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Siegel modular forms vanishing on the moduli space of curves

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Introduction

Let M_g be the moduli space of curves of genus g and let A_g be the moduli space of principally polarized Abelian varieties of dimension g. The morphism

$$j\colon M_g \to A_g$$

given by

$$\mathscr{C} \to J(\mathscr{C}) = \text{Jacobian of } \mathscr{C}$$

is injective (on geometrical points, Torelli's theorem) and the Schottky problem is to find equations or characterizations of the locus $j(M_g)$ or of J_g, its closure in A_g. (Note that $\dim M_g = 3g-3$ $(g \geqq 2)$ and $\dim A_g = \frac{1}{2}g(g+1)$.)

In case we work over the complex number field the quasi projective variety A_g is a Zariski open subset of its Satake compactification $\bar{A}_g$:

$$\bar{A}_g = \operatorname{Proj}(A(\Gamma_g))$$

where $A(\Gamma_g)$ is the graded ring generated by Siegel modular forms on $\Gamma_g = Sp(2g, \mathbb{Z})$. A solution of the Schottky problem would be a set of Siegel modular forms which generate the ideal of the variety $\bar{J}_g$, the closure of J_g in $\bar{A}_g$.

The goal of this paper is to give "explicitly" an ideal $S(\Gamma_g)$ of $A(\Gamma_g)$ whose zero locus $\bar{S}_g$ contains $\bar{J}_g$ as an irreducible component.

In case $g=4$ this ideal, which is then generated by one element, was given by Schottky [S]. It was shown by Igusa [I3] (see also Freitag [Fr1]) that $\bar{S}_4$ is irreducible, which gave a solution to the Schottky problem for $g=4$. For $g=5$ a set of Siegel modular forms which define a subset of $\bar{A}_5$ having $\bar{J}_g$ as an irreducible component has been given by Accola [A]. These Siegel modular forms are elements of the ideal $S(\Gamma_5)$.

The ideal $S(\Gamma_g)$ is defined using the Schottky-Jung relations between the values of the theta constants for a Jacobian and those of its Prym varieties. To

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prove that $\bar{J}_g$ is a component of $\bar{S}_g$ we use an induction argument (with respect to g). This is suggested by the fact that $\bar{A}_g = A_g \cup \bar{A}_{g-1}$ and $\bar{J}_g \cap \bar{A}_{g-1} = \bar{J}_{g-1}$. Moreover we show (in Sect. 3) that $\bar{S}_g \cap \bar{A}_{g-1} \subset \bar{S}_{g-1}$. However we do not intersect with the boundary $\bar{A}_{g-1}$ of $\bar{A}_g$ but with the boundary of A_g^*, the blow up of $\bar{A}_g$ along its boundary. This idea is already implicit in the paper of Schottky [S], §11, see also Frobenius [Fro], §14. For technical reasons we work on finite coverings of $\bar{A}_g$. In the last section we will prove some facts on theta relations and their consequences for the structure of $S(\Gamma_g)$.

Recently there has been much progress on the characterization of Jacobians by differential equations, i.e. by polynomials in the theta constants and their derivatives. In [A-DeC] Arbarello and DeConcini gave a set of such equations defining $\bar{J}_g$. Moreover, the Novikov Hypothesis which states that a theta function satisfying the Kadomtsev-Petviasvili (K-P) (differential) equation is the theta function of a Jacobian, was proved by T. Shiota (Harvard thesis). (The K-P equation is included in the system of [A-DeC].) Mulase has also found simple differential equations characterizing theta functions of Jacobians. The relation between differential equations and $S(\Gamma_g)$ is not yet clear.

Notations

(0.1) We write $\mathbb{H}_g$ for the Siegel upper halfplane:

$$\mathbb{H}_g = \{\tau\colon\ g \times g \text{ complex matrix, } {}^t\tau = \tau,\ \operatorname{Im}\tau > 0\}.$$

The group $\Gamma_g = Sp(2g, \mathbb{Z})$ acts on $\mathbb{H}_g$ by:

$$M \cdot \tau = (A\tau + B)(C\tau + D)^{-1}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g$ and $\tau \in \mathbb{H}_g$.

We define congruence subgroups of Γ_g:

$$\Gamma_g(n) = \left\{ M \in \Gamma_g\colon\ M \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{mod.}\, n \right\}$$

$$\Gamma_g(n, 2n) = \{M \in \Gamma_g(n)\colon\ \operatorname{diag}(A\,{}^tB) \equiv \operatorname{diag}(C\,{}^tD) \equiv 0 \bmod \cdot\, 2n\}.$$

We denote by $\bar{A}_{g,n}$ resp. $\bar{A}_{g,(n,2n)}$ the Satake compactification of $A_{g,n} = \Gamma_g(n)\backslash\mathbb{H}_g$ resp. $A_{g,(n,2n)} = \Gamma_g(n, 2n)\backslash\mathbb{H}_g$, cf. [Fr2]. The boundary $\bar{A}_{g,n} - A_{g,n}$ of $\bar{A}_{g,n}$ is a disjoint union of a finite number of copies of $A_{k,n}$, with $0 \leqq k \leqq g-1$, each of these is called a boundary component of $\bar{A}_{g,n}$.

(0.2) A Siegel modular form on a subgroup $\Gamma \subset \Gamma_g$, $g \geqq 2$, is a holomorphic function $f\colon \mathbb{H}_g \to \mathbb{C}$ which satisfies:

$$f(M \cdot \tau) = \det(C\tau + D)^k f(\tau)$$

for all $M \in \Gamma$ (with components A, B, C, D) and k is called the weight of f.

(0.3) We define the theta functions (with half integral characteristics):

$$\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(\tau, z)=\sum \exp \pi i\left[{}^t\left(m+\frac{\varepsilon}{2}\right)\tau\left(m+\frac{\varepsilon}{2}\right)+2\,{}^t\left(m+\frac{\varepsilon}{2}\right)\left(z+\frac{\varepsilon'}{2}\right)\right]$$

where we sum over $m\in\mathbb{Z}^g$, where $\tau\in\mathbb{H}_g$, $z\in\mathbb{C}^g$ and $\varepsilon, \varepsilon'\in\mathbb{Z}^g$ have all their components in $\{0, 1\}$, in fact it is often convenient to write $\varepsilon, \varepsilon'\in(\mathbb{Z}/2)^g$, and add the $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$ as elements of $(\mathbb{Z}/2)^{2g}$. Note that

$$\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(\tau, -z)=(-1)^{{}^t\varepsilon\varepsilon'}\,\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(\tau, z),$$

hence there are $2^{g-1}(2^g+1)$ even (i.e. ${}^t\varepsilon\varepsilon'=0$) and $2^{g-1}(2^g-1)$ odd theta functions.

(0.4) The theta constants $\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(\tau,0)$, $\theta^2\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(\tau,0)$ and $\theta^4\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(\tau,0)$ are Siegel modular forms of weight $\frac{1}{2}$, 1 and 2 resp. on the groups $\Gamma_g(4,8)$, $\Gamma_g(2,4)$ and $\Gamma_g(2)$ resp. (use [I2], V.1, Cor. to Th. 2).

(0.5) Let X_τ be the Abelian variety $\mathbb{C}^g/\langle I\,\tau\rangle$ where we divide out the lattice generated by the columns of the identity matrix and those of τ. Line bundles on X_τ can be described by automorphy factors (cf. [I2], Chap. II, [M3], Chap. 1). The automorphy factor

$$e(\tau, z)=\exp -\pi i({}^t\lambda'\tau\lambda'+2\,{}^t\lambda' z)$$

where $\lambda=\tau\lambda'+\lambda''\in\langle I\,\tau\rangle$, defines symmetric line bundle L_τ (i.e. $\iota^* L\cong L$, where $\iota: X\to X$ with $\iota(z)=-z$) on X_τ, and (X_τ, L_τ) is a principally polarized Abelian variety. The theta function $\theta\begin{bmatrix}0\\ 0\end{bmatrix}(\tau, z)$ corresponds to the (up to a scalar multiple unique) global section of L_τ. The other $2^{2g}-1$ symmetric line bundles on X_τ algebraically equivalent with L_τ, are defined by the automorphy factor $\chi\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(\lambda)\cdot e_\lambda(\tau, z)$, where:

$$\chi\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(\lambda)=\exp \pi i({}^t\lambda''\varepsilon-{}^t\lambda'\varepsilon').$$

The corresponding line bundles are denoted by $M\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$, $\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(\tau, z)$ corresponds to a global section of $M\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$.

(0.6) Let $\mathscr{C}$ be a curve (in this paper $\mathscr{C}$ is in fact a compact Riemann surface) and let D be a divisor on $\mathscr{C}$. Then we write $[D]$ for the linear equivalence class of D and we write $h^0([D])$ for $\dim H^0(\mathscr{C}, \mathcal{O}_{\mathscr{C}}(D))$.

(1.1) We recall some facts from the theory of Prym varieties. Let $\mathscr{C}$ be a curve of genus g, $g\geqq 1$, then $\mathscr{C}$ admits an étale covering of degree two π: $\tilde{\mathscr{C}}\to\mathscr{C}$, and the genus of $\tilde{\mathscr{C}}$ is $2g-1$. The Jacobian of $\tilde{\mathscr{C}}$ is isogeneous to a product $J(\mathscr{C})\times P$, where $J(\mathscr{C})$ is the Jacobian of $\mathscr{C}$ and P is a principally polarized Abelian variety of dimension $g-1$, the Prym variety.

