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Interpolation, Amalgamation and Combination (the non-disjoint signatures case)

Extended Version

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Abstract. In this paper, we study the conditions under which existence of interpolants (for quantifier-free formulae) is modular, in the sense that it can be transferred from two first-order theories T_1, T_2 to their combination $T_1 \cup T_2$. We generalize to the non-disjoint signatures case the results from [3]. As a surprising application, we relate the Horn combinability criterion of this paper to superamalgamability conditions known from propositional logic and we use this fact to derive old and new results concerning fusions transfer of interpolation properties in modal logic.

1 Introduction

Craig's interpolation theorem [5] applies to first order formulae and states that whenever the formula $\phi \to \psi$ is valid, then it is possible to find a formula θ such that (i) $\phi \to \theta$ is valid; (ii) $\theta \to \psi$ is valid, and (iii) θ is defined over the common symbols of ϕ and ψ. Interpolation theory has a long tradition in non-classical logics (see for instance the seminal papers by L.L. Maksimova [11],[12]) and has been recently introduced also in verification, after the work of McMillan (see, e.g., [14]). Intuitively, the interpolant θ can be seen as an over-approximation of ϕ with respect to ψ: thus, for example, in the abstraction-refinement phase of software model checking [9], interpolants are used to compute increasingly precise over-approximations of the set of reachable states.

Of particular importance for verification techniques are those algorithms capable of computing *quantifier-free* interpolants in presence of some background theory. This is so because several symbolic verification problems are formalized by representing sets of states and transitions as quantifier-free formulae. Unfortunately, Craig's interpolation theorem does not guarantee that it is always possible to compute quantifier-free interpolants when reasoning *modulo a first-order theory*: in fact, for certain first-order theories, it is known that quantifiers must occur in interpolants of quantifier-free formulae [10]. Even when quantifier-free interpolants exist for single theories, this might not be anymore the case when considering their combinations (see e.g. Example 3.5 below). Since verification techniques frequently require to reason in combinations of theories, methods to modularly combine available interpolation algorithms are indeed desirable.

The study of the modularity property of quantifier-free interpolation was first started in [26], where the disjoint signatures convex case was solved; in [3] - the journal version of [2] - the non-convex (still disjoint) case was also thoroughly investigated. The analysis in [3] is large-spectrum: combinability of quantifier-free interpolation is first semantically analyzed (where it is related to strong sub-amalgamability), then it is syntactically characterized and finally suitable combination algorithms are designed.

This paper intends to be a first contribution for an extension to the non-disjoint signatures case. Given the complexity of the problem, we shall limit to semantic investigations, leaving for future research the subsequent, algorithmically oriented aspects. However, we show that our semantic techniques can be quite effective in practice: in fact, we show how to use them in order to establish that some theories combining integers and common datatypes (lists, trees, etc.) indeed enjoy quantifier-free interpolation. In addition, we employ our results in order to get interesting information concerning the transfer of interpolation properties to the fusion of modal logics: in fact, not only we show how to obtain Wolter's interpolation fusion transfer theorem [25] for normal modal logics, but we also identify a modular interpolation property for the non-normal case.

In attacking combination problems for non-disjoint signatures, we follow the model-theoretic approach successfully employed in [6], [8], [19], [16], [17], [18]; this approach relies on the notion of T_0-compatibility, in order to identify modular conditions for combinability. The reason why this approach works can roughly be explained as follows. In combining a model of a theory T_1 with a model of a theory T_2, one needs to produce a superstructure of both of them: in such a superstructure, additional constraints in the shared subsignature might turn out to be satisfied and T_0-compatibility is meant to keep satisfiability of constraints in superstructures under control inside T_1 and T_2. This is because T_0-compatibility refers to model-completeness and model-completeness is the appropriate technique [4] to talk about satisfiability of quantifier-free formulae in extended structures.

The paper is organized as follows: in Section 2, we introduce notations and basic ingredients from the literature; in Section 3 we obtain a first general result (Theorem 3.2) and show how to use it in examples taken from verification theories. In the final Section 4, we apply our results to modal logic (Corollary 4.3 and Theorem 4.7); the proofs of the results from this last section require some algebraic logic background, so they are moved to the Appendix.

2 Formal Preliminaries

We adopt the usual first-order syntactic notions of signature, term, atom, (ground) formula, sentence, and so on. Let Σ be a first-order signature; we assume the binary equality predicate symbol '=' to be added to any signature (so, if $\Sigma = \emptyset$, then Σ just contains equality). The signature obtained from Σ by adding it a set $\underline{a}$ of new constants (i.e., 0-ary function symbols) is denoted by $\Sigma^{\underline{a}}$. A *positive clause* is a disjunction of atoms. A *constraint* is a conjunction of literals. A formula is *quantifier-free* (or open) iff it does not contain quantifiers. A *Σ-theory* T is a set of sentences (called the axioms of T) in the signature Σ and it is *universal* iff it has universal closures of open formulae as axioms.

We also assume the usual first-order notion of interpretation and truth of a formula, with the proviso that the equality predicate $=$ is always interpreted as the identity relation. We let $\bot$ denote a ground formula which is true in no structure. A formula φ is *satisfiable* in $\mathcal{M}$ iff its *existential* closure is true in $\mathcal{M}$. A Σ-structure $\mathcal{M}$ is a *model* of a Σ-theory T (in symbols $\mathcal{M} \models T$) iff all the sentences of T are true in $\mathcal{M}$. If φ is a formula, $T \models \varphi$ ('*φ is a logical consequence of T*') means that the universal closure of φ is true in all the models of T. T is *consistent* iff it has a model, i.e., if $T \not\models \bot$. A sentence φ is T-consistent iff $T \cup \{\varphi\}$ is consistent. A Σ-theory T is *complete* iff for every Σ-sentence φ, either φ or $\neg\varphi$ is a logical consequence of T. T admits *quantifier elimination* iff for every formula $\varphi(\underline{x})$ there is a quantifier-free formula $\varphi'(\underline{x})$ such that $T \models \varphi(\underline{x}) \leftrightarrow \varphi'(\underline{x})$ (notations like $\varphi(\underline{x})$ mean that φ has free variables only among the tuple $\underline{x}$).

If $\Sigma_0 \subseteq \Sigma$ is a subsignature of Σ and if $\mathcal{M}$ is a Σ-structure, the *Σ_0-reduct* of $\mathcal{M}$ is the Σ_0-structure $\mathcal{M}_{|\Sigma_0}$ obtained from $\mathcal{M}$ by forgetting the interpretation of function and predicate symbols from $\Sigma \setminus \Sigma_0$. A *$\Sigma$-homomorphism* (or, simply, a homomorphism) between two Σ-structures $\mathcal{M}$ and $\mathcal{N}$ is any mapping $\mu : |\mathcal{M}| \longrightarrow |\mathcal{N}|$ among the support sets $|\mathcal{M}|$ of $\mathcal{M}$ and $|\mathcal{N}|$ of $\mathcal{N}$ satisfying the condition

$$\mathcal{M} \models \varphi \quad \Rightarrow \quad \mathcal{N} \models \varphi \tag{1}$$

for all $\Sigma^{|\mathcal{M}|}$-atoms φ (here $\mathcal{M}$ is regarded as a $\Sigma^{|\mathcal{M}|}$-structure, by interpreting each additional constant $a \in |\mathcal{M}|$ into itself and $\mathcal{N}$ is regarded as a $\Sigma^{|\mathcal{M}|}$-structure by interpreting each additional constant $a \in |\mathcal{M}|$ into $\mu(a)$). In case condition (1) holds for all $\Sigma^{|\mathcal{M}|}$-literals, the homomorphism μ is said to be an *embedding* and if it holds for all first order formulae, the embedding μ is said to be *elementary*. If $\mu : \mathcal{M} \longrightarrow \mathcal{N}$ is an embedding which is just the identity inclusion $|\mathcal{M}| \subseteq |\mathcal{N}|$, we say that $\mathcal{M}$ is a *substructure* of $\mathcal{N}$ or that $\mathcal{N}$ is an *extension* of $\mathcal{M}$. A Σ-structure $\mathcal{M}$ is said to be *generated by* a set X included in its support $|\mathcal{M}|$ iff there are no proper substructures of $\mathcal{M}$ including X.

Given a signature Σ and a Σ-structure $\mathcal{A}$, we indicate with $\Delta_\Sigma(\mathcal{A})$ the *diagram* of $\mathcal{A}$: this is the set of sentences obtained by first expanding Σ with a fresh constant $\bar{a}$ for every element a from $|\mathcal{A}|$ and then taking the set of ground $\Sigma^{|\mathcal{A}|}$-literals which are true in $\mathcal{A}$ (under the natural expanded interpretation mapping $\bar{a}$ to a).

Finally, we point out that all the above definitions can be extended in a natural way to many-sorted signatures (we shall use many-sorted theories in some examples).

2.1 Model completion and T_0-compatibility

We recall a standard notion in Model Theory, namely the notion of a *model completion* of a first order theory [4] (we limit the definition to universal theories, because we shall use only this case):

Definition 2.1. *Let T_0 be a universal Σ_0-theory and let $T_0^\star \supseteq T_0$ be a further Σ_0-theory; we say that $T_0^\star$ is a model completion of T_0 iff:* (i) *every model of T_0*

can be embedded into a model of $T_0^\star$; (ii) *for every model $\mathcal{M}$ of T_0, we have that $T_0^\star \cup \Delta_{\Sigma_0}(\mathcal{M})$ is a complete theory in the signature $\Sigma_0^{|\mathcal{M}|}$.*

Being T_0 universal, condition (ii) is equivalent to the fact that $T_0^\star$ *has quantifier elimination*; we recall also that the model completion $T_0^\star$ of a theory T_0 is unique, if it exists (see [4] for these results and for examples).

We also recall the concept of T_0-compatibility [6,8], which is crucial for our combination technique.

Definition 2.2. *Let T be a theory in the signature Σ and let T_0 be a universal theory in a subsignature $\Sigma_0 \subseteq \Sigma$. We say that T is* T_0-compatible *iff $T_0 \subseteq T$ and there is a Σ_0-theory $T_0^\star$ such that:*

(i) *$T_0 \subseteq T_0^\star$;*
(ii) *$T_0^\star$ is a model completion of T_0;*
(iii) *every model of T can be embedded, as a Σ-structure, into a model of $T \cup T_0^\star$.*

Notice that if T_0 is the empty theory over the empty signature, then T_0^* is the theory axiomatizing an infinite domain, and the requirement of T_0-compatibility is equivalent to the stably infinite requirement of the Nelson-Oppen schema [15,23] (in the sense that T is T_0-compatible iff it is stably infinite). We remind that a theory T is stably infinite iff every T-satisfiable quantifier-free formula (from the signature of T) is satisfiable in an infinite model of T. By compactness, it is possible to show that T is stably infinite iff every model of T embeds into an infinite one.

