An exceptional isomorphism between modular varieties

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This article aims to give a modular construction of a correspondence between modular varieties whose existence was suspected in [Ge-Ny]. The modular varieties in question are on the one hand a Siegel modular threefold X^o, that is a moduli space of 2 dimensional abelian varieties, which is a quotient of the moduli space with level 8 structure and on the other hand the self-product (over the base curve) V^o of $\mathcal{E} \to Y_0(8)$, the universal (smooth) elliptic curve with a cyclic subgroup of order 8. These varieties have smooth compactifications X and V respectively and it is shown in (loc. cit.) that the 2 dimensional Galois representation $G_{\mathbf{Q}} = Gal(\bar{\mathbf{Q}}/\mathbf{Q}) \to Aut(H^3(X, \mathbf{Q}_\ell))$ is a subrepresentation of $H^3(V, \mathbf{Q}_\ell)$. This fact can be explained by the theory of lifting of elliptic modular cusp forms to Siegel modular forms, but the Tate conjecture relating algebraic cycles and Galois invariant subspaces in the étale cohomology groups predicts the existence of an algebraic cycle Z on $X \times V$ inducing the injection of Galois representations $H^3(X) \to H^3(V)$. Using explicit equations, J. Stienstra found a dominant rational map $X \to V$ whose graph gives the desired cycle. In this paper we give in fact an isomorphism of stacks (cf. Cor 2.10) which also induces the desired inclusion. It seems however that our construction does not directly work for higher levels, whereas the lifting theory is quite general.

Roughly speaking the isomorphism is constructed as follows. One starts with the space of elliptic curves (i.e. a smooth genus 1 curves with a section) with 2 distinguished points. Given a point of that space one gets 6 points on a projective line; the line is the quotient of the curve by ± 1 and the 6 points are the 4 ramification points of the quotient map and the images of the 2 distinguished points. To these 6 points one associates the double cover ramified at them and finally one obtains a principally polarised abelian surface; the Jacobian of the double cover. This gives an element of $\mathcal{A}_2$ the moduli space of principally polarised 2-dimensional abelian varieties. In order to get an equivalence one first needs to keep track of various choices. We get for instance not just 6 points on a projective line but rather one point, corresponding to the zero section of the elliptic curve, one set of 3 and one of 2 points. Furthermore, only in the case when the 6 points are distinct do we get a smooth double cover, which corresponds to the condition that the Jacobian of the double cover being an abelian surface *and* to the theta divisor of this principally polarised abelian surface being irreducible. (It should also be noted that the genus 1 curve associated to a genus 2 curve C is simply the Prym variety associated to a double cover of C and so this article can also be seen as an attempt to compare certain level structures on C (or its Jacobian) with level structures on the associated Prym variety.)

The article is therefore divided into several parts. First we get an equivalence of data with the minimal number of choices and with a situation where the 6 points are distinct. The proof of the equivalence follows almost exactly the sketch given. The only difference is that we make the constructions in families of elliptic curves etc. This forces us to deal with one choice which was implicit in the above description. Namely that a double cover is not determined by the choice of a degree 6 divisor in a $\mathbf{P}^1$-bundle, one also needs the choice of a square root line bundle of the associated line bundle. The next step is to compare some further choices (i.e. level structures) on either side of the equivalence such as a level 2 structure on the elliptic curve. This is done by first comparing extra data for the elliptic curve and the double cover genus 2 curve and then to translate the data for the genus 2 curve with data involving only its Jacobian. Finally, we make a brief study of compactifications of moduli spaces. It turns that when we do this extension we obtain not an equivalence but only a map from the modular variety of genus 2 curves to the one for elliptic curves.

We generally formulate our results in terms of algebraic stacks rather than (coarse) moduli spaces. There are two reasons for this choice. Firstly, we have started with an elementary construction of an elliptic curve from the 6 Weierstrass points of a genus 2 curve, which gives us the searched for map between modular varieties on geometric points. With some care it is seen that this construction can be done in any family. Working with algebraic stacks immediately shows that the map on geometric points comes from an algebraic map. Secondly, the equivalence of stacks gives us the isomorphism of coarse moduli spaces by passing to the associated algebraic spaces. The equivalence of stacks, however, contains more information. To see an example of this let us recall that the moduli stack of elliptic curves is the quotient of the upper half plane by $SL_2(\mathbf{Z})$ as analytic *stacks* whereas the (coarse) moduli stack is the quotient as analytic *spaces*. The difference between those two quotients is that the former behaves as if the action of $SL_2(\mathbf{Z})$ is free whereas the latter doesn't as some elements do have fixed points. Hence the topological fundamental group of the stack is equal $SL_2(\mathbf{Z})$ with the upper half plane as universal cover while the fundamental group of the space is trivial, the space being simply the affine line. Similar phenomena will hold for many of the other stacks we will encounter (an example where the fundamental groups are equal is the stack of irreducible genus 1 curves with a section of the smooth locus; it as well as its associated analytic stack is simply-connected).

0. Preliminaries and conventions.

We will use the equality sign to denote a canonically defined isomorphism. Unless otherwise mentioned 2 will be invertible in all our schemes and S will denote such a scheme. A multisection of a map $T \to S$ of schemes is a closed subscheme V of T such that the composed map $V \to T \to S$ is finite and flat; a multisection everywhere of degree m will also be refered to as an m-section; a 1-section is then simply a section. If $\pi: T \to S$ is a map and V a subscheme of S then a section of π over V is a section of the restriction of π to V. Unless otherwise mentioned $\mathbf{P}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$, T, U and V will have the meaning given to them in Theorem 1.2 and the paragraph after it. By an *elliptic curve* over S we will mean a smooth proper map $\pi: \mathcal{E} \to S$ with fibers curves of genus 1 together with a chosen section of π. The *scheme of Weierstrass points* of an elliptic curve is then the fixed points of multiplication by -1 wrt the S-group structure on $\mathcal{E}$ with the chosen section as origin.

Definition 0.1. Let $\pi: X \to S$ be a smooth map everywhere of relative dimension 1, $\rho: T \to S$ a finite flat map and $\tau: T \to X$ an S-map. The *divisorial image* of τ is the subscheme defined by the invertible ideal which is the image under the norm map (i.e. the ideal generated by all norms of elements in the ideal) for the map $X \times_S T \to X$ induced by ρ of the ideal of the T-section of $X \times_S T$ induced by τ and π.

Recall that a finite flat map of degree 2 $\pi: T \to S$ is determined by a line bundle $\mathcal{L}$ on S and a map $\rho: \mathcal{L}^{\otimes -2} \to \mathcal{O}_S$; $\pi_* \mathcal{O}_T$ decomposes as $\mathcal{O}_S \bigoplus \mathcal{L}^{-1}$, where $\mathcal{L}^{-1}$ is the kernel of the trace map and the multiplication in $\pi_* \mathcal{O}_T$ determines and is determined by ρ. The cokernel of ρ is the structure sheaf of the branch locus R of the cover. If R is everywhere of codimension 1 then ρ gives also an isomorphism $\mathcal{L}^{\otimes 2} \cong \mathcal{O}_S(R)$ and conversely if R is any Cartier divisor on S giving a double cover with

R as its ramification locus is equivalent to giving a line bundle $\mathcal{L}$ and an isomorphism $\mathcal{L}^{\otimes 2} \cong \mathcal{O}_S(R)$. Furthermore, we have a global section s of $\mathcal{L}^{\otimes 2}$ given by the transpose of ρ and the double cover represents the problem of finding a section t of $\mathcal{L}$ such that $t^{\otimes 2} = s$. On the double cover itself there is therefore a canonical section of (the pullback of $\mathcal{L}$) which we will denote $t_{\mathcal{L}}$.