(1.2) Let $\tau_g\in\mathbb{H}_g$ be a period matrix of $J(\mathscr{C})$. Then there exists an étale covering of degree two of $\mathscr{C}$ such that the Prym variety has a period matrix $\pi_{g-1}\in\mathbb{H}_{g-1}$ which satisfies the following equations for all $\varepsilon,\varepsilon'\in(\mathbb{Z}/2)^{g-1}$:

$$\lambda\cdot\theta^2\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(\pi_{g-1},0)=\theta\begin{bmatrix}\varepsilon & 0\\ \varepsilon' & 0\end{bmatrix}(\tau_g,0)\cdot\theta\begin{bmatrix}\varepsilon & 0\\ \varepsilon' & 1\end{bmatrix}(\tau_g,0),$$

where $\lambda\in\mathbb{C}$, $\lambda\neq 0$, is a constant independent of $\varepsilon,\varepsilon'\in(\mathbb{Z}/2)^{g-1}$. For proofs see: [S-J], [F-R], (also [R-F], Chap. VI). [Fa], form. (80) and [M2], p. 340.

(1.3) The theta constants $\theta^2\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(\pi,0)$, $\pi\in\mathbb{H}_{g-1}$, are modular forms on $\Gamma_{g-1}(2,4)$, and they define a map:

$$\theta_2\colon A_{g-1,(2,4)}=\Gamma_{g-1}(2,4)\backslash\mathbb{H}_{g-1}\to\mathbb{P}_N$$

given by: $\theta_2(\pi)=\left(\dots:\theta^2\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(\pi,0):\dots\right)$, where ${}^t\varepsilon\varepsilon'=0$, $N+1=2^{g-2}(2^{g-1}+1)$ and where we ordered the even theta characteristics in some fixed way. (Occasionally we make this ordering more explicit by writing $X_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]}$ for the coordinate functions on $\mathbb{P}_N$.)

The map θ_2 extends to a map of the Satake compactification $\bar{A}_{g-1,(2,4)}$ which we also denote by θ_2. The closure of $\theta_2(A_{g-1,(2,4)})$ in $\mathbb{P}_N$ is $\theta_2(\bar{A}_{g-1,(2,4)})$.

(1.4) Let $I_{g-1}\subset\mathbb{C}[X_0,\dots,X_N]$ be the ideal defining the projective variety $\theta_2(\bar{A}_{g-1,(2,4)})$. For homogeneous $F\in I_{g-1}$ we define $\sigma(F)\colon\mathbb{H}_g\to\mathbb{C}$ by:

$$\sigma(F)(\tau)=F\left(\dots,\theta\begin{bmatrix}\varepsilon & 0\\ \varepsilon' & 0\end{bmatrix}(\tau,0)\,\theta\begin{bmatrix}\varepsilon & 0\\ \varepsilon' & 1\end{bmatrix}(\tau,0),\dots\right)$$

where $\tau\in\mathbb{H}_g$, and we have substituted: $X_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]}=\theta\begin{bmatrix}\varepsilon & 0\\ \varepsilon' & 0\end{bmatrix}(\tau,0)\,\theta\begin{bmatrix}\varepsilon & 0\\ \varepsilon' & 1\end{bmatrix}(\tau,0)$. By (1.2), $\sigma(F)(\tau)=0$ if τ is the period matrix of a curve. As F is homogeneous and the theta constants are modular forms on $\Gamma_g(4,8)$, the $\sigma(F)$ are also modular forms on $\Gamma_g(4,8)$.

(1.5) Let $A(\Gamma_g(4,8))$ be the graded ring generated by the modular forms on $\Gamma_g(4,8)$. We define $S(\Gamma_g(4,8))\subset A(\Gamma_g(4,8))$ to be the ideal generated by the $\sigma(F)$, $F\in I_{g-1}$ and F homogeneous, and the conjugates of $\sigma(F)$ under the action of $\Gamma_g/\Gamma_g(4,8)$ on $A(\Gamma_g(4,8))$. Let $\bar{S}_{g,(4,8)}$ be the subset of $\bar{A}_{g,(4,8)}$ defined by $S(\Gamma_g(4,8))$. The canonical map $\bar{A}_{g,(4,8)}\to\bar{A}_g$ maps $\bar{S}_{g,(4,8)}$ onto a set $\bar{S}_g$. Obviously, $\bar{J}_g\subset\bar{S}_g$.

(1.6) **Theorem.** *Let* $g \geqq 2$. *Then* $\bar{J}_g$, *the closure of* J_g *in* $\bar{A}_g$, *is an irreducible component of* $\bar{S}_g$.

(1.7) *Remark.* We can define the ideal $S(\Gamma_g)$, mentioned in the introduction, to be the ideal of $A(\Gamma_g)$ generated by the elements of $S(\Gamma_g(4,8))$ invariant under the action of $\Gamma_g/\Gamma_g(4,8)$ on $S(\Gamma_g(4,8))$.

(1.8) *Remark.* For $g \leqq 3$ the theorem follows from (1.5), as $\bar{J}_g = \bar{S}_g = \bar{A}_g$ for $g = 2, 3$. This will give the starting point for the induction argument.

(1.9) For technical reasons it is easier to work on $\bar{A}_{g,8}$, the Satake compactification of $A_{g,8} = \Gamma_g(8) \backslash \mathbb{H}_g$. As $\Gamma_g(8) \subset \Gamma_g(4,8)$, the $\sigma(F)$ are also modular forms on $\Gamma_g(8)$. Let $\bar{S}_{g,8}$ be the subset of $\bar{A}_{g,8}$ which they define. (Obviously, $\bar{S}_{g,8}$ is the inverse image of $\bar{S}_g$ under the canonical map $\bar{A}_{g,8} \to \bar{A}_g$.)

(1.10) Let $A^*_{g,8}$ be the monoidal transform of $\bar{A}_{g,8}$ along its boundary, and let $\beta\colon A^*_{g,8} \to \bar{A}_{g,8}$ be the projection map. If $A_{g-1,8} \subset \bar{A}_{g,8}$ is a boundary component and $\pi \in A_{g-1,8}$ then $\beta^{-1}(\pi)$ is the Abelian variety $\mathbb{C}^{g-1}/8\langle I\,\pi\rangle$ ([I1], p. 249).

Let $S_{g,8}$ be the intersection of $A_{g,8}$ and $\bar{S}_{g,8}$ in $\bar{A}_{g,8}$ and let $S^*_{g,8}$ be the closure of $\beta^{-1}(S_{g,8})$ in $A^*_{g,8}$, i.e. $S^*_{g,8}$ is the strict transform of $\bar{S}_{g,8}$. The following proposition will be proved in Sect. 2:

(1.11) **Proposition.** *Let* $\mathscr{C}$ *be a hyperelliptic curve of genus* $g-1$, *and let* $\pi \in A_{g-1,8}$ *be a period matrix of* $\mathscr{C}$. *Then:*

$$\dim(S^*_{g,8} \cap \beta^{-1}(\pi)) \leqq 2.$$

(1.12) **Lemma.** *Theorem* (1.6) *follows from Proposition* (1.11).

(1.13) *Proof* (of Lemma (1.12)). Let $J_{g,8} \subset A_{g,8}$ be the inverse image of $J_g \subset A_g$ under the canonical map $A_{g,8} \to A_g$ (or, equivalently, $J_{g,8}$ is the image of the moduli space of curves of genus g with level eight structure in $A_{g,8}$), and let $J^*_{g,8}$ be the closure of $J_{g,8}$ in $A^*_{g,8}$. Let V^* be an irreducible component of $S^*_{g,8}$ which contains $J^*_{g,8}$. It suffices to show that $\dim V^* \leqq 3g-3$.

Let $V^* \cap \beta^{-1}(A_{g-1,8}) = W_1 \cup \ldots \cup W_n$, where the W_i are irreducible. As every point in $\beta^{-1}(A_{g-1,8})$ is smooth on $A^*_{g,8}$, [I1] Theorem 2, p. 246, and $\beta^{-1}(A_{g-1,8})$ is a divisor on $A^*_{g,8}$ we have $\dim W_i = \dim V^* - 1$ (cf. [M4], (3.28)). Hence it suffices to show:

$$\dim W_i \leqq 3g-4 \quad \text{for some } i.$$

As $J^*_{g,8} \subset V^*$ there is a k such that $J_{g-1,8} \subset \beta(W_k)$. From Proposition (1.14), to be proved in Sect. 3, and an induction hypothesis it follows that $J_{g-1,8} = \beta(W_k)$.

(1.14) **Proposition.** *Let* $\bar{A}_{g-1,8}$ *be the closure of a boundary component of* $\bar{A}_{g,8}$. *Then:*

$$\bar{S}_{g,8} \cap \bar{A}_{g-1,8} \subset \bar{S}_{g-1,8}.$$

(1.15) *Induction hypothesis.* The closure $\bar{J}_{g-1}$ of J_{g-1} in $\bar{A}_{g-1}$ is an irreducible component of $\bar{S}_{g-1}$.

(The hypothesis is true for $g=4$, cf. (1.8).)

(1.16) Using Proposition (1.11) to estimate the dimension of the general fiber of $W_k \to \beta(W_k)$, we find

$$\dim W_k \leqq (3(g-1)-3)+2=3g-4,$$

hence Lemma (1.12) is proved.

(1.17) *Remark.* The intersection of J_g^*, the closurê of J_g in A_g^*, with the boundary of A_g^*, was studied by Namikawa [N].

(2.0) In this section we prove Proposition (1.11). First we give a basic property of $A_{g,8}^*$.

(2.1) Let f be a Siegel modular form on $\Gamma_g(8)$, $g \geqq 2$, and let

$$(f)=\{\tau \in A_{g,8}\colon f(\tau)=0\}.$$

Let $\overline{(f)}$ be the closure of (f) in $\bar{A}_{g,8}$, and let $(f)^*$ be the closure of (f) in $A_{g,8}^*$, i.e. $(f)^*$ is the strict transform of $\overline{(f)}$.

