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Schottky-Jung Relations and Vectorbundles on Hyperelliptic Curves

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Introduction

Schottky and Jung have proved that there are very simple relations between the theta constants of the Jacobian of a curve and those of (any) one of the Prym varieties of the curve. The theta constants in the Schottky-Jung relations are obtained from second order theta functions. For a principally polarized abelian variety (ppav) A, let M be a line bundle defining the polarization and let $L=M\otimes[-1]^*M$, where $[n]$, for $n\in\mathbb{Z}$, stands for multiplication by n on A. The line bundle L depends only on the polarization of A and not on the choice of M. We write $V=V_A=\Gamma(A,L)$ for the space of second order theta functions. The group of two torsion points of A, $A[2]$, acts on $\mathbb{P}V$. The image of the canonical map

$$K:A\to\mathbb{P}V$$

is called the Kummer variety of A and for indecomposable A (that is the ppav A is not a product of ppav's of lower dimensions) one has that $K(A)\cong A/[-1]$.

The projective geometry of $K(A)\subset\mathbb{P}A$ (its trisecants for example) plays an important role in the Schottky problem, which asks for a characterization of the Jacobians among the ppav's. The geometry of $K(A)$ is of course determined by the equations which define it. It was shown by Wirtinger that the general Kummer variety is defined by quartic equations. Among the coefficients of these equations are the same theta constants that appear in the Schottky-Jung relations. We have not yet been able to deduce interesting results on the geometry of the Kummer variety from this. However, assuming that A satisfies Schottky-Jung relations we can relate the equations defining the Kummer variety of A and those defining the Kummer variety of a Prym of A.

A ppav P of dimension $g-1$ is called a Prym of A if there is a nonzero $x\in A[2]$ such that for (some) $y\in A$ with $2y=x$ one has

$$K_A(y)=K_P(0) \tag{I.1}$$

with

$$\begin{array}{c} K_A:A\to\mathbb{P}V \\ \cup \\ K_P:P\to\mathbb{P}V_x. \end{array}$$

Here $\mathbb{P}V_x$ is an eigenspace for the action of x on $\mathbb{P}V$, an isomorphism $\mathbb{P}V_x \cong \mathbb{P}V_P$ is obtained by relating the actions of $A[2]$ and $P[2]$ (as in [M 2, Sect. 4]). The ppav A satisfies a Schottky-Jung relation if (I.1) holds (for some P and some x).

Let S_0^4V be the space of those homogeneous polynomials of degree four on V which are invariant w.r.t. the action of $A[2]$. We denote by $Z(F)$ the zero locus of a polynomial F and by $\operatorname{Sing}Z(F)$ the singular locus of $Z(F)$. Theorem 1′ is a weaker version of Theorem 1:

Theorem 1′. a) *Let* $F \in S_0^4V$ *and assume that*

$$K(A) \subset \operatorname{Sing}Z(F). \tag{I.2}$$

Then for every Prym P *of* A *we have:*

$$K(P) \subset Z(F) \tag{I.3}$$

(where, as above, $K(P) \subset \mathbb{P}V_x \subset \mathbb{P}V \supset K(A)$*).*

b) *Assume that* A *satisfies all Schottky-Jung relations (i.e. assume that* A *has a* Prym P_x *for every nonzero* $x \in A[2]$*). Let* $F \in S_0^4V$ *and assume that*

$$K(P_x) \subset Z(F).$$

for all nonzero $x \in A[2]$. *Then*

$$K(A) \subset \operatorname{Sing}Z(F).$$

For any ppav A of dimension $\geqq 3$ there exist $F \in S_0^4V$ satisfying (I.2). It seems that the locus in $\mathbb{P}V$ defined by all such polynomials has nice properties besides containing all Pryms of A. For example, in case $A = J\mathscr{C}$, with $\mathscr{C}$ a curve of genus three, there is one such polynomial. For nonhyperelliptic $\mathscr{C}$ it was found by Coble, [C, Sect. 33 (5)], who also suggested (I.3) in this case. Narasimhan and Ramanan proved that the locus defined by this polynomial is isomorphic to $SU_{\mathscr{C}}(2)$, the moduli space of semistable bundles of rank 2 and trivial determinant on $\mathscr{C}$ [N-R]. In case $\mathscr{C}$ is hyperelliptic the locus is a double quadric, the quadric itself is isomorphic to $SU_{\mathscr{C}}(2)/i$, where i is the hyperelliptic involution.

In fact, for any g Narasimhan and Ramanan have defined a map [N-R]:

$$\delta: SU_{\mathscr{C}}(2) \to \mathbb{P}V_{J\mathscr{C}} = \mathbb{P}V.$$

For $g \geqq 3$ the Kummer variety $J\mathscr{C}/[-1]$ is the singular locus of $SU_{\mathscr{C}}(2)$, it corresponds to the nonstable bundles $j \oplus j^{-1}$ with $j \in \operatorname{Pic}^0(\mathscr{C})$. It is mapped onto $K(J\mathscr{C}) \subset \mathbb{P}V_{J\mathscr{C}}$. The image of δ is stable under the action of $J\mathscr{C}[2]$, in fact $\alpha \in J\mathscr{C}[2]$ acts on $SU_{\mathscr{C}}(2)$ by $E \mapsto E \otimes \alpha$ and the map δ is equivariant for this action. Moreover, if P is a Prym of $J\mathscr{C}$ then there are two copies of P in $SU_{\mathscr{C}}(2)$. Indeed, if $\pi: \tilde{\mathscr{C}} \to \mathscr{C}$ is the étale double covering corresponding to $\alpha \in J\mathscr{C}[2]$ then $Nm^{-1}(\alpha)$, with $Nm: \operatorname{Pic}^0(\tilde{\mathscr{C}}) \to \operatorname{Pic}^0(\mathscr{C})$ the norm map, consists of two copies of P. For any $\tilde{j}$ with $Nm(\tilde{j}) = \alpha$, the rank two vectorbundle $\pi_*\tilde{j}$ has trivial determinant [because $Nm\tilde{j} = \det(\pi_*\tilde{j}) \otimes (\det \pi_*\mathcal{O}_{\tilde{\mathscr{C}}})^{-1} = \det(\pi_*\tilde{j}) \otimes \alpha$], and hence defines a point of $SU_{\mathscr{C}}(2)$. The map δ restricted to a copy of P is just the canonical map $K_P: P \to \mathbb{P}V_\alpha \subset \mathbb{P}V_{J\mathscr{C}}$. Ramanan recently proved that the trisecants of $K(J\mathscr{C})$ are also contained in $\delta(SU_{\mathscr{C}}(2))$.

Theorem 1′b) implies that any $F \in S_0^4V$ whose zero locus contains $\delta(SU_{\mathscr{C}}(2))$ has $K(J\mathscr{C}) \subset \operatorname{Sing}Z(F)$. Conversely, one might hope that any $F \in S_0^4V$ with $K(J\mathscr{C})$

$\subset \mathrm{Sing}\, Z(F)$ also has $\delta(SU_{\mathscr{C}}(2)) \subset Z(F)$. We have been unable to prove this, even in the hyperelliptic case. Note that at least the Pryms and the trisecants are in $Z(F)$, the latter because they intersect $Z(F)$ with multiplicity $\geqq 6$.

For hyperelliptic $\mathscr{C}$ we do construct a map $\varphi: SU_{\mathscr{C}}(2) \to \mathbb{P}V$, which embeds $SU_{\mathscr{C}}(2)/i$ into $\mathbb{P}V$ and we show that $\varphi(SU_{\mathscr{C}}(2))$ is defined, at least as a set, by quartic equations. We could not show that $\delta = \varphi$, but recently Beauville gave a proof of this [Be]. It seems that for $g \geqq 5$ one also finds quartic equations for $\varphi(SU_{\mathscr{C}}(2))$ which are not in $S_0^4 V$. We use the (half) spin representation of an orthogonal group to prove these results. The relation with the orthogonal group and that $\delta = \varphi$ was observed independently by Ramanan.

1

The quartic polynomials on $\mathbb{P}V$ are elements of $S^4 V$, where $S^n V$ is the n-fold symmetric product of V. In this section we will decompose $S^4 V$ w.r.t. the action of the Heisenberg group. We also give some relations between $S^3 V$ and $S^4 V$ arising from this action. Most of the results here generalize [C, Sect. 33].

We fix a theta structure α for L, that is an isomorphism between $G(L) = \{(x, \varphi) \mid \varphi: T_x^* L \xrightarrow{\cong} L\}$ and the Heisenberg group

$$H = \mathbb{C}^* \times (\mathbb{Z}/2)^g \times \mathrm{Hom}(\mathbb{Z}/2, \mathbb{C}^*)^g \quad \text{(as sets)}$$

with multiplication

$$(t, x, x^*)(s, y, y^*) = (tsy^*(x), x+y, x^*+y^*),$$

once and for all [M 1]. H acts on the vector space V and there is a basis $\{X_\sigma\}$, with $\sigma \in (\mathbb{Z}/z)^g$, for V such that:

$$(t, x, x^*) X_\sigma = t x^*(x+\sigma) X_{\sigma+x} \tag{1.1}$$

such a basis is called a canonical basis.

The theta structure α gives an isomorphism $\bar{\alpha}: A[2] \to H/\mathbb{C}^* \cong (\mathbb{Z}/2)^g \times \mathrm{Hom}(\mathbb{Z}/2, \mathbb{C}^*)^g$. The group $A[2]$ acts on $\mathbb{P}V$ (induced by $a \circ s = T_a^* s$, $a \in A[2]$). Moreover $a \in A[2]$ and any lift of $\bar{\alpha}(a) \in H/\mathbb{C}^*$ to H give the same map on $\mathbb{P}V$.

A basis of $S^n V$ is given by the products $X_{\sigma_1} X_{\sigma_2} \dots X_{\sigma_n}$, and H acts by

$$h \cdot (X_{\sigma_1} \dots X_{\sigma_n}) := (h X_{\sigma_1}) \dots (h X_{\sigma_n}). \tag{1.2}$$

The commutator of (t, x, x^*) and (s, y, y^*) is given by $(y^*(x) x^*(y), 0, 0)$ and $y^*(x) x^*(y) \in \{\pm 1\}$. Note that (1.1) and (1.2) imply that the action of H on $S^n V$ for even n factors over the abelian quotient $H^{ab} \cong \mathbb{C}^* \times (\mathbb{Z}/2)^g \times \mathrm{Hom}(\mathbb{Z}/2, \mathbb{C}^*)^g$ of H. For n odd however the commutator subgroup $\{(t, 0, 0) : t \in \{\pm 1\}\}$ of H acts non-trivially.