Lemma 0.2. *Suppose that U and V are two disjoint Cartier divisors, $\mathcal{L}$ and $\mathcal{M}$ line bundles on S and $\rho: \mathcal{L}^{\otimes 2} \to \mathcal{O}_S(U)$ and $\tau: \mathcal{M}^{\otimes 2} \to \mathcal{O}_S(V)$ isomorphisms. Let $\alpha: S_1 \to S$, $\beta: S_2 \to S$ and $\gamma: S_3 \to S$ be the double covers associated to ρ, τ and $\rho \otimes \tau$ respectively and $\delta: S_4 := S_1 \times_S S_2 \to S_3$ the map for which $\delta^* t_{\mathcal{L} \bigotimes \mathcal{M}} = t_{\mathcal{L}} \otimes t_{\mathcal{M}}$. Then δ is an étale double cover with associated line bundle equal to $\gamma^*\mathcal{L}(-U')$ (and $\gamma^*\mathcal{M}(-V')$) where U' is the intersection between the zero set of $t_{\mathcal{L} \bigotimes \mathcal{M}}$ and the inverse image of U. Furthermore, the étale cover $\beta^{-1}U \to U$ is isomorphic to the étale cover $\delta^{-1}U' \to U'$.*

Proof. The map δ induces a map $S_1 \times_S S_2 \to \gamma^*(\alpha)$ over S_3. Let us consider the situtation locally with U defined by $f = 0$ and V defined $g = 0$ so that the affine algebras of S_1, S_2, S_3 and S_4 are equal to $\mathcal{O}_S[s]/(s^2 - f)$, $\mathcal{O}_S[t]/(t^2 - g)$, $\mathcal{O}_S[y]/(y^2 - fg)$ and $\mathcal{O}_S[s,t]/(s^2 - f, t^2 - g)$ respectively and δ is given by $y \mapsto st$. Then the affine algebra of δ is given by $\mathcal{O}_{S_3}[s,t]/(s^2 - f, t^2 - g, y - st)$. If f is invertible then $\mathcal{O}_{S_4}$ is free as $\mathcal{O}_{S_3}$-module with a basis $\{1, s\}$ and $s^2 = f$ is invertible, if g is invertible then $\mathcal{O}_{S_4}$ is free as $\mathcal{O}_{S_3}$-module with a basis $\{1, t\}$ and $t^2 = g$ is invertible. This shows that δ is étale and it is clear that the annihilator of the cokernel of $\mathcal{O}_{S_3}[s]/(s^2 - f) \to \mathcal{O}_{S_4}$ is exactly the ideal defining U'. This last fact remains true globally and so if we write $\delta_*\mathcal{O}_{S_4} = \mathcal{O}_{S_3} \bigoplus \mathcal{N}^{-1}$ we see that the annihilator of the cokernel of the map $\gamma^*\mathcal{L}^{-1} \to \mathcal{N}^{-1}$ induced from $S_1 \times_S S_2 \to \gamma^*(\alpha)$ is the ideal defining U' and hence $\mathcal{N} \bigotimes \gamma^*\mathcal{L}^{-1} = \mathcal{O}_{S_3}(-U')$. Finally the last statement is clear since γ induces an isomorphism between U' and U and together with the projection $S_4 \to S_2$ it induces a map between étale covers $\delta^{-1}U' \to U'$ to $\beta^{-1}U \to U$. Q.E.D.

Definition 0.3. Let $\mathcal{A}$ be a principally polarised abelian S-scheme and α a section of order 2. On the scheme $2^{-1}\alpha := \{\beta \in \mathcal{A} : 2\beta = \alpha\}$ we have the equivalence relation $\beta_1 \sim \beta_2 \iff e_2(\beta_1 - \beta_2, \alpha) = 1$, where $e_2(-,-)$ is the pairing on the kernel of multiplication by 2. A $\frac{1}{2}\alpha$-*structure* on $\mathcal{A}$ is the choice of one of the equivalence classes of this equivalence relation.

If $\mathcal{A}$ is a principally polarised abelian S-scheme, then a section α of order 2 of $\mathcal{A}$ gives rise to an étale cover of $\mathcal{A}$ in the following way. By the polarisation α corresponds to a line bundle $\mathcal{L}$ on $\mathcal{A}$ rigidified along the 0-section. As α is of order 2 so is the rigidified line bundle and hence there is a unique isomorphism $\mathcal{L}^{\otimes 2} \cong \mathcal{O}_S$ compatible with the rigidifications. This isomorphism is then used to construct a double cover (with affine algebra $\mathcal{O}_S \bigoplus \mathcal{L}$ and multiplication given by the isomorphism). An alternative way of constructing this cover is as follows. One considers the orthogonal complement G_α of α in ${}_2\mathcal{A}$ under the Weil pairing $e_2()$ induced by the polarisation. Multiplication by 2 on $\mathcal{A}$ factors as $\mathcal{A} \to \mathcal{A}/G_\alpha =: \mathcal{A}_\alpha \to \mathcal{A}$ for a unique map $\mathcal{A}_\alpha \to \mathcal{A}$. This map is the required double cover. (This is seen for instance by considering the action G_α on the $\mathcal{O}_{\mathcal{A}}$-algebra $2_*\mathcal{O}_{\mathcal{A}}$ and dividing it into eigenspaces.) As the equivalence classes of $2^{-1}\alpha$ are exactly the orbits under G_α one gets the following lemma.

Lemma 0.4. *Using the notations of (0.3) and the preceding paragraph a $\frac{1}{2}\alpha$-structure is equivalent to choosing a section of the map $\mathcal{A}_\alpha \to \mathcal{A}$ over α.*

Proof. Q.E.D.

Lemma 0.5. *(No conditions on S.) Let $\tau: T \to S$ be a finite, flat S-scheme of degree 3 which is relatively Gorenstein (i.e. the dualising sheaf $\omega_{T/S}$ is a line bundle).*

i) There is up to unique isomorphism a unique S-embedding of T into a $\mathbf{P}^1$-bundle over S.

ii) If T has a section R then it and T itself give relative Cartier divisors in the $\mathbf{P}^1$-bundle $\mathbf{P}$ of i). Their difference is a 2-section of T which we will call the (schematic) complement of R in T. Its ideal is a line bundle $\mathcal{I}$ and there is a canonical choice of a line bundle $\mathcal{M}$ and an isomorphism $\mathcal{I} \cong \mathcal{M}^{\otimes 2}$.

Proof. Assume, to begin with, that T is embedded in a $\mathbf{P}^1$-bundle $\pi: \mathbf{P} \to S$. Then T is a relative Cartier divisor of degree 3 in $\mathbf{P}$ and so $\mathcal{L} := \omega_{\mathbf{P}/S}(T)$ is a line bundle which fiber by fiber is of degree 1 and so $\mathbf{P}$ can be identified with $\mathbf{P}(\mathcal{E})$, where $\mathcal{E} := \pi_*\mathcal{L}$. We have the adjunction exact sequence

$$0 \to \omega_{\mathbf{P}/S} \longrightarrow \mathcal{L} \longrightarrow \omega_T \to 0,$$

which gives the short exact sequence

$$0 \to \mathcal{E} \to \tau_*\omega_T \to R^1\pi_*\omega_{\mathbf{P}/S} \to 0.$$

The map $\tau_*\omega_T \to R^1\pi_*\omega_{\mathbf{P}/S}$ composed with the trace map $R^1\pi_*\omega_{\mathbf{P}/S} \to \mathcal{O}_S$ is the trace map for τ and so $\mathcal{E}$ is identified with the kernel of the trace map $\tau_*\omega_T \to \mathcal{O}_S$. The embedding of T into $\mathbf{P}$ is determined by a line bundle quotient of $\tau^*\mathcal{E}$; the correspondance is given by associating to an embedding the restriction of the map $\tau^*\tau_*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1) \to \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ to T. This shows that the line bundle quotient map is simply the composite of the pullback to τ of the inclusion $\mathcal{E} \to \tau_*\omega_T$ and the adjunction map $\tau^*\tau_*\omega_T \to \omega_T$. Hence $\mathbf{P}$ as well as the embedding of T into it is canonically determined by T itself (as an S-scheme). This proves the uniqueness part of i). To prove existence we need to show that, without assuming an embedding, the composite $\tau^*\mathcal{E} \to \tau^*\tau_*\omega_T \to \omega_T$ is surjective, where $\mathcal{E}$ is defined as the kernel of the trace map and then that the obtained S-map from T to $\mathbf{P}(\mathcal{E})$ is an embedding. This is something which can be checked geometric fiber by geometric fiber so we may assume that S is an algebraically closed field. As we have seen, what we want to show is true as soon as T embeds in any $\mathbf{P}^1$-bundle so what remains to be shown is that any T so embeds. If $\mathbf{k}$ is an algebraically closed field there are up to isomorphism exactly four 3-dimensional commutative $\mathbf{k}$-algebras; $\mathbf{k}^3$, $\mathbf{k}\bigoplus \mathbf{k}[x]/(x^2)$, $\mathbf{k}[x]/(x^3)$ and $\mathbf{k}[x,y]/(x,y)^2$. The last is not Gorenstein (having a 2-dimensional socle) and the others evidently embed into $\mathbf{P}^1$.

