and take Sym^4 of this identification. Similarly:

$$\begin{tikzcd}
\ker(p_{x,\bar\mu}) \arrow[r] \arrow[d, "\cong"'] & S^4_{\bar\mu} V_x \arrow[r, "p_{x,\bar\mu}"] \arrow[d, "\cong"'] & {[2]^* H^0(P_x, L_{P_x,\bar\mu})_+} \\
W \arrow[r] & S^4_0 V_{x,y} & \\
\ker(p_{y,\bar\chi}) \arrow[r] \arrow[u, "\cong"] & S^4_{\bar\chi} V_y \arrow[r, "p_{y,\bar\chi}"] \arrow[u, "\cong"] & {[2]^* H^0(P_y, L_{P_y,\bar\chi})_+ .}
\end{tikzcd}$$

Lemma. *Let $x, y \in X[2]$ correspond to characters $\chi, \mu \in Ch(H_g)$. Assume $E(x, y) = 0$, and let $\bar\chi, \bar\mu \in Ch(H_{g-1})$ be the characters induced on $P_x[2]$, $P_y[2]$.*
Then under the isomorphisms given by restriction:

$$S^4_{\bar\mu} V_x \cong S^4_0 V_{x,y} \cong S^4_{\bar\chi} V_y ,$$

the multiplication maps

$$p_4 : S^4_{\bar\mu} V_x \to [2]^* H^0(P_x, L_{P_x,\bar\mu})_+ \quad \text{and} \quad p_4 : S^4_{\bar\chi} V_y \to [2]^* H^0(P_y, L_{P_y,\bar\chi})_+$$

differ by a nonzero multiplicative constant. (Note that we should write $p_{4,x,\bar\mu}$, $p_{4,y,\bar\chi}$.)
In particular, $W := \ker(p_{4,x,\bar\mu}) = \ker(p_{4,y,\bar\chi})$.

Proof. By a suitable identification $X[2] \cong H/\mathbb{C}^*$ we can assume that χ and μ correspond to the points $x = e_{2g}, y = e_{2g-1}$ in the "standard basis" $\{e_J\}$ of $X[2]$. The linearized eigenspaces of $U(x)$, $U(y)$ are coordinatized by:

$$[X_0, \dots, X_{2^{g-1}-1}; 0, \dots, 0]$$

and

$$[Y_0, \dots, Y_{2^{g-2}-1}, 0, \dots, 0; Y_{2^{g-1}}, \dots, Y_{2^{g-1}+2^{g-2}-1}, 0, \dots, 0]$$

and their intersection, which is a $\mathbb{P}^{2^{g-2}-1}$, has coordinates $[Z_0, \dots, Z_{2^{g-2}-1}]$ which correspond to $X_{(\sigma''00)}$, $\sigma'' \in (\mathbb{Z}/2)^{g-2}$. The $X_{(\sigma'0)}$, $\sigma' \in (\mathbb{Z}/2)^{g-1}$, and $Y_{(\sigma''0\varepsilon)}$, $\varepsilon = 0, 1$, are acted on by the obvious Heisenbergs H_{g-1} in the obvious way.

Thus we can use ([vG, Proposition 4], see also above) to express the matrices of the multiplication maps p_4 on $S^4_{\bar{\mu}} V_\chi$ and $S^4_{\bar{\chi}} V_\mu$, resp. ($\bar{\mu}, \bar{\chi} \in H_{g-1}$), where we use the (analog of the) basis R_J on the left $\theta_{(\sigma''0)\left(0\frac{1}{2}\right)}(2\pi_x, 2z)$, and $\theta_{(\sigma''0)\left(0\frac{1}{2}\right)}(2\pi_y, 2z)$ resp. on the right. The theta constants in the matrices are the coordinates of 4-torsion points on the Pryms, and the Donagi relations say precisely that the coefficients of the multiplication matrices are the same up to a (nonzero) multiplicative constant.

The last statement follows because the R_J's from $S^4_{\bar{\mu}} V_\chi$ and $S^4_{\bar{\chi}} V_\mu$ restrict to the same elements in $S^4_0 V_{x,y}$, which are polynomials in the Z_i, $0 \leqq i \leqq 2^{g-2} - 1$. $\square$

Finally, using the same ideas as in [vG], we get:

Theorem 1. *Let $F \in S^4_\chi V$, $\chi \neq 0$, and let $x \in X[2]$ correspond with $\chi \in Ch(H)$. Then F vanishes on all Pryms if and only if $K(P_x) \subset \operatorname{Sing} Z(F)$.*

Proof. We will take x and V_x as in the proof above. Using the basis of $S^4 V_x$ given by the R_J's one sees:

$$K(P_x) \subset \operatorname{Sing} Z(F) \quad \text{iff} \quad K(P_x) \subset \operatorname{Sing}(F_\chi), \quad \text{with } F_\chi := F|_{V_x}.$$

