Unification in Propositional Logic

SILVIO GHILARDI

Università degli Studi di Milano
Dipart. di Scienze dell'Informazione

Dresden, October 2004

PART I: FOUNDATIONS

1. MAIN AIM

It is well-known that standard propositional logics have algebraic counterparts into special lattice-based varieties, for instance:

classical logic	*versus*	Boolean algebras
intuitionistic logic	*versus*	Heyting algebras
modal logics	*versus*	Boolean algebras with operators

...

Thus it makes sense to apply *E-Unification Theory* to a propositional logic L: we can define various kinds of unification problems (elementary, with constants, general), speak about unifiers, unification types, etc., *in the logic* L: when we do that, we are simply using standard definitions (see e.g. Baader-Snyder survey in the 'Handbook of Automated Reasoning') for the corresponding algebraic theories.

Of course, definitions can be 'translated back' directly to logic: we shall do that for elementary unification in intuitionistic logic, give results and applications for that specific case and mention parallel extensions to some standard modal logics over $K4$.

N.B.: in the 'logical translation' below, some obvious simplifications are directly applied.

2. PROBLEMS AND UNIFIERS

- Intuitionistic *formulas* are built from variables $x, y, \ldots$ by using connectives $\wedge, \vee, \rightarrow, \bot, \top$. $F(\underline{x})$ are the formulas containing at most the variables $\underline{x} = x_1, \ldots, x_n$.
- $\vdash A$ means that A is provable in intuitionistic propositional calculus (IPC);
- A *substitution* $\sigma : F(\underline{x}) \longrightarrow F(\underline{y})$ is a map commuting with the connectives.
- Two substitutions of same domain $F(\underline{x})$ and codomain $F(\underline{y})$ are *equivalent* (written $\sigma_1 \sim \sigma_2$) iff for all $x \in \underline{x}$,

$$\vdash \sigma_1(x) \leftrightarrow \sigma_2(x).$$

- $\sigma_1 : F(\underline{x}) \longrightarrow F(\underline{y}_1)$ is *more general* than $\sigma_2 : F(\underline{x}) \longrightarrow F(\underline{y}_2)$ iff there is $\tau : F(\underline{y}_1) \longrightarrow F(\underline{y}_2)$ such that $(\tau \circ \sigma_1) \sim \sigma_2$.

- A unification problem is a formula $A(\underline{x})$; a solution to it (i.e. a *unifier*) is any substitution $\sigma : F(\underline{x}) \longrightarrow F(\underline{y})$ such that

$$\vdash \sigma(A).$$

- The remaining definitions (complete sets of unifiers, bases, unification types, etc.) are the usual ones.

Theorem 1. *Unification type is finite: that is, for every unification problem A there is a finite (possibly empty) set S_A of solutions, such that any other possible solution for A is less general than a member of S_A.*

Also, S_A is computable from A (we shall describe an algorithm later on).

3. KRIPKE MODELS

A finite *Kripke model* over $\underline{x}$ is a triple $\langle \underline{x}, P, u \rangle$, where $\underline{x}$ is a finite tuple of variables, P is a finite rooted poset and $u : P \longrightarrow \mathcal{P}(\underline{x})$ is a map satisfying the monotonicity requirement

$$q \leq p \quad \Rightarrow \quad u(q) \subseteq u(p).$$

If $u : P \longrightarrow \mathcal{P}(\underline{x})$ is a Kripke model, $A \in F(\underline{x})$ and $p \in P$, the forcing relation $u \models_p A$ is defined as:

$$\begin{array}{lll} u \models_p x & \text{iff} & x \in u(p); \\ u \models_p \top; & & \\ u \not\models_p \bot; & & \\ u \models_p A_1 \wedge A_2 & \text{iff} & u \models_p A_1 \text{ and } u \models_p A_2; \\ u \models_p A_1 \vee A_2 & \text{iff} & u \models_p A_1 \text{ or } u \models_p A_2; \\ u \models_p A_1 \rightarrow A_2 & \text{iff} & \forall q \geq p \, (u \models_q A_1 \Rightarrow \\ & & \quad \Rightarrow u \models_q A_2). \end{array}$$

(Bounded) bisimulation. Given two models over $\underline{x}$, say $u : P \longrightarrow \mathcal{P}(\underline{x})$ and $v : Q \longrightarrow \mathcal{P}(\underline{x})$, the Ehrenfeucht-Fraissé-Fine game on them has two players. Player I chooses either P or Q and a point in it; Player II picks a point in the other poset and so on. The only rule is that, if w (either in P or in Q) has been chosen, then only points $w' \geq w$ can be chosen in the successive move. The game has a preassigned number n of moves. Player II wins iff it succeeds in keeping the forcing of propositional variables pairwise identical (i.e. it wins iff for every $i = 1, \ldots, n$, if $p_i \in P$ and $q_i \in Q$ have been chosen in the i-th round, then we have $u(p_i) = u(q_i)$).