To prove ii) we note first that it follows from the construction that $\mathcal{O}_{\mathbf{P}}(T) = \omega^{-1}_{\mathbf{P}/S}(1) = \pi^* \det \mathcal{E}^{-1}(3)$. The next step is to compute $\mathcal{O}_{\mathbf{P}}(R)$. To do that we are required to compute the line bundle quotient of $\mathcal{E}$ which corresponds to R. This is the restriction to R of the map $\tau^*\mathcal{E} \to \tau^*\tau_*\omega_T \to \omega_T$ so that the searched for line bundle quotient is the restriction to R of ω_T. Now, by duality, ω_T can be identified with $\underline{Hom}_{\mathcal{O}_S}(\mathcal{O}_T, \mathcal{O}_S)$ whose restriction to R is $\underline{Hom}_{\mathcal{O}_S}(\mathcal{O}_R, \mathcal{O}_S) = \mathcal{O}_R$. Hence the restriction of ω_T to R is canonically trivial and $\mathcal{O}_{\mathbf{P}}(R) = \pi^* \det \mathcal{E}(1)$. Therefore if R' is the schematic complement of R in T then $\mathcal{O}_{\mathbf{P}}(R') = \mathcal{O}_{\mathbf{P}}(T-R) = \pi^* \det \mathcal{E}^{-2}(2) = (\pi^* \det \mathcal{E}^{-1}(1))^{\otimes 2}$. This proves ii). Q.E.D.

Remark. i) When T consists of 3 disjoint copies of S part i) simply says that up to projective equivalence there is a unique way of choosing 3 points on the projective line. Intuitively our result confirms that this remains valid even when the points may be infinitely close. In a case when all 3 points coincide up to first order this was proven by P. Deligne [De:I,5.2].

ii) As $\mathcal{O}_{\mathbf{P}}(T) = \pi^* \det \mathcal{E}^{-1}(3)$, T gives rise to (and is determined by) an everywhere non-zero section of $S^3\mathcal{E} \bigotimes \det \mathcal{E}^{-1}$. Conversely such a section gives a T. When 6 is invertible this is encompassed by a result of R. Miranda [Mi] who shows, more generally, that a section of $S^3\mathcal{E} \bigotimes \det \mathcal{E}^{-1}$ gives rise to a degree 3-cover, the zero locus of the section being the locus of "non-Gorenstein-ness" of the cover.

iii) Note that in general the choice of "square root" of $\mathcal{O}_{\mathbf{P}}(R')$ depends on R and not just on the embedding of R' in $\mathbf{P}$. For instance, when S is the spectrum of a field $\mathbf{k}$ and T consists of 3 distinct points then the choice of this square root amounts to a choice, up to the square of a scalar, of a quadratic form having $T \setminus R$ as its zero set. This form can then seen to be determined by the condition that its value on T (this value is well-defined up to a square) is 1 (again up to a square). In other words, the double cover associated to the square root is split over T.

In order to be able to compare levels properly we now want to formulate the well-known description of the kernel of multiplication by 2 on the Jacobian of a hyper-elliptic curve in terms of its Weierstrass in a way so that no choices choices are made. This leads to the following definition.

If P is a finite set of even cardinality we define an $\mathbf{F}_2$-vector space $A(P)$ associated to P as follows. We look at the set of subsets of P of even cardinality with the symmetric difference as group operation and then divide out by the subgroup consisting of the empty set and all of P. The group of permutations of P clearly acts as a group of group automorphisms $\mathcal{S}(P)$ of $A(P)$. Note also that the two representative subsets of an element of $A(P)$ are a set and its complement. Recall that a bilinear form on an abelian group is alternating if the scalar product of every element with itself is zero.

Remark. As we will see, if P is the set of Weierstrass points of a hyper-elliptic curve then $A(P)$ is indeed the kernel of multiplication by 2 on the Jacobian of the curve. The pairing to be defined will also be seen to be the Weil pairing.

Lemma 0.6. *The form $\langle U,V\rangle_P := |U \bigcap V| \pmod 2$ is the unique non-zero alternating $\mathbf{F}_2$-valued form on $A(P)$ invariant under the action of $\mathcal{S}(P)$.*

Proof. One first verifies that the given form is well-defined and bilinear which is left to the reader. That it is alternating is clear and so it suffices to show that it is unique. As $A(P)$ is generated by 2-element subsets it is enough to show that it is unique on such. The only invariant under the action of $\mathcal{S}(P)$ on pairs of 2-element subsets is the cardinality of the intersection so there are only 3 cases. The case of cardinality of intersection equal to 2 is determined by the condition that the form be alternating. For a given 2-element subset we can write any 2-element subset as the sum of 2 other 2-element subsets meeting the given one in 1 point. Hence if the intersection is empty the scalar product is zero. In order for it to be non-trivial the scalar product of two 2-element subsets meeting in one point must be 1. Q.E.D.

Remark. For uniqueness we never used the fact that the form was alternating or even symmetric.

For each $s \in P$ we can define a quadratic form γ_s on $A(P)$ whose associated bilinear form is $\langle -,- \rangle_P$. Indeed, for an element α of $A(P)$ we let U be its unique representative which does not contain s. We then put $\gamma_s(\alpha) := |U|/2 \pmod 2$ and leave as an exercise to show that this is a quadratic form with $\langle -,- \rangle_P$ as its associated bilinear form. This gives a map from P into the set of quadratic forms on $A(P)$ whose associated bilinear form is $\langle -,- \rangle_P$ and it is easily verified that if $|P|$ is not divisible by 4 the this map is injective.

Remark. If P is the set of Weierstrass points of a hyperelliptic curve then γ_s is the quadratic form associated by Mumford (cf. [Mu:IIIa,Prop. 6.3 b]) to the theta characteristic $(g-1)s$, where g is the genus of the curve. Recall now that up to equivalence there are only 2 non-degenerate quadratic forms on an even-dimensional $\mathbf{Z}/2$-vector space (of dimension $2n$) and that the 2 possibilities are distinguished by the existence (the *split* case) or non-existence (the *non-split* case) of an isotropic subspace of dimension n. The split form can also be distinguished by there being $2^{2n-2} - 1 + (-1)^n 2^{n-1}$ isotropic elements in the vector space.

Lemma 0.7. *i) The quadratic form γ_s is split if $|P|/2$ is congruent to 1 or 2 modulo 4 and non-split if not.*

ii) If $|P| = 6$ then the association $s \mapsto \gamma_s$ gives a bijection between P and the set of non-split quadratic forms on $A(P)$ whose associated bilinear form is $\langle -,- \rangle_P$.

Proof. For i) one needs to compute the number of subsets of $P \setminus \{s\}$ of cardinality divisible by 4. Using the binomial theorem this is equal to

$$\frac{1}{4}\left((1+1)^{n-1} + (1-1)^{n-1} + (1+i)^{n-1} + (1-i)^{n-1}\right)$$

where $n := |P|$. Expanding this, using $(1+i)^2 = 2i$, gives the needed computation. As for ii), as the association is an injection it suffices to show that the set of such quadratic forms contains at most 6 elements. However the linear group of $A(P)$ acts transitively on all quadratic forms of given type and so the symplectic group of $\langle -,- \rangle_P$ acts transitively on quadratic forms of given type and associated bilinear form $\langle -,- \rangle_P$. However, $Sp(4, \mathbf{F}_2)$ has 720 elements and the stabiliser, in the symplectic group, of γ_s contains the symmetric group on $P \setminus \{s\}$ which contains 120 elements. Hence there are at most $720/120 = 6$ such forms. Q.E.D.

Remark. There are several ways to prove ii) (or equivalently that the map from the symmetric group on P to the symplectic group of $A(P)$ is surjective).

i) A transposition maps to a transvection and one gets all transvections this way. Transvections generate the symplectic group.

ii) Modeled on even theta characteristics one associates quadratic forms of Arf invariant 1 to 3-element subsets. By counting ($16 = \binom{6}{1} + \binom{6}{3}/2$) one sees that one gets all quadratic forms this way and hence all of Arf invariant -1 are of the form γ_s.

iii) The full monodromy action on the kernel of multiplication by 2 on the Jacobian of a curve is always the full symplectic group. In the case of genus 2 all curves are hyperelliptic so the monodromy group is the symmetric group.

As $A(P)$ is natural in P, for a finite étale cover of even degree $T \to S$ we can define another étale cover $A(T) \to S$ which is an étale group scheme with a canonical alternating form. If $T \to S$ is the branch locus of a double cover of a $\mathbf{P}^1$-bundle then we will want to identify $A(T) \to S$ with the kernel by multiplication by 2 on the Jacobian of the cover. This is straightforward if $T \to S$ is a split cover (and even if it has a section) but some care has to be taken in the general case.