We shall see many examples of T_0-compatible theories (for various T_0) during the paper, here we just underline that T_0-compatibility is a modular condition. The following result is proved in [6] (as Proposition 4.4):

Proposition 2.3. *Let T_1 be a Σ_1-theory and let T_2 be a Σ_2-theory; suppose they are both compatible with respect to a Σ_0-theory T_0 (where $\Sigma_0 := \Sigma_1 \cap \Sigma_2$). Then $T_1 \cup T_2$ is T_0-compatible too.*

2.2 Interpolation and Amalgamation

We say that a theory T *has quantifier-free interpolation* iff the following hold, for every pair of quantifier free formulae $\varphi(\underline{x}, \underline{y}), \psi(\underline{y}, \underline{z})$: if $T \models \varphi(\underline{x}, \underline{y}) \to \psi(\underline{y}, \underline{z})$, then there exists a quantifier-free formula $\theta(\underline{y})$ such that $T \models \varphi(\underline{x}, \underline{y}) \to \theta(\underline{y})$ and $T \models \theta(\underline{y}) \to \psi(\underline{y}, \underline{z})$. We underline that the requirement that θ is quantifier-free is essential: in general such a $\theta(\underline{y})$ exists by the Craig interpolation theorem, but it is not quantifier-free even if φ, ψ are such. [1] Quantifier-free interpolation property can be semantically characterized using the following notions, introduced in [1,3]:

[1] Notice that in the above definition free function and predicate symbols (not already present in the signature Σ of T) are not allowed; allowing them (and requiring that only shared symbols occur in the interpolant θ) produces a different stronger definition, which is nevertheless reducible to quantifier-free interpolation in the combination with the theory of equality with uninterpreted function symbols (see [3]).

Definition 2.4. *A theory T has the* sub-amalgamation property *iff, for given models $\mathcal{M}_1$ and $\mathcal{M}_2$ of T sharing a common substructure $\mathcal{A}$, there exists a further model $\mathcal{M}$ of T endowed with embeddings $\mu_1 : \mathcal{M}_1 \longrightarrow \mathcal{M}$ and $\mu_2 : \mathcal{M}_2 \longrightarrow \mathcal{M}$ whose restrictions to the support of $\mathcal{A}$ coincide. The triple $(\mathcal{M}, \mu_1, \mu_2)$ (or, by abuse, $\mathcal{M}$ itself) is said to be a T*-sub-amalgama *of $\mathcal{M}_1, \mathcal{M}_2, \mathcal{A}$.*

Definition 2.5. *A theory T has the* strong sub-amalgamation property *if the T-sub-amalgama $(\mathcal{M}, \mu_1, \mu_2)$ of $\mathcal{M}_1, \mathcal{M}_2, \mathcal{A}$ can be chosen so as to satisfy the following additional condition: if for some m_1, m_2 we have $\mu_1(m_1) = \mu_2(m_2)$, then there exists an element a in $|\mathcal{A}|$ such that $m_1 = a = m_2$.*

If T is universal, then every substructure of a model of T is itself a model of T: in these cases, we shall drop the prefix sub- and directly speak of 'amalgamability', 'strong amalgamability' and 'T-amalgama'. The following fact is proved in [3], as Theorem 3.3:

Theorem 2.6. *A theory T has the sub-amalgamation property iff it admits quantifier-free interpolants.*

3 Conditions for Combination

The main result from [3] says that if T_1, T_2 have disjoint signatures, are both stably infinite and both enjoy the strong sub-amalgamation property, then the combined theory $T_1 \cup T_2$ also has the strong sub-amalgamation property[2] (and so it has quantifier-free interpolation).

In this paper, we try to extend the above results to the non-disjoint signatures case. The idea, already shown to be fruitful for combined satisfiability problems in [6], is to use T_0-compatibility as the proper generalization of stable infiniteness.

We shall first obtain a rather abstract sufficient condition for transfer of quantifier-free interpolation property to combined theories; nevertheless, we show that such sufficient condition generalizes the disjoint signatures result from [3] and is powerful enough to establish the quantifier-free interpolation property for some natural combined theories arising in verification. Then we move to the case in which the shared theory T_0 is Horn and obtain as a corollary a specialized result which is quite effective in modal logic applications.

3.1 Sub-amalgamation schemata

Let T_0, T be theories in their respective signatures Σ_0, Σ such that $\Sigma_0 \subseteq \Sigma$, T_0 is universal and $T_0 \subseteq T$. If $\mathcal{M}_1$ and $\mathcal{M}_2$ are Σ-models of T with a common substructure $\mathcal{A}$, we call the triple $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})$ a *T-fork* (or, simply, a *fork*).

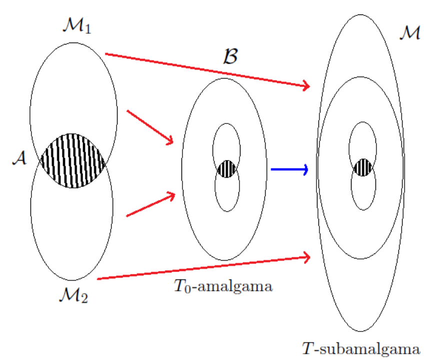
[2] It is possible to characterize syntactically strong sub-amalgamability in terms of a suitable 'equality interpolating' condition [3]. That sub-amalgamability needs to be strenghtened to strong sub-amalgamability in order to get positive combination results is demonstrated by converse facts also proved in [3].

The *sub-amalgamation schema* $\sigma^T_{T_0}$ (of T over T_0) is the following function, associating sets of T_0-amalgama with T-forks: [3]

$$\sigma^T_{T_0}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})] := \left\{ \begin{array}{l} \text{the set of all } (\mathcal{B}, \nu_1, \nu_2) \text{ s.t.} \\ (i)\ (\mathcal{B}, \nu_1, \nu_2) \text{ is a } T_0\text{-amalgama of the } \Sigma_0\text{-reducts of} \\ \qquad \mathcal{M}_1 \text{ and } \mathcal{M}_2 \text{ over the } \Sigma_0\text{-reduct of } \mathcal{A}; \\ (ii)\ \mathcal{B} \text{ is generated, as } \Sigma_0\text{-structure, by the union of} \\ \qquad \text{the images of } \nu_1 \text{ and } \nu_2; \\ (iii)\ (\mathcal{B}, \nu_1, \nu_2) \text{ is embeddable in the } \Sigma_0\text{-reduct of a} \\ \qquad T\text{-sub-amalgama of the fork } (\mathcal{M}_1, \mathcal{M}_2, \mathcal{A}). \end{array} \right\}$$

Condition (iii) means that there is a T-sub-amalgama $(\mathcal{M}, \mu_1, \mu_2)$ such that $\mathcal{B}$ is a Σ_0-substructure of $\mathcal{M}$ and that μ_1, μ_2 coincide with ν_1, ν_2 on their domains.

Condition (ii) ensures that, disregarding isomorphic copies, $\sigma^T_{T_0}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})]$ is a set and not a proper class. Recall that T_0 is universal, so that substructures of models of T_0 are also models of T_0. This ensures that the following Proposition trivially holds:



Proposition 3.1. *T is sub-amalgamable iff $\sigma^T_{T_0}$ is not empty (i.e. iff we have that $\sigma^T_{T_0}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})] \neq \emptyset$, for all forks $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})$).*

One side of the inclusion of the following Theorem is also immediate; for the other one, T_0-compatibility is needed (we shall prove the theorem in Subsection 3.2 below).

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[3] It is not difficult to realize (using well-known Löwenheim-Skolem theorems [4]) that one can get all the results in the paper by limiting this definition to forks among structures whose cardinality is bounded by the cardinality of set of the formulae in our signatures (signatures are finite or countable in all practical cases).

Theorem 3.2. *Let T_1 and T_2 be two theories in their respective signatures Σ_1, Σ_2; assume that they are both T_0-compatible, where T_0 is a universal theory in the signature $\Sigma_0 := \Sigma_1 \cap \Sigma_2$. The following hold for the amalgamation schema of $T_1 \cup T_2$ over T_0:*

$$\sigma_{T_0}^{T_1 \cup T_2}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})] \ = \ \sigma_{T_0}^{T_1}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})_{|\Sigma_1}] \cap \sigma_{T_0}^{T_2}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})_{|\Sigma_2}]$$

for every $(T_1 \cup T_2)$-fork $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})$ (here, with $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})_{|\Sigma_i}$ we indicate the T_i-fork obtained by taking reducts to the signature Σ_i).

Despite its abstract formulation, Theorem 3.2 is powerful enough to imply the main disjoint signatures result of [3] and also to work out interesting examples.

Example 3.3 (The disjoint signature case). Let S_0, S_1, S_2 be sets such that $S_0 \subseteq S_1, S_0 \subseteq S_2$; the amalgamated sum $S_1 +_{S_0} S_2$ of S_1, S_2 over S_0 is just the set-theoretic union $S_1 \cup S_2$ in which elements from $S_1 \setminus S_0$ are renamed away so as to be different from the elements of $S_2 \setminus S_0$. With this terminology, a theory T is strongly sub-amalgamable iff its sub-amalgamation schema over the empty theory T_0 is such that $\sigma_{T_0}^{T}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})]$ always contains the amalgamated sum of the supports of $\mathcal{M}_1, \mathcal{M}_2$ over the support of $\mathcal{A}$. Thus, Theorem 3.2 says in particular that if T_1, T_2 are both stably infinite and strongly sub-amalgamable, then so is $T_1 \cup T_2$ (and the last is in particular quantifier-free interpolating).

Example 3.4 (Lists and Trees). Consider $T_0 := T_S$, the 'theory of increment' [19]; T_S has the monosorted signature $\Sigma_S := \{0 : NUM,\ s : NUM \longrightarrow NUM\}$ and it is axiomatized by the following sentences:

$$\forall x \forall y \ s(x) = s(y) \rightarrow x = y \quad \text{(injectivity)}$$

$$\forall x \ s^n(x) \neq x \ \text{ for all } n \in \mathbb{N},\ n > 0$$

This theory is universal and it admits as a model-completion T_S^* the theory obtained by adding the axiom $\forall x \exists y \ x = s(y)$. Hence, T_S is amalgamable for general reasons [4] (but notice that it is not strongly amalgamable).