We say that u and v are *n-bisimilar* (written $u \sim_n v$) iff Player II has a winning strategy in the game with n moves. It turns out that u and v are n-bisimilar iff they force (in the root) precisely the same formulas whose nested implications degree is at most n.

The relation $u \leq_n v$ is defined as $\sim_n$, the only difference is that Player I is now allowed to play the first move only in the domain of u.

The bisimulation relation $u \sim_\infty v$ is defined as $\sim_n$, but now the game has infinitely many moves.

To get the appropriate *geometric* intuition, one has to consider formulas as subspaces of the spaces of models and substitutions as transformations among such spaces. Formally, define:

- for a tuple $\underline{x}$, $F(\underline{x})^*$ is the set ('space') of models over $\underline{x}$;

- for a formula $A \in F(\underline{x})$, let

$$A^* = \{u \in F(\underline{x})^* \mid u \models A\},$$

  where $u \models A$ means that A is true at all points of P (or, equivalently, at the root).

- for a substitution $\sigma : F(\underline{x}) \longrightarrow F(\underline{y})$, define

$$\sigma^* : F(\underline{y})^* \longrightarrow F(\underline{x})^*$$

  by associating with $u : P \longrightarrow \mathcal{P}(\underline{y})$, the Kripke model $\sigma^*(u) : P \longrightarrow \mathcal{P}(\underline{x})$ given by:

$$x \in \sigma^*(u)(p) \ \text{ iff } \ u \models_p \sigma(x)$$

  for all $x \in \underline{x}$ and $p \in P$.

Easy but important facts:

- $\vdash A \to B$ iff $A^* \subseteq B^*$;
- $u \models_p \sigma(A)$ iff $\sigma^*(u) \models_p A$;
- $\vdash \sigma(A)$ iff the image of σ^* is contained in A^*;
- $(\sigma \circ \tau)^* = \tau^* \circ \sigma^*$.

It can be shown that the sets of models of the kind A^* are precisely the sets of models closed under $\leq_n$ for sufficiently large n (see the book (G.-Zawadowski 2002), for a full duality theory).

Digression. A very remarkable (absolutely non trivial) fact is *Pitts' theorem*, that can be reformulated by saying that $[\sigma^*(A^*)]_{\sim_\infty}$ is always of the kind B^*, for some B (in modal logic, Pitts' theorem holds for GL, Grz, but not for $K4, S4$). Here $[\sigma^*(A^*)]_{\sim_\infty}$ is the closure of $\sigma^*(A^*)$ under bisimulation.

Pitts' theorem has many meanings: proof-theoretically, it means that second order IPC can be interpreted in IPC, model-theoretically it means that the theory of Heyting algebras has a model completion and categorically that the opposite of the category of finitely presented Heyting algebras is a Heyting category.

To prove Pitts' theorem semantically, one basically has to show that $[\sigma^*(A^*)]_{\sim_\infty}$ is closed under $\leq_N$, for sufficiently large N (depending - recursively ! - on the implicational degree of A and of the formulas $\sigma(x)$).

4. PROJECTIVE FORMULAS

The notion of a projective (finitely presented) algebra is well-known and, once 'translated' into symbolic logic means the following. A formula $A \in F(\underline{x})$ is *projective* iff there is a unifier $\mu_A : F(\underline{x}) \longrightarrow F(\underline{x})$ for it such that we have

$$(*) \qquad \vdash A \to (x \leftrightarrow \mu_A(x))$$

for all $x \in \underline{x}$ (such a μ_A is easily seen to be automatically an mgu for A).

Geometrically, A is projective iff A^* is a *contractible* subspace of $F(\underline{x})^*$: that is, there is a substitution μ_A, whose associated transformation μ_A^* maps $F(\underline{x})^*$ onto A^*, while keeping A^* fixed.