Proposition 0.8. *Suppose $C \to S$ is a hyperelliptic curve and $R \to S$ its scheme of Weierstrass points. Then there is a canonical isomorphism of étale S-group schemes between $A(R)$ and ${}_2\underline{Pic}^0(C/S)$ taking the canonical alternating form to the Weil pairing.*

Proof. The line bundle of degree 2 giving the hyperelliptic linear system may not exist as such, it does, however, give a well-defined element ℓ in $\underline{Pic}(C/S)$. Let now $\mathcal{B}(R)$ be the étale cover of subsets of R of even cardinality. If M is a section of $\mathcal{B}(R)$ (over some S-scheme W) then we get an element $\mathcal{O}(M) - (\deg M/2)\ell$ of $\underline{Pic}^0(\mathcal{C}/S)(W)$. In this way we get a morphism of group schemes $\sigma\colon \mathcal{B}(R) \to {}_2Pic^0(\mathcal{C}/S)$. The line bundle associated to R is a multiple of ℓ in $\underline{Pic}(\mathcal{C}/S)$ and so σ factors to give a map $A(R) \to {}_2Pic^0(\mathcal{C}/S)$. To verify that it is an isomorphism it suffices to check on geometric fibers where it is well known. To see the correspondence on pairings one immediately reduces to the universal situation where S is the scheme of unordered $\deg R$-tuples of points on the projective line. As the space of ordered such tuples is irreducible, the monodromy action on the universal $\deg R$-divisor on the projective line over S is the full symmetric group and as the Weil pairing is non-trivial (non-degenerate even) we conclude by (0.6). Q.E.D.

1. An equivalence of stacks.

Definition 1.1. i) $\mathcal{F}$ is the stack for which the objects of $\mathcal{F}(S)$ consist of an elliptic curve $\mathcal{E}$ (with zero section T) over S together with a 2-section U of $\mathcal{P} := \mathcal{E}/\{\pm 1\}$ disjoint from the branch locus of $\mathcal{E} \to \mathcal{P}$. The morphisms are isomorphisms between curves preserving U.

ii) $\mathcal{M}_2^*$ is the stack for which the objects of $\mathcal{M}_2^*(S)$ consist of a smooth genus 2 curve $\mathcal{C}$ and a division of the branch locus of $\mathcal{C} \to \mathcal{C}/\iota =: \mathcal{P}$, where ι is the hyperelliptic involution, into three disjoint subschemes T, V, U which are multisections of degree 1, 3 and 2 respectively. The morphisms are isomorphisms between curves preserving the decompositions of the branch loci.

Remark. It is immediate from Artin's criterion [Ar:5.3] that $\mathcal{F}$ and $\mathcal{M}_2^*$ are algebraic stacks.

Theorem 1.2. *The two stacks $\mathcal{F}$ and $\mathcal{M}_2^*$ are equivalent. Under this equivalence the T's and U's correspond and V corresponds to the the complement of T in the Weierstrass points of $\mathcal{E}$.*

Proof. Using the discussion above on double covers we see that an object of $\mathcal{M}_2^*(S)$ is specified by a $\mathbf{P}^1$-bundle $\mathbf{P}$ over S, three disjoint multisections T, V and U of degree 1, 3 and 2 respectively, a line bundle $\mathcal{N}$ on $\mathbf{P}$ and an isomorphism $\mathcal{N}^{\otimes 2} \cong \mathcal{O}_{\mathbf{P}}(T+V+U)$. Using lemma 0.5 one sees that this is equivalent to giving a double étale cover $\tau\colon V \to S$ of S, a multisection U of $\mathbf{P}' \setminus T \bigcup V$, where $\mathbf{P}'$ is the $\mathbf{P}^1$-bundle associated to the disjoint union of V and a copy T of S, a line bundle $\mathcal{L}$ on S and an isomorphism $\mathcal{L}(3) \cong \mathcal{O}_{\mathbf{P}'}(V)$. On the other hand, an object of $\mathcal{F}(S)$ is specified by $\mathbf{P}^1$-bundle $\mathbf{P}$ over S, three disjoint multisections T, $V :=$ branch locus $\setminus T$ and U of degree 1, 3 and 2 respectively, a line bundle $\mathcal{N}$ on $\mathbf{P}$ and an isomorphism $\mathcal{N}^{\otimes 2} \cong \mathcal{O}_{\mathbf{P}}(T+V)$ which by a similar reasoning is equivalent to the same data. This correspondence clearly is natural for isomorphisms giving an equivalence of stacks. Q.E.D.

From the proof of the theorem it follows that if we have an object of $\mathcal{F}(S)$ (or equivalently of $\mathcal{M}_2^*(S)$) there is a canonical choice of square root of U on $\mathbf{P}$ (using the notations of the proof) and hence a canonical double cover of $\mathbf{P}$ ramified at U. Using the construction of section 0 and (0.2) we get a double étale cover $\mathcal{D} \to \mathcal{C}$. We will call $\mathcal{D}$ the *associated genus 3 curve* and the map $\mathcal{D} \to \mathcal{C}$ the *associated double cover*. By construction there are canonical sections of $\mathcal{C} \to \mathbf{P}$ over T, U and V and we will use the same letters to denote the images under these sections. Similarly for V and $\mathcal{E} \to \mathbf{P}$.

Proposition 1.3. *Let $(\mathcal{C}, T, U, V)$ be an element of $\mathcal{M}_2^*(S)$. The associated double cover is nontrivial and in particular the associated genus 3 curve is a curve of genus 3 (over S).*

Proof. To see this it suffices to check that it is non-trivial over one geometric point of S which is obvious. Q.E.D.

2. Comparing levels.

We will now see what happens with the equivalence of the previous section when we introduce various level structures on the two sides.

Definition 2.1. The superscript *sp* on $\mathcal{F}$ or $\mathcal{M}_2^*$ or any of the modifications to be defined presently means the stack obtained by adding the choice of an S-isomorphism between U and $S\times\mathbf{2}$ ($\mathbf{2} = \{\mathbf{0},\mathbf{1}\}$). Similarly, the superscript *se* on $\mathcal{F}$ (and its modifications) will denote the choice of a section of $\mathcal{E} \to \mathbf{P}$ over U and on $\mathcal{M}_2^*$ the choice of a section of the associated double cover $\mathcal{D} \to \mathcal{C}$ over U.

i) $\mathcal{F}(n)$ is the stack, on schemes on which n is invertible, whose objects are those of $\mathcal{F}$ plus a level n-structure on the elliptic curve.

$\mathcal{F}_0(n)$ is the stack, on schemes on which n is invertible, whose objects are those of $\mathcal{F}$ plus a choice of a cyclic subgroup of order n in the elliptic curve.

$\mathcal{F}(2,4)$ is the stack whose objects are those of $\mathcal{F}$ plus a level 2-structure and a choice of one of the cyclic groups of order 4 of the elliptic curve containing the first of the elements of order 2.

ii) $\mathcal{M}_2^*(2)$ is the stack whose objects are those of $\mathcal{M}_2^*$ plus an identification of V with $S \times \mathbf{3}$.

$\mathcal{M}_2^*(0,4)$ is the stack whose objects are those of $\mathcal{M}_2^*$ plus the following data:

b) A section over the image R in $\mathcal{C}$ of $S \times \mathbf{0}$, of the double cover of $\mathcal{C}$ associated to the decomposition of the branch locus of $\mathcal{C} \to \mathbf{P}$ into T, $S \times \mathbf{0} \bigcup U$ and $S \times \{\mathbf{1},\mathbf{2}\}$ (which gives an element of $\mathcal{M}_2^*(S)$ different from the one associated to the decomposition T, V and U).

$\mathcal{M}_2^*(2,4)$ is the stack whose objects are those of $\mathcal{M}_2^*$ plus the following data:

a) An identification of V with $S \times \mathbf{3}$.
b) A section over the image R in $\mathcal{C}$ of $S \times \mathbf{0}$, of the double cover of $\mathcal{C}$ associated to the decomposition of the branch locus of $\mathcal{C} \to \mathbf{P}$ into T, $S \times \mathbf{0} \bigcup U$ and $S \times \{\mathbf{1},\mathbf{2}\}$ (which gives an element of $\mathcal{M}_2^*(S)$ different from the one associated to the decomposition T, V and U).

Remark. The stack $\mathcal{F}^{se,sp}$ is equivalent to an open substack of the fibre product of the moduli stack $\mathcal{X}$ classifying elliptic curves with a distinguished point with itself over the moduli stack $\mathcal{M}$ classifying elliptic curves. The equivalence is obtained by mapping $(\mathcal{E}, U, V, U \leftarrow S \times \mathbf{2}, U \to \mathcal{E})$ to $((\mathcal{E}, \mathrm{im}(S \times \mathbf{0} \to U \to \mathcal{E})), (\mathcal{E}, \mathrm{im}(S \times \mathbf{1} \to U \to \mathcal{E})))$.