Now consider the theory T_{LS} of 'lists endowed with length' [19]. This is a many-sorted theory; its signature Σ_{LS} contains, besides Σ_S-symbols, the additional sorts $LISTS, ELEM_L$, the additional set of function symbols $\{nil : LISTS,\ car : LISTS \longrightarrow ELEM_L,\ cdr : LISTS \longrightarrow LISTS,\ cons : ELEM_L \times LISTS \longrightarrow LISTS,\ l : LISTS \longrightarrow NUM\}$ and a single unary relation symbol $atom : LISTS$. The axioms of T_{LS} are the following:

1. $car(cons(x, y)) = x$	5. $\neg atom(x) \rightarrow cons(car(x), cdr(x)) = x$
2. $cdr(cons(x, y)) = y$	6. $\neg atom(cons(x, y))$
3. $l(nil) = 0$	7. $atom(nil)$
4. $l(cons(x, y)) = s(l(y))$	

This theory is T_S-compatible ([19]); below, we show that *every* T_S-amalgama of the T_S-reducts of two models of T_{LS} (sharing a common submodel) can be embedded in a T_{LS}-amalgama (since T_{LS} is universal we can speak of amalgams instead of sub-amalgams).

Let a T_{LS}-fork $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})$ be given and let $\mathcal{B}$ be any amalgam of the T_S-reducts of $\mathcal{M}_1, \mathcal{M}_2$. We sketch the definition of a T_{LS}-amalgam $\mathcal{M}$ of the fork (based on $\mathcal{B}$). The support $NUM^{\mathcal{M}}$ is the support of $\mathcal{B}$ and $ELEM_L^{\mathcal{M}_1} \cup ELEM_L^{\mathcal{M}_2}$ is the support of $ELEM_L^{\mathcal{M}}$. It remains to define $LISTS^{\mathcal{M}}$;[4] we take $LISTS^{\mathcal{M}}$ to be the union of $LISTS^{\mathcal{M}_1}$, $LISTS^{\mathcal{M}_2}$ and of LT, where LT is the set containing the pairs (x, l), with $x \in LIST^{\mathcal{M}_{3-j}} \setminus LIST^{\mathcal{M}_j}$ and l a finite list of elements from $ELEM_L^{\mathcal{M}_1} \cup ELEM_L^{\mathcal{M}_2}$ which begins with an element in $ELEM_L^{\mathcal{M}_j}$ $(j = 1, 2)$. In other words, an element in LT has the form:

$$(x, (e_1, e_2, ..., e_n))$$

where (1) $j = 1, 2$; (2) e_1 is in $ELEM_L^{\mathcal{M}_j}$; (3) x is in $LISTS^{\mathcal{M}_{3-j}}$; and (4) e_i $(i > 1)$ is in $ELEM_L^{\mathcal{M}_1} \cup ELEM_L^{\mathcal{M}_2}$. Σ_{LS}-operations and relations can be defined in the obvious way so that axioms 1-7 above hold and so that the inclusions $\mathcal{M}_1 \subseteq \mathcal{M}$ and $\mathcal{M}_2 \subseteq \mathcal{M}$ are embeddings.

Let us now consider the theory T_{BS} of binary trees endowed with size functions [19]. This is also a many-sorted theory: its signature Σ_{BS} has the symbols of the signature Σ_S of the theory of increment plus the set of function symbols $\{null : TREES,\ bin : ELEM_T \times TREES \times TREES \longrightarrow TREES,\ l_L : TREES \longrightarrow NUM,\ l_R : TREES \longrightarrow NUM\}$. The axioms of T_{LS} are the following:

1. $l_L(null) = 0$ 2. $l_R(null) = 0$
3. $l_L(bin(e, t_1, t_2)) = s(l_L(t_1))$ 4. $l_R(bin(e, t_1, t_2)) = s(l_R(t_2))$

It can be showed that this theory is T_S-compatible ([19]). By arguments similar to those we employed for T_{LS}, it is possible to show that *every* T_S-amalgama of the T_S-reducts of two models of T_{BS} (sharing a common submodel) can be embedded in a T_{BS}-amalgama.

In conclusion, by (the multi-sorted version of) Theorem 3.2 we get that for every $(T_{LS} \cup T_{BS})$-fork $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})$, the amalgamation schema for this fork $\sigma_{T_S}^{T_{LS} \cup T_{BS}}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})]$, being equal to the intersection of $\sigma_{T_S}^{T_{LS}}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})_{|\Sigma_{LS}}]$ and of $\sigma_{T_S}^{T_{BS}}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})_{|\Sigma_{BS}}]$, contains all the amalgams of the Σ_S-reduced fork $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})_{|\Sigma_S}$ and hence it is trivially not empty. This guarantees that $T_{LS} \cup T_{BS}$ has quantifier-free interpolation by Proposition 3.1.

Example 3.5 (Where combined quantifier-free interpolation fails). Let T_0 be the theory of linear orders (its signature Σ_0 has just a binary relation symbol $<$ and the axioms of T_0 say that $<$ is irreflexive, transitive and satisfies the trichotomy condition $x < y \vee x = y \vee y < x$). This is a universal theory and admits a model completion T_0^*, which is the theory of dense linear orders without endpoints [4]; it is easily seen also that T_0 is strongly sub-amalgamable. We consider the signature Σ_1 of linear orders endowed with an extra unary relation symbol P and we let let T_1 be the theory obtained by adding to T_0 the following axiom:

$$\forall x \forall y\ (P(x) \wedge \neg P(y) \rightarrow x < y)$$

[4] We can freely assume that $ELEM_L^{\mathcal{M}_1} \cap ELEM_L^{\mathcal{M}_2} = ELEM_L^{\mathcal{A}}$ and $LIST^{\mathcal{M}_1} \cap LIST^{\mathcal{M}_2} = LIST^{\mathcal{A}}$.

It is not difficult to see that T_1 is T_0-compatible and also strongly-sub-amalgamable. We shall be interested in the combination of T_1 with a partially renamed copy of itself: this is the $\Sigma_2 := \Sigma_0 \cup \{Q\}$-theory T_2 axiomatized by the axioms of T_0 and

$$\forall x \forall y\ (Q(x) \wedge \neg Q(y) \to x < y)$$

Quantifier-free interpolation fails in $T_1 \cup T_2$, because sub-amalgamability fails: to see this fact, just consider a fork $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})$ such that there exists an element $a \in |\mathcal{M}_1| \setminus |\mathcal{A}|$ which satisfies $P \wedge \neg Q$ and another element $b \in |\mathcal{M}_2| \setminus |\mathcal{A}|$ that satisfies Q and $\neg P$. Notice that we have $\sigma^{T_1}_{T_0}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})_{|\Sigma_1}] \cap \sigma^{T_2}_{T_0}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})_{|\Sigma_2}] = \emptyset$ although both $\sigma^{T_i}_{T_0}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})_{|\Sigma_i}]$ are not empty (the sub-amalgamation schemata here 'do not match').

3.2 Proof of Theorem 3.2

This subsection is entirely devoted to the proof of Theorem 3.2. We begin by recalling some standard results from model theory and by introducing some preliminary lemmata. The following easy fact is proved in [3], as Lemma 3.7:

Lemma 3.6. *Let Σ_1, Σ_2 be two signatures and $\mathcal{A}$ be a $\Sigma_1 \cup \Sigma_2$-structure; then $\Delta_{\Sigma_1 \cup \Sigma_2}(\mathcal{A})$ is logically equivalent to $\Delta_{\Sigma_1}(\mathcal{A}) \cup \Delta_{\Sigma_2}(\mathcal{A})$.*

An easy but nevertheless important basic result, called *Robinson Diagram Lemma* [4], says that, given any Σ-structure $\mathcal{B}$, the embeddings $\mu : \mathcal{A} \longrightarrow \mathcal{B}$ are in bijective correspondence with expansions of $\mathcal{B}$ to $\Sigma^{|\mathcal{A}|}$-structures which are models of $\Delta_\Sigma(\mathcal{A})$. The expansions and the embeddings are related in the obvious way: $\bar{a}$ is interpreted as $\mu(a)$.

The following Lemma is proved using this property of diagrams:

Lemma 3.7. *Let T_0, T be theories in their respective signatures Σ_0, Σ such that $\Sigma_0 \subseteq \Sigma$ and $T_0 \subseteq T$; let $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})$ be a T-fork. For a T_0-amalgam $(\mathcal{B}, \nu_1, \nu_2)$ the following conditions are equivalent (we suppose that the support of $\mathcal{B}$ is disjoint from the supports of $\mathcal{M}_1, \mathcal{M}_2$):*

(i) $(\mathcal{B}, \nu_1, \nu_2) \in \sigma^T_{T_0}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})]$;
(ii) *the following theory* $(*)$ *is consistent*

$$\begin{aligned} &T \cup \Delta_\Sigma(\mathcal{M}_1) \cup \Delta_\Sigma(\mathcal{M}_2) \cup \Delta_{\Sigma_0}(\mathcal{B}) \cup \\ &\cup \{\bar{a}_1 = \bar{b} \parallel b \in |\mathcal{B}|, a_1 \in |\mathcal{M}_1|, \nu_1(a_1) = b\} \cup \\ &\{\bar{a}_2 = \bar{b} \parallel b \in |\mathcal{B}|, a_2 \in |\mathcal{M}_2|, \nu_2(a_2) = b\}. \end{aligned}$$

Furthermore, in case T is T_0-compatible, we can equivalently put $T \cup T_0^\star$ instead of T in the theory $(*)$ *mentioned in* (ii) *above.*

Proof. By the above mentioned property of diagrams, the consistency of $(*)$ means that there is a model $\mathcal{N} \models T$ and there are three embeddings

$$\mu_1 : \mathcal{M}_1 \longrightarrow \mathcal{N}, \quad \mu_2 : \mathcal{M}_2 \longrightarrow \mathcal{N}, \quad \nu : \mathcal{B} \longrightarrow \mathcal{N}$$

(the last one is a Σ_0-embedding, the first two are Σ-embeddings) such that $\nu \circ \nu_1 = \mu_1$ and $\nu \circ \nu_2 = \mu_2$. Since μ_1, μ_2 agree on the support of $\mathcal{A}$, the triple $(\mathcal{N}, \mu_1, \mu_2)$ is a T-sub-amalgam of the fork. To make $\mathcal{B}$ a substructure of $\mathcal{N}$, it is sufficient to make a renaming of the elements in the image of ν (so that ν becomes an inclusion). Thus consistency of $(*)$ means precisely that $(\mathcal{B}, \nu_1, \nu_2) \in \sigma^T_{T_0}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})]$.