To decompose $S^n V$ it is useful to introduce two subgraphs of H:

$$K = \{(t, x, x^*) : t = 1,\ x = 0\}$$

$$K^* = \{(t, x, x^*) : t = 1,\ x^* = 0\}.$$

Note that $\mathbb{C}^*$, K, and K^* generate H and that K acts by scalar multiplication on the basis X_σ of V, whereas K^* acts by permuting these basis elements. The following proposition gives the decomposition of S^nV for $n=2,3,4$.

Proposition 1. (i) *A basis of eigenvectors for the action of H on S^2V is given by the $2^{g-1}(2^g+1)$ elements*

$$Q\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}:=\sum(-1)^{{}^t\sigma\varepsilon'}X_\sigma X_{\sigma+\varepsilon}\qquad(\varepsilon,\varepsilon',\sigma\in(\mathbb{Z}/2)^g,\ {}^t\varepsilon\varepsilon'=0),$$

where σ runs over $(\mathbb{Z}/2)^g$. The action of H is given by:

$$(t,x,x^*)Q\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}=t^2(-1)^{{}^tx\varepsilon'}x^*(\varepsilon)Q\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}.$$

(ii) *The space S^3V is a direct sum of $(2^g+1)(2^{g-1}+1)/3$ copies of a 2^g dimensional irreducible representation of H. A basis for S^3V as H-module is given by a basis of the K-invariant elements. An example of such a basis is:*

$$\begin{array}{l}X_0^3\\ X_0X_\varrho^2\qquad\text{with}\quad \varrho\in(\mathbb{Z}/2)^g-\{0\}\quad\text{and with}\quad T=\{0,\varrho,\tau,\varrho+\tau\},\\ X_\varrho X_\tau X_{\varrho+\tau}\end{array}$$

for a subgroup $T\cong(\mathbb{Z}/2)^2\subset(\mathbb{Z}/2)^g$.

(iii) *Let $X(H)$ be the group of characters of $(\mathbb{Z}/2)^g\times\operatorname{Hom}(\mathbb{Z}/2,\mathbb{C}^*)^g$ and let $0\in X(H)$ be the trivial character. Then*

$$S^4V=\oplus S^4_\chi V\qquad(\chi\in X(H))$$

and for every $F_\chi\in S^4_\chi V$ one has:

$$(t,x,x^*)F_\chi=t^4\chi(x,x^*)F_\chi.$$

The dimension of $S^4_\chi V$ is $(2^g+1)(2^{g-1}+1)/3$ if $\chi=0$ and $(2^{g-1}+1)(2^{g-2}+1)/3$ if $\chi\neq0$. A basis for S^4_0V is given by the P_I with $I=\{0,\varrho,\nu,\varrho+\nu\}\subset(\mathbb{Z}/2)^g$:

$$P_I=\sum_\sigma X_\sigma X_{\sigma+\varrho}X_{\sigma+\nu}X_{\sigma+\varrho+\nu},$$

where we sum over $\sigma\in(\mathbb{Z}/2)^g$. Let $\chi_1\in X(H)$ be the character with $\chi_1((x_1,Y,x_g),(x_1^*,\dots,x_g^*))=(-1)^{x_g}$. A basis for $S^4_{\chi_1}V$ is given by the R_J, with $J=\{0,\beta,\gamma,\beta+\gamma\}\subset(\mathbb{Z}/2)^{g-1}$:

$$R_J=\sum_\alpha X_{(\alpha\ 0)}X_{\alpha+\beta\ 0)}X_{(\alpha+\gamma\ 0)}X_{(\alpha+\beta+\gamma\ 0)}-X_{(\alpha\ 1)}X_{(\alpha+\beta\ 1)}X_{(\alpha+\gamma\ 1)}X_{(\alpha+\beta+\gamma\ 1)},$$

where we sum over $\alpha\in(\mathbb{Z}/2)^{g-1}$.

Example. In case $g=1$, $\dim S^4_0V=2$, a basis is given by $P_{I_1}=X_0^4+X_1^4$, $P_{I_2}=2X_0^2X_1^2$ (with $I_1=\{0,0,0,0\}$, $I_2=\{0,0,1,1\}$) and $\dim S^4_{\chi_1}V=1$, a basis being $R_J=X_0^4-X_1^4$. A basis of $S^4_{\chi_1}$ in case $g=2$ is given by $R_{J_1}=X_{00}^4+X_{10}^4-X_{01}^4-X_{11}^4$ and $R_{J_2}=2(X_{00}^2X_{10}^2-X_{01}^2X_{11}^2)$ (with $J_1=I_1$, $J_2=I_2$) and in S^4_0V we find $P_I=4X_{00}X_{01}X_{10}X_{11}$ with $I=\{00,01,10,11\}$.

Proof. *(i)* It is straightforward to check the action of H on the elements given. These elements are independent hence they span a space of dimension $2^{g-1}(2^g+1)$, which is also the dimension of S^2V.

(ii) The center of the Heisenberg group, $\{(t,0,0)\colon t\in\mathbb{C}^*\}$ acts by multiplication by t^3 on S^3V. As in the proof of [M 1, Proposition 3] one finds that all irreducible subrepresentations of S^3V are isomorphic and have dimension 2^g. Moreover, each irreducible subrepresentation is generated by a nontrivial K-invariant element, which itself is unique up to scalar multiple. The rest easily follows from (1.1), (1.2) and the definition of K.

(iii) As the action of H factors over H^{ab} the existence of the decomposition with the H-action is obvious. An element of S^4V is in S_0^4V iff it is both K and K^* invariant. Every monomial of a K-invariant element must be K-invariant (so its indices sum up to zero) and the monomials must be summed in such a way as to be K^* invariant. This shows that we indeed gave a basis, the dimension follows from: $1+(2^g-1)+(2^g-1)(2^g-2)/6=(2^g+1)(2^{g-1}+1)/3$ (corresponding to $I=\{0,0,0,0\}, \{0,0,\varrho,\varrho\}$ with $\varrho\neq0$, $\{0,\varrho,\lambda,\varrho+\lambda\}$ with $\varrho\neq0, \lambda\neq0, \varrho\neq\lambda$). To find a basis of $S_{\chi_1}^4V$ we observe that each monomial of an element in it must be K invariant, and the sum must be such that it is invariant under the subgroup $\{(1,x,0)\in K^*\colon x_g=0\}$ and such that it changes sign under $(1,(0,\dots,0,1),0)$. As all the S_χ^4V with $\chi\neq0$ obviously have the same dimension, the proposition follows. QED

We define maps between the various spaces we encountered.

Definition. Let S_0^3V be the subspace of K-invariant elements of S^3V. For $F\in S_0^3V$ and $\sigma\in(\mathbb{Z}/2)^g$ let

$$F_\sigma=(1,\sigma,0)F.$$

[Note that $F_\sigma\notin S_0^3V$ for $\sigma\neq0$, but that $(1,0,x^*)F_\sigma=x^*(\sigma)F_\sigma$ by the commutation relation in H.] For $\chi\in X(H)$, with $\chi(x^*,x)=y^*(x)x^*(y)$, with y,y^* defining χ, we define:

$$M(\chi)\colon S_0^3V\to S^4V$$
$$M(\chi)F=\sum_\sigma y^*(\sigma)X_{\sigma+y}F_\sigma,$$

where we sum over $\sigma\in(\mathbb{Z}/2)^g$.

Proposition 2. (i) *The image of* $M(\chi)$ *is contained in* S_χ^4V:

$$M(\chi)\colon S_0^3V\to S_\chi^4V.$$

(ii) *In case* $\chi=0$, *the map* $M(\chi)$ *is an isomorphism:*

$$M(0)\colon S_0^3V\xrightarrow{\cong}S_0^4V.$$

(iii) *The inverse of* $M(0)$ *is given by* $4^{-1}\partial/\partial X_0$. *Moreover,*

$$(\partial/\partial X_\sigma)(M(0)F)=4F_\sigma \quad \textit{for all} \quad \sigma\in(\mathbb{Z}/2)^g.$$

(iv) *The map* $M(\chi_1)\colon S_0^3V\to S_{\chi_1}^4V$ *is a surjection.*

Proof. First we show that $(t,x,x^*)M(\chi)P=t^4y^*(x)x^*(M)(\chi)P$, with χ,y,y^* as above. Note $(t,x,x^*)=(t,0,0)(1,0,x^*)(1,x,0)$. Now

$$(1,x.0)M(\chi)P=\sum_\sigma y^*(\sigma)X_{\sigma+y+x}P_{\sigma+x}=\sum_\varrho y^*(\varrho+x)X_{\varrho+y}P_\varrho=y^*(x)M(\chi)P.$$

Furthermore

$$(1,0,x^*)M(\chi)P=\sum_\sigma y^*(\sigma)x^*(\sigma+y)X_{\sigma+y}(1,0,x^*)P_\sigma=\sum_\sigma y^*(\sigma)x^*(\sigma+y)X_{\sigma+y}x^*(\sigma)P_\sigma,$$

as already noted before, now because $x^*(\sigma+y)=x^*(\sigma)\cdot x^*(y)$ and $x^*(\sigma)^2=1$ we find that the last expression equals $x^*(y)M(\chi)P$. Hence $M(\chi):S_0^3\to S_\chi^4V$.