(2.2) Let

$$f(\tau)=\sum_{k=k_0}^{\infty} \theta_k(\pi, z)\, \xi^k$$

where $\xi=\exp 2\pi i w/8$, $\theta_{k_0}(\pi, z) \not\equiv 0$ and

$$\tau=\begin{pmatrix} \pi & z \\ {}^t z & w \end{pmatrix}$$

where $\pi \in \mathbb{H}_{g-1}$, $w \in \mathbb{H}_1$, and $z \in \mathbb{C}^{g-1}$, be the Fourier-Jacobi series of f. The functions $z \to \theta_k(\pi, 8z)$ are classical theta functions of order $16k$, i.e. they correspond to global sections of the linĕ bundle L_π^{16k} on X_π.

(2.3) From Igusa's study of the blow up we have [I1]:

Let $\pi \in \overline{(f)} \cap A_{g-1,8}$, where $A_{g-1,8}$ is a boundary component of $\bar{A}_{g,8}$. Then the intersection of $(f)^*$ and the Abelian variety $\beta^{-1}(\pi) \cong \mathbb{C}^{g-1}/8\langle I\, \pi\rangle$ is the divisor of the theta function $\theta_{k_0}(\pi, z)$ on $\beta^{-1}(\pi)$.

(2.4) **Lemma.** *The Fourier-Jacobi series of a theta constant* $\theta\begin{bmatrix} \varepsilon & 0 \\ \varepsilon' & \delta \end{bmatrix}(\tau, 0)$, $\delta \in \mathbb{Z}/2$. $\varepsilon, \varepsilon' \in (\mathbb{Z}/2)^{g-1}$, ${}^t\varepsilon\,\varepsilon'=0$, *is given by:*

$$\theta\begin{bmatrix} \varepsilon & 0 \\ \varepsilon' & \delta \end{bmatrix}(\tau, 0)=\theta\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}(\pi, 0)+2\sum_{n=1}^{\infty}(-1)^{\delta n}\,\theta\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}(\pi, n z)\, \xi^{4n^2}.$$

Proof. Substituting for τ the matrix given above in the power series defining the theta constants we find:

$$\theta\begin{bmatrix}\varepsilon & 0\\ \varepsilon' & \delta\end{bmatrix}(\tau,0)=\sum_{(m,n)}\exp\pi i\left[{}^t\!\left(m+\frac{\varepsilon}{2}\right)\pi\left(m+\frac{\varepsilon}{2}\right)\right.$$
$$\left.+2\,{}^t\!\left(m+\frac{\varepsilon}{2}\right)n z+n^2 w+2\,{}^t\!\left(m+\frac{\varepsilon}{2}\right)\frac{\varepsilon'}{2}+n\delta\right]$$

where we sum over $(m,n)\in\mathbb{Z}^{g-1}\times\mathbb{Z}$. From this the lemma easily follows.

If $F\in I_{g-1}\subset\mathbb{C}[X_0,\dots,X_N]$, and F is homogeneous then the modular form $\sigma(F)$ was obtained by substituting:

$$X_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]}=\theta\begin{bmatrix}\varepsilon & 0\\ \varepsilon' & 0\end{bmatrix}(\tau,0)\,\theta\begin{bmatrix}\varepsilon & 0\\ \varepsilon' & 1\end{bmatrix}(\tau,0).$$

(2.5) **Lemma.** *Let $\pi\in\overline{(\sigma(F))}\cap A_{g-1,8}$, where $A_{g-1,8}$ is a boundary component of $\bar{A}_{g,8}$. Then the intersection $(\sigma(F))^*\cap\beta^{-1}(\pi)$ is contained in the divisor of the theta function θ_F:*

$$\theta_F(z)=\sum_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]}\frac{\partial F}{\partial X_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]}}\left(\dots,\theta^2\begin{bmatrix}\eta\\ \eta'\end{bmatrix}(\pi,0),\dots\right)\theta^2\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(\pi,z).$$

Proof. From Lemma (2.4) we have:

$$\theta\begin{bmatrix}\varepsilon & 0\\ \varepsilon' & 0\end{bmatrix}(\tau,0)\,\theta\begin{bmatrix}\varepsilon & 0\\ \varepsilon' & 1\end{bmatrix}(\tau,0)=\theta^2\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(\pi,0)-4\theta^2\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(\pi,z)\,\xi^8+O(\xi^{16}).$$

Substitute this in F and expand it in a power series in ξ. As $F\in I_{g-1}$ the constant term vanishes and the lemma follows by applying (2.3).

To find $S^*_{g,8}\cap\beta^{-1}(\pi)$ we determine the $\mathbb{C}$-vector space spanned by the θ_F.

(2.6) Let $\pi\in\mathbb{H}_{g-1}$ and let $\Gamma=\Gamma(X_\pi,L_\pi^{\otimes 2})$, i.e. Γ is the $\mathbb{C}$-vector space of theta functions with functional equation:

$$\theta(z+\lambda)=e_\lambda^2(\pi,z)\,\theta(z).$$

Then $\dim\Gamma=2^{g-1}$ and a basis for Γ is given by the 2^{g-1} theta functions ([I2], Chap. II):

$$\Theta[\sigma](\pi,z):=\theta\begin{bmatrix}\sigma\\ 0\end{bmatrix}(2\pi,2z)\qquad(\sigma\in(\mathbb{Z}/2)^{g-1}).$$

(2.7) **Lemma.** *The even theta functions $\theta^2\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(\pi,z)$ (i.e. ${}^t\varepsilon\,\varepsilon'=0$) span the vector space Γ.*

Proof. The relation between the $\theta^2\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$ and the basis functions $\Theta[\sigma]$ is given by ([I2], Chap. IV, Th. 2):

$$\theta^2\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(\tau,z)=\sum(-1)^{{}^t\sigma\varepsilon'}\,\Theta[\sigma+\varepsilon](\tau,0)\,\Theta[\sigma](\tau,z),$$

where we sum over all σ. As $\theta^2\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(\tau,z)$ is not identically zero as function of z, at least one of the $\Theta[\sigma](\tau,0)$ is not zero. Using this fact and the following formula, derived from the one above, the lemma follows:

$$\begin{aligned}&2^g(\Theta[\sigma+\varepsilon](\tau,0)\,\Theta[\sigma](\tau,z)+\Theta[\sigma](\tau,0)\,\Theta[\sigma+\varepsilon](\tau,z))\\ &\quad=2\sum(-1)^{{}^t\sigma\varepsilon'}\theta^2\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(\tau,z),\end{aligned}$$

where we sum over all ε' such that ${}^t\varepsilon\varepsilon'=0$, and we took $\tau\in\mathbb{H}_g$.

(2.9) Let

$$\Gamma_{00}=\left\{\theta\in\Gamma:\ \theta(0)=0,\ \frac{\partial^2\theta}{\partial z_i\,\partial z_j}(0)=0\ \text{ for all }\ 1\leqq i,j\leqq g-1\right\}.$$

Note that $\theta(z)=\theta(-z)$ for all $\theta\in\Gamma$, hence $\dfrac{\partial\theta}{\partial z_i}(0)=0$, all i.

(2.10) **Proposition.** Let $\pi\in\mathbb{H}_{g-1}$, and assume that $\theta_2(\pi)$ is a smooth point of $\theta_2(A_{g-1,(2,4)})$. Then the subspace of Γ spanned by the θ_F, $F\in I_{g-1}$, is Γ_{00}.

Proof. First we show that $\theta_F\in\Gamma_{00}$. As $F\in I_{g-1}$, substituting $z=0$ in θ_F and applying Euler's relation we find $\theta_F(\pi,0)=0$. Differentiating the function $F(\pi)=F\left(\dots,\theta^2\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(\pi,0),\dots\right)$, which is zero for all $\pi\in\mathbb{H}_{g-1}$, with respect to π_{ij} we find:

$$0=\frac{\partial F}{\partial\pi_{ij}}=\sum_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]}\frac{\partial F}{\partial X_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]}}\left(\dots,\theta^2\begin{bmatrix}\eta\\ \eta'\end{bmatrix}(\pi,0),\dots\right)\frac{\partial\theta^2\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}}{\partial\pi_{ij}}(\pi,0).$$

From the Heat equations:

$$2\pi i(1+\delta_{ij})\frac{\partial\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}}{\partial\pi_{ij}}(\pi,z)=\frac{\partial^2\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}}{\partial z_i\,\partial z_j}(\pi,z)$$

and ${}^t\varepsilon\varepsilon'=0$ we find:

$$2\pi i(1+\delta_{ij})\frac{\partial\theta^2\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}}{\partial\pi_{ij}}(\pi,0)=\frac{\partial^2\theta^2\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}}{\partial z_i\,\partial z_j}(\pi,0),$$

hence $\theta_F\in\Gamma_{00}$.

As $\theta_2(\pi)$ is a smooth point of $\theta_2(A_{g-1,(2,4)})$ we have in particular that the following $2^{g-2}(2^{g-1}+1)\times\frac{1}{2}g(g-1)+1$ matrix has maximal rank:

$$\left(\theta^2\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(\pi,0)\dots\frac{\partial\theta^2\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}}{\partial\pi_{ij}}(\pi,0)\dots\right).$$

Using the Heat equations again we find that $\dim\Gamma_{00}=\dim\Gamma-(1+\frac{1}{2}g(g-1))$.