Since, by T_0-compatibility, every model of T can be embedded into a model of $T \cup T_0^\star$, the consistency of $(*)$ is the same of the consistency of $T_0^\star \cup (*)$. ⊣

We need a further result from model theory to be found in textbooks like [4]; it can be seen as a combination result 'ante litteram':

Lemma 3.8. *[Joint Consistency] Let Θ_1, Θ_2 be two signatures and let $\Theta_0 := \Theta_1 \cap \Theta_2$; suppose that the Θ_1-theory U_1 and the Θ_2-theory U_2 are both consistent and that there is a Θ_0-theory U_0 which is complete and included both in U_1 and in U_2. Then, $U_1 \cup U_2$ is also consistent.*

Proof. There are basically two proofs of this result, one by Craig's interpolation Theorem and another one by a double chain argument. The interested reader is referred to [4]. ⊣

We can now *prove Theorem 3.2;* the Theorem concerns theories T_1, T_2 (in their respective signatures Σ_1, Σ_2) which are both T_0-compatible with respect to a universal theory T_0 in the shared signature $\Sigma_0 := \Sigma_1 \cap \Sigma_2$.

Fix a $T_1 \cup T_2$-fork $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})$. On one side, it is evident that if $(\mathcal{B}, \nu_1, \nu_2)$ belongs to $\sigma^{T_1 \cup T_2}_{T_0}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})]$, then it also belongs to $\sigma^{T_1}_{T_0}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})_{|\Sigma_1}] \cap \sigma^{T_2}_{T_0}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})_{|\Sigma_2}]$.

Vice versa, suppose that $(\mathcal{B}, \nu_1, \nu_2)$ belongs to $\sigma^{T_1}_{T_0}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})_{|\Sigma_1}]$ and to $\sigma^{T_2}_{T_0}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})_{|\Sigma_2}]$; in order to show that it belongs to $\sigma^{T_1 \cup T_2}_{T_0}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})]$, in view of Lemmas 3.6 and 3.7 (recall also Proposition 2.3), we need to show that the following theory (let us call it U) is consistent:

$$\begin{aligned} & T_1 \cup T_2 \cup T_0^\star \cup \Delta_{\Sigma_1}(\mathcal{M}_1) \cup \Delta_{\Sigma_1}(\mathcal{M}_2) \cup \Delta_{\Sigma_0}(\mathcal{B}) \cup \\ & \cup \Delta_{\Sigma_2}(\mathcal{M}_1) \cup \Delta_{\Sigma_2}(\mathcal{M}_2) \cup \\ & \cup \{\bar{a}_1 = \bar{b} \parallel b \in |\mathcal{B}|, a_1 \in |\mathcal{M}_1|, \nu_1(a_1) = b\} \cup \\ & \{\bar{a}_2 = \bar{b} \parallel b \in |\mathcal{B}|, a_2 \in |\mathcal{M}_2|, \nu_2(a_2) = b\}. \end{aligned}$$

The idea is to use Robinson Joint Consistency Lemma 3.8 and split U as $U_1 \cup U_2$. Now U is a theory in the signature $\Sigma_1 \cup \Sigma_2 \cup |\mathcal{M}_1| \cup |\mathcal{M}_2| \cup |\mathcal{B}|$; we let (for $i = 1, 2$) U_i be the following theory in the signature $\Sigma_i \cup |\mathcal{M}_1| \cup |\mathcal{M}_2| \cup |\mathcal{B}|$:

$$\begin{aligned} & T_i \cup T_0^\star \cup \Delta_{\Sigma_i}(\mathcal{M}_1) \cup \Delta_{\Sigma_i}(\mathcal{M}_2) \cup \Delta_{\Sigma_0}(\mathcal{B}) \cup \\ & \cup \{\bar{a}_1 = \bar{b} \parallel b \in |\mathcal{B}|, a_1 \in |\mathcal{M}_1|, \nu_1(a_1) = b\} \cup \\ & \{\bar{a}_2 = \bar{b} \parallel b \in |\mathcal{B}|, a_2 \in |\mathcal{M}_2|, \nu_2(a_2) = b\}. \end{aligned}$$

Notice that U_i is consistent by Lemma 3.7 because our assumption is that $(\mathcal{B}, \nu_1, \nu_2)$ belongs to $\sigma_{T_0}^{T_i}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})_{|\Sigma_i}]$. We now only have to identify a complete theory U_0 included in $U_1 \cap U_2$. The shared signature of U_1 and U_2 is $\Sigma_0 \cup |\mathcal{M}_1| \cup |\mathcal{M}_2| \cup |\mathcal{B}|$ and we take as U_0 the theory

$$
\begin{aligned}
&T_0^\star \cup \Delta_{\Sigma_0}(\mathcal{M}_1) \cup \Delta_{\Sigma_0}(\mathcal{M}_2) \cup \Delta_{\Sigma_0}(\mathcal{B}) \cup \\
&\cup \{\bar{a}_1 = \bar{b} \parallel b \in |\mathcal{B}|, a_1 \in |\mathcal{M}_1|, \nu_1(a_1) = b\} \cup \\
&\{\bar{a}_2 = \bar{b} \parallel b \in |\mathcal{B}|, a_2 \in |\mathcal{M}_2|, \nu_2(a_2) = b\}.
\end{aligned}
$$

By the definition of a model-completion ($T_0^\star$ is a model-completion of T_0), we know that $T_0^\star \cup \Delta_{\Sigma_0}(\mathcal{B})$ is a complete theory in the signature $\Sigma_0 \cup |\mathcal{B}|$. Now it is sufficient to observe that every $\Sigma_0 \cup |\mathcal{M}_1| \cup |\mathcal{M}_2| \cup |\mathcal{B}|$-sentence is equivalent, modulo $U_0 \supseteq T_0^\star \cup \Delta_{\Sigma_0}(\mathcal{B})$, to a $\Sigma_0 \cup |\mathcal{B}|$-sentence: this is clear because U_0 contains the sentences

$$
\begin{aligned}
&\{\bar{a}_1 = \bar{b} \parallel b \in |\mathcal{B}|, a_1 \in |\mathcal{M}_1|, \nu_1(a_1) = b\} \cup \\
&\{\bar{a}_2 = \bar{b} \parallel b \in |\mathcal{B}|, a_2 \in |\mathcal{M}_2|, \nu_2(a_2) = b\}.
\end{aligned}
$$

which can be used to eliminate the constants from $|\mathcal{M}_1| \cup |\mathcal{M}_2|$. $\dashv$

3.3 When the shared theory is Horn

Theorem 3.2 gives modular information to determine the combined sub-amalgamation schema, but it is not a modular result itself. In fact, a modular result should identify a condition C on a single (standing alone) theory such that whenever T_1, T_2 satisfy C, then $T_1 \cup T_2$ is sub-amalgamable and also satisfies C. To get a modular sufficient condition, we need to specialize our framework. In doing that, we are still guided by what happens in the disjoint signatures case. Although we feel that suitable conditions could be identified without Horn hypotheses, we prefer to assume that the shared theory is universal Horn to simplify the statement of the results below.

Recall that a Σ-theory T is *universal Horn* iff it can be axiomatized via Horn clauses (i.e. via formulae of the form $A_1 \wedge \cdots \wedge A_n \to B$, where the A_i are atoms and B is either an atom or $\bot$). In universal Horn theories, it is possible to show that if amalgamation holds, then there is always a minimal amalgama, as stated in the following fact (which is basically due to the universal property of pushouts, see the Appendix for a proof):

Proposition 3.9. *Let T be a universal Horn theory having the amalgamation property; given a T-fork $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})$, there exists a T-amalgam $(\mathcal{M}, \mu_1, \mu_2)$ of $\mathcal{M}_1$ and $\mathcal{M}_2$ over $\mathcal{A}$ such that for every other T-amalgam $(\mathcal{M}', \mu_1', \mu_2')$ there is a unique homomorphism $\nu : \mathcal{M} \longrightarrow \mathcal{M}'$ such that $\nu \circ \mu_i = \mu_i'$ ($i = 1, 2$).*

We call the amalgam mentioned in the above Proposition (which is unique up to isomorphism) the *minimal T-amalgam* of the T-fork $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})$; the homomorphism ν (which needs not to be an embedding) is called the *comparison* homomorphism.

Let now T be a Σ-theory and let $T_0 \subseteq T$ be a universal Horn Σ_0-theory having the amalgamation property (with $\Sigma_0 \subseteq \Sigma$). We say that T is T_0*-strongly sub-amalgamable* if the sub-amalgamation schema $\sigma^T_{T_0}$ always contains the minimal T_0-amalgama (meaning that for every T-fork $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})$, we have that the minimal T_0-amalgama of $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})$ belongs to $\sigma^T_{T_0}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})]$). Notice that, whenever T_0 is the empty theory in the empty signature, being T_0-strongly sub-amalgamable is the same as being strongly sub-amalgamable.

Theorem 3.2 immediately implies the following:

Theorem 3.10. *If T_1, T_2 are both T_0-compatible and T_0-strongly sub-amalgamable (over an amalgamable universal Horn theory T_0 in their common subsignature Σ_0), then so it is $T_1 \cup T_2$.*

Proof. Since T_1 and T_2 are T_0-strongly sub-amalgamable, their sub-amalgamation schemata $\sigma^{T_i}_{T_0}$ $(i = 1, 2)$ always contain minimal T_0-amalgamas. By Theorem 3.2 (T_1 and T_2 are also T_0-compatible), this implies that for every $T_1 \cup T_2$-fork $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})$, the minimal amalgama $\mathcal{B}$ of $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})_{|\Sigma_0}$ belongs to the set $\sigma^{T_1 \cup T_2}_{T_0}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})]$. Using Proposition 2.3, we conclude that also $T_1 \cup T_2$ is T_0-compatible and T_0-strongly sub-amalgamable. ⊣

4 Applications to modal logic

Theorem 3.10 (obtained as a generalization of the analogous result from [3] for the disjoint signatures case) has surprising applications to modal logic. To get such applications, we need to reformulate it in the case of Boolean algebras with operators: the reformulation needs a further Theorem, showing that T_0-strong sub-amalgamability, in case T_0 is the theory of Boolean algebras, is nothing but the superamalgamability property known from algebraic logic. Let us recall the last property and state the Theorem we are still missing. For space reasons, all proofs in this section are deferred to the Appendix.

In the following, we let BA be the theory of Boolean algebras; a *BAO-equational theory*[5] is any theory T whose signature extends the signature of Boolean algebras and whose axioms are all equations and include the Boolean algebra axioms. In subsection 6.2 below, we shall recall in detail how BAO-equational theories are related to modal propositional logics via Lindenbaum constructions. The *fusion* of two BAO-equational theories T_1 and T_2 is just their combination $T_1 \cup T_2$ (when speaking of the fusion of T_1 and T_2, we assume that T_1 and T_2 share only the Boolean algebras operations and no other symbol).