If $\chi=0$, then $M(0)P=\sum X_\sigma P_\sigma$. Considering a monomial of $M(0)P$, we note that: $\partial/\partial X_\alpha X_\sigma(X_{\sigma+\varrho}X_{\sigma+\tau}X_{\sigma+\varrho+\tau})\neq 0$ (any $\varrho,\tau\in(\mathbb{Z}/2)^g$) iff $\sigma=\alpha$ or $\sigma+\varrho=\alpha$ or $\sigma+\tau=\alpha$ or $\sigma+\varrho+\tau=\alpha$. In each case one easily checks that, if nonzero, the answer is always $X_{\varrho+\alpha}X_{\tau+\alpha}X_{\varrho+\tau+\alpha}$. In case $\alpha=0$, monomials of the last type give a basis for S_0^3V and we conclude that $(4^{-1}\partial/\partial X_0)M(0)$ is the identity on S_0^3V, hence $M(0)$ is injective and surjective (the last because the dimensions of S_0^3V and S_0^4V are equal). It is easy to check (iii). For (iv) we observe that $M(\chi_1)$ maps the elements X_0^3, $X_0X_{(\alpha,0)}^2$, $X_{(\beta,0)}X_{(\gamma,0)}X_{(\beta+\gamma,0)}$ of S_0^3V to the basis elements of $S_{\chi_1}^4V$ listed in Proposition 1((*iii*). QED

Finally we want to consider the restrictions of the various elements in S^4V to eigenspaces of H, i.e. we study the canonical map $S^4V\to S^4W$, with $W\subset V$ an eigenspace.

Proposition 3. *Let $W\subset V$ be an eigenspace of $(s,y,y^*)\in H$. Let $\chi\in X(H)$ be defined by $\chi(x,x^*)=y^*(x)x^*(y)$.*

(i) *For every $P\in S_0^3V$ the restrictions of $M(0)P$ and $M(\chi)P$ to W differ by multiplication by a constant.*

(ii) *The restriction map:*

$$S_\chi^4V\to S_0^4W$$

is an isomorphism. In case $\chi=\chi_1$ one has that the restriction of $R_J\in S_{\chi_1}^4$, $J=\{0,\alpha,\beta,\alpha+\beta\}\subset(\mathbb{Z}/2)^{g-1}$, is $\pm P_J\in S_0^4W$, the sign depends on the choice of W.

Proof. In case $y^*(y)=+1$ the two eigenspaces of (t,y,y^*) are defined by:

$$X_\sigma+y^*(\sigma)X_{\sigma+y}\quad\text{and}\quad X_\sigma-y^*(\sigma)X_{\sigma+y}$$

($\sigma\in(\mathbb{Z}/2)^g$) respectively. [Indeed,

$$(t,y,y^*)(X_\sigma\pm y^*(\sigma)X_{\sigma+y})=t(y^*(\sigma+y)X_{\sigma+y}\pm y^*(\sigma)y^*(\sigma)X_\sigma)=\pm t(X_\sigma\pm y^*(\sigma)X_{\sigma+y}).]$$

Now $(M(0)-\lambda M(\chi))P=\sum(X_\sigma-\lambda y^*(\sigma)X_{\sigma+y})P_\sigma$, hence $M(0)P=-M(\chi)P$ on the first eigenspace and $M(0)P=M(\chi)P$ on the second eigenspace. If $y^*(y)=-1$, then the eigenspaces are given by

$$X_\sigma+iy^*(\sigma)X_{\sigma+y}\quad\text{and}\quad X_\sigma-iy^*(\sigma)X_{\sigma+y}$$

($\sigma\in(\mathbb{Z}/2)^g$) respectively, where $i^2=-1$. Now one takes $\lambda=\pm i$.

(ii) Straightforward. QED

2

In this section we compute the canonical multiplication maps

$$p: S^n V \to \Gamma(A, L^{\otimes n})$$

for $n=2,4$. [This map sends a "formal product" of theta functions to the theta function given by the (actual) product.] The kernel of p are then the relations of degree n among the theta functions in $\Gamma(A, L)$, i.e. they are the equations of degree n on $K(A) \subset \mathbb{P}V$. In case $n=3$ an explicit expression for p seems to be unknown.

Choosing a period matrix τ for A, a canonical basis of $\Gamma(A, L)$ is given by the second order theta functions $\theta_{\sigma/2\ 0}(2\tau, 2z)$, with $\sigma \in (\mathbb{Z}/2)^g$ (I, IV, Sect. 2, p. 145). A basis of $\Gamma(A, L^{\otimes 2})_+$, the subspace of even theta functions of order 4, is given by the $\theta_{\varepsilon/2\ \varepsilon'/2}(\tau, 2z)$ with $\varepsilon, \varepsilon' \in (\mathbb{Z}/2)^g$ and ${}^t\varepsilon\varepsilon' = 0$. The multiplication p is given by [I, IV, Sect. 1, a corollary of Theorem 2]:

$$Q\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}(\ldots, \theta_{\sigma/2\ 0}(2\tau, 2z), \ldots) = \theta_{\varepsilon/2\ \varepsilon'/2}(\tau, 0)\theta_{\varepsilon/2\ \varepsilon'/2}(\tau, 2z),$$

where $Q\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} \in S^2 V$ is as in Proposition 1. So if $\theta_{\varepsilon/2\ \varepsilon'/2}(\tau, 0) \neq 0$, then the quadric (defined by) $Q\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}$ intersects $K(A)$ in $K([2]^*\Theta_{\varepsilon,\varepsilon'})$, where $\Theta_{\varepsilon,\varepsilon'} = Z(\theta_{\varepsilon/2\ \varepsilon'/2}(\tau, \cdot))$ is a symmetric divisor in A giving the principal polarization. We also conclude that a basis of the space of those quadrics which contain $K(A)$ is given by the $Q\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}$ with $\theta_{\varepsilon/2\ \varepsilon'/2}(\tau, 0) = 0$, i.e. the divisor $\Theta_{\varepsilon,\varepsilon'}$ has a singularity at $O \in A$ of even multiplicity.

Now assume that $A = J\mathscr{C}$. An even theta characteristic on $\mathscr{L}$ on $\mathscr{C}$ is a linebundle $\mathscr{L}$ on $\mathscr{C}$ with $\mathscr{L}^{\otimes 2} \cong \Omega^1_{\mathscr{C}}$ and with $\dim H^0(\mathscr{C}, \mathscr{L}) \equiv 0 \pmod 2$. The divisors $\Theta_{\varepsilon,\varepsilon'}$ with ${}^t\varepsilon\varepsilon' = 0$ correspond canonically to the even theta characteristics. If $\Theta \subset \mathrm{Pic}^{g-1}(\mathscr{C})$ denotes the divisor consisting of effective divisor classes, then $\Theta \otimes \mathscr{L}^{-1} = \{\mathscr{O}_{\mathscr{C}}(\mathscr{D}) \otimes \mathscr{L}^{-1} \in \mathrm{Pic}^0(\mathscr{C}) = J\mathscr{C} \colon \mathscr{D} \in \Theta\}$ is the corresponding divisor. Riemann's vanishing theorem shows that $O \in A$ is a singular point with multiplicity $2k$ iff $\dim H^0(\mathscr{C}, \mathscr{L}) = 2k$.

In case $\mathscr{C}$ is a hyperelliptic curve, let h denote the divisor class of degree 2 with $h^0(h) = 2$. Let $P_1, \ldots, P_{2g+2}$ be the Weierstrass points of $\mathscr{C}$. Then the even theta characteristics are given by [M 3, Proposition (6.1)]:

$$\begin{array}{ll}
P_{i_1} + P_{i_2} + \ldots + P_{i_g} - P_{i_{g+1}} & \dim H^0 = 0 \\
h + P_{i_1} + \ldots + P_{i_{g-3}} & \dim H^0 = 2 \\
3h + P_{i_1} + \ldots + P_{i_{g-7}} & \dim H^0 = 4 \\
\vdots & \\
(2k-1)h + P_{i_1} + \ldots + P_{i_{g-4k+1}} & \dim H^0 = 2k. \\
\vdots &
\end{array}$$

Hence the number of even theta characteristics with $H^0 \neq 0$ is $\binom{2g+2}{g-3} + \binom{2g+2}{g-7} + \ldots + \binom{2g+2}{g-4k+1} + \ldots$, and this is also the dimension of the space of quadrics which contain $K(J\mathscr{C})$. We will determine the intersection of these quadrics in $\mathbb{P}V_{J\mathscr{C}}$ in Theorem 2.

To compute the multiplication map $p: S^4V \to \Gamma(A, L^{\otimes 4})$ we use Riemann's theta formula to find the image of the bases of $S_0^4 V$ and $S_{\chi_1}^4 V$ given in Proposition 1 *(iii)*. Rather than using the standard basis of $\Gamma(A, L^{\otimes 4})$, we use that for every $\chi \in X(H)$ we have maps:

$$p_\chi : S_\chi^4 V \to [2]^* \Gamma(A, L_\chi) \subset \Gamma(X, L^{\otimes 4}).$$

The symmetric line bundle L_χ is defined as follows. To the character χ corresponds, via pull-back along $\bar{\alpha}$ and the duality given by the Weil pairing on $A[2]$, a point $x(\chi) \in A[2]$. Take (any) $y \in A$ with $2y = x(\chi)$ and define $L_\chi = T_y^* L$, where T_y is translation by y on A. A basis of $[2]^*\Gamma(A, L)$ is obviously given by the $\theta_{\sigma/2\ 0}(2\tau, 4z)$, $\sigma \in (\mathbb{Z}/2)^g$. The 2^{g-1} functions $\theta_{(\sigma/2,0)(0,\frac{1}{2})}$ span $[23^*\Gamma(A, L_\chi)_+$, where $(\alpha/2, 0)$, $(0, \frac{1}{2}) \in (\frac{1}{2}\mathbb{Z}/\mathbb{Z})^{g-1} \times (\frac{1}{2}\mathbb{Z}/\mathbb{Z})$.