Boolean Unification. In the Boolean case, a formula $A \in F(\underline{x})$ is projective iff it is satisfiable (from this, unitarity of Boolean unification is immediate). Here you are the proof.

Let $A \in F(\underline{x})$ be satisfiable by the assignment a. Define θ_a^A by:

$$\theta_a^A(x) = \begin{cases} A \to x, & \text{if } a(x) = 1; \\ A \wedge x, & \text{if } a(x) = 0. \end{cases}$$

That θ_a^A satisfy $(*)$ is easy; to show that $\theta_a^A(A)$ is provable in classical logic, one may use the following argument. We need to show that $(\theta_a^A)^*(u) \in A^*$ (here u is any one-point model - remember we are in classical logic), but from the definition

$$(\theta_a^A)^*(u) \models x \;\text{ iff }\; u \models \theta_a^A(x),$$

we realize that $(\theta_a^A)^*(u)$ is u (if $u \in A^*$), or it is equal to a (if $u \notin A^*$): in both cases $(\theta_a^A)^*(u) \in A^*$.

The substitutions θ_a^A used in the Boolean case, contribute to the construction of minimal bases of unifiers in IPC too. θ_a^A is indexed by a formula $A \in F(\underline{x})$ and by a classical assignment a over $\underline{x}$.

How does the transformation $(\theta_a^A)^*$ act on a Kripke model $u : P \longrightarrow \mathcal{P}(\underline{x})$? First, it does not change the forcing in the points $p \in P$ such that $u \models_p A$. In the other points q, θ_a^A tries to make the forcing 'as close as possible' to a: if $a(x) = 0$, then $x \notin (\theta_a^A)^*(u)(q)$ and if $a(x) = 1$, then $x \in (\theta_a^A)^*(u)(q)$, unless this is impossible because in some point $p \geq q$, we have $u \models_p A$ and $u \not\models_p x$ (recall that forcing in such p's is not changed by $(\theta_a^A)^*$).

Thus, applying any iteration of transformations of the kind $(\theta_a^A)^*$ keeps models in A^* fixed and (in principle) pushes further models into A^*.

On the other hand, it is easily seen that a contractible subspace A^* is *extensible*: if $u \notin A^*$, then it is possible to change the forcing in the points of u which do not force A in such a way that the model so modified belongs *in toto* to A^* (this is mainly because the transformations induced by a substitution commute with restrictions to generated submodels).

By exploiting the above mentioned effect of transformations of the kind $(\theta_a^A)^*$ on Kripke models, one can show that a carefully built iteration θ_A of substitutions of the kind θ_a^A can in fact always act as the contraction transformation of a contractible A^* (that is, either such θ_A unifies A and consequently works as a contraction, or A is not projective).[1]

This leads to the following

[1]For simplicity, we skip the precise definition of θ_A, see the papers in the references below.

Theorem 2. *For a formula $A \in F(\underline{x})$ the following are equivalent:*

(i) *A is projective;*

(ii) *A^* is extensible;*

(iii) *θ_A unifies A.*

In particular, it is decidable whether A is projective or not and mgus of projective formulas are computable.

For modal logics over $K4$ with finite model property, the above theorem holds too: here only formulas of the kind $\Box^+ A$ are candidate projective and the substitution $\theta_{\Box^+ A}$ is defined in a slightly different way (much more iterations of the $\theta_a^{\Box^+ A}$ are needed). The proof of the Theorem uses a more sophisticated argument (based on bounded bisimulation ranks) which is not necessary in intuitionistic logic, where more elementary considerations suffice.

4. UNIFICATION IS FINITARY

We are not far from the general result. If $\sigma : F(\underline{x}) \longrightarrow F(\underline{y})$ unifies $A \in F(\underline{x})$, then A^* contains the (bisimulation closure of the) image of σ^*; the latter is an extensible set which, by Pitts theorem[2] is of the kind B^*. Then B^* is projective, it has an mgu μ_B which is a better substitution than σ; also $B^* \subseteq A^*$ implies that μ_B is a unifier for A too (better than the original σ). Thus, *in order to unify A we do not loose in generality if we restrict to mgus μ_B of projective formulas B implying A.*

Unfortunately, however, there are usually infinitely many projective formulas in $F(\underline{x})$ implying a given $A \in F(\underline{x})$. Hence we need a refinement of the argument: bounded bisimulations will help.