The following Proposition is proved in [6] (proof is reported in the Appendix):

Proposition 4.1. *Every BAO-equational theory is BA-compatible.*

We say that a BAO-equational theory T has the *superamalgamation* property iff for every T-fork $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})$ there exists a T-amalgam $(\mathcal{M}, \mu_1, \mu_2)$ such that

[5] BAO stands for 'Boolean algebras with operators'

for every $a_1 \in |\mathcal{M}_1|, a_2 \in |\mathcal{M}_2|$ such that $\mu_1(a_1) \leq \mu_2(a_2)$ there exists $a_0 \in |\mathcal{A}|$ such that $a_1 \leq a_0$ holds in $\mathcal{M}_1$ and $a_0 \leq a_2$ holds in $\mathcal{M}_2$.[6]

We can now state our second main result (see the Appendix for the proof):

Theorem 4.2. *A BAO-equational theory T has the superamalgamation property iff it is BA-strongly amalgamable.*

As an immediate consequence, from Theorem 3.10, we get:

Corollary 4.3. *If two BAO-equational theories T_1 and T_2 both have the superamalgamability property, so does their fusion.*

4.1 Superamalgamability and interpolation in propositional logic

Corollary 4.3 immediately implies Wolter's result [25] on fusion transfer of Craig interpolation property for normal modal logics and says something new for non-normal modal logics too. To see all this, we only need to recall some background from propositional logic. For simplicity, we deal only with unary modalities (and, consequently, we shall consider only BAO-theories whose non-Boolean symbols are unary function symbols), however we point out that the extension to n-ary modalities is straightforward.

A *modal signature* Σ_M is a set of unary operation symbols; from Σ_M, propositional *modal formulae* are built using countably many propositional variables, the operation symbols in Σ_M, the Boolean connectives $\cap, \cup, \sim$ and the constants 1 for truth and 0 for falsity. We use the letters $x, x_1, \dots, y, y_1, \dots$ to denote propositional variables and the letters $t, t_1, \dots, u, u_1, \dots$ to denote propositional formulae; $t \Rightarrow u$ and $t \Leftrightarrow u$ are abbreviations for $(\sim t) \cup u$ and for $(t \Rightarrow u) \cap (u \Rightarrow t)$, respectively. We use notations like $t(\underline{x})$ (resp. $\Gamma(\underline{x})$) to say that the modal formula t (the set of modal formulae Γ) is built up from a set of propositional variables included in the tuple $\underline{x}$.

The following definition is taken from [20], pp. 8–9:

Definition 4.4. *A* classical modal logic L *based on a modal signature Σ_M is a set of modal formulae that*

(i) *contains all classical propositional tautologies;*
(ii) *is closed under uniform substitution of propositional variables by propositional formulae;*
(iii) *is closed under the modus ponens rule ('from t and $t \Rightarrow u$ infer u');*
(iv) *is closed under the replacement rules, which are specified as follows. We have one such rule for each $o \in \Sigma_M$, namely:*

$$\frac{t \Leftrightarrow u}{o(t) \Leftrightarrow o(u)}$$

[6] We recall that in every Boolean algebra (more generally, in every semilattice) $x \leq y$ is defined as $x \cap y = x$, where $\cap$ is the meet operation.

A classical modal logic L is said to be normal *iff for every modal operator $o \in \Sigma_M$, L contains the modal formulae $o(1)$ and $o(y \Rightarrow z) \Rightarrow (o(y) \Rightarrow o(z))$.*

Since classical modal logics (based on a given modal signature) are closed under intersections, it makes sense to speak of the least classical modal logic $[S]$ containing a certain set of propositional formulae S. If $L = [S]$, we say that S is a set of *axiom schemata* for L.

If L_1 is a classical modal logic over the modal signature Σ_M^1 and L_2 is a classical modal logic over the modal signature Σ_M^2 and $\Sigma_M^1 \cap \Sigma_M^2 = \emptyset$, the *fusion* $L_1 \oplus L_2$ is the modal logic $[L_1 \cup L_2]$ over the modal signature $\Sigma_M^1 \cup \Sigma_M^2$.

Given a modal logic L, a set of modal formulae Γ and a modal formula t, the *global consequence relation* $\Gamma \vdash_L t$ holds iff there is a finite list of modal formulae $t_0, \dots, t_n$ such that: (i) t_n is t; (ii) each t_i is either a member of L or a member of Γ or is obtained from previous member of the list by applying one of the two inference rules from Definition 4.4 (i.e. modus ponens and replacement).

Global consequence relation should be contrasted with *local consequence relation*, to be indicated with $\vdash_L \Gamma \Rightarrow t$: this holds iff there are $g_1, \dots, g_n \in \Gamma$ such that $\bigcap_{i=1}^n g_i \Rightarrow t$ belongs to L. If Γ consists of a single modal formula g, below we write $g \vdash_L t$ and $\vdash_L g \Rightarrow t$ instead of $\{g\} \vdash_L t$ and of $\vdash_L \{g\} \Rightarrow t$.

In case L is normal, one can reduce the global consequence relation to the local one: in fact, it is not difficult to see by induction that the following fact ('deduction theorem') holds:

$$\Gamma \vdash_L t \qquad \text{iff} \qquad \vdash_L o\Gamma \Rightarrow t$$

where $o\Gamma$ is some finite set of modal formulae (depending on t) obtained from Γ by prefixing a string of modal operators (i.e. elements of $o\Gamma$ are modal formulae of the kind $o_1(o_2 \cdots o_n(g) \cdots)$, for $g \in \Gamma$ and $n \geq 0$, $o_1, \dots, o_n \in \Sigma_M$).

Due to the presence of local and global consequence relations, we can formulate two different versions of the Craig's interpolation theorem:

Definition 4.5. *Let L be a classical modal logic in a modal signature Σ_M.*

(i) *We say that L enjoys the* local *interpolation property iff whenever we have $\vdash_L t_1(\underline{x}, \underline{y}) \Rightarrow t_2(\underline{x}, \underline{z})$ for two modal formulae t_1, t_2, then there is a modal formula $u(\underline{x})$ such that $\vdash_L t_1 \Rightarrow u$ and $\vdash_L u \Rightarrow t_2$.*
(ii) *We say that L enjoys the* global *interpolation property iff whenever we have $t_1(\underline{x}, \underline{y}) \vdash_L t_2(\underline{x}, \underline{z})$ for two modal formulae t_1, t_2, then there is a modal formula $u(\underline{x})$ such that $t_1 \vdash_L u$ and $u \vdash_L t_2$.*

For *normal* modal logics, in view of the above deduction theorem, it is easy to see that the local interpolation property implies the global one (but it is not equivalent to it, see [12]). In the non-normal case, there is no deduction theorem available, so that in order to have an interpolation property encompassing both the local and the global versions, it seems that a different notion needs to be introduced. This is what we are doing now.

Given a modal logic L and two sets of modal formulae $\Gamma_1(\underline{x}, \underline{y}), \Gamma_2(\underline{x}, \underline{z})$, let us call an *$\underline{x}$-residue chain* a tuple of modal formulae $C(\underline{x}) = g_1(\underline{x}), \dots, g_k(\underline{x})$

such that we have $\Gamma_1 \cup \{g_1, \dots, g_{2i}\} \vdash_L g_{2i+1}$ and $\Gamma_2 \cup \{g_1, \dots, g_{2j-1}\} \vdash_L g_{2j}$, for all i such that $0 \leq 2i < n$ and for all j such that $0 < 2j \leq n$.

Definition 4.6. *Let L be a classical modal logic in a modal signature Σ_M.*

(iii) *We say that L enjoys the* comprehensive *interpolation property iff whenever we have $\Gamma_1(\underline{x}, \underline{y}), \Gamma_2(\underline{x}, \underline{z}) \vdash_L t_1(\underline{x}, \underline{y}) \Rightarrow t_2(\underline{x}, \underline{z})$ for two modal formulae t_1, t_2 and for two finite sets of modal formulae Γ_1, Γ_2, there are an $\underline{x}$-residue chain $C(\underline{x})$ and a modal formula $u(\underline{x})$ such that we have $\Gamma_1, C \vdash_L t_1 \Rightarrow u$ and $\Gamma_2, C \vdash_L u \Rightarrow t_2$.*

Notice that the comprehensive interpolation property implies both the local and the global interpolation properties; moreover, in the normal case, via deduction theorem, it is easily seen that the comprehensive interpolation property is equivalent to the local interpolation property. Our final result, giving an extension of Wolter's result [25] to non-normal case, is the following:

Theorem 4.7. *If the modal logics L_1 and L_2 both have the comprehensive interpolation property, so does their fusion $L_1 \oplus L_2$.*

The proof of the above Theorem is reported in the Appendix for space reasons; in fact, it requires some background, but only routine work. The idea is the following. One first recall that classical modal logics are in bijective correspondence with BAO-equational theories. Under this correspondence, in the normal case, global interpolation property coincides with quantifier-free interpolation (alias amalgamation property) and local interpolation property coincides with superamalgamability [12] (see [7] for a proof operating in a general context). Using similar techniques as in the above mentioned papers, in the non-normal general case, we show that *the comprehensive interpolation property coincides with superamalgamability.* Now it is sufficient to apply Corollary 4.3.

5 Conclusions and future work

In this paper we considered the problem of transferring the quantifier-free interpolation property from two theories to their union, in the case where the two theories share symbols other than pure equality.

We are not aware of previous papers attacking this problem. One should however mention a series of papers (e.g. [21,22,24]) analyzing the problem of transferring, in a hierarchical way, interpolation properties to theory extensions. This problem is related to ours, but it is different because there interpolation is assumed to hold for a basic theory T_0 and conditions on super-theories $T \supseteq T_0$ are analyzed in order to be able to extend interpolation to them. In our case, we are given interpolation properties for component theories T_1, T_2 and we are asked for modular conditions in order to transfer the property to $T_1 \cup T_2$.

To this aim, we obtained a sufficient condition (Theorem 3.2) in terms of sub-amalgamation schemata; we used such result to get a modular condition in case the shared theory is universal Horn (Theorem 3.10). For equational theories extending the theory of Boolean algebras, this modular condition turns out to be equivalent to the superamalgamability condition known from algebraic logic [13]. Thus, our results immediately imply the fusion transfer of local interpolation property [25] for classical normal modal logics. In the general non-normal case, the modularity of superamalgamability can be translated into a fusion transfer result for a new kind of interpolation property (which we called 'comprehensive interpolation property').

Still, many problems need to be faced by future research. Our combinability conditions should be characterizable from a syntactic point of view and, from such syntactic characterizations, we expect to be able to design concrete combined interpolation algorithms. Concerning modal logic, besides the old question about modularity of local interpolation property in the non-normal case, new questions arise concerning the status of the new comprehensive interpolation property: is it really stronger than other forms of interpolation property (e.g. than the local one)? Are there different ways of specifying it? Is it modular also for modal logics on a non-classical basis?