Proposition 4. *For every $I = \{0, \varrho, v, \varrho + v\} \subset (\mathbb{Z}/2)^g$ and every $\tau \in \mathbb{H}_g$, the (image under p of the) theta function P_I ($\in S_0^4 V$) is given by*

$$P_I = \sum_\lambda C_{I,\lambda} \theta_{\lambda/2\ 0}(2\tau, 4z),$$

where we sum over $\lambda \in (\mathbb{Z}/2)^g$ and the $C_{I,\lambda} \in \mathbb{C}$, which depend only on τ, are given by:

$$C_{I,\lambda} = (\theta_{(\lambda+\varrho)/2\ 0}\theta_{(\lambda+v)/2\ 0}\theta_{(\lambda+\varrho+v)/2\ 0})(2\tau, 0).$$

For every $J = \{0, \beta, \gamma, \beta + \gamma\} \subset (\mathbb{Z}/2)^{g-1}$ and every $\tau \in \mathbb{H}_g$, the (image of the) theta function R_J ($\in S_{\chi_1}^4 V$) is given by

$$R_J = \sum_\sigma D_{J,\delta} \theta_{(\delta/2,0)(0,\frac{1}{2})}(2\tau, 4z),$$

where we sum over $\delta \in (\mathbb{Z}/2)^{g-1}$ and the $D_{J,\delta} \in \mathbb{C}$, which depend only on τ, are given by:

$$D_{J,\delta} = (\theta_{((\delta+\beta)/2,0)(0,\frac{1}{2})}\theta_{((\delta+\gamma)/2,0)(0,\frac{1}{2})}\theta_{((\delta+\beta+\gamma)/2,0)(0,\frac{1}{2})})(2\tau, 0).$$

In particular, the multiplication maps p_p and p_{χ_1} on $S_0^4 V$ and $S_{\chi_1}^4 V$ are given by the matrices $(C_{I,\lambda})_{I,\lambda}$ and $(D_{J,\delta})_{J,\delta}$ respectively.

Proof. Recall the following consequence of Riemann's theta formula [I, IV, Sect. 1, p. 141 first formula]:

$$(\theta_{\sigma\ 0}\theta_{\sigma+\varrho\ 0}\theta_{\sigma+\varrho+v\ 0})(\tau, z) = \frac{1}{2^g}\sum (-1)^{\sigma^t\eta'}\theta_{2\sigma+\eta\ \eta'}(\tau, 2z)(\theta_{\eta+\varrho\ \eta'}\theta_{\eta+v\ \eta'}\theta_{\eta+\varrho+v\ \eta'})(\tau, 0), \tag{1}$$

where we sum over $\eta, \eta' \in (\frac{1}{2}\mathbb{Z}/\mathbb{Z})^g$ and, for convenience, we also took $\sigma, \varrho, v \in (\frac{1}{2}\mathbb{Z}/\mathbb{Z})^g$. We replaced the minus signs in the last characteristics on both sides by plus signs, using $(\theta . 2)$ p. 39. For the same reason we may omit the term 2σ in the first characteristic on the right-hand side. Now sum both sides over $\sigma \in (\mathbb{Z}/2)^g$. This gives:

$$P_I = 2^{-g} \sum_{\eta,\eta'} \left(\sum_\sigma (-1)^{4\sigma^t\eta'}\right) \theta_{\eta\ \eta'}(\tau, 2z)\theta_{\eta+\varrho\ \eta'}\theta_{\eta+v\ \eta'}\theta_{\eta+\varrho+v\ \eta'})(\tau, 0).$$

The sum over σ gives zero unless $\eta' = 0$, and then it gives 2^g. Hence we get the first formula, upon replacing τ by 2τ, z by $2z$, and η by λ.

To get the second formula, we multiply both sides of (1) by $(-1)^{2\varrho_g}$ and then sum over $\sigma \in (\mathbb{Z}/2)^g$. The left-hand side gives R_J, when replacing ϱ by β, v by γ. The sum over σ in the right side is just $\sum(-1)^{4(\bar{\sigma},0)^t(\bar{\eta}',\eta'_g)} - \sum(-1)^{4(\bar{\sigma},\frac{1}{2})^t(\bar{\eta}',\eta'_g)}$ where we sum over $\bar{\sigma} \in (\mathbb{Z}/2)^{g-1}$ and $\sigma = (\bar{\sigma}, \sigma_g)$, $\eta' = (\bar{\eta}', \eta'_g)$. Both of the sums are zero if $\bar{\eta}' \neq 0$. In case $\bar{\eta}' = 0$ and $\eta'_g = 0$ the sums cancel. Finally, if $\bar{\eta}' = 0$, $\eta'_g = \frac{1}{2}$ the sums add up to 2^g, and we get the second formula. QED

To relate these results to Schottky-Jung relations we use the expression of these relations in theta functions.

Proposition 5. *Let P be a ppav of dimension $g-1$. Then P is a Prym for (A, x), some $x \in A[2]$, that is:*

$$K_A(y) = K_P(0) \tag{2.1}$$

for (some) $y \in A$ with $2y = x$ iff there are period matrices τ for A and π for P such that for every $\varepsilon \in (\mathbb{Z}/2)^{g-1}$ one has:

$$\theta_{(\varepsilon/2,0)(0,\frac{1}{2})}(2\tau, 0) = c\,\theta_{\varepsilon/2\ 0}(2\pi, 0) \tag{2.2}$$

for some non zero $c \in \mathbb{C}$ and with $(\varepsilon/2, 0), (0, \frac{1}{2}) \in (\frac{1}{2}\mathbb{Z}/\mathbb{Z})^{g-1} \times (\frac{1}{2}\mathbb{Z}/\mathbb{Z})$.

Proof. Choose a period matrix for A such that $x = \frac{1}{2}e_g + \Lambda_\tau \in \mathbb{C}^g/\Lambda_\tau \cong A$, where e_g is the last basis vector of $\mathbb{C}^g$. The map K_A is given by the $\theta_{\sigma/2\ 0}(2\tau, 2z)$ with $\sigma \in (\mathbb{Z}/2)^g$, so (in $\mathbb{P}V$):

$$K_A(\tfrac{1}{4}e_g)_{(\alpha,0)} = \theta_{(\alpha/2,0)(0,\frac{1}{2})}(2\tau, 0), \qquad K_A(\tfrac{1}{4}e_g)_{(\alpha,1)} = 0,$$

where $\alpha \in (\mathbb{Z}/2)^{g-1}$. Using the action of $A[2]$ on the points y with $2y = x$ and on $\mathbb{P}V$, we see that we may assume that $K_A(\frac{1}{4}e_g) = K_P(0)$. Now $K_A(\frac{1}{4}e_g)$ is in the eigenspace V_x of x defined by $X_{(\alpha,1)} = 0$, all $\alpha \in (\mathbb{Z}/2)^{g-1}$, and canonical coordinates on $\mathbb{P}V_x$ are given by the $X_{(\alpha,0)}$ so

$$K_P(0)_{(\alpha,0)} = \theta_{\alpha/2}(2\pi, 0), \qquad K_P(0)_{(\alpha,1)} = 0$$

hence (2.1) implies (2.2). The converse is now easy. QED

A glance at the Propositions 4 and 5 shows that the assumption that A and P satisfy a Schottky-Jung relation (i.e. that P is a Prym for A), gives a relation between the multiplication maps for A and P.

Proposition 6. *Assume that A and P satisfy a Schottky-Jung relation so that (2) holds. Then for every $J = \{0, \beta, \gamma, \beta+\gamma\} \subset (\mathbb{Z}/2)^{g-1}$ and every $\delta \in (\mathbb{Z}/2)^{g-1}$ one has:*

$$D_{J,\delta}(\tau) = c^3 C_{J,\delta}(\pi),$$

where c is the constant in (2.2). In particular, the multiplication maps $p_{\chi_1}: S^4_{\chi_1}\Gamma(A, L_A^{\otimes 4}) \to \Gamma(A, L_A^{\otimes 4})$ and $p_0: S^4_0\Gamma(P, L_P) \to \Gamma(P, L_P^{\otimes 4})$ are defined by the same matrices.

Proof. Straightforward. QED

This proposition, combined with the relations between various subspaces of S^*V, has some strange consequences.

Proposition 7. *Let $\chi \in X(H)$ be a nontrivial character, and let $x(\chi) \in A[2]$ be the point corresponding to χ. Assume that P is a Prym for $(A, x(\chi))$. Then for every $F_\chi \in S^4_\chi V$ one has:*

$$K(A) \subset Z(F_\chi)$$

if and only if

$$K(P)\subset Z(F_\chi)$$

(with $K(P)\subset \mathbb{P}V_{x(\chi)}\subset \mathbb{P}V\supset K(A)$*).*

Proof. Using the action of $\mathrm{Aut}(H)$ we may assume that $\chi=\chi_1$. Then $F_\chi=\sum a_J R_J$, sum over $J\subset(\mathbb{Z}/2)^{g-1}$. F_χ is an equation for $K(A)$ iff $\sum D_{J,\delta}a_J=0$, sum over J, for all $\delta\in(\mathbb{Z}/2)^{g-1}$. The restriction of F_χ to $V_{x(\chi)}$ is $G=\sum a_J P_J$ [Proposition 3(ii)], so F_χ is an equation for $K(P)$ iff $\sum C_{J,\delta}a_J=0$. So Proposition 6 gives the equivalence of the statements. QED

Theorem 1. a) *Let* $F\in S_0^4V$ *be an equation for* $K(A)$ *and assume that one even has*

$$K(A)\subset \operatorname{Sing} Z(F).$$

Then for every Prym P *of* A*:*

$$K(P)\subset Z(F).$$

We give two partial converses to this.

b) *Assume that* A *has a Prym for every point* $x\in B$ $(\subset A[2])$, *where* $B\cong\mathbb{F}_2^g$ *is a maximal isotropic subspace (w.r.t. the Weil pairing). Let* $F\in S_0^4V$ *be an equation for* $K(A)$ *and assume that*

$$K(P_x)\subset Z(F)$$

for every nonzero $x\in B$. *Then*

$$K(A)\subset \operatorname{Sing} Z(F).$$

c) *Assume that* A *has a Prym for every* x *in the (affine) subspace* $y+B$ *with* B *as above and* $y\notin B$. *Let* $F\in S_0V$ *and assume that*

$$K(P_x)\subset Z(F)$$

for every $x\in y+B$. *Then*

$$K(A)\subset \operatorname{Sing} Z(F).$$

Proof. a) As $F\in S_0^4V$, Proposition 2 shows that $F=M(0)P_0=\sum X_\alpha P_\alpha$ [sum over $\alpha\in(\mathbb{Z}/2)^g$] with $4P_\alpha=(\partial/\partial X_\alpha)F$. Because $K(A)\subset \operatorname{Sing}Z(F)$ each P_α is a cubic equation for $K(A)$. Let $x\in A[2]$ be the point defined by P. Define $F_\chi=M(\chi)P_0$, then Proposition 2 shows that $F_\chi\in S_\chi^4V$ and F_χ is also an equation for $K(A)$ because each P_α is. Now Proposition 7 shows that $K(P)\subset Z(F_\chi)$. By Proposition 3(i) F_χ and F differ only by a constant on $V_{x(\chi)}$ hence also $K(P)\subset Z(F)$.