On the other hand, if $\theta_2(\pi)$ is a smooth point of $\theta_2(A_{g-1,(2,4)})$, the linear forms:

$$\sum_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]} \frac{\partial F}{\partial X_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]}} \left(\dots, \theta^2 \begin{bmatrix}\eta\\ \eta'\end{bmatrix}(\pi, 0), \dots\right) X_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]}$$

where $F \in I_{g-1}$, F homogeneous, define the (projective) tangent space to $\theta_2(A_{g-1,(2,4)}) \subset \mathbb{P}_N$ which has (linear) dimension $1+\frac{1}{2}g(g-1)$. As the functions $\theta^2 \begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(\pi, z)$, ${}^t\varepsilon\,\varepsilon'=0$ span Γ the proposition follows.

To apply the proposition we need the following lemma, due to R. Sasaki [Sas]:

(2.11) **Lemma.** *Let $\tau \in \mathbb{H}_g$ be the period matrix of a hyperelliptic curve. Then $\theta_2(\tau)$ is a smooth point of $\theta_2(A_{g,(2,4)})$.*

We use the results obtained thus far to prove Proposition (1.11).

(2.12) *Proof* (of Proposition (1.11)). As we have seen:

$$S^*_{g,8} \cap \beta^{-1}(\pi) \subset \{z \in \beta^{-1}(\pi)\colon\ \theta(z)=0,\ \text{all}\ \theta \in \Gamma_{00}\},$$

where π is the period matrix of a hyperelliptic curve $\mathscr{C}$. The projection map

$$\beta^{-1}(\pi) = \mathbb{C}^{g-1}/8\langle I\,\pi\rangle \to \mathbb{C}^{g-1}/\langle I\,\pi\rangle = J(\mathscr{C})$$

is finite, hence it suffices to show:

$$\dim\{z \in J(\mathscr{C})\colon\ \theta(z)=0,\ \text{all}\ \theta \in \Gamma_{00}\} \leqq 2.$$

Note that $J(\mathscr{C}) = \mathrm{Pic}^0(\mathscr{C})$, where $\mathrm{Pic}^0(\mathscr{C})$ is the group of divisor classes of degree zero modulo linear equivalence. For convenience sake we let the genus of $\mathscr{C}$ be equal to g. Let

$$\Theta = \{\alpha \in \mathrm{Pic}^{g-1}(\mathscr{C})\colon\ h^0(\alpha) \geqq 1\}$$

where $\mathrm{Pic}^{g-1}(\mathscr{C})$ is the algebraic variety isomorphic to $J(\mathscr{C})$ which parametrizes divisor classes of degree $g-1$ modulo linear equivalence. Then Θ is a divisor on $\mathrm{Pic}^{g-1}(\mathscr{C})$, and its singular points are:

$$\mathrm{Sing}\,\Theta = \{\alpha \in \mathrm{Pic}^{g-1}(\mathscr{C})\colon\ h^0(\alpha) \geqq 2\}.$$

For $\alpha \in \mathrm{Pic}^{g-1}(\mathscr{C})$ we define:

$$\Theta_\alpha = \{z \in J(\mathscr{C}) = \mathrm{Pic}^0(\mathscr{C})\colon\ h^0(z+\alpha) \geqq 1\},$$

then Θ_α is the zero locus of the theta function $\theta \begin{bmatrix}0\\ 0\end{bmatrix}(\pi, z-u_\alpha)$, for some $u_\alpha \in \mathbb{C}^g$. One can verify that the zero locus of $f_\alpha(z) = \theta \begin{bmatrix}0\\ 0\end{bmatrix}(\pi, z-u_\alpha)\,\theta \begin{bmatrix}0\\ 0\end{bmatrix}(\pi, z+u_\alpha)$ is $\Theta_\alpha \cup \Theta_{K-\alpha}$, where K is the canonical divisor class of $\mathscr{C}$, and that $f_\alpha \in \Gamma$. Moreover, if $\alpha\in\operatorname{Sing}\Theta$, then $f_\alpha\in\Gamma_{00}$, and $f_\alpha(z)=0$ if and only if: $h^0(z+\alpha)\geqq 1$ or $h^0(z+K-\alpha)=h^0(-z+\alpha)\geqq 1$.

We conclude that the following lemma proves Proposition (1.11):

(2.13) **Lemma.** *Let $\mathscr{C}$ be a hyperelliptic curve of genus g. Let $z\in J(\mathscr{C})$ be a divisor class of degree zero and assume:*

$$f_\alpha(z)=0 \qquad \text{for all } \alpha\in\operatorname{Sing}\Theta.$$

Then $z=[P-Q]$ for some $P,Q\in\mathscr{C}$.

Proof (of Lemma (2.13)). For $g\leqq 2$ there is nothing to prove, so assume $g\geqq 3$. Let $[D]\in\operatorname{Pic}^0(\mathscr{C})$ and write $\deg[D]=[D_+-D_-]$, where we choose D_+,D_- effective and such that $\deg D_+=\deg D_-$ is minimal. It suffices to show:

If $\deg D_+\geqq 2$, then there is an $\alpha\in\operatorname{Sing}\Theta$ such that $h^0([D]+\alpha)=h^0(-[D]+\alpha)=0$.

Let h be the divisor class of degree 2 on $\mathscr{C}$ with $h^0(h)=2$. Assume $\deg D_+=r\geqq 2$, then $h^0(h+[D])=0$, because else $h+[D]=[P+Q]$ for some $P,Q\in\mathscr{C}$. As $h=[Q+Q']$ for a $Q'\in\mathscr{C}$ we find $[D]=[P-Q']$, hence $\deg D_+=1$ in contradiction with our assumption.

By Riemann-Roch: $h^0(K-h-[D])=g-3$. So if $g=3$ take $\alpha=h$. For $g>3$ there is a Zariski open subset $U\subset\mathscr{C}^{(g-3)}$ such that for $E\in U$ we have: $h^0(K-(h+[D+E]))=0$. The same argument for $-D$ gives an $U'\subset\mathscr{C}^{(g-3)}$ such that for $E'\in U'$ we have $h^0(K-(h+[-D+E']))=0$, hence for $E\in U\cap U'$ we find: $\alpha=h+[E]\in\operatorname{Sing}\Theta$, $h^0(K-\alpha-[D])=h^0(\alpha+[D])=0$ and $h^0(\alpha-[D])=0$ which proves the lemma.

(2.14) *Remark.* G. Welters [We] recently generalized (2.13) to non hyperelliptic curves.

(3.0) In this section we prove Proposition (1.14). The technical result which we need is Lemma (3.8), to prove it we recall some well known facts.

(3.1) Let X be an Abelian variety of dimension g. A symmetric sheaf on X is called totally symmetric if e_*^L, the function on the two torsion points of X with values in $\{-1,1\}$ associated to L, [M1] §2, is trivial. Let L be a totally symmetric sheaf with $\dim H^0(X,L)=2^g$. The theta group of L, $G(L)$, is isomorphic to the Heisenberg group $H(2)$, which, as a set, is just $\mathbb{C}^*\times(\mathbb{Z}/2)^g\times\operatorname{Hom}(\mathbb{Z}/2,\mathbb{C}^*)^g$ with multiplication:

$$(t,x,x^*)\cdot(s,y,y^*)=(t\,s\,y^*(x),x+y,x^*+y^*).$$

A theta structure is an isomorphism α: $G(L)\to H(2)$, which is the identity on the center $\mathbb{C}^*$ of both groups (in this case any theta structure is symmetric). With the definitions $A_{g,(2,4)}$ is the moduli space of triples (X,L,α).

The group $G(L)$, resp. $H(2)$, has a linear representation U', resp. U, on $H^0(X,L)$ resp. on the $\mathbb{C}$ vector space of functions $(\mathbb{Z}/2)^g\to\mathbb{C}$, i.e. on $\mathbb{C}^{M+1}$ where $M+1=2^g$. Given a theta structure α, we get equivalent irreducible representations $U'\circ\alpha^{-1}$ and U of $H(2)$, hence by Schur's Lemma α defines an, up to scalar multiple unique, isomorphism ϕ_α: $H^0(X,L)\to\mathbb{C}^{M+1}$. In particular, it defines a canonical map Φ_α: $X\to\mathbb{P}_M$, which gives a map:

$$\Theta\colon A_{g,(2,4)}\to\mathbb{P}_M \quad\text{by}\quad \Theta\colon (X,L,\alpha)\to\Phi_\alpha(0),$$

where $0\in X$ is the identity element. We always take $L=L_\tau^{\otimes 2}$, and the map Θ is defined by the theta constants, cf. (2.6):

$$\Theta\colon \Gamma_g(2,4)\backslash\mathbb{H}_g\to\mathbb{P}_M, \qquad \Theta(\tau)=(\ldots\colon \Theta[\sigma](\tau,0)\colon\ldots).$$

(3.2) Let G be the group of automorphisms of $H(2)$ inducing the identity on the center $\mathbb{C}^*$ of $H(2)$. Then obviously G acts transitively and faithfully on the set of symmetric theta structures by $g\cdot\alpha{:}=g\circ\alpha$. Hence G acts on $A_{g,(2,4)}$ by $g\cdot(X,L,\alpha)=(X,L,g\cdot\alpha)$.

The irreducible representations U and $U\circ g$ of $H(2)$ ($g\in G$) are equivalent, hence by Schur's Lemma there is a $\rho(g)\in Gl(M+1,\mathbb{C})$ (unique upto scalar multiple) such that:

$$\rho(g)\,U(x)\,\rho(g)^{-1}=U(g\cdot x) \qquad \text{for all } x\in H(2).$$

The map ρ: $G\to Gl(M+1,\mathbb{C})$ defines a projective representation of G and, up to scalar multiple, we have: $\rho(g)\,\Phi_\alpha=\Phi_{g\cdot\alpha}$, for all theta structures α. In particular, $\rho(g)\,\Theta(X,L,\alpha)=\Theta(X,L,g\cdot\alpha)$, so Θ is a G equivariant map.