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References

1. P. D. Bacsich. Amalgamation properties and interpolation theorems for equational theories. *Algebra Universalis*, 5:45–55, 1975.
2. R. Bruttomesso, S. Ghilardi, and S. Ranise. From strong amalgamability to modularity of quantifier-free interpolation. In *Proc. of IJCAR 2012*, pages 118–133, 2012.
3. R. Bruttomesso, S. Ghilardi, and S. Ranise. Quantifier-free interpolation in combinations of equality interpolating theories. *ACM Trans. Comput. Log.*, 15(1):5:1–5:34, 2014.
4. C.-C. Chang and J. H. Keisler. *Model Theory*. North-Holland, Amsterdam-London, third edition, 1990.
5. W. Craig. Three uses of the Herbrand-Gentzen theorem in relating model theory and proof theory. *J. Symb. Log.*, pages 269–285, 1957.
6. S. Ghilardi. Model theoretic methods in combined constraint satisfiability. *Journal of Automated Reasoning*, 33(3-4):221–249, 2004.
7. S. Ghilardi and G. C. Meloni. Modal logics with n-ary connectives. *Z. Math. Logik Grundlag. Math.*, 36(3):193–215, 1990.
8. S. Ghilardi, E. Nicolini, and D. Zucchelli. A comprehensive framework for combined decision procedures. *ACM Transactions on Computational Logic*, 9(2):1–54, 2008.
9. T. Henzinger and K. L. McMillan R. Jhala, R. Majumdar. Abstractions from Proofs. In *POPL*, pages 232–244, 2004.
10. D. Kapur, R. Majumdar, and C. Zarba. Interpolation for Data Structures. In *SIGSOFT'06/FSE-14*, pages 105–116, 2006.
11. L. L. Maksimova. Craig's theorem in superintuitionistic logics and amalgamable varieties. *Algebra i Logika*, 16(6):643–681, 741, 1977.
12. L. L. Maksimova. Interpolation theorems in modal logics and amalgamable varieties of topological Boolean algebras. *Algebra i Logika*, 18(5):556–586, 632, 1979.

13. L. L. Maksimova. Interpolation theorems in modal logics. Sufficient conditions. *Algebra i Logika*, 19(2):194–213, 250–251, 1980.
14. K. L. McMillan. Applications of Craig Interpolation to Model Checking. In *Proc. of TACAS*, LNCS, pages 1–12, 2005.
15. G. Nelson and D. C. Oppen. Simplification by cooperating decision procedures. *ACM Transaction on Programming Languages and Systems*, 1(2):245–257, 1979.
16. E. Nicolini, C. Ringeissen, and M. Rusinowitch. Combinable extensions of abelian groups. In *Proc. of CADE-22*, pages 51–66, 2009.
17. E. Nicolini, C. Ringeissen, and M. Rusinowitch. Data structures with arithmetic constraints: A non-disjoint combination. In *Proc. of FroCoS 2009*, pages 319–334, 2009.
18. E. Nicolini, C. Ringeissen, and M. Rusinowitch. Satisfiability procedures for combination of theories sharing integer offsets. In *Proc. of TACAS 2009*, pages 428–442, 2009.
19. E. Nicolini, C. Ringeissen, and M. Rusinowitch. Combining satisfiability procedures for unions of theories with a shared counting operator. *Fundam. Inform.*, 105(1-2):163–187, 2010.
20. K. Segerberg. *An Essay in Classical Modal Logic*, volume 13 of *Filosofiska Studier*. Uppsala Universitet, 1971.
21. V. Sofronie-Stokkermans. Interpolation in local theory extensions. *Logical Methods in Computer Science*, 4(4), 2008.
22. V. Sofronie-Stokkermans. On interpolation and symbol elimination in theory extensions. In *Proc. of IJCAR 2016*, pages 273–289, 2016.
23. C. Tinelli and M. T. Harandi. A new correctness proof of the Nelson-Oppen combination procedure. In *Proc. of FroCoS 1996*, Applied Logic, pages 103–120. Kluwer Academic Publishers, 1996.
24. N. Totla and T. Wies. Complete instantiation-based interpolation. *J. Autom. Reasoning*, 57(1):37–65, 2016.
25. F. Wolter. Fusions of modal logics revisited. In *Advances in modal logic, Vol. 1 (Berlin, 1996)*, CSLI Lecture Notes, pages 361–379. 1998.
26. G. Yorsh and M. Musuvathi. A combination method for generating interpolants. In *Proc. of CADE-20*, LNCS, pages 353–368. 2005.

6 Appendix

In this Appendix we supply some background (Subsections 6.1 and 6.2) and the missed proofs (Subsection 6.3) of our original results from Section 4.

6.1 Minimal amalgama and presentations

Here we show how to prove Proposition 3.9: this is essentially a consequence of well-known co-completeness results for our categories of models [27], nevertheless we shall use the explicit construction of the minimal amalgam in the sequel, so we report it.

For simplicity (and because our applications to modal logic do not require more), in this section we shall consider only universal Horn theories axiomatized by clauses containing *exactly one* positive literal.[7] We fix such a T for the remaining part of this subsection (we let also Σ be the signature of T).

The key feature of T is that it *admits presentations*, in the following sense. A T-presentation is a pair (X, Γ), where X is a set of fresh constants and Γ is a set of Σ^X-atoms. To a T-presentation we associate the T-model $F_T(X, \Gamma)$ built as follows:

(i) the support $|F_T(X, \Gamma)|$ of $F_T(X, \Gamma)$ is formed by the equivalence classes of Σ^X-ground terms with respect to the equivalence relation $\sim_\Gamma$ defined by: $t \sim_\Gamma u$ iff $T \cup \Gamma \models t = u$;
(ii) function and relation symbols are interpreted so as to have that $F_T(X, \Gamma) \models A$ holds iff $T \cup \Gamma \models A$, for every Σ^X-atom A (clearly $F_T(X, \Gamma)$ is a Σ^X-structure, with constants from X interpreted as their own equivalence classes).[8]

Thanks to the current assumption on T (i.e. that the axioms of T are all Horn clauses with exactly one positive literal), we can easily check that $F_T(X, \Gamma) \models T$.

The fundamental property of $F_T(X, \Gamma) \models T$ is the following (this is very similar to the diagrams property): *for every T-model $\mathcal{M}$, there is a bijective correspondence between Σ-homomorphisms $F_T(X, \Gamma) \longrightarrow \mathcal{M}$ and expansions of $\mathcal{M}$ to Σ^X-structures which are models of Γ.*

Every model $\mathcal{M}$ of T is isomorphic to a model of the kind $F_T(X, \Gamma)$: this is because it is easily seen that $\mathcal{M} \simeq F_T(|\mathcal{M}|, \Delta^+_\Sigma(\mathcal{M}))$, where $\Delta^+_\Sigma(\mathcal{M})$ (the *positive diagram* of $\mathcal{M}$) is given by $\{A \mid A \text{ is an atomic formula of } \Sigma^{|\mathcal{M}|} \text{ s.t. } \mathcal{M} \models A\}$. We call $F_T(|\mathcal{M}|, \Delta^+_\Sigma(\mathcal{M}))$ the *canonical presentation of* $\mathcal{M}$.

Let $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})$ be a T-fork; a *pushout of the fork* is a triple $(\mathcal{M}, \mu_1, \mu_2)$, where $\mathcal{M}$ is a T-model and $\mu_1 : \mathcal{M}_1 \longrightarrow \mathcal{M}$, $\mu_2 : \mathcal{M}_2 \longrightarrow \mathcal{M}$ are Σ-homomorphisms whose restrictions to the support of $\mathcal{A}$ coincide, such that for every other

[7] The extension of the proof below of Proposition 3.9 to the case of universal Horn theories whose axioms include clauses with no positive literals is not difficult, though.

[8] In more detail, in $F_T(X, \Gamma)$ an n-ary function symbol f is interpreted as the function mapping the tuple of equivalence classes $[t_1], \dots, [t_n]$ to the equivalence class $[f(t_1, \dots, t_n)]$; an n-ary relation symbol R is interpreted as the set of tuples of equivalence classes $[t_1], \dots, [t_n]$ such that $T \cup \Gamma \models R(t_1, \dots, t_n)$.

triple $(\mathcal{M}', \mu'_1, \mu'_2)$ with the same properties, there is a unique homomorphism $\theta : \mathcal{M} \longrightarrow \mathcal{M}'$ such that $\theta \circ \mu_i = \mu'_i$ $(i = 1, 2)$.

Proposition 3.9 is an immediate consequence of the following:

Proposition 6.1. *Every T-fork $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})$ has a pushout $(\mathcal{M}, \mu_1, \mu_2)$; if T has the amalgamation property, the pushout is the minimal T-amalgam of the T-fork.*

Proof. Up to renamings, we can freely suppose that $|\mathcal{M}_1| \cap |\mathcal{M}_2| = |\mathcal{A}|$. Let us take

$$\mathcal{M} \;:=\; F_T(|\mathcal{M}_1| \cup |\mathcal{M}_2|, \Delta^+_\Sigma(\mathcal{M}_1) \cup \Delta^+_\Sigma(\mathcal{M}_2)) \tag{2}$$

and let μ_1 (resp. μ_2) be the map associating an element from $|\mathcal{M}_1|$ (resp. $|\mathcal{M}_2|$) to its own equivalence class in $\mathcal{M}$. The fact that this is a pushout is guaranteed by the fundamental property of the presentations: indeed, given a T-model $\mathcal{M}'$, the Σ-homomorphisms $\mathcal{M} \longrightarrow \mathcal{M}'$ are in bijective correspondence with expansions of $\mathcal{M}'$ to a $\Sigma^{|\mathcal{M}_1| \cup |\mathcal{M}_2|}$-structure modeling $\Delta^+_\Sigma(\mathcal{M}_1) \cup \Delta^+_\Sigma(\mathcal{M}_2)$ and the latter are in bijective correspondence with pairs of Σ-homomorphisms $\mathcal{M}_1 \longrightarrow \mathcal{M}'$ and $\mathcal{M}_2 \longrightarrow \mathcal{M}'$, agreeing on $|\mathcal{M}_1| \cap |\mathcal{M}_2| = |\mathcal{A}|$.[9]

If T has the amalgamation property, then there exists a T-amalgama $\mathcal{N}$, with embeddings $\nu_i : \mathcal{M}_i \longrightarrow \mathcal{N}$ $(i = 1, 2)$; by the above property of the pushout, there is $\theta : \mathcal{M} \longrightarrow \mathcal{N}$ with $\nu_i = \theta \circ \mu_i$ $(i = 1, 2)$. Since the ν_i are injective and $\nu_i = \theta \circ \mu_i$, then also the μ_i are injective; the same argument shows that the μ_i reflect relations and hence they are embeddings: if a relation R holds in $\mathcal{M}$ (when applied to some parameters from $\mathcal{M}_i$), then it holds in $\mathcal{N}$ because θ is a homomorphism, hence it holds also in $\mathcal{M}_i$, since the ν_i are embeddings. Thus, if T has the amalgamation property, the pushout (2) is an amalgam (minimal by the definition of pushout) of the fork $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})$. ⊣

It is useful to have a formula like (2) also in terms of T-presentations of $\mathcal{M}_1, \mathcal{M}_2$ which might not be canonical. To this aim, we introduce T-presentations of embeddings and of T-forks.