Theorem 2.2. *The equivalence of Theorem 1.2 extends to an equivalence between*

$\mathcal{F}(2)$	*and*	$\mathcal{M}_2^*(2)$,
$\mathcal{F}_0(4)$	*and*	$\mathcal{M}_2^*(0,4)$,
$\mathcal{F}(2,4)$	*and*	$\mathcal{M}_2^*(2,4)$,
$\mathcal{F}^{se}$	*and*	$\mathcal{M}_2^{*se}$,
$\mathcal{F}^{sp}$	*and*	$\mathcal{M}_2^{*sp}$.

Proof. For the level 2-structure it is immediate by the construction of the equivalence. As for a cyclic subgroup of order 4 of $\mathcal{E}$, as every element of order 2 is orthogonal to itself under the Weil pairing it is clear that a subgroup of order 4 is equivalent to giving an element α of order 2 (the unique element of order 2 in the subgroup) and a $\frac{1}{2}\alpha$-structure. Giving α is the same thing as giving a section the scheme of elements of order 2 in $\mathcal{E}$ hence giving a section of V. By (0.4) a $\frac{1}{2}\alpha$-structure is equivalent to finding a section over α of the double cover associated to α. By (0.2) this covering $\mathcal{E}'$ is the fiber product, over $\mathbf{P}$, of the double cover $\mathbf{P}'$ ramified over the complement V' of α in V and the double cover ramified over the union of T and α (the first has a canonical meaning by (0.5) and the second as E is given and the first cover has canonical meaning). Equivalently, it is obtained as the double cover ramified over the inverse image of $T \bigcup \alpha$ in $\mathbf{P}'$. Hence choosing a section of $\mathcal{E}'$ over α is equivalent to choosing a section over α of the covering $\mathbf{P}' \to \mathbf{P}$. On the other hand, we get a decomposition of the branch locus of $\mathcal{C} \to \mathbf{P}$ as T, $S \times \mathbf{0} \bigcup U$ and $S \times \{1,2\}$ and by (0.2) the covering $\mathbf{P}' \to \mathbf{P}$ restricted to α is isomorphic to the covering $\mathcal{D} \to \mathcal{C}$ restricted to α. The case of a $(2,4)$-structure is the combination of the two previous cases, the case of an sp-structure is immediate, and the case of an se-structure follows again from (0.2). Q.E.D.

On the side of the genus 2 curve $\mathcal{C}$ it turns out that the different supplementary structures we have put on the curve may be interpreted in terms of more familiar structures on the Jacobian. Let us then first note that if $\mathcal{A} \to S$ is a principally polarised abelian scheme then as a defining line bundle is determined up to translation there is a well-defined $\mathcal{A}$-torsor $\mathcal{P}$ such that the polarisation gives a line bundle on $\mathcal{P}$. The theta divisor – the common support of the sections of this line bundle – is then a divisor in $\mathcal{P}$. Let us also say that a quadratic form on ${}_2\mathcal{A}$ is *compatible* if its associated bilinear form is the Weil pairing and its Arf invariant is that of (0.7) for $|P| = 2\dim\mathcal{A} + 2$. If the dimension of $\mathcal{A}$ is 2 and γ is a compatible form then for each non-isotropic (i.e. $\gamma(\alpha) = 1$) element α of ${}_2\mathcal{A}$ there is a unique (unordered) pair of isotropic elements $\{\alpha_1, \alpha_2\}$ such that $\alpha = \alpha_1 + \alpha_2$. We will call this pair the *isotropic pair associated to* α.

Lemma 2.3. If P is a set of 6 elements then for every choice of a compatible quadratic fom γ on $A(P)$ and every non-isotropic (i.e. $\gamma(\alpha) = 1$) element α of $A(P)$ there is a unique (unordered) pair of isotropic elements $\{\alpha_1, \alpha_2\}$ such that $\alpha = \alpha_1 + \alpha_2$.

By (0.7 ii) γ corresponds to an element s of P and from the definition it follows that an element of $A(P)$ considered as a 2-element subset of P is non-isotropic for γ iff it is disjoint from s. Hence what the lemma says is that any 2-element subset of P disjoint from s is the symmetric difference of a unique pair of 2-element subsets containing s which is obvious. Q.E.D.

Definition 2.4. Using the notations of (2.3) the pair $\{\alpha_1, \alpha_2\}$ will be called the *isotropic pair associated to* α.

Definition 2.5. With the notations of the preceding paragraph, a $(\gamma, \frac{1}{2}\alpha)$-structure is the choice of a section of $\mathcal{A}_\alpha \to \mathcal{A}$ over $\{\alpha_1, \alpha_2\}$.

Remark. The significance of a $\frac{1}{2}\alpha$-structure in this context is that if given it makes the problem of finding a section above α_1 equivalent to finding one above α_2; one simply requires that the sum in $\mathcal{A}_\alpha$ be equal to the given lifting of α.

We can now formulate the abelian surface analogs of the level structures we have defined for genus 2 curves.

Definition 2.6. $\mathcal{A}_2^*$ is the stack whose objects are principally polarised abelian surfaces $\mathcal{A}$ with smooth theta divisor and a choice of a section α of order 2 of $\mathcal{A}$ and a compatible quadratic form γ on ${}_2\mathcal{A}$ for which α is not isotropic.

The superscript *sp* on $\mathcal{A}_2^*$ or any of the modifications to be defined presently means the stack obtained by adding an ordering of the two isotropic elements whose sum is α. Similarly, the superscript *se* on $\mathcal{A}_2^*$ (and its modifications) will denote the choice of a $(\gamma, \frac{1}{2}\alpha)$-structure on $\mathcal{A}$.

$\mathcal{A}_2^*(2)$ is the stack whose objects are those of $\mathcal{A}_2^*$ plus three γ-isotropic elements $(\beta_1, \beta_2, \beta_3)$ all of which are orthogonal to α.

$\mathcal{A}_2^*(0,4)$ is the stack whose objects are those of $\mathcal{A}_2^*$ plus a choice of a section β of $\mathcal{A}$ different from but orthogonal, under the Weil pairing, to α and not isotropic for γ and a $(\gamma, \frac{1}{2}\beta)$-structure on $\mathcal{A}$.

$\mathcal{A}_2^*(2,4)$ is the stack whose objects are those of $\mathcal{A}_2^*$ plus the choices of both (2) and (0,4).

The following result then comes as no surprise.

Theorem 2.7. *There is an equivalence of stacks between $\mathcal{M}_2^*$ and $\mathcal{A}_2^*$ which extends to equivalences between*

$\mathcal{A}_2^*(2)$	*and*	$\mathcal{M}_2^*(2)$
$\mathcal{A}_2^*(0,4)$	*and*	$\mathcal{M}_2^*(0,4)$
$\mathcal{A}_2^*(2,4)$	*and*	$\mathcal{M}_2^*(2,4)$
$\mathcal{A}_2^{*se}$	*and*	$\mathcal{M}_2^{*se}$
$\mathcal{A}_2^{*sp}$	*and*	$\mathcal{M}_2^{*sp}$.

Proof. To begin with one passes back and forth between principally polarised abelian surfaces and curves of genus 2 using the Jacobian and the theta divisor respectively. From (0.8) it follows that 2-sections of the scheme of Weierstrass points R of the curve correspond to elements of order 2 of the abelian surface. The section T of R allows us to embed the curve in its Jacobian (and not just a torsor over it) so that T maps to the origin. Sections of R disjoint from T then map to elements of order 2 in the Jacobian which correspond to the 2-section which is the union of T and the section. On the other hand, the element α of order 2 corresponding to a 2-section of R disjoint from T equals the sum of the images α_1 and α_2 of the two sections of the 2-section (this sum is well-defined as it is independent of the order of the 2 sections). From the proof of (2.3) it follows that the pair $\{\alpha_1, \alpha_2\}$ is the isotropic pair associated to the isotropic element α. The equivalence between $\mathcal{M}_2^*$ and $\mathcal{A}_2^*$ follows because the choice of T and U corresponds, by (0.7 ii) and (0.8), to an element of order 2 of the Jacobian and a compatible quadratic form on the kernel of multiplication by 2.

To prove the (2)-equivalence one simply notices that sections of V map to elements of order 2, isotropic for γ and orthogonal to α and that they give all such elements (which is immediately seen from the $A(P)$-description).