According to [G] or [W] there is an exact sequence:

$$0\to\mathbb{F}_2^{2g}\to G\to Sp(2g,\mathbb{F}_2)\to 1$$

where $\mathbb{F}_2^{2g}\simeq H(2)/\mathbb{C}^*$ is the group of inner automorphisms of $H(2)$. The elements $\rho(g)$, for $g\in\mathbb{F}_2^{2g}-\{0\}\subset G$ have, in $PGl(M+1,\mathbb{C})$, order two, hence they have two eigenspaces in $\mathbb{P}_M$.

(3.3) **Lemma.** *The* $2(2^{2g}-1)$ *eigenspaces of the projective transformations* $\rho(g)$, *g as above, are permuted transitively by the projective transformations* $\rho(h)$, $h\in G$.

Proof. As $\mathbb{F}_2^{2g}$ is a normal subgroup of G it is obvious that the eigenspaces of the $\rho(g)$ are permuted by the $\rho(h)$. If $\tilde g=(t,x,x^*)\in H(2)$ is such that $\tilde g \bmod \mathbb{C}^* =g\in H(2)/\mathbb{C}^*$ then we can take $\rho(g)=U(t,x,x^*)$. Let $\tilde h=(s,y,y^*)\in H(2)$ with $y^*(x)\cdot x^*(y)=-1$, then $\rho(h)$ permutes the two eigenspaces of $\rho(g)$. The group $G/\mathbb{F}_2^{2g}\simeq Sp(2g,\mathbb{F}_2)$ acts as group of automorphisms on $H(2)/\mathbb{C}\simeq\mathbb{F}_2^{2g}$. If $x,y\in\mathbb{F}_2^{2g}-\{0\}$, there is a $\bar h\in Sp(2g,\mathbb{F}_2)$ such that $\bar h x=y$. The images of the eigenspaces of $\rho(x)$ under $\rho(h)$ are the eigenspaces of $\rho(y)$. This proves the lemma.

(3.4) **Lemma.** *The Satake compactification* $\bar A_{g,(2,4)}$ *of* $A_{g,(2,4)}$ *has* $2(2^{2g}-1)$ *boundary components which are isomorphic to* $A_{g-1,(2,4)}$.

Proof. The proof is a computation similar to the example in [Sat].

(3.5) **Lemma.** *Let* $V\subset\mathbb{P}_M$ *be an eigenspace of* $\rho(g)$, *with* $g\in\mathbb{F}_2^{2g}-\{0\}\subset G$. *Then:*

$$V\cap\Theta(\bar A_{g,(2,4)})=\Theta(\bar A_{g-1,(2,4)}).$$

Proof. The image of a boundary component $A_{g-1,(2,4)}$ of $\bar{A}_{g,(2,4)}$ can be obtained as the image of the map Θ':

$$\Theta'\colon A_{g-1,(2,4)} \to \mathbb{P}_M, \qquad \Theta'(\tau_{g-1}) = \lim \Theta \begin{pmatrix} it & 0 \\ 0 & \tau_{g-1} \end{pmatrix},$$

where we take the limit of $t \to \infty$, $t \in \mathbb{R}$. From the definition of $\Theta[\sigma]$ it follows that $\Theta[\sigma](\tau,0) = \Theta[\sigma_1](it,0)\,\Theta[\sigma_{g-1}](\tau_{g-1},0)$, where τ is as above and $\sigma = (\sigma_1, \sigma_{g-1}) \in \mathbb{Z}/2 \times (\mathbb{Z}/2)^{g-1}$. It is easy to see that $\lim \Theta[0](it,0) = 1$ and $\lim \Theta[1](it,0) = 0$, hence $\Theta'(A_{g-1,(2,4)})$ is contained in the linear subspace $V \subset \mathbb{P}_M$ defined by $X_{[1\,\sigma_{g-1}]} = 0$ for all $\sigma_{g-1} \in (\mathbb{Z}/2)^{g-1}$. Note that V is an eigenspace of the element $\rho(g) = U(t, 0, x^*)$, where $x^*(\sigma_1, \rho_{g-1}) = (-1)^{\sigma_1}$. Moreover, the map Θ' is just the canonical map Θ on $A_{g-1,(2,4)}$ to $\mathbb{P}_{M'} \simeq V (M' = 2^{g-1} - 1)$.

According to formula (1) below, $\Theta[1\,\sigma_{g-1}](\tau,0) = 0$ (all σ) implies $\theta\begin{bmatrix} 1 & \varepsilon \\ \alpha & \varepsilon' \end{bmatrix}(\tau,0) = 0$, for all $\alpha \in \mathbb{Z}/2$, all $\varepsilon, \varepsilon' \in (\mathbb{Z}/2)^{g-1}$. Hence by formula (2): $\Theta[\rho](\tau,z)\,\Theta[\rho + (1,\varepsilon)](\tau,z) = 0$ for all $z \in \mathbb{C}^g$. As the $\Theta(\sigma)$ are not identically zero, this is impossible, hence $V \cap \Theta(A_{g,(2,4)}) = \emptyset$ and $V \cap \Theta(\bar{A}_{g,(2,4)})$ consists of boundary points.

The group G acts transitively on the eigenspaces and Θ is G equivariant. As each eigenspace is spanned by a boundary component the lemma is proven.

(3.6) We return to the map $\theta_2\colon A_{g,(2,4)} \to \mathbb{P}_N$. To apply Lemma (3.5) we need the following (equivalent) formulas ([R-F], Cor. IIA 2.3 or [I2], VI.1, Th. 2):

$$(1) \quad \theta\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}(\tau,0)\,\theta\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}(\tau,2z) = \sum_{\sigma} (-1)^{{}^t\sigma\varepsilon'}\,\Theta[\sigma](\tau,z)\,\Theta[\sigma+\varepsilon](\tau,z),$$

$$(2) \quad 2^g\,\Theta[\rho](\tau,z)\,\Theta[\rho+\varepsilon](\tau,z) = \sum_{\varepsilon'} (-1)^{{}^t\rho\varepsilon'}\,\theta\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}(\tau,0)\,\theta\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}(\tau,2z)$$

where we sum over all σ resp. all ε' in $(\mathbb{Z}/2)^g$. (Note that $\theta\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}(\tau,0) = 0$ if ${}^t\varepsilon\,\varepsilon' \neq 0$. The equalities are between global sections of $2^*M\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} \simeq L \otimes L (L \simeq L_\tau^{\otimes 2})$.)

These formulas show that the following diagram is commutative (substitute $z=0$ in (1)):

$$\begin{tikzcd} \bar{A}_{g,(2,4)} \arrow[r, "\Theta"] \arrow[dr, "\theta_2"'] & \mathbb{P}_M \arrow[d, "\Psi"] \\ & \mathbb{P}_N \end{tikzcd}$$

where Ψ is the second Veronese map, i.e. the canonical map $\mathbb{P}_M \to \mathbb{P}H^0(\mathbb{P}_M, \mathcal{O}_{\mathbb{P}_M}(2))$, with respect to the basis of quadratic polynomials given by formula (1).

(3.7) **Definition.** Let $W \subset \mathbb{P}_N$ be the linear subspace defined by:

$$X_{\left[\begin{smallmatrix}1 & \varepsilon\\ 1 & \varepsilon'\end{smallmatrix}\right]}=0 \quad \text{for all } \varepsilon, \varepsilon' \ (\mathbb{Z}/2)^{g-1} \text{ with } {}^t\varepsilon\varepsilon'=1,$$

$$X_{\left[\begin{smallmatrix}1 & \varepsilon\\ 0 & \varepsilon'\end{smallmatrix}\right]}=0 \quad \text{and } X_{\left[\begin{smallmatrix}0 & \varepsilon\\ 0 & \varepsilon'\end{smallmatrix}\right]}=X_{\left[\begin{smallmatrix}0 & \varepsilon\\ 1 & \varepsilon'\end{smallmatrix}\right]} \text{ all } \varepsilon, \varepsilon' \text{ with } {}^t\varepsilon\varepsilon'=0.$$

(3.8) **Lemma.** *$W \cap \theta_2(\bar{A}_{g,(2,4)})=\theta_2(\bar{A}_{g-1,(2,4)})$, where $\bar{A}_{g-1,(2,4)}$ is the Satake compactification of a boundary component $A_{g-1,(2,4)}$ of $\bar{A}_{g,(2,4)}$.*

Proof. According to Lemma (3.5) and its proof, the subspace $V \subset \mathbb{P}_M$ defined by $X_{[1\ \sigma_{g-1}]}=0$ for all $\sigma_{g-1} \in (\mathbb{Z}/2)^{g-1}$, intersects $(\bar{A}_{g,(2,4)})$ in the closure of a boundary component. Using formulas (1) and (2) the lemma follows.

By the definition of $\bar{S}_{g,8}$, the following lemma implies Proposition (1.14):

(3.9) **Lemma.** $\bar{S}_{g,(4,8)} \cap \bar{A}_{g-1,(4,8)} \subset \bar{S}_{g-1,(4,8)}$.

Proof. To obtain a suitable set of equations defining $\bar{S}_{g-1,(4,8)} \subset \bar{A}_{g-1,(4,8)}$ we use Lemma (3.8) with g replaced by $g-1$. Let $I'_{g-2} \subset \mathbb{C}[X_0, \ldots, X_N]$ $(N+1 = 2^{g-2}(2^{g-1}+1))$ be the ideal generated by I_{g-1} (the ideal defining $\theta_2(\bar{A}_{g-1,(2,4)})$ in $\mathbb{P}_N$) and the ideal of $W \subset \mathbb{P}_N$. Then I'_{g-2} defines, set theoretically, the variety $\theta_2(\bar{A}_{g-2,(2,4)}) \subset W \subset \mathbb{P}_N$.