Suppose that we are given an embedding among T-models; up to an isomorphism, we can assume that it is a substructure inclusion $\mathcal{M}_1 \subseteq \mathcal{M}_2$. Passing to canonical presentations, again up to isomorphisms, we have an embedding

$$F_T(|\mathcal{M}_1|, \Delta^+_\Sigma(\mathcal{M}_1)) \longrightarrow F_T(|\mathcal{M}_2|, \Delta^+_\Sigma(\mathcal{M}_2)) \tag{3}$$

which is also 'canonical', in the sense that for every $a \in |\mathcal{M}_1|$, the embedding (3) maps the equivalence class of a in $F(|\mathcal{M}_1|, \Delta^+_\Sigma(\mathcal{M}_1))$ into the equivalence class of a in $F(|\mathcal{M}_2|, \Delta^+_\Sigma(\mathcal{M}_2))$. Notice that the reason why we have an embedding here is that $\Delta^+_\Sigma(\mathcal{M}_2)$ is *conservative* over $\Delta^+_\Sigma(\mathcal{M}_1)$, meaning that for every $\Sigma^{|\mathcal{M}_1|}$-atom A we have $T \cup \Delta^+_\Sigma(\mathcal{M}_2) \models A$ iff $T \cup \Delta^+_\Sigma(\mathcal{M}_1) \models A$ (it is so, because (3) is obtained from the substructure inclusion $\mathcal{M}_1 \subseteq \mathcal{M}_2$).

[9] A slight modification of this construction shows the existence of pushouts also when the maps of $\mathcal{A}$ into $\mathcal{M}_i$ are Σ-homomorphisms (not just substructure inclusions).

Vice versa, a *T-presentation of an embedding* is a pair of T-presentations $(X_1, \Gamma_1), (X_2, \Gamma_2)$ with $X_1 \subseteq X_2$ and with Γ_2 conservative over Γ_1; to this presentation, it corresponds the embedding (which we also call *canonical*) $F_T(X_1, \Gamma_1) \longrightarrow F_T(X_2, \Gamma_2)$ given by the map associating, for every $a \in X_1$, the equivalence class of a in $F_T(X_1, \Gamma_1)$ with the equivalence class of a in $F_T(X_2, \Gamma_2)$.

Similar considerations apply to T-forks: given a T-fork $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})$, we can associate with it a pair of canonical embeddings among canonical presentations

$$F_T(|\mathcal{M}_1|, \Delta^+_\Sigma(\mathcal{M}_1)) \longleftarrow F_T(|\mathcal{A}|, \Delta^+_\Sigma(\mathcal{A})) \longrightarrow F_T(|\mathcal{M}_2|, \Delta^+_\Sigma(\mathcal{M}_2))$$

A *T-presentation of a fork* is a triple of T-presentations $(X_0, \Gamma_0), (X_1, \Gamma_1), (X_2, \Gamma_2)$ with $X_1 \cap X_2 = X_0$, $X_0 \subseteq X_i$, and Γ_i conservative over Γ_0 $(i = 1, 2)$. To this T-presentation it corresponds a pair of canonical embeddings

$$F_T(X_1, \Gamma_1) \longleftarrow F_T(X_0, \Gamma_0) \longrightarrow F_T(X_2, \Gamma_2) \tag{4}$$

(which is a T-fork, up to renamings). An argument analogous to that used in the proof of Proposition 6.1 shows that the pushout of (4) is given by

$$F_T(X_1, \Gamma_1) \xrightarrow{\mu_1} F_T(X_1 \cup X_2, \Gamma_1 \cup \Gamma_2) \xleftarrow{\mu_2} F_T(X_2, \Gamma_2) \tag{5}$$

Notice that (for $i = 1, 2$) the map μ_i still associates, for every $a \in X_i$, the equivalence class of a in $F_T(X_i, \Gamma_i)$ with the equivalence class of a in $F_T(X_1 \cup X_2, \Gamma_1 \cup \Gamma_2)$, but this map *needs not* to be an embedding, because one cannot in general infer that $\Gamma_1 \cup \Gamma_2$ is conservative over Γ_i from the fact that Γ_i is conservative over Γ_0: this is indeed the key property leading to amalgamability (in the case of our Horn theories, if the property fails, amalgamability fails and vice versa).

6.2 Algebraic logic background

Below, we use as signature for Boolean algebras the signature Σ_{BA} comprising two unary operations $\cap$ and $\cup$ (for meet and join) and a unary operation $\sim$ for complement. We abbreviate $(\sim u) \cup t$ as $u \Rightarrow t$; $u_1 \Leftrightarrow u_2$ is defined as $(u_1 \Rightarrow u_2) \cap (u_2 \Rightarrow u_1)$. The atom $u \leq t$ is defined as $u \cap t = u$ and $u < t$ stands for $u \leq t \wedge u \neq t$.

Proposition 4.1 is proved in [6]; we report the proof because this is an important result for this paper too:

Proposition 4.1 *Every BAO-equational theory is BA-compatible.*

Proof. Let T be a BAO-equational theory;[10] the theory of Boolean algebras has as model completion the theory of *atomless* Boolean algebras [4],[11] so it is

[10] The argument works also for universal Horn theories extending the theory of Boolean algebras.

[11] Recall that a Boolean algebra is atomless iff for every nonzero element a from its support there is a nonzero b such that $b < a$.

sufficient to show how to embed a model $\mathcal{M}$ of T into a model $\mathcal{M}'$ of T which is based on an atomless Boolean algebra. Define a sequence of models of T by: $\mathcal{M}_0 := \mathcal{M}$, $\mathcal{M}_{k+1} := \mathcal{M}_k \times \mathcal{M}_k$; define also embeddings $\delta_k : \mathcal{M}_k \longrightarrow \mathcal{M}_{k+1}$ by $\delta_k(a) := \langle a, a\rangle$. Now take as $\mathcal{M}'$ the union (more precisely, the inductive limit) of this chain: clearly $\mathcal{M}'$ is atomless as a Boolean algebra (no non-zero element is minimal in it, as any $a \in \mathcal{M}_k$ gets identified with $\langle a, a\rangle = \langle a, 0\rangle \cup \langle 0, a\rangle$ in $\mathcal{M}_{k+1}$). $\dashv$

We now revisit key notion from algebraic logic and recall the *bijective correspondence between modal logics and BAO-equational theories* [29]. The correspondence works as follows.

Given a logic L with modal signature Σ_M, we define the BAO-equational theory T_L as the theory having as signature $\Sigma_L := \Sigma_M \cup \Sigma_{BA}$ and as set of axioms the set

$$BA \cup \{t = 1 \ \mid \ t \in L\}.$$

Notice that, from our notational conventions, it follows that *Σ_M-modal formulae are the same as Σ_L-terms.* Models of T_L will be called *L-algebras* in the following.

Vice versa, given an equational extension T of BA over the signature Σ, we define L_T as the classical modal logic over the modal signature $\Sigma \setminus \Sigma_{BA}$ axiomatized by the formulae

$$\{t \ \mid \ T \models t = 1\} \quad .$$

Notice that under the above bijection, we have $T_{L_1 \oplus L_2} = T_{L_1} \cup T_{L_2}$, i.e. *the fusion of BAO's* (as defined in Section 4) *corresponds to the fusion of modal logics* (as defined in Subsection 4.1).

Classical modal logics (in our sense) and equational extensions of BA are equivalent formalisms. For our purposes, it is important to revisit presentations, as introduced in the previous subsection, in terms of Lindenbaum-Tarski algebras.

Recall that a T_L-presentation (we fix a logic L) is pair (X, Γ) given by a set of fresh constants X and a set of Σ_L^X-atoms. Now, in the current situation, we can view the X as propositional variables and notice that every atom is equivalent to an atom of the kind $t = 1$, where t is a Σ_L^X-term (alias modal formula in which at most the X occur). This is so because there are no predicate symbols other than equality in Σ_L and because we can transform an atom $t_1 = t_2$ into the atom $t_1 \Leftrightarrow t_2 = 1$, modulo T_L equivalence. Thus, from now on, *a presentation will be just a pair (X, Γ) where the elements of Γ are Σ_L^X-terms.* We recall that the L-algebra $F_{T_L}(X, \Gamma)$ corresponding to the presentation (X, Γ) is built up from equivalence classes of Σ_L^X-terms under the equivalence relation $u_1 \sim_\Gamma u_2$ given by $T \cup \{t = 1 \mid t \in \Gamma\} \models u_1 = u_2$, by defining all operations on representative elements of equivalence classes.

Now, it is well-known that it is possible to build $F_{T_L}(X, \Gamma)$ in another equivalent way (the Lindenbaum-Tarski construction), directly via the global consequence relation $\vdash_L$ of the logic L. We recall how to do it and show the equivalence with the old construction of $F_{T_L}(X, \Gamma)$.

Given a presentation (X, Γ), the *Lindenbaum-Tarski algebra* $LT(X, \Gamma)$ is built as follows: we take the set of the modal formulae containing at most the propositional variables X and introduce the equivalence relation defined by $u_1 \sim_{\Gamma}^{LT} u_2$ iff $\Gamma \vdash_L u_1 \Leftrightarrow u_2$. The latter is a congruence because, for istance, if $u_1 \sim_{\Gamma}^{LT} u_2$ (i.e. $\Gamma \vdash_L u_1 \Leftrightarrow u_2$), then, applying the Replacement Rule, we also have $\Gamma \vdash_L o(u_1) \Leftrightarrow o(u_2)$, which means $o(u_1) \sim_{\Gamma}^{LT} o(u_2)$. The boolean cases are analogous. Defining each operation on representatives of equivalence classes, we obtain our algebra $LT(X, \Gamma)$; by construction, $LT(X, \Gamma)$ is a model of T_L and (under the natural evaluation of each variable in X as its own equivalence class) we have that all $u \in \Gamma$ evaluates to 1.

Proposition 6.2. *We have* $LT(X, \Gamma) \simeq F_{T_L}(X, \Gamma)$*, for every presentation* (X, Γ)*.*

Proof. It is sufficient to show that we have

$$\Gamma \vdash_L t \qquad \text{iff} \qquad T_L \cup \{u = 1 \mid u \in \Gamma\} \models t = 1$$

for every t. We start proving the right-to-left implication. Since, by hypothesis $T_L \cup \{u = 1 \mid u \in \Gamma\} \models t = 1$, and all $u \in \Gamma$ evaluate to 1 in $LT(X, \Gamma)$ (under the natural evaluation), we have that t also evaluates to 1 in $LT(X, \Gamma)$, which means, by definition of the congruence $\sim_{\Gamma}^{LT}$, that $\Gamma \vdash_L t \Leftrightarrow 1$, i.e. that $\Gamma \vdash_L t$ as required.