b) Choosing a suitable theta structure, we may assume that $B=\{x(\chi)\}$ with $\chi(x,x^*)=y^*(x)$ and y^* runs over $\mathrm{Hom}(\mathbb{Z}/2,\mathbb{C}^*)$. Writing $F=M(0)P_0$ we have $F_\chi:=M(\chi)P_0=\sum y^*(\alpha)X_\alpha P_\alpha$, sum over $\alpha\in(\mathbb{Z}/2)^g$. As F and F_χ differ only by a constant on $V_{x(\chi)}$ we see that F_χ is also an equation for $K(P_{x(\chi)})$. Then Proposition 7 shows that F_χ is an equation for $K(A)$. Taking suitable linear combinations of the F_χ's we see that each $X_\alpha P_\alpha$ is an equation. Hence $4(\partial/\partial X_\alpha)F=P_\alpha$ is an equation for $K(A)$.

c) Now choose $y=x(\chi')$ with $\chi'(x,x^*)=x^*(z)$, some z, and take B as above. The proof is then similar to b). QED

Proposition 2 implies that the $F \in S_0^4 V$ with $K(A) \subset \operatorname{Sing} Z(F)$ correspond to the $P \in S_0^3 V$ with $K(A) \subset Z(P)$ [the H-invariance of $K(A)$ implies that if $K(A) \subset Z(\partial F/\partial X_0)$, then $K(A) \subset Z(\partial F/\partial X_\alpha)$, all α]. So these F correspond to the kernel of the multiplication map $p: S_0^3 V \to \Gamma(A, L^{\otimes 3})_+$. One has $\dim S_0^3 = (2^g+1)(2^{g-1}+1)/3$ and $\dim \Gamma(A, L^{\otimes 3})_+ = (6^g + 2^g)/2 = 2^g(3^g+1)/2$. As p is H-equivariant, the kernel has dimension $\geqq (2^g+1)(2^{g-1}+1)/3 - (3^g+1)/2$.

3

We recall the description of $SU_{\mathscr{C}}(2)/i$, where $\mathscr{C}$ is a hyperelliptic curve with involution i [D-R]. Let

$$Q: X_1^2 + X_2^2 + \dots + X_{2g+2}^2$$

$$Q_w: w_1 X_1^2 + w_2 X_2^2 + \dots + w_{2g+2} X_{2g+2}^2,$$

with $w = (w_1, \dots, w_{2g+2}) \in \mathbb{C}^{2g+2}$, and we assume the w_i to be distinct. Let $\mathscr{C}_w$ be the hyperelliptic curve of genus g defined by

$$y^2 = (x - w_1) \dots (x - w_{2g+2}).$$

We will also use the symbol Q to denote the quadric in $\mathbb{P}^{2g+1}$ defined by Q. This quadric has two rulings, i.e. two families of linear $\mathbb{P}^g$'s lying on Q. We fix one of them and call it R^+. Let

$$Gr_{SO} = \{\mathbb{P}^g \subset \mathbb{P}^{2g+1} : \mathbb{P}^g \in R^+\}.$$

The Grassmannian Gr_{SO} is a smooth variety of dimension $\frac{1}{2}g(g+1)$, [G-H, Chap. 6], and it is a subvariety of

$$Gr_{SL} = G(g+1, 2g+2) = \{\mathbb{P}^g \subset \mathbb{P}^{2g+1} : \mathbb{P}^g \text{ linear subvariety of } \mathbb{P}^{2g+1}\}.$$

For $0 \leqq k \leqq g+1$ let

$$B_k(w) = \{\mathbb{P}^g \in Gr_{SL} : \operatorname{rank}(Q_w \text{ restricted to } \mathbb{P}^g) = k\},$$

then $B_k(w)$ is a quasi projective subvariety of Gr_{SL}. The Zariski closure of $B_k(w)$ is denoted by $\bar{B}_k(w)$ and one has

$$\bar{B}_k(w) = B_k(w) \cup B_{k-1}(w) \cup \dots \cup B_0(w).$$

The closed subvariety $B_0(w)$ has two irreducible components, which correspond to the two rulings of Q_w. Witt's extension theorem [J, 6.5] shows that $O(Q_w)$, the orthogonal group of Q_w, acts transitively on each of the $B_k(w)$'s. Its subgroup of index two $SO(Q_w)$ still acts transitively on each $B_k(w)$ with $k \geqq 1$, and it acts transitively on each ruling, but it cannot permute the two rulings.

Theorem 3 of [D-R] states:

$$SU_{\mathscr{C}_w}(2)/i \cong Gr_{SO} \cap \bar{B}_4(w),$$

and it also describes the subvarieties $Gr_{SO} \cap \bar{B}_k(w)$, $1 \leqq k \leqq 3$, of $SU_{\mathscr{C}_w}(2)/i$, for example $Gr_{SO} \cap \bar{B}_2(w) = K(J\mathscr{C}_w) \subset SU_{\mathscr{C}_w}(2)/i$. We observe that if we embed Gr_{SO} into $\mathbb{P}V$, then we also get an embedding of $SU_{\mathscr{C}_w}(2)/i$. To construct such an embedding φ we use the halfspin representation of $\widetilde{SO}$, the universal covering of $SO = SO(Q)$.

The map φ is described explicitly at the end of this section. In [Bh] it is shown that Gr_{SO} itself is a moduli space of certain vectorbundles on (any) $\mathscr{C}_w$. A description of φ in terms of bundles is not yet known.

To construct the halfspin representation, we consider a hyperelliptic curve $\mathscr{C}$ of genus $g+1$. Let $P_1, \ldots, P_{2g+4}$ be the Weierstrass points of $\mathscr{C}$, then the $x_i = (P_i - P_{2g+4})$, $1 \leqq i \leqq 2g+3$, are points of order two in $J\mathscr{C}$ and one has

$$E(x_i, x_j) = 1 \quad \text{if} \quad i \neq j, \tag{3.1}$$

where E is the Weil pairing (with values in $\mathbb{Z}/2\mathbb{Z}$), [M 3, Proposition 6.3]. Fixing a theta structure α for $\mathscr{C}$, we have a commutative diagram;

$$\begin{array}{ccc} G(L) & \xrightarrow[\cong]{\bar{\alpha}} & H_{g+1} \\ \downarrow & & \downarrow \\ J(\mathscr{C})[2] & \xrightarrow[\cong]{\bar{\alpha}} & (\mathbb{Z}/2)^{g+1} \times \widehat{(\mathbb{Z}/2)}^{g+1}, \end{array}$$

where the vertical maps are quotients by $\mathbb{C}^*$. Choose elements $h_i \in H_{g+1}$ such that $\bar{h}_i = \bar{\alpha}(x_i)$ and such that $h_i^2 = 1$ [i.e. $h_i = (t, x, x^*)$ with $t = \pm 1$ if $x^*(x) = 1$ and $t = \pm \sqrt{-1}$ if $x^*(x) = -1$]. As the Weil pairing corresponds to commutators in H_{g+1} [M 1] one has $[h_i, h_j] = (-1, 0, 0)$ for $i \neq j$. Recall that H_{g+1} acts on $\tilde{V} = V_{J\tilde{\mathscr{C}}}$, a vector space of dimension 2^{g+1}, and let let $\tilde{E}_i \in \operatorname{Aut}(\tilde{V})$ be the map given by h_i. Then one has:

$$\tilde{E}_i^2 = 1, \qquad 1 \leqq i \leqq 2g+3 \tag{3.2.a}$$

$$\tilde{E}_i \tilde{E}_j = -\tilde{E}_j \tilde{E}_i, \qquad i \neq j. \tag{3.2.b}$$

Hence, over $\mathbb{C}$, the $\tilde{E}_i$ generate the Clifford algebra of the quadratic form Q (in fact in $\operatorname{End}(V) \otimes \mathbb{C}[X]$ one has $(X_1^2 + \ldots + X_{2g+2}^2)I = (X_1 \tilde{E}_1 + \ldots + X_{2g+2} \tilde{E}_{2g+2})^2$). Following Brauer and Weyl [B-W], we get the spin representation

$$\tilde{\sigma} : \widetilde{SO}(Q) \to \operatorname{Aut}(\tilde{V}),$$

and for $\tilde{g} \in \widetilde{SO}(Q))$ mapping to $g = (g_{ij}) \in SO(Q)$ one has [B-W, (12)]:

$$\tilde{\sigma}(\tilde{g}) \tilde{E}_i \tilde{\sigma}(\tilde{g})^{-1} = \sum_{j=1}^{2g+2} g_{ji} \tilde{E}_j \qquad (1 \leqq i \leqq 2g+2). \tag{3.3}$$

To get the halfspin representation we observe that the linear equivalence of divisors on $\mathscr{C}: P_1 + \ldots + P_{2g+4} \equiv (2g+4) P_{2g+4}$ implies that $\tilde{E}_1 \tilde{E}_2 \cdot \tilde{E}_{2g+2} = c \tilde{E}_{2g+3}$ with a $c \in \mathbb{C}$, $c^4 = 1$. From this, (3.2.a) and (3.3) one finds that

$$\tilde{\sigma}(\tilde{g}) \tilde{E}_{2g+3} \tilde{\sigma}(\tilde{g})^{-1} = \det(g) \tilde{E}_{2g+3}.$$

As $\tilde{E}_{2g+3}$ has two eigenspaces, each of dimension 2^g, in $\tilde{V}$ the representation $\tilde{\sigma}$ is reducible. Let V be one of these eigenspaces and let σ denote the restriction of $\tilde{\sigma}$ to V. Then

$$\sigma : \widetilde{SO} \to \operatorname{Aut}(V)$$

is the halfspin representation. (Lie group theory shows that if g is even, then σ is injective [and $\widetilde{SO} \to SO/(\pm 1)$ is a $\mathbb{Z}/4$ covering]. In case $g = 2$ one has $\widetilde{SO}_6 \cong SL_4$ and σ is the standard 4 dimensional representation (or its dual). If g ($\geqq 3$) is odd, then the 4:1 covering $\widetilde{SO}\to SO/(\pm 1)$ has group $\mathbb{Z}/2\times\mathbb{Z}/2=\{0,a,a_+,a_-\}$, with $\widetilde{SO}/a_+=SO$. Then $\widetilde{SO}/a_+\cong\widetilde{SO}/a_-\cong\sigma(\widetilde{SO})$. Only in case $g=3$ (triality) one has $\sigma(\widetilde{SO})\cong SO$: in this case the halfspin representation may be identified with the standard representation.)