[2] The whole argument can be made independent on Pitts theorem (this is important for modal logic where Pitts theorem can fail).

First, notice that for projective P (see $(*)$)

$$(+) \qquad \vdash P \to B \quad \text{iff} \quad \vdash \mu_P(B).$$

Hence, if P_1, P_2 are both projective, $\vdash P_1 \to P_2$ means that μ_{P_1} is less general than μ_{P_2} ('provability comparison is the same as mgus comparison for projective formulas').

Secondly, for any n, the $\leq_n$-closure of an extensible set is still extensible. In particular, if P is projective, A has implicational degree n and $P^* \subseteq A^*$, we have $P^* \subseteq [P^*]_{\leq n} \subseteq A^*$ and $[P^*]_{\leq n} = Q^*$, for some Q projective of degree at most n.

These two facts, once combined, say that *in order to unify A we do not loose in generality if we restrict to mgus μ_P of projective formulas P implying A and having at most the same implicational degree as A.*

This shows finitarity of intuitionistic unification and gives a type conformal unification algorithm. The arguments in this section can be repeated for the modal logics $K4, S4, Grz, GL$ without any essential modification.

As a corollary, we also get that a formula of degree n is *unifiable* iff there is a projective formula implying it and having at most degree n (this gives an effective test for solvability of unification problems - the test is precious in the modal case, where unifiability does not reduce to satisfiability, as it happens in the intuitionistic case).

4. ADMISSIBLE RULES

Let A be a formula of implicational degree n. A *projective approximation* Π_A of A is any finite set of projective formulas implying A and having at most implicational degree n; Π_A must be such that any further projective formula implying A implies a member of Π_A too.

Clearly, for any projective approximation Π_A of A, the set

$$S_A \;=\; \{\mu_P \mid P \in \Pi_A\}$$

is a finite complete set of unifiers for A.

An inference rule

$$\frac{A}{B}$$

is *admissible* in IPC iff we have $\vdash \sigma(B)$ for all σ such that $\vdash \sigma(A)$.

From the above considerations and from (+), it follows that the rule A/B is admissible iff we have

$$\vdash P \to B$$

for all $P \in \Pi_A$, where Π_A is any projective approximation of A.

The same result holds for modal logics $K4, S4, Grz, GL$.

PART II: ALGORITHMS

5. TWO PROBLEMS

From Part I, it is evident that there are two main computational problems for a given formula A:

- check whether A is projective or not;
- in the negative case, compute a projective approximation of A.

The explicit computation of mgus or of complete sets of unifiers seems to be less important (see the application to admissible rules) and, in any case, it is only a question of writing down explicitly defined substitutions (namely the θ_P's for $P \in \Pi_A$).

The algorithms for solving our two problems that are suggested by the proofs of Theorems 2 and 1 turn out to be very inefficient. In particular, Theorem 1 gives a non elementary procedure (because the number of non provably equivalent formulas up to a certain implicational degree is non-elementary).

We shall provide an *exponential* algorithm for the first problem and a *double exponential* one for the second.

6. CHECK-PROJECTIVITY

The algorithm analyzes 'reasons' why a formula A can be *false in the root* of a Kripke model and *true in the context* of the model (namely, in all the points different from the root). If such reasons 'cannot be repaired' by changing the forcing in the root of the model, A is not projective; otherwise A is projective.

The algorithm is a mixture of tableaux and of resolution methods. It deals with sets of *signed subformulas* of A, where a signed subformula is a 'modality' followed by a subformula of A. We have truth, context and atomic modalities, whose meaning is explained as follows:

Truth Modalities

TB ‘B is true in the root’

FB ‘B is false in the root’

Context Modalities

T_cB ‘B is true in the context’

F_cB ‘B is false in the context’

Atomic modalities

x^+ ‘x is true in the root’

x^- ‘x is false in the root and true in the context’

The algorithm is initialized to

$$\{FA\}.$$

It applies the following rules in a dont'care non-deterministic way, till no rule applies anymore. To apply a rule, replace (in the current state) the set(s) of signed formulas in the premise by the set(s) of signed formulas in the conclusion.

The algorithm terminates in exponentially many steps.[3]

A is projective iff all output sets contain atomic modalities.

[3] Provided some control device forbids repeated applications of the same instance of the Resolution Rule.