In particular, if we substitute:

$$X_{\left[\begin{smallmatrix}1 & \varepsilon\\ 1 & \varepsilon'\end{smallmatrix}\right]}=X_{\left[\begin{smallmatrix}1 & \varepsilon\\ 0 & \varepsilon'\end{smallmatrix}\right]}=0$$

$$X_{\left[\begin{smallmatrix}0 & \varepsilon\\ 0 & \varepsilon'\end{smallmatrix}\right]}=X_{\left[\begin{smallmatrix}0 & \varepsilon\\ 1 & \varepsilon'\end{smallmatrix}\right]}=\theta\begin{bmatrix}0 & \varepsilon\\ 0 & \varepsilon'\end{bmatrix}(\tau_{g-1}, 0)\,\theta\begin{bmatrix}0 & \varepsilon\\ 1 & \varepsilon'\end{bmatrix}(\tau_{g-1}, 0)$$

in the polynomials $F \in I'_{g-2}$ we obtain a set of Siegel modular forms on $\Gamma_{g-1,(4,8)}$ which, together with their conjugates under Γ_{g-1}, define the Schottky locus $\bar{S}_{g-1,(4,8)}$ in $\bar{A}_{g-1,(4,8)}$.

Let $\tau_{g-1} \in \bar{S}_{g,(4,8)} \cap \bar{A}_{g-1,(4,8)}$, then

$$\lim \sigma(F)(\tau)=\lim F\left(\ldots, \theta\begin{bmatrix}\varepsilon & 0\\ \varepsilon' & 0\end{bmatrix}(\tau, 0)\,\theta\begin{bmatrix}\varepsilon & 0\\ \varepsilon' & 1\end{bmatrix}(\tau, 0), \ldots\right)=0,$$

for all $F \in I_{g-1}$ and $\tau \in \mathbb{H}_g$ is as in the proof of (3.5). As $\lim \theta\begin{bmatrix}\varepsilon & 0\\ \varepsilon' & 0\end{bmatrix}(\tau, 0)=0$ if $\varepsilon'_1=1$ and $\lim \theta\begin{bmatrix}0 & \eta & 0\\ \alpha & \eta' & \delta\end{bmatrix}(\tau, 0)=\theta\begin{bmatrix}\eta & 0\\ \eta' & \delta\end{bmatrix}(\tau_{g-1}, 0)$ $(\alpha, \delta \in \mathbb{Z}/2,\ \eta, \eta' \in (\mathbb{Z}/2)^{g-2})$ we find that $\lim \sigma(F)$ is one of the Siegel modular forms on $\Gamma_{g-1}(4,8)$ defined above. Using that $\Gamma_{g-1}(4,8) \subset \Gamma_g(4,8)$ we conclude that τ_{g-1} in fact satisfies all equations for $\bar{S}_{g-1,(4,8)}$ given above, which proves the lemma.

(4.0) In this section we study the ideal I_g of relations between the Siegel modular forms $\theta^2\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$. We give a description of all quadratic relations and we show that the Siegel modular form $\sigma(F)$ is identically zero if F is such a quadratic relation.

The only other relations which are explicitly known are obtained from rationalizing quartic relations between the $\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$. It is well known that not all $\sigma(F)$ obtained from these relations are identically zero, in fact Schottky's relation is of this type. We describe some of these quartic relations. R. Salvati Manni has recently shown that we have in fact found all the quartic relations between the $\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$'s. It is not known whether the ideal of relations between the $\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$ is generated by quartic relations.

(4.1) We use the action of $\Gamma_g/\Gamma_g(2,4)$ on the $\theta^2\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(\tau):=\theta^2\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(\tau,0)$ to find suitable generators for the quadratic relations. Let

$$G(\tau)=\sum\lambda_i\theta^2\begin{bmatrix}\varepsilon_i\\ \varepsilon'_i\end{bmatrix}(\tau)\,\theta^2\begin{bmatrix}\eta_i\\ \eta'_i\end{bmatrix}(\tau)$$

where $\lambda_i\in\mathbb{C}$, be a quadratic relation (i.e. $G(\tau)=0$ for all $\tau\in\mathbb{H}_g$). Then we can write:

$$G=\sum G_{\binom{\sigma}{\sigma'}}$$

where we sum over all $\binom{\sigma}{\sigma'}\in(\mathbb{Z}/2)^{2g}$ and

$$G_{\binom{\sigma}{\sigma'}}=\sum\lambda_i\theta^2\begin{bmatrix}\varepsilon_i\\ \varepsilon'_i\end{bmatrix}\theta^2\begin{bmatrix}\eta_i\\ \eta'_i\end{bmatrix}$$

where we sum over those i's for which:

$$\begin{pmatrix}\varepsilon_i+\eta_i\\ \varepsilon'_i+\eta'_i\end{pmatrix}=\begin{pmatrix}\sigma\\ \sigma'\end{pmatrix}.$$

(4.2) **Lemma.** *Let $G=\sum G_{\binom{\sigma}{\sigma'}}$ be a quadratic relation. Then $G_{\binom{\sigma}{\sigma'}}$ is a quadratic relation.*

Proof. Let $x\in\mathbb{Z}^g$, with components $x_i\in\{0,2\}$, let

$$M(x)=\begin{pmatrix}I & B(x)\\ 0 & I\end{pmatrix}$$

be the $2g\times 2g$ matrix with $B(x)_{ij}=0$ if $i\neq j$ and $B(x)_{ii}=x_i$. Then $M(x)\in Sp(2g,\mathbb{Z})$ and

$$\theta^2\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(M(x)\,\tau)=(-1)^{{}^t x\varepsilon}\cdot c\cdot\theta^2\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(\tau)$$

where c is independent of $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$ ([I2], p. 176, p. 49). Replacing $M(x)$ by ${}^tM(x)$ we find $(-1)^{{}^t x\varepsilon'}$ in stead of $(-1)^{{}^t x\varepsilon}$. From this the lemma easily follows.

(4.3) **Lemma.** *Let* $G_{\binom{\sigma}{\sigma'}}$ *be a quadratic relation and assume that* $\binom{\sigma}{\sigma'} \neq 0$. *Then there exists an* $M \in \Gamma_g$ *such that:*

$$G_{\binom{\sigma}{\sigma'}}(M\tau) = G_\beta(\tau).$$

where G_β *is a quadratic relation with* $\beta = \binom{\beta'}{\beta''}$, $\beta' = 0$ *and* $\beta'' = (1, 0, \ldots, 0) \in (\mathbb{Z}/2)^g$.

Proof. The group Γ_g acts non-linearly on the $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$ but it acts linearly and transitively on the $\binom{\sigma}{\sigma'} \in (\mathbb{Z}/2)^{2g} - \{0\}$, in fact $\Gamma_g/\Gamma_g(2) \simeq Sp(2g, \mathbb{F}_2)$ acts. Choosing $M^{-1} \in \Gamma_g$ such that $M^{-1}\binom{\sigma}{\sigma'} = \beta$ and using the transformation formula again ([I2], p. 176), the lemma is proved.

As a consequence of the lemmas we only have to study the quadratic relations of type G_α and G_β, where $\alpha = 0$ and β is as in Lemma (4.3). First we study the G_α's. To obtain relations of this type we use formula (1), (3.6).

(4.4) *Example.* Let $g = 1$. From formula (1) we have:

$$\theta^2\begin{bmatrix}0\\0\end{bmatrix} = \Theta^2[0] + \Theta^2[1], \quad \theta^2\begin{bmatrix}1\\0\end{bmatrix} = 2\Theta[0]\,\Theta[1] \quad \text{and}$$

$$\theta^2\begin{bmatrix}0\\1\end{bmatrix} = \Theta^2[0] - \Theta^2[1].$$

Hence we find the relation:

$$\theta^4\begin{bmatrix}0\\0\end{bmatrix}(\tau) - \theta^4\begin{bmatrix}1\\0\end{bmatrix}(\tau) - \theta^4\begin{bmatrix}0\\1\end{bmatrix}(\tau) = 0.$$

(4.5) *Definition.* Let T_g be the $\mathbb{C}$-vector space of Siegel modular forms of weight two on $\Gamma_g(2)$ spanned by the $\theta^4\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$.

Using only formula (1) we get the estimate:

(4.6) **Lemma.** $\dim T_g \leqq (2^g + 1)(2^{g-1} + 1)/3$.

Proof. Using formula (1) it is sufficient to show that the dimension of the vector space spanned by the homogeneous polynomials $P^2_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]}$ of degree 4 in 2^g variables X_σ:

$$P_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]} = \sum_\sigma (-1)^{{}^t\sigma\varepsilon'} X_\sigma X_{\sigma+\varepsilon} \qquad (\varepsilon, \varepsilon', \sigma \in (\mathbb{Z}/2)^g,\ {}^t\varepsilon\,\varepsilon' = 0)$$

is less then or equal to $(2^g + 1)(2^{g-1} + 1)/3$.