To identify V with $V_{J\mathscr{C}_w}$ it suffices to give a (standard) representation of H_g on V. (In fact we will also give a level two structure on $J\mathscr{C}_w$.) Let $C(\tilde{E}_{2g+3})$ be the centralizer of $\tilde{E}_{2g+3}$ in the (finite) Heisenberg group generated by the $\tilde{E}_i$. Then $C(\tilde{E}_{2g+3})$ acts on V. Choosing $c\in\mathbb{C}$ such that $c\tilde{E}_{2g+3}$ acts trivially on V, the group $C(\tilde{E}_{2g+3})/\langle c\tilde{E}_{2f+3}\rangle$, which is isomorphic to H_g, acts on V in the standard way. Define $E_i\in \mathrm{Aut}(V)$ to be the restriction of $\tilde{E}_i\tilde{E}_{2g+2}$ to V:

$$E_i=\tilde{E}_i\tilde{E}_{2g+2}|_V \tag{3.4}$$

then one has

$$E_i^2=1$$

$$E_iE_j=-E_jE_i,\qquad 1\leqq i<j\leqq 2g+1,$$

and the E_i and $\mathbb{C}^*$ generate $H_g\subset \mathrm{Aut}(V)$. To get a level two structure $\bar{\alpha}_w: J\mathscr{C}_w[2]\to(\mathbb{Z}/2)^g\times(\mathbb{Z}/2)^g$ we choose the ramification points of $\mathscr{C}\to\mathbb{P}^1$, the hyperelliptic curve of genus $g+1$ we started with, to be $w_1,\ldots,w_{2g+2}$ and (any) two other points $w_{2g+3},w_{2g+4}\in\mathbb{C}\subset\mathbb{P}^1$. Then the point of order two $x_{2g+3}=P_{2g+3}-P_{2g+4}$ defines an étale double cover $\tilde{\mathscr{C}}\to\mathscr{C}$ with Prym variety $P=J\mathscr{C}_w$ [M 2, p. 346]. As $P[2]\cong x_{2g+3}^{\perp}/\langle x_{2g+3}\rangle$ (with $\perp$ taken w.r.t. the Weil pairing), the level two structure $\bar{\alpha}$ on $J\mathscr{C}[2]$ induces a level two structure $\bar{\alpha}_w$ on $J\mathscr{C}_w[2]$. Note that $x_i+x_{2g+2}\in x_{2g+3}^{\perp}$ will correspond to $(P_i-P_{2g+2})\in J\mathscr{C}_w[2]$ [where now $P_i\in\mathscr{C}_w$ $(1\leqq i\leqq 2g+2)$ lies over w_i] and that E_i is an element of H_g lying over $\bar{\alpha}_w(P_i-P_{2g+2})\in H_g/\mathbb{C}^*$.

From [D-R] we recall that $J\mathscr{C}_w[2]$ acts on $\mathbb{P}^{2g+1}$. The action is defined on the generators (P_i-P_{2g+2}) by

$$\begin{aligned}(P_i-P_{2g+2})\circ X_j&=X_j &\quad j\notin\{i,2g+2\}\\ &=-X_j &\quad j\in\{i,2g+2\}.\end{aligned}$$

This action leaves both Q and Q_w stable, and in fact the generators given are in $SO(Q)\cap SO(Q_w)$. Hence $J\mathscr{C}_w[2]$ acts on $Gr_{SO}\cap\bar{B}_4(w)=SU_{\mathscr{C}_w}(2)$ and the action coincides with the action of $J\mathscr{C}_w[2]$ on $SU_{\mathscr{C}_w}(2)$ given by tensoring. Formula's (3.3), (3.2) show that $\tilde{\sigma}(P_i-P_{2g+2})=c\cdot\tilde{E}_i\tilde{E}_{2g+2}$ (some $c\in\mathbb{C}^*$), hence for some $c'\in\mathbb{C}^*$:

$$\sigma(P_i-P_{2g+2})=c'E_i.$$

Note that $\sigma(P_i-P_{2g+2})$ is also an element of H_g lying over $\bar{\alpha}_w(P_i-P_{2g+2})$.

Theorem 2. *Let $\mathscr{C}_w$ be a hyperelliptic curve of genus g. Let $K: J\mathscr{C}_w\to\mathbb{P}V$ be defined by a theta structure lying over $\bar{\alpha}_w$. Then the subvariety of $\mathbb{P}V$ defined by all quadrics containing $K(J\mathscr{C}_w)$ $(\subset\mathbb{P}V)$ is isomorphic to Gr_{SO}. In particular we have an embedding*

$$\varphi: Gr_{SO}\to\mathbb{P}V.$$

The map φ is equivariant w.r.t. the action of $\widetilde{SO}$ on both spaces (on the left via the SO action on $\mathbb{C}^{2g+2}$, on the right via σ).

Proof. We will first construct φ and show that it is $\widetilde{SO}$-equivariant. Let $P \subset SO(Q)$ be the stabilizer of a linear $\mathbb{P}^g \subset Q$, then $Gr_{SO} \cong SO/P$. (For computations it is convenient to choose a basis $\{e_i\}$ of $\mathbb{C}^{2g+2}$ such that Q is given by

$$Q_s: X_1 X_{g+2} + X_2 X_{g+3} + \dots + X_{g+1} X_{2g+2} \tag{3.5}$$

and to take the stabilizer of the $\mathbb{P}^g$ spanned by $e_1, \dots, e_{g+1}$.) Then P is a semidirect product of GL_{g+1} and an unipotent subgroup of dimension $\frac{1}{2}g(g+1)$. So the charactergroup of P, $\mathrm{Hom}(P, \mathbb{C}^*)$, is isomorphic to $\mathbb{Z}$ and it is generated by the determinant on GL_{g+1}. With respect to the basis $\{e_i\}$ a Cartan subalgebra of *so*, the Lie algebra of SO, is given by the $h(t) = \mathrm{diag}(t_1, \dots, t_{g+1}, -t_1, \dots, -t_{g+1})$ $(t_i \in \mathbb{C})$, and the characters of P correspond to integral multiples of the weight $2\omega_{g+1} = \varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_{g+1}$ [i.e. $2\omega_{g+1}(h(t)) = t_1 + t_2 + \dots + t_{g+1}$]. The weight ω_{g+1} is a fundamental weight of *so* (which is of type D_{g+1}) and it corresponds to a generator of the character group of $\tilde{P}$, the inverse image of P in $\widetilde{SO}$. Now the representation with highest weight ω_{g+1} is a halfspin representation (the other one having highest weight ω_g). Hence in V there will be a unique one dimensional subspace on which $\tilde{P}$ acts (via σ) via the character ω_{g+1}. The $\widetilde{SO}$ orbit of the corresponding point in $\mathbb{P}V$ will be $Gr_{SO} \subset \mathbb{P}V$. So we have found φ. By construction the map φ is equivariant w.r.t. the $\widetilde{SO}$ action.

The variety $\varphi(Gr_{SO})$ is defined by quadrics [S]. We show that those are precisely the quadrics which contain $K(J\mathscr{C}_w)$. Let L be the linebundle on $Gr_{SO} = \widetilde{SO}/\tilde{P}$ defined by $\tilde{\omega}_{g+1}$ (L is a quotient of $\widetilde{SO} \times \mathbb{C}$ by $\tilde{P}$, with the $\tilde{P}$ action on $\mathbb{C}$ given by ω_{g+1}, [Bo]). Then, as $\widetilde{SO}$ representations, one has:

$$H^0(Gr_{SO}, L^{\otimes n}) \cong V(n\omega_{g+1}),$$

where the right-hand side is the irreducible $\widetilde{SO}$ representation with highest weight $n\omega_{g+1}$.

In particular, the quadrics which vanish on Gr_{SO} are the kernel of $S^2 V = S^2 V(\omega_{g+1}) \to V(2\omega_{g+1})$. The dimension formula shows that $\dim V(2\omega_{g+1}) = \binom{2g+2}{g+1}$, but we still must show that we have the right quadrics. So we explicitly find $\widetilde{SO}$-stable subspaces in $S^2 V$. As the (finite) Heisenberg group, generated by the E_i, is in $\sigma(\widetilde{SO})$, any $\widetilde{SO}$ stable subspace is a direct sum of (one dimensional) H_g eigenspaces. Let π be an even permutation on $2g+2$ objects. Then π acts on $\mathbb{C}^{2g+2}$ by permuting the basis vectors and one gets $\pi \in SO(Q)$. If $\tilde{\pi} \in \widetilde{SO}$ lies over π, then $\tilde{\sigma}(\tilde{\pi}) \tilde{E}_i \tilde{\sigma}(\tilde{\pi})^{-1} = \tilde{E}_{\pi(i)}$ $(1 \leqq i \leqq 2g+2)$. To find $\sigma(\tilde{\pi})$ we write, in case $\pi(2g+2) \neq 2g+2$, $\tilde{E}_{\pi(i)} \tilde{E}_{\pi(2g+2)} = -\tilde{E}_{\pi(i)} \tilde{E}_{2g+2} \tilde{E}_{\pi(2g+2)} \tilde{E}_{2g+2}$, so in this case $\sigma(\tilde{\pi}) E_i \sigma(\tilde{\pi})^{-1} = -E_{\pi(i)} E_{\pi(2g+2)}$ by (3.4). If $\pi(2g+2) = 2g+2$ we simply get $\sigma(\tilde{\pi}) E_i \sigma(\tilde{\pi})^{-1} = E_{\pi(i)}$. So we have that $\sigma(\tilde{\pi}) E_i \sigma(\tilde{\pi})$ lies over $\bar{\alpha}_\omega(P_{\pi(i)} - P_{\pi(2g+2)})$. Hence π gives a permutation of the P_i, so

$$\sigma(\tilde{\pi}) Q_{i_1 \dots i_k} = c \cdot Q_{\pi(i_1) \dots \pi(i_k)},$$

where $Q_{i_1 \dots i_k} \in S^2 V$ is the quadric corresponding to the theta characteristic with the same indices (cf. Sect. 2) and $c \in \mathbb{C}^*$.