Tableaux Rules

$$\frac{\Delta \cup \{TB_1 \wedge B_2\}}{\Delta \cup \{TB_1, TB_2\}}$$

$$\frac{\Delta \cup \{T\top\}}{\Delta}$$

$$\frac{\Delta \cup \{TB_1 \vee B_2\}}{\Delta \cup \{TB_1\}, \Delta \cup \{TB_2\}}$$

$$\frac{\Delta \cup \{T\bot\}}{\times}$$

$$\frac{\Delta \cup \{FB_1 \wedge B_2\}}{\Delta \cup \{FB_1\}, \Delta \cup \{FB_2\}}$$

$$\frac{\Delta \cup \{F\top\}}{\times}$$

$$\frac{\Delta \cup \{FB_1 \vee B_2\}}{\Delta \cup \{FB_1, FB_2\}}$$

$$\frac{\Delta \cup \{F\bot\}}{\Delta}$$

$$\frac{\Delta \cup \{TB_1 \to B_2\}}{\Delta \cup \{FB_1, T_cB_1 \to B_2\}, \Delta \cup \{TB_2\}}$$

$$\frac{\Delta \cup \{FB_1 \to B_2\}}{\Delta \cup \{F_cB_1 \to B_2\}, \Delta \cup \{TB_1, FB_2\}}$$

$$\frac{\Delta \cup \{Tx\}}{\Delta \cup \{x^+\}}$$

$$\frac{\Delta \cup \{Fx\}}{\Delta \cup \{x^-\}, \Delta \cup \{F_cx\}}$$

Resolution Rule

$$\frac{\Delta \cup \{x^+\}, \Gamma \cup \{x^-\}}{\Delta \cup \{x^+\}, \Gamma \cup \{x^-\}, \Delta \cup \Gamma \cup \{T_c x\}}$$

Simplification Rule

$$\frac{\Delta}{\times}$$

provided there exists some $F_c C \in \Delta$ such that

$$\vdash A \wedge \bigwedge \{B \mid T_c B \in \Delta\} \to C$$

(the application of this rule requires calling for an IPC prover, hence solving a PSPACE-complete problem).

Refinements are possible: e.g. the calculus is complete, if we use ordered resolution instead of plain resolution. It is compatible with simplifications like

Optional Simplification Rule

$$\frac{\Delta}{\times}$$

provided Δ is subsumed by some other Γ or provided it contains a pair of contradictory modalities.

PROBLEMS:

- there might be a PSPACE algorithm (?);
- use ideas from DPLL, like 'truth-value branching', etc. (?);
- define useful strategies (the obvious one is that of giving priority to tableaux rules);
- make a thorough comparison with the hypersequent calculus for admissibility of (Iemhoff, 2003) (this is theoretically based on an eleboration of the above results of mine and of previous results from the dutch school).

Extensions to modal logics $K4, S4, GL, Grz$ are not difficult (Zucchelli, 2004): basically, it is sufficient to appropriately modify the tableaux rules.

7. ITERATIONS

If A is not projective, there is an output set Δ for A which does not contain neither truth (rules are exhaustively applied) nor atomic modalities. If there are no signed subformulas of the kind F_cB in Δ, A has empty projective approximation (consequently, it is not unifiable). Otherwise, *for each* $F_cC \in \Delta$, we re-run the check-projectivity algorithm on the new formula

$$A \wedge \bigwedge\{B \mid T_cB \in \Delta\} \to C.$$

We can in fact re-initialize the algorithm by simply replacing, in the final state, the set Δ by

$$\{TB \mid T_cB \in \Delta\} \cup \{FC\}.$$

Such iterations stop whenever the projectivity test is positive: *the projective formulas so collected are a projective approximation of A*. Notice that

- only sets of signed subformulas of the original A occur in any step of any iteration;
- the branches in the iterations tree are at most exponentially long (the same Δ cannot be used twice for re-initialization along the same branch, because once Δ is used, Simplification Rule can remove any further occurrence of it);
- only formulas which are implications of a conjunction of subformulas of A versus a subformula of A can be in the final projective approximation of A.

Starting from the above observations, it can be shown that computing projective approximations requires double exponential time; moreover projective approximations themselves seem to contain double exponentially many exponentially long formulas.

Obvious open problem: is it possible to do anything better? Practical examples examined so far (even by computer) do not confirm that projective approximations can be that large ....