First we cut down the number of variables by restriction, recall that restriction gives an isomorphism $S^4_\chi V \to S^4_0 V_x$ [vG, Proposition 3]. Now in $S^4_0 V_x$ we proceed as in [vG, Theorem 1], only with respect to the Heisenberg H_{g-1}: to minimize confusion (rather than to avoid *another* abuse of notation!) we denote the maps $M(\chi)$ of [vG] by $N(\bar{\mu})$ here. So if $\bar{\mu} = (\varepsilon^*, \varepsilon) \in \operatorname{Hom}((\mathbb{Z}/2)^{g-1}, \mathbb{C}^*) \times (\mathbb{Z}/2)^{g-1} = Ch(H_{g-1})$ is defined by: $\bar{\mu}((\sigma, \sigma^*)) = \varepsilon^*(\sigma)\sigma^*(\varepsilon)$, then we have (surjective) maps:

$$N(\bar{\mu}) : S^3_0 V_x \to S^4_{\bar{\mu}} V_x, \quad N(\bar{\mu})G := \sum_\sigma \varepsilon^*(\sigma) X_\sigma G_{\sigma+\varepsilon}, \quad \text{with } G_\sigma = \frac{1}{4} \frac{\partial H}{\partial X_\sigma},$$

here $H := N(0)G$ and since the space $S^3_0 V_x$ consists of cubics invariant under the action of the subgroup $K_{g-1} := \{(1, \sigma, \sigma^*) \in H_{g-1} : \sigma = 0\}$. For $\bar{\mu} = 0$, the trivial character, we have an isomorphism:

$$N(0) : S^3_0 V_x \xrightarrow{\cong} S^4_0 V_x, \quad N(0)^{-1} : H \mapsto \frac{1}{4} \frac{\partial H}{\partial X_0}.$$

(a) Assume $K(P_x) \subset \operatorname{Sing} F$, for $F \in S^4_\chi V$. We have to show that for all y we have $K(P_y) \subset Z(F)$.

In case $E(x, y) = 1$, it is easy to check that in fact the restriction of $F \in S^4_\chi V$ to V_y is identically zero. So F vanishes on all these $K(P_y)$'s. It remains to consider the $y \in x^\perp$.

The restriction map $S^4V \to S^4V_x$ gives an isomorphism $S^4_\chi V \to S^4_0 V_x$, and for

$$\mapsto F_\chi \in S^4_0 V_x\,, \quad \text{let } F_\chi = N(0)G\,, \quad \text{with} \quad G \in S^3_0 V_x\,.$$

Since $K(P_x) \subset \mathbb{P}V_x$, all $\dfrac{\partial F_\chi}{\partial X_\sigma}$, $\sigma \in (\mathbb{Z}/2)^{g-1}$, also vanish on $K(P_x)$. Thus all G_σ are equations for $K(P_x)$ and so, for all $\overline{\mu} \in Ch(H_{g-1})$, we have:

$$N(\overline{\mu})G \in \ker(S^4_{\overline{\mu}} V_x \longrightarrow H^0(P_x, L_x^{\otimes 4}))\,.$$

Any $y \in x^\perp$, $y \notin \langle x \rangle$ corresponds to a $\mu \in Ch(H)$ which defines a non trivial $\overline{\mu} \in Ch(H_{g-1})$. We can apply the lemma to these x, y and find that the element $F_\mu \in S^4_{\overline{\chi}} V_\mu$ corresponding to $N(\overline{\mu})G$ is an equation for $K(P_y) \subset \mathbb{P}V_y$:

$$F_\mu \in \ker(S^4_{\overline{\chi}} V_y \longrightarrow H^0(P_y, L_y^{\otimes 4}))\,.$$

Now F_μ is indeed, as the notation suggested, the restriction of F to V_y and since $K(P_y) \subset \mathbb{P}V_y$, we thus get $K(P_y) \subset Z(F)$, as desired. To see this, we must check that $N(\overline{\mu})G$ and F_μ have the same restriction to $V_{x,y} = V_x \cap V_y$. Recall that $V_x \cap V_y$ is an eigenspace the Heisenberg group H_{g-1}, then by [vG, Proposition 3i]:

$$\operatorname{res} N(\overline{\mu})G|_{V_{x,y}} = \operatorname{res} N(0)G|_{V_{x,y}} = \operatorname{res} F_\chi|_{V_{x,y}} = \operatorname{res} F|_{V_x \cap V_y} = \operatorname{res} F_\mu|_{V_{x,y}}\,.$$