Furthermore, from (0.2) it follows that the covering $\mathcal{D} \to \mathcal{C}$ is the pullback to $\mathcal{C}$ of the double cover of the Jacobian of $\mathcal{C}$ associated to the 2-section U and under the embedding. Hence a section of $\mathcal{D} \to \mathcal{C}$ over U corresponds precisely to a $(\gamma_T, \frac{1}{2}\alpha)$-structure. This gives the *se*-part. The *sp*-part is obvious, and the (0,4)-part is similar to the *se*-part. Q.E.D.

We now summarise some of our results in the following corollary. Let us recall that a level n-structure on a principally polarised abelian variety A is a *symplectic* isomorphism between the kernel of n on A and $(\mathbf{Z}/n\mathbf{Z})^{2\dim A}$ (which has the standard symplectic structure).

Corollary 2.8. *i) Let $\mathcal{A}_2^\dagger$ be the algebraic stack whose objects are principally polarised abelian surfaces $\mathcal{A}$ together with; a quadratic form γ on ${}_2\mathcal{A}$ whose associated bilinear form is the Weil pairing and whose Arf invariant is -1, two sections α_1 and α_2 of ${}_2\mathcal{A}$ for which $\gamma(\alpha_1) = \gamma(\alpha_2) = 0$ and a $(\gamma, \frac{1}{2}\alpha)$-structure, where $\alpha := \alpha_1 + \alpha_2$, and whose morphisms are iso-*

morphisms preserving all structures. Let $\mathcal{M}$ be the algebraic stack whose objects are elliptic curves and whose morphisms are isomorphisms. Let $\mathcal{X}$ be the stack whose objects are elliptic curves with a section and whose morphisms are isomorphisms preserving the section and let $\mathcal{X} \to \mathcal{E}$ be the forgetful functor. Then $\mathcal{A}_2^{\dagger}$ is equivalent to the open substack $\mathcal{U}$ of $\mathcal{X} \times_{\mathcal{M}} \mathcal{X}$ obtained by removing $\{(x,y) : 2x = 0 \vee 2y = 0\} \bigcup \{(x,x)\} \bigcup \{(x,-x)\}$.

ii) Let $\mathcal{A}_2^{\ddagger}$ be the algebraic stack whose objects are principally polarised abelian surfaces $\mathcal{A}$ together with:

a) A level 2-structure.

b) Sections over the first and second basis element of the double cover associated to their sum (which is an element of order 2).

c) Sections over the third and fourth basis element of the double cover associated to their sum.

Then $\mathcal{A}_2^{\ddagger}$ is equivalent to the pullback over $\mathcal{M}(2,4) \to \mathcal{M}$ of $\mathcal{U}$, where $\mathcal{M}(2,4)$ is the algebraic stack whose objects are elliptic curves with a level 2-structure and a choice of one of the subgroups of order 4 containing the first element of order 2.

Proof. i) is simply a reformulation of the theorem using the remark after (2.1). As for ii) it will be proved as soon as we have identified $\mathcal{A}_2^{\ddagger}$ with $\mathcal{A}_2^{(se,sp)}(2,4)$. To do this let us for a moment go back to $\mathcal{M}_2^{(se,sp)}(2,4)$. As part of the data we have a total ordering of the Weierstrass points of $\mathcal{C}$. Let us first show that such an ordering is equivalent to a level 2-structure. Indeed, one direction is clear and by (0.7 ii) one goes the other way by associating to a symplectic non-degenerate 4-dimensional space the set of compatible quadratic forms. Now the identification is obtained by interpreting the supplementary data on both sides. Q.E.D.

Remark. The topological fundamental group of the base extension of $\mathcal{A}_2^{\dagger}$ to $\mathbf{C}$, the complex numbers, is a subgroup of finite index of the mapping class group, in fact the inverse image of the appropriate congruence subgroup of $Sp(4,\mathbf{Z})$. As $\mathcal{X} \times_{\mathcal{M}} \mathcal{X}$ is normal and $\mathcal{A}_2^{\dagger}$ is open in it, this group maps surjectively onto the fundamental group of $\mathcal{X} \times_{\mathcal{M}} \mathcal{X}$ and, more precisely, of $\mathcal{X} \times_{\mathcal{M}} \mathcal{X} \setminus \{(x,y) : 2x = 0 \vee 2y = 0\}$ which is an extension of $SL_2(\mathbf{Z})$ by $\mathbf{Z}^2$ (resp. by a free group on 5 generators). In particular, as there are subgroups of finite index of $SL_2(\mathbf{Z})$ which are free of rank greater than 1 we see that there are subgroups of finite index of the mapping class group which map onto free groups of any finite rank and so onto free abelian groups of any finite rank. As all subgroups of $Sp_4(\mathbf{Z})$ of finite index have finite abelianisations we see that this map to $SL_2(\mathbf{Z})$ is very different from the natural map of the mapping class group onto $Sp_4(\mathbf{Z})$. In section 5 we will see a further vindication of this fact.

Lemma 2.9. *Let $\mathcal{M}_0(8)$ be the algebraic stack whose objects are elliptic curves with a chosen cyclic subgroup of order 8. Then $\mathcal{M}_0(8)$ is equivalent to $\mathcal{M}_0(2,4)$.*

Proof. Indeed, if (E, C_4, β) is an object of $\mathcal{M}_0(2,4)$ one associates to it $(E_\beta := E/\langle\beta\rangle, \ker : E_\beta \xrightarrow{{}^t\pi_\beta} E \to E/C_4)$, where ${}^t\pi_\beta$ is the transpose of the projection map $\pi_\beta : E \to E_\beta$. The inverse is given by associating to (E, C_8) the triple $(E/4C_8, C_8/4C_8, \beta)$, where β is the generator of ${}_2E/4C_8$. Q.E.D.

Combining we get

Proposition 2.10. *Let $\mathcal{X}_0'(8)$ be the algebraic stack of elliptic curves with a section, a cyclic subgroup C_8 of order 8 and a lifting of the section to the covering of the elliptic curve associated to $4C_8$. Then $\mathcal{A}_2^{\ddagger}$ is equivalent to an open substack of $\mathcal{X}_0'(8) \times_{\mathcal{M}_0(8)} \mathcal{X}_0'(8)$.*

Proof. Q.E.D.

3. Compactifications.

We will now extend the arguments presented to stable curves. It is easy to see that the picture must differ somewhat when the stable curve consists of 2 genus 1 curves. In fact when a smooth situation degenerates (in a 1-parameter family, say) to the union of 2 smooth genus 1 curves then 3 of the 6 Weierstrass points come together. The resulting 4 points (one of which is triple) have only 1 moduli whereas 2 smooth genus 1 curves have 2. Hence we can at most hope for a map from the space of stable genus 2 curves to the space of stable genus 1 curves with 2 distinguished points. This is in fact what we will obtain. To see this we will first need the following result on the bicanonical system of a stable curve.

Lemma 3.1. *Let $\mathcal{C}$ be a stable curve of genus 2 (over S). Then the (relative) bicanonical system gives a map of degree 2 onto a (fiber by fiber) reduced conic in a $\mathbf{P}^2$-bundle over S.*

Proof. As the bicanonical system commutes with base change [De-Mu:Thm. 1.2], this is something which can be checked fiber by fiber and then [Ca:Theorem A] shows that the bicanonical system is basepoint free. A simple case by case study then shows that the map is of degree 2 and maps onto a conic. Q.E.D.

This lemma already gives a clue to what is happening in the case mentioned above; the union of two elliptic curves is not the double cover of a projective line but rather of 2 projective lines meeting in a point and the triple point is blown up into 3 distinct points. Except for this things will work in a way close to the smooth situation. Our first vindication of this claim is the following definition-lemma.

Lemma-Definition 3.2. *Let $\pi: \mathcal{C} \to S$ be a family of stable genus 2 curves.*

i) $\mathbf{P}(\mathcal{C}) \to S$ is the conic fibration which is the image of $\mathcal{C}$ under the relative bicanonical map.

ii) If $\pi: \mathcal{C} \to S$ is a family of stable genus 2 curves then the disconnecting locus of $\mathbf{P}(\mathcal{C})$ is the closed subset of $\mathbf{P}(\mathcal{C})$ of points lying in the intersection of two components of any fiber of $\mathbf{P}(\mathcal{C}) \to S$. The disconnecting locus of $\mathcal{C}$ is the inverse image under the map $\mathcal{C} \to \mathbf{P}(\mathcal{C})$ of the disconnecting locus of $\mathbf{P}(\mathcal{C})$.

iii) The Weierstrass scheme of $\mathcal{C}$ is the the ramification locus of the map from the complement in $\mathcal{C}$ of its disconnecting locus to its image under the bicanonical map of $\mathcal{C}$. It is a 6-section of $\mathcal{C}$ as well as of $\mathbf{P}(\mathcal{C}) \to S$. When a fiber of π is reducible it meets each component three times.