The dimensions of the subspaces of $S^2 V$ stable under H_g and the even permutations are thus: $\frac{1}{2}\binom{2g+2}{g+1}, \binom{2g+2}{g-3}, \dots, \binom{2g+2}{g-4k+1}, \dots$. For most g it already follows that $H^0(Gr_{SO}, L^{\otimes 2})$ is the subrepresentation spanned by the quadrics in $\mathbb{P}(V)$ which do not vanish on $K(J\mathscr{C}_w)$. To get the conclusion in general we show:

$$S^2 V = V(2\omega_{g+1}) \oplus V(\omega_{g-3}) \oplus \ldots \oplus V(\omega_{g-4k+1}) \oplus \ldots \tag{3.6}$$

and one has $V(\omega_n) \cong \bigwedge^n \mathbb{C}^{2g+2}$ $(1 \leqq n \leqq g-1)$, in particular $\dim V(\omega_n) = \binom{2g+2}{n}$. Combining this result with the one above, we see that the quadrics which vanish on $K(J\mathscr{C}_w)$ are all in $\widetilde{SO}$ subrepresentations which are in the kernel of $S^2 V \to H^0(Gr_{SO}, L^{\otimes 2})$, so the theorem follows. To prove the decomposition one may for example compare the weights and their multiplicities in the representations on both sides. Applying the Dynkin diagram automorphism of D_{g+1} to both sides and adding the result to (3.6) and noting that $\bigwedge^{g+1} \mathbb{C}^{2g+2} = V(2\omega_g) \oplus V(2\omega_{g+1})$ (cf. the proof of Proposition 8), one does not need to determine the weights of $V(2\omega_{g+1})$. The weights of $\bigwedge^n \mathbb{C}^{2g+2}$ are easily found as the weightvectors are $e_{i_1} \wedge \ldots \wedge e_{i_m} \wedge e_{j_1} \wedge \ldots \wedge e_{j_k}$ with $m+k=n$, $1 \leqq i_1, \ldots, i_m \leqq g+1$, $g+2 \leqq j_1 \leqq \ldots \leqq 2g+2$, with weight $\varepsilon_{i_1} + \ldots + \varepsilon_{i_m} - \varepsilon_{j_1} - \ldots - \varepsilon_{j_k}$. The weights of $S^2 V(\omega_g)$ and $S^2 V(\omega_{g+1})$ are easily determined by noting that the weights of $V(\omega_g)$ and $V(\omega_{g+1})$ are $\frac{1}{2}(\pm\varepsilon_1 \ldots \pm\varepsilon_{g+1})$ with an odd, resp. even, number of minus signs. QED

In Theorem 2 we showed that Gr_{SO} can be embedded in $\mathbb{P}V$. Another projective embedding of Gr_{SO} is obtained by the inclusion $Gr_{SO} \subset Gr_{SL}$ and by embedding Gr_{SL} into the projective space $\mathbb{P}\left(\bigwedge^{g+1} \mathbb{C}^{2g+2}\right)$ by the Plücker embedding. The relation between the embeddings is given by:

Proposition 8. *The diagram below commutes. The (rational) map* ψ *is given by (a basis of) the quadrics in the* $\widetilde{SO}$ *subrepresentation* $H^0(Gr_{SO}, \varphi^* \mathcal{O}_{\mathbb{P}V}(2)) \subset S^2 V$

$$\begin{array}{ccc} Gr_{SL} & \longrightarrow & \mathbb{P}\left(\bigwedge^{g+1} \mathbb{C}^{2g+2}\right) \\ \uparrow & & \uparrow \psi \\ Gr_{SO} & \longrightarrow & \mathbb{P}V . \end{array}$$

Proof. The representation of SO on $\mathbb{C}^{2g+2}$ induces a representation of SO on $\bigwedge^{g+1} \mathbb{C}^{2g+2}$. This representation is reducible and one has (see below):

$$\bigwedge^{g+1} \mathbb{C}^{2g+2} = V(2\omega_g) \oplus V(2\omega_{g+1}) . \tag{3.7}$$

As $\mathrm{Pic}(Gr_{SO}) = \mathbb{Z}$, with generator $L = \varphi^* \mathcal{O}_{\mathbb{P}V}(1)$ (as in the proof of Theorem 2), and because $H^0(Gr_{SO}, L^{\otimes n}) \cong V(n\omega_{g+1})$, the composition of the embeddings will factor as described above. This result allow us to describe φ explicitly in terms of "square roots" of certain coordinate functions of the Plücker embedding. To see (3.7) note that the weights of the SO representation $\mathbb{C}^{2g+2}$ are the $\pm\varepsilon_i$ $(1 \leqq i \leqq g+1)$, hence a maximal weight of $\bigwedge^k \mathbb{C}^{2g+2}$ is $e_1 + \ldots + e_k$ for $1 \leqq k \leqq g+1$, the weight vector being

$e_1 \wedge \ldots \wedge e_k$. In particular $V(2\omega_{g+1})$ is a subrepresentation $\bigwedge^{g+1} \mathbb{C}^{2g+2}$. The weight of $e_1 \wedge \ldots \wedge e_g \wedge e_{2g+2}$ is $2\omega_g = \varepsilon_1 + \ldots + \varepsilon_g - \varepsilon_{g+1}$, which is also a maximal weight of $\bigwedge^{g+1} \mathbb{C}^{2g+2}$, so $V(2\omega_g)$ is also a subrepresentation. As $\dim V(2\omega_g) = \dim V(2\omega_{g+1}) = \frac{1}{2}\binom{2g+2}{g+1}$ [the first equality is immediate from the (Dynkin) diagram automorphism of SO] we also proved (3.7). QED

We recall that Desale and Ramanan showed that $SU_{\mathscr{C}_w}(2)/i \cong Gr_{SO} \cap \bar{B}_4(\omega)$.

Theorem 3. *For any k, $1 \leqq k \leqq g$ the algebraic variety*

$$\varphi(Gr_{SO} \cap \bar{B}_k(w)) \subset \mathbb{P}V$$

is defined (at least as a set) by quartic equations.

To prove this theorem we use the diagram in Proposition 8. As the rational map ψ is given by quadratic polynomials the following theorem implies Theorem 3.

Theorem 3′. *For any k, $1 \leqq k \leqq g$ the algebraic subvariety $\bar{B}_k(w) \subset Gr_{SL}$ is defined (as a set) by quadrics in the Plücker embedding*

$$\bar{B}_k(w) \subset Gr_{SL} \subset \mathbb{P}\left(\bigwedge^{g+1} \mathbb{C}^{2g+2}\right).$$

Proof. The group $SL = SL(\mathbb{C}^{2g+2})$ acts on $S^2\left(\bigwedge^{g+1} \mathbb{C}^{2g+2}\right)$ and the subrepresentation corresponding to the quadrics which do not vanish on Gr_{SL} is $H^0(Gr_{SL}, \mathcal{O}(2)) \cong V(2\lambda_{g+1})$, where λ_{g+1} is the fundamental SL weight $\varepsilon_1 + \ldots + \varepsilon_{g+1} - \frac{1}{2}(\varepsilon_1 + \ldots + \varepsilon_{2g+2})$ (note that a weight $\sum a_i \varepsilon_i$ of SL must have $\sum a_i = 0$). The SL representation $V(2\lambda_{g+1})$ will be reducible when restricted to SO. If W is any SO subrepresentation of $V(2\lambda_{g+1})$, then the locus in Gr_{SL} defined by W is of course a union of SO orbits. The splitting of $V(2\lambda_{g+1})$ into irreducible SO subrepresentations is:

$$V(2\lambda_{g+1})_{SO} = V(4\omega_{g+1}) \oplus V(4\omega_g) \oplus V(2(\omega_{g+1} + \omega_g)) \oplus V(2\omega_{g-1}) \oplus \ldots$$
$$\ldots \oplus V(2\omega_k) \oplus \ldots \oplus V(2\omega_1) \oplus V(0), \tag{3.8}$$

where $V(0)$ stands for the trivial one dimensional representation. Moreover,

$$\bar{B}_k = Z(V(0) \oplus V(2\omega_1) \oplus \ldots \oplus V(2\omega_{g-k})) \tag{3.9}$$

for $1 \leqq k \leqq g$ and we put $\omega_0 = 0$. The proof of (3.8) and (3.9) gives a proof of the theorem and will be given now.

As $SO(Q_w)$ is conjugate to $SO(Q_s)$ in SL, it suffices to prove (3.8) and (3.9) for $SO = SO(Q_s)$ and its orbits B_k, with Q_s as in (3.5). To decompose $V(2\lambda_{g+1})$ into SO representations we use the contraction operator (w.r.t. the indices $N, 2N$):

$$Q_s^{(1)}: \bigotimes^{2N} \mathbb{C}^{2g+2} \to \bigotimes^{2N-2} \mathbb{C}^{2g+2}$$

$\left(\text{we put } \bigotimes^{0} \mathbb{C}^{2g+2} = \mathbb{C}, \bigotimes^{k} \mathbb{C}^{2g+2} = 0 \text{ if } k < 0\right)$ defined by:

$$Q_s^{(1)}(e_{i_1} \otimes \ldots \otimes e_{i_{2N}}) = \delta_{i_{N+g+1} i_{2N}} e_{i_q} \otimes \ldots \otimes \hat{e}_{i_N} \otimes \ldots \otimes \hat{e}_{i_{2N}},$$

i.e. we omit e_{i_N} and $e_{i_{2N}}$, and δ_{kl} is Kronecker's delta. Iterating the contraction maps gives:

$$Q_s^{(k)}: \bigotimes^{2g+2} \mathbb{C}^{2g+2} \to \bigotimes^{2g+2-2k} \mathbb{C}^{2g+2}. \tag{3.10}$$

Both the kernel and the image of $Q_s^{(k)}$ restricted to $V(2\lambda_{g+1})$ are SO subrepresentations of $V(2\lambda_{g+1})$. Denoting them by $\mathrm{Ker}(Q_s^{(k)})$ and $\mathrm{Im}(Q_s^{(k)})$ we have:

$$V(2\lambda_{g+1})_{SO} = \mathrm{Ker}(Q_s^{(k)}) \oplus \mathrm{Im}(Q_s^{(k)}).$$