(b) Conversely, assume that F vanishes on all $K(P_y)$. Then F_μ is an equation for $K(P_y) \subset \mathbb{P}V_y$ and thus, by the reasoning above, $N(\overline{\mu})G$, for all $\overline{\mu} \in Ch(H_{g-1})$, is an equation for $K(P_x)$. Considering only the $\overline{\mu} = (\varepsilon', 0)$, we have the following equations for $K(P_x)$, with σ running over $(\mathbb{Z}/2)^{g-1}$:

$$\sum_\sigma \varepsilon'(\sigma) X_\sigma \frac{\partial F_\chi}{\partial X_\sigma} \quad (\forall \varepsilon' \in \operatorname{Hom}((\mathbb{Z}/2)^{g-1}, \mathbb{C}^*))\,.$$

Taking suitable linear combinations it follows that all $\dfrac{\partial F_\chi}{\partial X_\sigma}$ vanish on $K(P_x)$ and thus $K(P_x) \subset \operatorname{Sing}(F_\chi) = \operatorname{Sing}(F) \cap \mathbb{P}V_x$. $\square$

3 Quartics and $\mathcal{SM}_C(2)$

The kernels of the multiplication maps are the spaces that encode the projective geometry, and their dimensions may depend on the choice of the Jacobian (unlike the dimensions of the source and target of p_k, which depend only on k and g). We will in fact have to restrict our attention to curves without a vanishing thetanull, although the statement of Theorem 2a might be true for all curves.

To get the results on $D_4 = \dim H^0(\mathcal{S}M_C(2), \mathcal{L}^{\otimes 4})$, we draw some consequences from Theorem 1. Let $d(g) = \dim S_0^4 V = \dim S_0^3 V$ (then $d(g-1) = \dim S_0^4 V_x = \dim S_0^3 V_x$ for $x \neq 0$), let $e(g) = 2^{-g} \dim H^0(X, L^3)_+$:

$$d(g) = \frac{(2^g+1)(2^{g-1}+1)}{3}, \quad e(g) = \frac{3^g+1}{2},$$

and define, for $x \in X[2]$ corresponding to $\chi \in Ch(H)$:

$$S_0^4 V_{\text{sing}} := \{F \in S_0^4 V : K \subset \operatorname{Sing} Z(F)\},$$
$$S_\chi^4 V_{\text{sing}} := \{F \in S_\chi^4 V : K(P_x) \subset \operatorname{Sing} Z(F)\}.$$

Theorem 2. *With the same notation as above:*
(a) *For a curve C with no vanishing thetanulls we have:*

$$\dim S_0^4 V_{\text{sing}} = d(g) - e(g), \quad \dim S_\chi^4 V_{\text{sing}} = d(g-1) - e(g-1)$$

and the number of independent quartics in $\mathbb{P}V$ vanishing on all Pryms is equal to:

$$v(g) := d(g) - e(g) + (2^{2g}-1)(d(g-1) - e(g-1)).$$

(b) *For all curves C:*

$$D_4 = \dim H^0(\mathcal{S}M_C(2), \mathcal{L}^{\otimes 4}) \geqq e(g) + (2^{2g}-1)e(g-1) = N_4.$$

Proof. First of all we use Theorems 1 from [vG] and from Sect. 2 to rephrase the condition that a polynomial $F \in S^4 V$ vanishes on all Pryms. Writing $F = \sum F_\chi$ and observing that the union of all Pryms is stable under the action of $X[2]$, we see that F vanishes on all Pryms (and K) iff each F_χ vanishes on all Pryms (and K).

For $\chi = 0$, this is equivalent to $M(0)^{-1} F \in \ker(p_3 : S_0^3 V \to H^0(X, L^{\otimes 3})_+)$ and for $\chi \neq 0$ $N(0)^{-1} F_\chi \in \ker(p_3 : S_0^3 V_x \to H^0(P_x, L_{P_x}^{\otimes 3})_+)$ where $F_\chi = F|_{V_x}$. Thus we have to determine the dimensions of these kernels.

We recall (cf. [vG] for the setting, the result is classical) that for a ppav (A, Θ_A) with $L_A := \mathcal{O}_A(2\Theta_A)$ the map

$$p_2 : S^2 V_A \to H^0(A, L_A^{\otimes 2})_+$$

is surjective if and only if the abelian variety A has no vanishing thetanulls. For any ample divisor D, the multiplication $H^0(A, \mathcal{O}(aD)) \otimes H^0(A, \mathcal{O}(bD)) \to H^0(A, \mathcal{O}(a+b)D)$ is onto if $a \geqq 2$, $b \geqq 3$ (cf. [K]). We take $D = \Theta_A$, $a = 2$, $b = 4$, then, since $H^0(A, L_A)$ consists of even sections, we see that $H^0(A, L_A) \otimes H^0(A, L_A^{\otimes 2})_+ \to H^0(A, L_A^{\otimes 3})_+$ is surjective. Thus, if none of the thetanulls of A vanish, it follows that $p_3 : S^3 V_A = S^3 H^0(A, L_A) \to H^0(A, L_A^{\otimes 3})_+$ is surjective.