Proof. Except for the last statement what needs to be shown is that the Weierstrass scheme is finite and flat of degree 6. It is flat and quasi-finite being the ramification locus of a double cover of a smooth S-scheme by a fiberwise reduced relatively Cohen-Macaulay scheme. We may therefore check that it is finite of degree 6 fiber by fiber. The only non-trivial case is when the stable curve canonical system of the curve has a base point (as in the other case the curve is a double cover of $\mathbf{P}^1$) and then it is clear as the bicanonical system reduces to the complete linear system given by twice the intersection point with the other fiber. This fact also gives the last statement. Q.E.D.

Definition 3.3. i) $\overline{\mathcal{F}}$ is the stack for which the objects of $\overline{\mathcal{F}}(S)$ consist of a stable genus 1 curve $\pi: \mathcal{E} \to S$ with a section T, not meeting the singular locus of π, together with a 2-section U of $\mathbf{P}(\pi_* \mathcal{O}_{\mathcal{E}}(2T)) \to S$, which is a $\mathbf{P}^1$-bundle. The morphisms of $\overline{\mathcal{F}}(S)$ are the isomorphisms of curves preserving T and U.

ii) $\overline{\mathcal{M}_2^*}$ is the stack for which the objects of $\overline{\mathcal{F}}(S)$ consist of a stable genus 2 curve $\mathcal{C}$ and an expression of the Weierstrass scheme of $\mathcal{C}$, thought of as a Cartier divisor of $\mathcal{C}$, as the sum of three multisections T, V and U of degree 1, 3, and 2 respectively.

Remark. i) $\mathcal{F}$ and $\mathcal{M}_2^*$ corresponds to the cases when $\mathcal{E}$ resp. $\mathcal{C}$ are smooth and T and U resp. T, V and U are disjoint and étale.

ii) By the semi-stable reduction theorem $\overline{\mathcal{F}}$ and $\overline{\mathcal{M}_2^*}$ are proper algebraic stacks.

With the notations of i) we will use the V to denote the ramification locus of the map defined by the relative linear system of $2T$ minus T itself.

Theorem 3.4. *The map $\mathcal{M}_2^* \to \mathcal{F}$ of theorem 1.2 extends to a map $\overline{\mathcal{M}_2^*} \to \overline{\mathcal{F}}$.*

Proof. Ideally we would like to continue in the previous manner by associating to each element of $\overline{\mathcal{M}_2^*}(S)$ an element of $\overline{\mathcal{F}}(S)$ extending the earlier construction. For technical reasons we have not been able to do that. However, as $\overline{\mathcal{F}}$ and $\overline{\mathcal{M}_2^*}$ are algebraic stacks it is enough to give such an association when we restrict ourselves to versal (or in fact miniversal) families in $\overline{\mathcal{M}_2^*}$. Let therefore $\pi: \mathcal{C} \to S$ be a stable genus 2 curve which is versal at all its points. Our first step is to contract one of the $\mathbf{P}^1$'s of each reducible fiber of $\mathbf{P}(\mathcal{C}) \to S$ so as to obtain a $\mathbf{P}^1$-bundle. Note that any line bundle $\mathcal{L}$ on a conic bundle which has degree 1 on each fiber gives a contraction map onto a $\mathbf{P}^1$-bundle where the contracted components are those to which the restriction of $\mathcal{L}$ has degree 0. Furthermore, the $\mathbf{P}^1$-bundle and the contraction map only depend on the contracted component and not on $\mathcal{L}$ (as two line bundles contracting the same component in each fiber differ by a line bundle from the base). The rule for which component to contract will depend on how T, V and U meet the 2 components. In fact we want to contract the component not meeting T except when that component meets V 3 times. (The reason for the last exception is that we want the divisorial images of V and T under the contraction to be the Weierstrass scheme of a stable genus 1 fibration and hence the divisorial image of V may not have triple points.) In the first case we may contract using the relative linear system of $\mathcal{O}_{\mathbf{P}(\mathcal{C})}(2T)$ and in the second the relative linear system of $\omega_{\mathbf{P}(\mathcal{C})/S}(T)^{-1}$. Hence whenever only one of the two cases occur we can do the contraction. On the other hand, as the contraction – if it exists – is unique it is sufficient to show the existence locally. As the conditions distinguishing the two cases only involve intersection numbers, the locus of points of S over which the fibers of π are geometrically reducible can be divided up into two open and disjoint subsets such that on each such component one of the cases occur. Therefore the contraction is always possible locally and hence globally.

We therefore get a $\mathbf{P}^1$-bundle $\mathbf{P} \to S$ and an S-map $\rho: \mathbf{P}(\mathcal{C}) \to \mathbf{P}$. We let T, V and U denote the divisorial images of the subschemes T, V resp. U in $\mathbf{P}(\mathcal{C})$. It seems reasonable that contraction the component of $\mathcal{C}$ lying over the contracted component of $\mathbf{P}(\mathcal{C})$ should be a double cover of $\mathbf{P}$. Indeed, away from the image K of the contracted components this is clear. Note now that as π is versal K has codimension at least 2 at all its points. On the other hand this contraction is the normalisation over $\mathbf{P}$ of the double cover over the complement of K. Hence its affine algebra over $\mathbf{P}$ is a rank 2 reflexive sheaf. The trace map splits this sheaf as a sum of $\mathcal{O}$ and a rank 1 sheaf which is also reflexive. However, as $\mathcal{C} \to S$ is versal, S and hence $\mathbf{P}$ is regular and so a reflexive rank 1 sheaf is a line bundle and we have a double cover $\lambda: \mathcal{C}' \to \mathbf{P}$.

Lemma 3.5. *The ramification locus of λ equals the sum of T, V and U.*

Proof. Indeed, outside of K this is true by definition. However, K has codimension 2 and $\mathbf{P}$ is regular. Q.E.D.

We may now proceed exactly as we did before. The cover λ gives a line bundle $\mathcal{L}$ and an isomorphism $\mathcal{L}^{\otimes 2} \cong \mathcal{O}_{\mathbf{P}}(T+V+U)$ and by (0.5 ii) there is a canonical "square root" of $\mathcal{O}_{\mathbf{P}}(U)$ so by subtracting it off we get a line bundle $\mathcal{L}'$ and an isomorphism $\mathcal{L}'^{\otimes 2} \cong \mathcal{O}_{\mathbf{P}}(T+V)$. This gives a stable curve with a section T of the smooth locus except over the points where T and V meet. We will see that by a sequence of elementary transformations of $\mathbf{P}$ we may separate T and V keeping the data for a double cover. Let us first note that the schematic intersection of $T+V$ meets a fiber at most in double points. Indeed, as $\mathcal{C}$ is stable it is clear that this is true for the Weierstrass scheme in $\mathbf{P}(\mathcal{C})$. We have now chosen which component to contract so that we will not create a triple of $T+V$ during contraction (note that in general $T+U+V$ in $\mathbf{P}$ will, however, have triple points). This means that locally, in the étale topology, around the intersection of T and V there will be a branch V' of V meeting T and the rest of V will not meet T. Because of versality V and T meet transversally and their intersection maps isomorphically onto its image S' in S which is smooth. We can now blow up this intersection and then blow down the strict transform of the inverse image in $\mathbf{P}$ of S' to obtain the $\mathbf{P}^1$-bundle $\mathbf{P}'$. (Locally the picture is exactly that of an ordinary elementary transform with a 1-dimensional base crossed with a regular scheme.) Looking again at things locally we see that T and V' have been separated and neither of them meets the image of the strict transform of the inverse image in $\mathbf{P}$ of S'. Hence now S is disjoint from V and V itself meets fibers in at most double points. Furthermore, the isomorphism $\mathcal{L}'^{\otimes 2} \cong \mathcal{O}_{\mathbf{P}}(T+V)$ transforms into an isomorphism $\mathcal{L}'^{\otimes 2} \cong \mathcal{O}_{\mathbf{P}'}(T+V+2\mathbf{P}'_{S'})$ so that we may choose $\mathcal{L}'(-\mathbf{P}'_{S'})$ as our new square root. Q.E.D.

4. Levels and compactifications.

There seem to be some technical problems in defining our various level structures for the non-smooth curves (more precisely those which are the union of two genus 1 curves). We will therefore confine ourselves to the problem which interests us; namely of extending our results on principally polarised abelian surfaces with irreducible theta divisor to all principally polarised abelian surfaces. We will do this by bypassing the moduli stack of genus 2 curves and work directly with the abelian surfaces.