The SL representation $V(2\lambda_{g+1})$ is a subrepresentation of $V(\lambda_{g+1}) \otimes V(\lambda_{g+1})$, with $V(\lambda_{g+1}) \cong \bigwedge^{g+1} \mathbb{C}^{2g+2}$, which itself is a subrepresentation of $\bigotimes^{2g+2} \mathbb{C}^{2g+2}$. We use the action of the symmetric group S_{2g+2}, by permuting the tensor components, to describe $V(2\lambda_{g+1})$ explicitly, cf. [W]. The highest weight vector of $V(2\lambda_{g+1})$ is $(e_1 \wedge \ldots \wedge e_{g+1}) \otimes (e_1 \wedge \ldots \wedge e_{g+1})$, so its symmetry type corresponds to the diagram [W, IV, Sect. 2]:

1	$g+2$
2	$g+3$
$\vdots$	$\vdots$
$g+1$	$2g+2$

So if

$$C: \bigotimes^{2g+2} \mathbb{C}^{2g+2} \to \bigotimes^{2g+2} \mathbb{C}^{2g+2}$$

is the Young symmetrizer corresponding to this diagram, then

$$V(2\lambda_{g+1}) = \mathrm{Im}\, C, \quad \text{the image of } C \text{ [W, Theorem (4.4D)]}.$$

A basis of $\mathrm{Im}\, Q_s^{(n)}$, considered as a subspace of $\bigotimes^{2g+2-2n} \mathbb{C}^{2g+2}$, is given by the $e(i_1, \ldots, i_{g+1-n}; i_{g+2}, \ldots, i_{2g+2-n})$, where the entries must satisfy:

$$1 \leqq i_1 < i_2 < \ldots < i_{g+1-n} \leqq 2g+2$$

$$1 \leqq i_{g+2} < i_{g+3} < \ldots < i_{2g+2-n} \leqq 2g+2$$

$$i_j \leqq i_{j+g+1} \quad \text{for all } j \text{ with } 1 \leqq j \leqq g+1-n,$$

and

$$e(i_1, \ldots, i_{g+1-n}; i_{g+2}, \ldots, i_{2g+2-n}) = \sum (e_{j_1} \wedge \ldots \wedge e_{j_{g+1-n}})(e_{j_{g+2}} \wedge \ldots \wedge e_{j_{2g+2-n}}), \tag{3.11}$$

where $\omega\omega'$ stands for $\omega \otimes \omega' + \omega' \otimes \omega$ and where we sum over all tuples $(j_1, \ldots, j_{g+1-n}; j_{g+2}, \ldots, j_{2g+2-n})$ satisfying $\{j_\ell, j_{\ell+g+1}\} = \{i_\ell, i_{\ell+g+1}\}$ for every ℓ with $1 \leqq \ell \leqq g+1-n$. The *sl* weight of the basis vector (3.11) is $\varepsilon_{i_1} + \ldots + \varepsilon_{g+1-n} + \varepsilon_{g+2} + \ldots + \varepsilon_{2g+2-n} - \frac{(2g+2-2n)}{2g+2}(\varepsilon_1 + \ldots + \varepsilon_{2g+2})$, its *so* weight is found by substituting $-\varepsilon_k$ for ε_{g+1+k} if $1 \leqq k \leqq g+1$. It is easy to see that $e(1, \ldots, g+1; 1, \ldots, g+1)$ and $e(1, \ldots, g, 2g+2; 1, \ldots, g, 2g+2)$ are in $\mathrm{Ker}\, Q_s^{(1)}$, and their *so* weights $4\omega_{g+1} = 2\varepsilon_1 + \ldots + 2\varepsilon_{g+1}$ and $4\omega_g = 2\varepsilon_1 + \ldots + 2\varepsilon_g - 2\varepsilon_{g+1}$ are maximal in $\mathrm{Ker}\, Q_s^{(1)}$.

Hence:

$$V(4\omega_{g+1})\oplus V(4\omega_g)\subset \operatorname{Ker} Q_s^{(1)}. \tag{3.12}$$

The tensor $e(1,\dots,g+1;\ 1,\dots,g,2g+2)$ is in $\operatorname{Ker} Q_s^{(2)}$, but not in $\operatorname{Ker}(Q_s^{(1)})$. Its weight $2(\omega_g+\omega_{g+1})=2\varepsilon_1+\dots+2\varepsilon_g$ is a maximal weight of $\operatorname{Ker}(Q_s^{(2)}/\operatorname{Ker} Q_s^{(1)})$ [in fact it is maximal in $V(2\lambda_{g+1})_{SO}$] hence

$$V(2(\omega_g+\omega_{g+1}))\subset \operatorname{Ker}(Q_s^{(2)}). \tag{3.13}$$

For $k\geqq 2$ put: $t_k=(1,\dots,g+1;\ 1,\dots,g+1-k,2g+3-k,2g+4-k,\dots,2g+2)$. Then $e(t_k)\in \operatorname{Ker}(Q^{(k+1)})$ but $e(t_k)\notin \operatorname{Ker} Q_s^{(k)}$. The weight of $e(t_k)$ is $2\omega_{g+1-k}=2\varepsilon_1+\dots+2\varepsilon_{g+1-k}$, which is a maximal weight of $\operatorname{Ker} Q^{(k+1)}/\operatorname{Ker} Q^{(k)}$, hence

$$V(2\omega_{g+1-k})\subset V(2\lambda_{g+1}). \tag{3.14}$$

From (3.12), (3.13), and (3.14) we conclude that the right-hand side of (3.8) is a subrepresentation of the left-hand side, so to prove equality it suffices to show that the dimensions are equal. The dimension formula gives

$$\dim V(2\lambda_{g+1})=\frac{1}{2g+3}\binom{2g+3}{g+1}^2.$$

The dimensions of the SO representations may be computed with the algorithm in [K]. To use it note that $V(2\omega_l)$, with $1\leqq l\leqq g-1$ corresponds to the partition $2+2+\dots+2=2l$, that $V(2(\omega_g+\omega_{g+1}))$ corresponds to $2+2+\dots+2=2g$ and that $V(4\omega_g)\oplus V(4\omega_{g+1})$, on which there is an irreducible O representation, corresponds to $2+2+\dots+2=2(g+1)$, cf. [W, Sect. 7]. The dimension d_k of the representation corresponding to $2+\dots+2=2k$ is

$$d_k=\frac{(2g+3)-2k}{k(k+1)}\binom{2g+2}{k-1}\binom{2g+4}{k},$$

$$\left[\text{note that } 1+d_1+\dots+d_k=\frac{1}{k+1}\binom{2g+2}{k}\binom{2g+3}{k}\right].$$

We will show that the sections in $\operatorname{Im} Q_s^{(k)}$ define $\bar{B}_{k-1}$ for $k\geqq 2$. The two closed orbits are defined by $\operatorname{Im}(Q_s^{(1)})\oplus V(4\omega_g)$ and $\operatorname{Im}(Q_s^{(1)})\oplus V(4\omega_{g+1})$ (cf. Proposition 8). To find $\operatorname{Im} Q_s^{(k)}$ as a subrepresentation of $V(2\lambda_{g+1})_{SO}$, i.e. to split (3.10), we tensor $\operatorname{Im} Q_s^{(k)}$ k times with Q_s^*, the dual of Q_s, and then apply the Young symmetrizer C to get a subrepresentation of $V(2\lambda_{g+1})$. So a basis of $\operatorname{Im} Q_s^{(n)}$ is given by:

$$f(i_1,\dots,i_{g+1-n};\ i_{g+1},\dots,i_{2g+2-n})=\sum\sum(\omega\wedge e_{k_1}\wedge\dots\wedge e_{k_n})(\omega'\wedge e_{l_1}\wedge\dots\wedge e_{l_n}), \tag{3.15}$$

where we sum over ω,ω' as in (3.11), and over all k_i with $1\leqq k_i\leqq 2g+2$, the l_i being determined by $|k_i-l_i|=g+1$ and $1\leqq l_i\leqq 2g+2$.

Let $\mathbb{P}_n^g\subset\mathbb{P}^{2g+1}$ be the linear subspace spanned by $e_1,\dots,e_{g+1-n}$, $e_{g+2-n}+e_{2g+3-n},\dots,e_{g+1}+e_{2g+2}$. Then Q_s restricted to $\mathbb{P}_n^g$ has rank n, so we have a point in B_n. Choosing $i_j=i_{j+g+1}=j$ for $1\leqq j\leqq g+1-n$ in (3.15) above, we see that this basis element doesn't vanish on $\mathbb{P}_n^g$, so $B_n\not\subset Z(\operatorname{Im} Q_s^{(n)})$. Every basis element of $\operatorname{Im} Q^{(n)}$ is easily seen to vanish on $\mathbb{P}_{n-1}^g$ hence $\bar{B}_{n-1}\subset Z(\operatorname{Im} Q_s^{(n)})$ [use $f(x\cdot\mathbb{P}_{n-1}^g)=(x^{-1}\overset{S}{\cdot}f)(\mathbb{P}_{n-1}^g)$ for every $x\in SO$] hence $\bar{B}_{n-1}=Z(\operatorname{Im} Q_s^{(n)})$. QED

We give a more explicit description of φ. Let X be an alternating $(g+1)\times(g+1)$ matrix. Then the rows of the matrix (IX), where I is the $(g+1)\times(g+1)$ identity matrix, span a $\mathbb{P}^g \subset Q_s$. Any Plücker coordinate is then a minor of X or $1=\det(I)$. The determinants of the alternating submatrices Y of X are perfect squares which are identically zero unless Y has an even number of rows (and columns). A square root of det Y is the Pfaffian of Y. The map φ, restricted to this Zariski open subset of Gr_{SO}, is given by (linear combinations of) all such pfaffians and the function 1. There are $1+\binom{g+1}{2}+\binom{g+1}{4}+\ldots=2^g$ such functions. Note that Proposition 8 implies that every minor of X is a quadratic polynomial in these pfaffians. In case $g=3$ one finds the pfaffian of X (a quadric in the entries of X), the six entries (pfaffians of 2×2 alternating submatrices), and 1. They satisfy one obvious quadratic relation which thus defines $\varphi(Gr_{SO})\subset\mathbb{P}V$. I am indebted to J. Stevens for having shown me this example.

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