If X has no vanishing thetanulls, then the (SJ) relations (in their classical form: $\left(\theta \begin{bmatrix} \varepsilon & 0 \\ \varepsilon' & 0 \end{bmatrix} \theta \begin{bmatrix} \varepsilon & 0 \\ \varepsilon' & 1 \end{bmatrix} (\tau, 0) = \theta \begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}^2 (\pi_x, 0)\right)$ show that the Pryms of X do not have any vanishing thetanull either. Applying the result above to $A = X$ and $A = P$ for a C

with no vanishing thetanulls (meaning those of X), the maps $S^3V \to H^0(X, L^{\otimes 3})_+$ and $S^3V_x \to H^0(P_x, L_x^{\otimes 3})_+$ are seen to be surjective. Thus the dimension of the kernels is $2^g(d(g) - e(g))$ and $2^{g-1}(d(g-1) - e(g-1))$, and the dimension of the kernel of p_3 restricted to S_0^3 is found by dividing by 2^g and 2^{g-1} respectively.

Now we prove (b). Recall that $\varphi(\mathscr{S}M_C(2))$ contains all $K(P_x)$ (and K). Thus, by Theorem 1, the kernel of the multiplication map:

$$m_4 : S^4H^0(\mathscr{S}M(2), \mathscr{L}) = S^4V \longrightarrow H^0(\mathscr{S}M(2), \mathscr{L}^{\otimes 4})$$

must be contained in $\bigoplus_\chi S_\chi^4 V_{\mathrm{Sing}}$. Then $\dim \ker m_4 \leqq v(g)$ by (i), and

$$\begin{aligned} D_4 &\geqq \dim S^4V - v(g) \\ &= e(g) + (2^{2g} - 1)e(g-1) \\ &= (2 \cdot 3^{g-1} + 3^{g-1} + 1)/2 + (2 \cdot 2^{2g-1} - 1)(3^{g-1} + 1)/2 \\ &= 2^{2g-1}3^{g-1} + 2^{2g-1} + 3^{g-1} \\ &= 3^{g-1}(2 \cdot 2^{2g-2} + 2 \cdot (2/\sqrt{3})^{2g-2} + 1) \\ &= N_4 , \end{aligned}$$

with N_4 the Verlinde number. Since D_k depends only on the genus of C, this result is valid for all C. □

Theorem 2b together with Bertram's result that $D_k \leqq N_k$, yields:

Corollary 1. *The Verlinde formula* $(*)$ *holds for* $k = 4$.

The results of Theorem 2 can be rephrased more geometrically:

Corollary 2. *Let C be a Riemann surface without vanishing thetanulls. Then a quartic vanishes on $\varphi(\mathscr{S}M_C(2))$ iff it vanishes on all the Pryms of C and K.*

Proof. We need only argue that a quartic that vanishes on all $K(P_x)$ vanishes on $\varphi(\mathscr{S}M_C(2))$. This is a dimension count; indeed, by Theorem 2a, there are exactly $v(g)$ independent quartics that vanish on all Pryms. On the other hand, since $D_4 = N_4$, there are at least $\dim S^4V - N_4 = v(g)$ quartics vanishing on $\varphi(\mathscr{S}M_C(2))$. □

Remark. Conjecturally, the Verlinde numbers can also be computed using the representation ring $R(SU(2))$, cf. [Bott]. We verify this for $k = 4$. For $r \in R(SU(2))$, define $\Psi_*(r) \in \mathbb{Z}$ by:

$$r \equiv \Psi_*(r)V_0 + a_1V_1 + \ldots + a_4V_4 \bmod V_5 ,$$

with V_i the irreducible representation of $SU(2)$ of dimension $i + 1$. Then we show that $\Psi_*((V_0^2 + \ldots + V_4^2)^g) = N_4$. One has:

$$(V_0^2 + \ldots + V_4^2) \equiv 4V_0 + 2V_2 + V_2^2 \quad \text{and} \quad \Psi_*(V_2^n) = (2^{n-1} + (-1)^n)/3 .$$

Using the binomial expansion, it follows that

$$\Psi_*((2V_2 + V_2^2)^n) = (2^{3n-1} + (-1)^n)/3 ,$$

and using the binomial expansion once more, we finally get

$$\Psi_*((4V_0 + (2V_2 + V_2^2))^g) = 2^{2g-1} + 2^{2g-1}3^{g-1} + 3^{g-1} = N_4 .$$

(Zagier proved this result for all k).

Acknowledgment. We would like to thank A. Bertram for sharing his ideas and results.

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Noted added in Proof. The equality $D_k = N_k$ was recently proved by Bertram and Szenes, a general proof (for arbitrary structure groups) was found by Faltings.