Definition 4.1. $\mathcal{A}_2$ is the stack whose objects are principally polarised abelian surfaces $\mathcal{A}$ a choice of a section α of order 2 of $\mathcal{A}$ and a compatible quadratic form γ on ${}_2\mathcal{A}$ for which α is not isotropic.

The superscript *sp* on $\mathcal{A}_2$ or any of the modifications to be defined presently means the stack obtained by adding an ordering of the two isotropic elements whose sum is α. Similarly, the superscript *se* on $\mathcal{A}_2$ (and its modifications) will denote the choice of a $(\gamma, \frac{1}{2}\alpha)$-structure on $\mathcal{A}$.

$\mathcal{A}_2(2)$ is the stack whose objects are those of $\mathcal{A}_2$ plus three γ-isotropic elements $(\beta_1, \beta_2, \beta_3)$ all of which are orthogonal to α.

$\mathcal{A}_2(0,4)$ is the stack whose objects are those of $\mathcal{A}_2$ plus a choice of a section β of $\mathcal{A}$ different from but orthogonal, under the Weil pairing, to α and not isotropic for γ and a $(\gamma, \frac{1}{2}\beta)$-structure on $\mathcal{A}$.

$\mathcal{A}_2(2,4)$ is the stack whose objects are those of $\mathcal{A}_2$ plus the choices of both (2) and $(0,4)$.

The following result, of course comes as no surprise.

Theorem 4.2. *The equivalences of (2.7) and (2.2) extend to a map from*

$$\mathcal{A}_2(2) \qquad \textit{to} \qquad \overline{\mathcal{F}}(2),$$

$$\begin{array}{lll} \mathcal{A}_{20}(4) & \textit{to} & \overline{\mathcal{F}}(0,4), \\ \mathcal{A}_2(2,4) & \textit{to} & \overline{\mathcal{F}}(2,4), \\ \mathcal{A}_2{}^{se} & \textit{to} & \overline{\mathcal{F}}^{se}, \\ \mathcal{A}_2{}^{sp} & \textit{to} & \overline{\mathcal{F}}^{sp} \end{array}$$

Proof. Indeed, it is enough to notice that all the variants of $\overline{\mathcal{F}}$ map by a finite map to $\overline{\mathcal{F}}$ itself and that all the variants of $\mathcal{A}_2$ are normal (in fact smooth over $\mathbf{Z}$). Q.E.D.

Remark. i) What stops us from defining extensions of the level structures in the the case of stable genus 2 curves is that one needs to copy the previous arguments and in particular one needs the definition of the associated double cover and needs to express it as a fibered product. In the case of a versal deformation this causes no problem which would be enough to prove things about the extension but not to define it. In the general case one needs only to define the associated double cover away from the disconnecting locus (as T, U and V never meet it). We assume that this would not cause any essential problems but have not felt motivated to carry such an argument through.

ii) The image of the map from $\mathcal{A}_2$ to $\overline{\mathcal{F}}$ does not lie in $\mathcal{F}$. Indeed, following the constructions through one sees that the genus 1 curve associated to a principally polarised abelian surface A is smooth exactly when both points of U meet the same component of $\mathbf{P}(C)$, where C is the theta divisor of A. (This is true as it is equivalent to V on $\mathbf{P}$ having no double points.)

5. Genus 1 levels vs. genus 2 levels.

We will here take the opportunity to note that in general putting a level structure on an elliptic curve is a condition completely independent from a level structure on the associated genus 2 curve. To simplify, let us show this for an odd prime order level p for the elliptic curve and an odd prime order level q (not necessarily distinct from p) for the genus 2 curve. We will use the following lemma (which the first named author learned from J.-P. Serre).

Lemma 5.1. *Let G and H be groups. Then there is a bijection between the set of subgroups M of $G\times H$ which map surjectively by the two projections onto G and H and isomorphisms between quotient groups of G and H.*

Proof. Given a subgroup M of $G \times H$ with the desired properties consider $G' := M \bigcap G \times \{e\}$ and $H' := M \bigcap \{e\} \times H$. These subgroups are normal in G and H respectively. Indeed, for any $g \in G$ there is, by assumption, an $h \in H$ such that $(g,h) \in M$ and hence G' is stable under conjugation by g. By assumption the composite $M \hookrightarrow G \times H \to G$ is surjective and so G/G' is the quotient of M by $M \bigcap G' \times H = G' \times H'$. As the same thing is true of H/H' we get an isomorphism between G/G' and H/H'. On the other hand, starting with an isomorphism $\phi: \overline{G} \to \overline{H}$ between quotient groups $\overline{G}$ and $\overline{H}$ of G resp. H we associate to it the subgroup $\{(g,h) : \phi(\overline{g}) = \overline{h}\}$ of $G \times H$. It is clear that these two constructions are inverses of each other. Q.E.D.

We assume that we are placed over an algebraically closed field of characteristic different from 2, p and q. Let Γ be the algebraic fundamental group of $\mathcal{F} \cong \mathcal{M}_2^*$ with respect to a fixed base point. We have 2 surjective maps $\Gamma \to SL_2(\mathbf{Z}/p)$ and $\Gamma \to Sp_4(\mathbf{Z}/q)$ given by the action on the kernel of multiplication by p on the Jacobian of the base point as an elliptic curve resp. by multiplication by p on the Jacobian of the base point as a genus 2 curve. That the 2 level conditions are independent is equivalent to saying that the induced map $\Gamma \to SL_2(\mathbf{Z}/p) \times Sp_4(\mathbf{Z}/q)$ is surjective. As the projection on both factors is surjective the image Λ is,by the lemma, described by giving a common quotient group of the 2 factors. However, except for $SL_2(\mathbf{Z}/3)$ which is solvable, $SL_2(\mathbf{Z}/p)$ and $Sp_4(\mathbf{Z}/q)$ modulo their center are simple and $PSL_2(\mathbf{Z}/p)$ and $PSp_4(\mathbf{Z}/q)$ are non-isomorphic groups (their Sylow 2-groups are different). Hence the only possible common quotient is the trivial group and $\Gamma \to SL_2(\mathbf{Z}/p) \times Sp_4(\mathbf{Z}/q)$ is surjective.

In particular, the pullback of the equivalence $\mathcal{F} \cong \mathcal{M}_2^*$, considered as a correspondence, to the spaces with odd level structures added is irreducible and hence induces the zero map on "new" modular forms. This shows that the fact that our equivalence induces a non-trivial correspondence in the case mentioned in the introduction is indeed coincidental.

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Chern Functors

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This is the second of four papers in which we try to come to terms with Deligne's problem of constructing a functorial Riemann-Roch isomorphism for the determinant line bundle of the cohomology of a proper smooth morphism $p: X \to S$

$$\det \mathbb{R}p_* \mathcal{E} \to (I_{X/S}\, \mathfrak{ch}(\mathcal{E}) \mathfrak{Td}(T_{X/S}))^{(1)}. \tag{1}$$

The first step in such a construction is to give life to the right hand side of (1). This was done by Deligne and Elkik ([D],[E]), who treated (1) as a global expression. It is our approach to give life to each ingredient of the right hand side of (1), i.e., we can not only integrate the Chern functors along the fibres, we can also say what the Chern functors themselves are. This allows us to approach (1) by copying Grothendieck's proof of Riemann-Roch via embeddings into projective spaces, as we shall see in a forthcoming paper.

As the first step in this program, Chow categories as target categories for the Chern functors have been introduced in [F1]. Here we study the Chern functors themselves. Because of difficulties with the intersection product for non-smooth schemes over Spec($\mathbb{Z}$), we introduce $\mathfrak{c}_k(\mathcal{E})$ not as a mere object of the Chow category $\mathfrak{CH}^k(X)$, but as a whole intersection product functor

$$\mathfrak{c}_k(\mathcal{E}) \cap -: \tilde{\mathfrak{CH}}^p(X) \to \tilde{\mathfrak{CH}}^{p+k}(X). \tag{2}$$

In the first five paragraphs of §1, we introduce $\mathfrak{c}_1(\mathcal{L}) \cap A$ for a line bundle $\mathcal{L}$, using a functorial version of the product

$$H^1(X, K_1) \otimes E_2^{p,q}(X) \to E_2^{p+1,q-1}(X),$$

where E_2 is the E_2-term of Quillen's spectral sequence. Starting from this point, in the remaining paragraphs of §1 we construct (2), copying Grothendieck's definition of the Chern classes. We also prove a Whitney isomorphism for the Chern functors. The second paragraph considers further properties of the Chern functors (like relation to the Gysin functor $\underline{f}^!$

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