8. A FIRST IMPLEMENTATION

(Zucchelli, 2004) realized a first implementation in his Master Thesis. His system is designed to check projectivity, compute projective approximations and decide admissibility of rules for modal systems $K4$ and $S4$ (this encompasses intuitionistic logic via modal translation).

For Simplification Rule provability tests, the well-performed RACER prover is used.

We report some experimental tests (NB: the actual system is still at a very prototypical level and even evident optimizations have not yet been implemented).

The system was able to recognize in less than one second simple well-known admissible IPC-rules like

$$\frac{\neg x \to y \vee z}{(\neg x \to y) \vee (\neg x \to z)}$$

$$\frac{((x \to y) \to z) \to ((x \to y) \vee \neg z)}{\neg\neg(x \to y) \vee \neg z}$$

Visser rules

$$\frac{\bigwedge_{i=1}^{n}[(x_i \to y_i) \to x_{i+1} \vee x_{i+2}]}{\bigvee_{i=1}^{n+2}[\bigwedge_{j=1}^{n}(x_j \to y_j) \to x_i]}$$

were proved to be admissible for $n \leq 11$.

The 6 formulas in the projective approximation of (the modal translation of) the formula

$$[[x_1 \to (y_1 \vee z_1)] \wedge [(x_1 \to x_2) \to y_2 \vee z_2] \to w] \to t_1 \vee t_2$$

from (G. 2002) were correctly computed in about 3 sec.

8. FURTHER READINGS

The content of Part I is covered by the papers

- S. Ghilardi *Unification in intuitionistic logic*, Journal of Symbolic Logic, vol.64, n.2, pp.859-880 (1999);
- S. Ghilardi *Best solving modal equations*, Annals of Pure and Applied Logic, 102, pp.183-198 (2000),

whereas the content of Part II is covered by

- S. Ghilardi *A resolution/tableaux algorithm for projective approximations in IPC*, Logic Journal of the IGPL, vol.10, n.3, pp.227-241 (2002);
- D. Zucchelli *Studio e realizzazione di algoritmi per l'unificazione nelle logiche modali*, Master Thesis, Università degli Studi di Milano (2004).[4]

[4]A regular english paper is in preparation.

Further topics on unification in propositional logic can be found in:

- S. Ghilardi *Unification through projectivity*, Journal of Logic and Computation, 7, 6, pp.733-752 (1997);
- S. Ghilardi *Unification, Finite Duality and Projectivity in Varieties of Heyting Algebras*, Annals of Pure and Applied Logic, vol. 127/1-3, pp.99-115 (2004);
- S. Ghilardi, L. Sacchetti *Filtering Unification and Most General Unifiers in Modal Logic*, Journal of Symbolic Logic, 69, 3, pp.879-906 (2004).

For unification and matching with constants in fragments of multimodal languages (within a description logics context), see

- F. Baader, P. Narendran *Unification of Concept Terms in Description Logics*, Journal of Symbolic Computation, 31(3), pp. 277-305 (2001);
- F. Baader, R. Küsters *Matching in description logics with existential restrictions*, in "Principles of Knowledge Representation and Reasoning (KR '00)", (2000).
- F. Baader, R. Küsters *Unification in a description logic with transitive closure of roles*, in "Logic for Programming, Artificial Intelligence and Reasoning (LPAR '01)", Lecture Notes in Artificial Intelligence, (2001).

For furter information on admissibility of inference rules, see

- V. Rybakov *Admissibility of Logical Inference Rules*, North Holland (1997);
- R. Iemhoff *On the admissible rules of intuitionistic propositional logic*, Journal of Symbolic Logic, 66, pp. 281-243 (2001);
- R. Iemhoff *Towards a proof system for admissibility*, in M. Baaz and A. Makowski (eds.) 'Computer Science Logic 03', Lecture Notes in Computer Science 2803, pp.255-270 (2003);
- G. Mints, A. Kojevnikov *Intuitionistic Frege systems are polynomially equivalent*, preprint (2004).

For more on Pitts theorem, (bounded) bisimulation, games and duality in propositional logic, see the book

- S.Ghilardi, M. Zawadowski *Sheaves, Games and Model Completions*, Trends in Logic Series, Kluwer (2002).

July 2006 (update): undecidability of unifiability in various modal systems *endowed with the universal modality* has been recently shown by F. Wolter and M. Zakharyaschev.