Let H be the subgroup of the Heisenberg group $H(2)$:

$$H = \{(t, x, x^*) \in H(2) :\ t^4 = 1\}.$$

The group H has a linear representation on $\mathbb{C}[\ldots, X_\sigma, \ldots]$, where $\sigma \in (\mathbb{Z}/2)^g$, defined by:

$$R(t, x, x^*) X_\sigma = t \cdot x^*(\sigma + x) \cdot X_{\sigma + x}.$$

An easy computation shows:

$$R(t, x, x^*) P_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]} = t^2 \cdot (-1)^{{}^t x \varepsilon'} \cdot x^*(\varepsilon) \cdot P_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]}.$$

In particular, the $P^2_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]}$'s are invariant under the action of H. We define:

$$P_0 = \sum_\sigma X_\sigma^2$$

$$P_\rho = \sum_\sigma X_\sigma^2 X_{\sigma+\rho}^2 \qquad \text{for } \rho \in (\mathbb{Z}/2)^g - 0,$$

$$P_T = \sum_\sigma X_\sigma X_{\sigma+\rho} X_{\sigma+\tau} X_{\sigma+\rho+\tau} \quad \text{for } T = \{0, \rho, \tau, \rho+\tau\}$$

with a group $T \simeq (\mathbb{Z}/2)^2 \subset (\mathbb{Z}/2)^g$, and we sum over $\sigma \in (\mathbb{Z}/2)^g$. It is easy to see that these polynomials are linearly independent, that they are invariant under the action of H and that they span a vector space of dimension

$$1 + (2^g - 1) + (2^g - 1)(2^{g-1} - 1)/3 = (2^g + 1)(2^{g-1} + 1)/3.$$

Conversely, if a homogeneous polynomial of degree 4 is invariant under H, then each of its monomials is invariant under the subgroup K of H:

$$K = \{(t, x, x^*) \in H : \; x = 0\}.$$

The action of H on a K-invariant monomial will give one of the polynomials above. From this it easily follows that they are a basis of the H-invariant polynomials of degree 4, hence the lemma is proved.

(4.7) *Remark.* It is not difficult to show that we found a basis of the space of invariant polynomials of degree 4, we find this result later in another way. The group H acts on the $\mathbb{C}$-vector space of homogeneous polynomials of degree 4, W, through its Abelian quotient $(\mathbb{Z}/2)^{2g}$. Hence $W = \oplus W_\chi$ where we sum over the characters of $(\mathbb{Z}/2)^{2g}$. One can show: $\dim W_\chi = (2^{g-1} + 1)(2^{g-2} + 1)/3$ for all non trivial χ (we also find this result later).

(4.8) The group $K = Sp(2g, \mathbb{F}_2) \simeq \Gamma_g/\Gamma_g(2)$ has a linear representation $\tilde{\mu}$ on the $\mathbb{C}$ vector space of Siegel modular forms weight 2 on $\Gamma_g(2)$ defined by:

$$(\tilde{\mu}(k^{-1}) f)(\tau) = \det(C\tau + D)^{-2} f((A\tau + B)(C\tau + D)^{-1})$$

where f is such a Siegel modular form and the matrix with components A, B, C and D is a representative in Γ_g of k. Using the transformation formula for theta functions one finds that T_g is an invariant subspace. Let μ be the representation of K on T_g obtained by restricting $\tilde{\mu}$.

Let $Q: \mathbb{F}_2^{2g} \to \mathbb{F}_2$ be a quadratic form with maximal index, (i.e. Q is zero on a linear subspace of dimension g) and whose associated bilinear form is the symplectic form for K (note: char$\cdot(\mathbb{F}_2)=2$). Let $O^+(2g,\mathbb{F}_2)$ be the orthogonal group of Q. This group has only one nontrivial one dimensional representation:

$$\varepsilon\colon O^+(2g,\mathbb{F}_2)\to\{-1,1\}.$$

Let $\operatorname{Ind}_O^K(\varepsilon)$ be the induced representation, according to [Fra] this representation is reducible:

$$\operatorname{Ind}_{O^+}^K(\varepsilon)=\nu\oplus\rho$$

where ν and ρ are irreducible representations of K, and $\dim\nu=(2^g+1)(2^{g-1}+1)/3$, $\dim\rho=(2^g+1)(2^g-1)/3$ and $\dim\nu\oplus\rho=2^{g-1}(2^g+1)$.

(4.9) **Proposition.** *The representation μ of K on T_g is equivalent to the irreducible representation ν.*

(4.10) **Corollary.** $\dim T_g=(2^g+1)(2^{g-1}+1)/3$. *In particular, all linear relations between the $\theta^4\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$'s are consequences of formula* (1).

(4.11) **Corollary.** *The representation of K on the $\mathbb{C}$ vector space of linear relations between the $\theta^4\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$'s is equivalent with the irreducible representation ρ. In particular, the conjugates of any linear relation span the vector space of linear relations.*

For practical purposes it is often convenient to have explicit relations with only a few nonzero coefficients. It is known that for $g=1,2,3$ resp. 4 the minimal number of nonzero coefficients is 3, 4, 6 resp. 10. Hence the following corollary can be used to derive relations with $10\cdot 2^{g-4}$ terms for $g\geqq 5$:

(4.12) **Corollary.** $\sum\lambda_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]}\theta^4\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(\tau_g)=0$ *for all $\tau_g\in\mathbb{H}_g$ if and only if:*

$$\sum\lambda_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]}\left(\theta^4\begin{bmatrix}0&\varepsilon\\0&\varepsilon'\end{bmatrix}(\tau_{g+1})+\theta^4\begin{bmatrix}0&\varepsilon\\1&\varepsilon'\end{bmatrix}(\tau_{g+1})\right)=0$$

for all $\tau_{g+1}\in\mathbb{H}_{g+1}$, where $\lambda_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]}\in\mathbb{C}$.

Proof (of Corollary (4.12)). Using formula (1) we find:

$$\begin{aligned}\theta^2\begin{bmatrix}0&\varepsilon\\ \delta&\varepsilon'\end{bmatrix}(\tau)&=(\textstyle\sum(-1)^{{}^t\sigma\varepsilon'}\Theta[0\ \ \sigma](\tau)\Theta[0\ \ \sigma+\varepsilon](\tau))\\&\quad+(-1)^{\delta}(\textstyle\sum(-1)^{{}^t\sigma\varepsilon'}\Theta[1\ \ \sigma](\tau)\Theta[1\ \ \sigma+\varepsilon](\tau))\\&=Q_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]}+(-1)^{\delta}R_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]}\end{aligned}$$

where we sum over $\sigma\in(\mathbb{Z}/2)^g$ and $\tau\in\mathbb{H}_{g+1}$. Hence:

$$\sum\lambda_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]}\left(\theta^4\begin{bmatrix}0&\varepsilon\\0&\varepsilon'\end{bmatrix}(\tau)+\theta^4\begin{bmatrix}0&\varepsilon\\1&\varepsilon'\end{bmatrix}(\tau)\right)=2\sum\lambda_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]}(Q^2_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]}-R^2_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]}).$$

Assuming the first relation and applying Corollary (4.10) we find that the polynomials $\sum \lambda_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]} Q^2_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]}$ and $\sum \lambda_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]} R^2_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]}$ are identically zero. Conversely, the second relation and Corollary (4.10) also show that these polynomials are identically zero. Hence by applying formula (1) to the first relation, we conclude that it also holds.

(4.13) *Remark*. The representation μ is in fact a real representation and can be realized on the real vector space spanned by the theta constants. These theta constants are, up to sign, permuted and this gives a permutation representation of K on the (real) lines spanned by them which is doubly transitive. As there is an invariant inner product on this real vector space, these lines are thus equiangular. Sets of equiangular lines can be described combinatorically by two-graphs [Se] (in this case the orthogonal two graph $\Omega^+(2g,2)$). These two-graphs can also be used to find equations between the theta constants.

(4.14) *Proof* (of Proposition (4.9)). It is sufficient to show that μ is a factor of $\mathrm{Ind}_{O^+}^K(\varepsilon)$, because then $\mu=\nu$ or $\mu=\rho$. As $\dim T_g \leqq (2^g+1)(2^{g-1}+1)/3$ it follows that $\mu=\nu$ if $g\geqq 3$ or $g=1$, and for $g=2$ it follows by explicitly comparing the representations.

Using Frobenius reciprocity ($\langle \mu, \mathrm{Ind}_{O^+}^K(\varepsilon)\rangle_K = \langle \mathrm{Res}(\mu), \varepsilon\rangle_{O^+}$) it suffices to find an $f\in T_g$, $f\neq 0$, such that $\mu(g)f=\varepsilon(g)f$ for all $g\in O^+(2g,\mathbb{F}_2)$.

From the transformation formula we find:

$$\mu(g)\,\theta\begin{bmatrix}0\\0\end{bmatrix}=\lambda(g)\,\theta\begin{bmatrix}0\\0\end{bmatrix} \quad \text{iff} \quad g\in\Gamma_g(1,2)/\Gamma_g(2) \quad (\lambda(g)\in\mathbb{C}).$$

Let Q be the quadratic form on $\mathbb{F}_2^{2g}$ defined by:

$$Q(x)=x_1x_{g+1}+\ldots+x_gx_{2g} \qquad \text{for } x\in\mathbb{F}_2^{2g}.$$

Then Q has maximal index, its associated bilinear form is invariant under K, and its orthogonal group is just $\Gamma_g(1,2)/\Gamma_g(2)$ (Q is in fact $e_*^{L_\tau}$). To show that the character λ is nontrivial, let $M\in\Gamma_g(1,2)$ be the matrix with components A, B, C and D with $A=D$, $B=-C$ and $A_{ij}=0$ unless $i=j\geqq 2$, and then $A_{ii}=1$; $B_{11}=1$ and $B_{ij}=0$ otherwise. It is easily verified that $\lambda(g)=-1$, where g is the image of M in $\Gamma_g(1,2)/\Gamma_g(2)$ $\Big($take τ a diagonal matrix and use $\theta\begin{bmatrix}0\\0\end{bmatrix}(-\tau_1^{-1}) = \sqrt{(i\tau_1)}\,\theta\begin{bmatrix}0\\0\end{bmatrix}(\tau_1)$, where $\tau_1=\tau_{11}\in\mathbb{H}_1$ and i is a primitive fourth root of unity$\Big)$.

Now that we know the linear relations between the $\theta^4\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$ we consider the quadratic relations of type G_β between the $\theta^2\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$. The following proposition describes them completely.

(4.15) **Proposition.** *All quadratic relations of type G_β are consequences of formula* (1). *Every relation of type G_β is of the form* (**) *below, and relations* (*)

and (**) *are equivalent. In particular, the vector spacd of Siegel modular forms spanned by the* $\theta^2\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}\theta^2\begin{bmatrix}\eta\\ \eta'\end{bmatrix}$, *with* $\begin{pmatrix}\varepsilon+\eta\\ \varepsilon'+\eta'\end{pmatrix}=\beta$, *has dimension* $(2^{g-1}+1)(2^{g-2}+1)/3$.

$$(*)\qquad \sum \lambda_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]}\theta^4\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(\tau)=0 \qquad \textit{for all } \tau\in\mathbb{H}_{g-1}$$

$$(**)\qquad \sum \lambda_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]}\theta^2\begin{bmatrix}0 & \varepsilon\\ 0 & \varepsilon'\end{bmatrix}(\tau)\,\theta^2\begin{bmatrix}0 & \varepsilon\\ 1 & \varepsilon'\end{bmatrix}(\tau)=0 \quad \textit{for all } \tau\in\mathbb{H}_g.$$

Proof. Let $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$ and $\begin{bmatrix}\eta\\ \eta'\end{bmatrix}$ be even and assume $\begin{pmatrix}\varepsilon+\eta\\ \varepsilon'+\eta'\end{pmatrix}=\beta$, i.e. $\varepsilon+\eta=0$, $\varepsilon'+\eta'=(1,0,\ldots,0)$. Then obviously $\left\{\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix},\begin{bmatrix}\eta\\ \eta'\end{bmatrix}\right\}=\left\{\begin{bmatrix}0 & \rho\\ 0 & \rho'\end{bmatrix},\begin{bmatrix}0 & \rho\\ 1 & \rho'\end{bmatrix}\right\}$ for some $\rho,\rho'\in(\mathbb{Z}/2)^{g-1}$ hence every relation of type G_β can be written in the form (**).

Assume that (**) holds. Substituting $\tau=\begin{pmatrix}\tau_1 & 0\\ 0 & \tau_{g-1}\end{pmatrix}$ we find that (*) holds. Conversely, if (*) holds, then by Corollary (4.10) the polynomial:

$$\sum \lambda_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]}\Big(\sum_\sigma(-1)^{{}^t\sigma\varepsilon'}\Theta[\sigma]\,\Theta[\sigma+\varepsilon]\Big)^2$$

($\sigma\in(\mathbb{Z}/2)^{g-1}$) is identically zero. Using form. (1), (**) gives:

$$\sum \lambda_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]}\{(\textstyle\sum(-1)^{{}^t\sigma\varepsilon'}\Theta[0\ \sigma]\,\Theta[0\ \sigma+\varepsilon])^2-(\sum(-1)^{{}^t\sigma\varepsilon'}\Theta[1\ \sigma]\,\Theta[1\ \sigma+\varepsilon])^2\}$$

which is thus identically zero, proving the first statement of the proposition. The other statements easily follow.

(4.16) *Remark.* It follows from Corollary (4.10) and Proposition (4.15) that the quadratic relations between the $\theta^2\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$ only define the image of $\mathbb{P}_N$ under the Veronese map Ψ. In case $g=1,2$ it is known that $\bar{A}_{g,(2,4)}=\mathbb{P}_N$ $(N=1,3)$. Hence in these two cases the ideal of relations between the $\theta^2\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$ will be generated by quadratic relations. From Proposition (4.18) it will follow that $\sigma(F)$, $F\in I_{g-1}$ with $g-1=1,2$, is identically zero. In particular for $g=2,3$ we find $\bar{S}_g=\bar{A}_g$ in agreement with (1.8).

(4.17) For further study we need a generalization of formula (1):

$$(3)\qquad \theta\begin{bmatrix}0 & \varepsilon\\ 0 & \varepsilon'\end{bmatrix}(\tau)\,\theta\begin{bmatrix}0 & \varepsilon\\ 1 & \varepsilon'\end{bmatrix}(\tau)=\sum(-1)^{{}^t\sigma\varepsilon'}\theta\begin{bmatrix}0 & \sigma\\ 1 & 0\end{bmatrix}(2\tau)\,\theta\begin{bmatrix}0 & \sigma+\varepsilon\\ 1 & 0\end{bmatrix}(2\tau)$$

where we sum over $\sigma,\varepsilon,\varepsilon'\in(\mathbb{Z}/2)^{g-1}$, ${}^t\varepsilon\varepsilon'=0$, and $\tau\in\mathbb{H}_g$. This formula is a specialization of Corollary II A 2.3 [R-F], or of [I2], VI,1, Theorem 2.

(4.18) **Proposition.** *Let F be a quadratic relation between the* $\theta^2\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$. *Then the Siegel modular form* $\sigma(F)$ *is identically zero.*

Proof. If F is a relation of type G_α the statement follows from Proposition (4.15), where we take $F=(*)$ and $(**)$ is then $\sigma(F)$. If F is of type G_β then F can be written as:

$$\sum \lambda_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]} \theta^2 \begin{bmatrix}0 & \varepsilon\\ 0 & \varepsilon'\end{bmatrix}(\tau)\, \theta^2 \begin{bmatrix}0 & \varepsilon\\ 1 & \varepsilon'\end{bmatrix}(\tau)=0 \qquad \text{for all } \tau\in\mathbb{H}_g.$$

Hence, by Proposition (4.15), the polynomial:

$$\sum \lambda_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]} \left(\sum (-1)^{{}^t\sigma(0\ \varepsilon')}\Theta[\sigma]\,\Theta[\sigma+(0\ \varepsilon)]\right)\left(\sum (-1)^{{}^t\sigma(1\ \varepsilon')}\Theta[\sigma]\,\Theta[\sigma+(0\ \varepsilon)]\right)$$

is zero, hence the polynomial

$$\sum \lambda_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]} \left(\sum (-1)^{{}^t\sigma(0\ \varepsilon')}\theta\begin{bmatrix}0 & \sigma\\ 1 & 0\end{bmatrix}\theta\begin{bmatrix}0 & \sigma+(0\ \varepsilon)\\ 1 & 0\end{bmatrix}\right)$$
$$\cdot\left(\sum (-1)^{{}^t\sigma(1\ \varepsilon')}\theta\begin{bmatrix}0 & \sigma\\ 1 & 0\end{bmatrix}\theta\begin{bmatrix}0 & \sigma+(0\ \varepsilon)\\ 1 & 0\end{bmatrix}\right)$$

is also zero. Using formula (3) we find (for all $\tau\in\mathbb{H}_{g+1}$):

$$\sigma(F)=\sum \lambda_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]}\theta\begin{bmatrix}0 & 0 & \varepsilon\\ 0 & 0 & \varepsilon'\end{bmatrix}\theta\begin{bmatrix}0 & 0 & \varepsilon\\ 1 & 0 & \varepsilon'\end{bmatrix}\theta\begin{bmatrix}0 & 0 & \varepsilon\\ 0 & 1 & \varepsilon'\end{bmatrix}\theta\begin{bmatrix}0 & 0 & \varepsilon\\ 1 & 1 & \varepsilon'\end{bmatrix}=0$$

(note that we found in fact a conjugate of $\sigma(F)$).

(4.19) Finally we describe the known quartic relations between the $\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$. The monomials occuring in such a relation are:

$$\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}\theta\begin{bmatrix}\varepsilon+\lambda\\ \varepsilon'+\lambda'\end{bmatrix}\theta\begin{bmatrix}\varepsilon+\mu\\ \varepsilon'+\mu'\end{bmatrix}\theta\begin{bmatrix}\varepsilon+\lambda+\mu\\ \varepsilon'+\lambda'+\mu'\end{bmatrix}$$

where $\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}\lambda\\ \lambda'\end{pmatrix}, \begin{pmatrix}\mu\\ \mu'\end{pmatrix}, \begin{pmatrix}\lambda+\mu\\ \lambda'+\mu'\end{pmatrix}$ is an isotropic subspace V of the symplectic vector space $\mathbb{F}_2^{2g}$, i.e. ${}^t\mu\lambda'+{}^t\mu'\lambda=0$. Using the action of Γ_g on the $\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$ we find:

(4.20) **Lemma.** *A quartic relation of type* (4.19) *is the sum of a quadratic relation and a sum of relations which are conjugates under the action of* Γ_g *of a relation of type:*

$$(***)\qquad \sum \lambda_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]}\theta\begin{bmatrix}0 & 0 & \varepsilon\\ 0 & 0 & \varepsilon'\end{bmatrix}\theta\begin{bmatrix}0 & 0 & \varepsilon\\ 0 & 1 & \varepsilon'\end{bmatrix}\theta\begin{bmatrix}0 & 0 & \varepsilon\\ 1 & 0 & \varepsilon'\end{bmatrix}\theta\begin{bmatrix}0 & 0 & \varepsilon\\ 1 & 1 & \varepsilon'\end{bmatrix}.$$

Proof. Similar to the proof of the Lemmas 1 and 2.

The following proposition describes the relations of type (***). Its proof is similar to the proof of Proposition (4.15) and we omit it. (Note that part of it is in fact given in the proof of Proposition (4.18).)

(4.21) **Proposition.** *All quartic relations of type* (4.19) *are consequences of the formula's* (1) *and* (3). *The relations* (*), (**) *and* (***) *are equivalent where we take* $\tau \in \mathbb{H}_{g-2}$, $\mathbb{H}_{g-1}$ *and* $\mathbb{H}_g$ *respectively and* $\varepsilon, \varepsilon' \in (\mathbb{Z}/2)^{g-2}$.

(Note that by Salvati Manni's result all quartic relations are of type (4.19).)

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