Conversely, suppose that $\Gamma \vdash_L t$, i.e. that there exists a proof in the modal logic L which uses the formulae in Γ and ends up with the formula t. We need to show that in any Σ_L^X-model $\mathcal{M}$ of $T_L \cup \Gamma$, we have $\mathcal{M} \models t = 1$. The statement is proved by an easy induction on the length of the proof witnessing that $\Gamma \vdash_L t$, recalling that in the logic L we have only two rules: Modus Ponens and Replacement Rule. Thus, for instance, for the inductive step concerning the latter rule, it is sufficient to observe that $\mathcal{M} \models (t_1 \Leftrightarrow t_2) = 1$ is the same as $\mathcal{M} \models t_1 = t_2$ and it implies $\mathcal{M} \models o(t_1) = o(t_2)$ and finally $\mathcal{M} \models (o(t_1) \Leftrightarrow o(t_2)) = 1$. ⊣

Due to the above Proposition, we shall feel free to use the constructions of $LT(X, \Gamma)$ and of $F_{T_L}(X, \Gamma)$ interchangeably.

6.3 Missed proofs from Section 4

We finally report the proof of our two main results from Section 4, namely Theorems 4.2 and 4.7.

Definition 6.3. *Let L be a classical modal logic. A commutative square of* T_L*-algebras*

$$\begin{array}{ccc} \mathcal{A}_1 & \xrightarrow{\varepsilon_1} & \mathcal{A} \\ {\scriptstyle \eta_1}\uparrow & & \uparrow{\scriptstyle \varepsilon_2} \\ \mathcal{A}_0 & \xrightarrow[\eta_2]{} & \mathcal{A}_2 \end{array}$$

is said to have the interpolation property *iff the following holds:*

$$\forall a_1 \in |\mathcal{A}_1|, \forall a_2 \in |\mathcal{A}_2| \ (\varepsilon_1(a_1) \leq \varepsilon_2(a_2) \Rightarrow \exists b \in |\mathcal{A}_0| \ (a_1 \leq \eta_1(b) \wedge \eta_2(b) \leq a_2))$$

Lemma 6.4. *In a commutative square having the interpolation property as above, if η_1 is injective (i.e. an embedding), so it is ε_2.*

Proof. Recall that a morphism μ among Boolean algebras is injective iff $1 \leq \mu(a)$ implies $1 \leq a$ for all a. Suppose that η_1 is injective and $1 \leq \varepsilon_2(a)$; then $\varepsilon_1(1) \leq \varepsilon_2(a)$, so there is $b \in |\mathcal{A}_0|$ such that $1 \leq \eta_1(b)$ and $\eta_2(b) \leq a$; this implies $b = 1$ and $1 \leq a$, as required. ⊣

Definition 6.5. *Let L be a classical modal logic. Given three algebras $\mathcal{A}_0$, $\mathcal{A}_1$ and $\mathcal{A}_2$ endowed with homomorphisms $\eta_1 : \mathcal{A}_0 \to \mathcal{A}_1$ and $\eta_2 : \mathcal{A}_0 \to \mathcal{A}_2$, we say that they are* superamalgamable *if there exists another algebra $\mathcal{A}$ with homomorphisms $\varepsilon_1 : \mathcal{A}_1 \to \mathcal{A}$ and $\varepsilon_2 : \mathcal{A}_2 \to \mathcal{A}$ such that the following square*

$$\begin{array}{ccc} \mathcal{A}_1 & \xrightarrow{\varepsilon_1} & \mathcal{A} \\ {\scriptstyle \eta_1}\uparrow & & \uparrow{\scriptstyle \varepsilon_2} \\ \mathcal{A}_0 & \xrightarrow[\eta_2]{} & \mathcal{A}_2 \end{array}$$

is commutative and has the interpolation property.

Next Proposition slightly restates the superamalgamation property (basically, it shows that in order to check superamalgamation property, we only need to fill a T_L-fork into a square having the interpolation property, without caring about the fact that the square is formed by embeddings):

Proposition 6.6. *The following conditions are equivalent for a modal logic L:*

(i) *T_L has the superamalgamation property;*
(ii) *every T_L-fork is superamalgamable;*
(iii) *the pushout of every T_L-fork has the interpolation property.*

Proof. The implication (i) $\Rightarrow$ (ii) is trivial, whereas the implication (ii) $\Rightarrow$ (iii) is immediate by the universal property of pushouts. To show that (iii) $\Rightarrow$ (i), assume (iii) and take a T_L-fork $(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_0)$; the related pushout

$$\begin{array}{ccc} \mathcal{A}_1 & \xrightarrow{\varepsilon_1} & \mathcal{A} \\ \uparrow & & \uparrow{\scriptstyle \varepsilon_2} \\ \mathcal{A}_0 & \rightarrowtail & \mathcal{A}_2 \end{array}$$

has the interpolation property and by Lemma 6.4, it follows that ε_2 is injective. Exchanging $\mathcal{A}_1$ and $\mathcal{A}_2$ (i.e. considering the fork $(\mathcal{A}_2, \mathcal{A}_1, \mathcal{A}_0)$), it follows that ε_1 is also injective (the pushout construction is symmetric). Thus, we have found a T-amalgam of the fork having the interpolation property. ⊣

Notice that, up to a renaming isomorphism, every triple $\mathcal{A}_0$, $\mathcal{A}_1$ and $\mathcal{A}_2$ of L-algebras endowed with embeddings $\eta_1 : \mathcal{A}_0 \rightarrowtail \mathcal{A}_1$ and $\eta_2 : \mathcal{A}_0 \rightarrowtail \mathcal{A}_2$ is a T_L-fork; as a consequence, T_L is superamalgamable iff *every such a triple of L-algebras connected by embeddings is superamalgamable.*

The following proposition relates comprehensive interpolation property and superamalgamability:

Proposition 6.7. *Let L be a modal logic. Then, T_L has the superamalgamability property iff L satisfies the comprehensive interpolation property.*

Proof. Suppose that L *has the comprehensive interpolation property.* Consider a T_L-fork; using presentations, we can suppose that the fork is given by a triple of T_L-presentations $(X_0, \Gamma_0), (X_0 \cup X_1, \Gamma_1), (X_0 \cup X_2, \Gamma_2)$ with $(X_0 \cup X_1) \cap (X_0 \cup X_2) = X_0$, and with Γ_i conservative over Γ_0 $(i = 1, 2)$. To this T_L-presentation it corresponds a pair of canonical embeddings (let us write X_0, X_i instead of $X_0 \cup X_i$ for simplicity):

$$F_{T_L}(X_0, X_1, \Gamma_1) \overset{\eta_1}{\longleftarrow} F_{T_L}(X_0, \Gamma_0) \overset{\eta_2}{\longrightarrow} F_{T_L}(X_0, X_2, \Gamma_2) \tag{6}$$

(we recall that we use the word 'canonical' to mean that η_i associates the equivalence class of t in $F_{T_L}(X_0, \Gamma_0)$ with the equivalence class of t in $F_{T_L}(X_0, X_i, \Gamma_i)$).[12] We are using Lindenbaum-Tarski construction for presentations, hence conservativity of Γ_i means that we have

$$\Gamma_i \vdash_L t(\underline{x}_0) \qquad \text{iff} \qquad \Gamma_0 \vdash_L t(\underline{x}_0) \tag{7}$$

for every modal formula (alias Σ_L-term) containing at most propositional variables $\underline{x}_0 \subseteq X_0$.

From (5), we know that the pushout of (6) is given by

$$F_{T_L}(X_0, X_1, \Gamma_1) \overset{\varepsilon_1}{\longrightarrow} F_{T_L}(X_0, X_1, X_2, \Gamma_1 \cup \Gamma_2) \overset{\varepsilon_2}{\longleftarrow} F_{T_L}(X_0, X_2, \Gamma_2) \tag{8}$$

with canonical maps $\varepsilon_1, \varepsilon_2$.

We use Proposition 6.6 and just show that the square formed by $(\eta_1, \eta_2, \varepsilon_1, \varepsilon_2)$ has the interpolation property. To this aim, let us suppose that for modal formulae t_1, t_2 (where t_1 is built up from the variables in $X_0 \cup X_1$ and t_2 is built up from the variables in $X_0 \cup X_2$) we have

$$\varepsilon_1([t_1]) = [t_1] \leq [t_2] = \varepsilon_2([t_2])$$

in $F_{T_L}(X_0, X_1, X_2, \Gamma_1 \cup \Gamma_2)$; the latter means that $\Gamma_1 \cup \Gamma_2 \vdash_L t_1 \Rightarrow t_2$ by the construction of the Lindenbaum-Tarski algebra. Since only finitely many formulae are used in a derivation, there exist two finite subsets $\Gamma_1' \subseteq \Gamma_1$ and $\Gamma_2' \subseteq \Gamma_2$ such that

$$\Gamma_1'(\underline{x}_0, \underline{x}_1), \Gamma_2'(\underline{x}_0, \underline{x}_2) \vdash_L t_1(\underline{x}_0, \underline{x}_1) \Rightarrow t_2(\underline{x}_0, \underline{x}_2)$$

[12] Notationally, the fact that η_i is canonical allows us to write equations like $\eta_i([t]) = [t]$; there is a bit of abuse in this notation, because we do not indicate where the equivalence class $[t]$ of t is taken from, but such missed information can be easily deduced from the context.

(here $\underline{x}_0$ is a tuple including all the variables from X_0 occurring in $\Gamma'_1, \Gamma'_2, t_1, t_2$, $\underline{x}_1$ is a tuple including all the variables from X_1 occurring in Γ'_1, t_1 and $\underline{x}_2$ is a tuple including all the variables from X_2 occurring in Γ'_2, t_2). By the comprehensive interpolation property, there exists a formula $u(\underline{x}_0)$ and a finite set of formulae $g_1(\underline{x}_0), ..., g_n(\underline{x}_0)$ such that:

$$\Gamma'_1(\underline{x}_0, \underline{x}_1) \vdash_L g_1$$

$$\Gamma'_2(\underline{x}_0, \underline{x}_2), g_1 \vdash_L g_2$$

$$\cdots$$

$$\Gamma'_1(\underline{x}_0, \underline{x}_1), g_1, ..., g_{2j-1} \vdash_L g_{2j}$$

$$\Gamma'_2(\underline{x}_0, \underline{x}_2), g_1, ..., g_{2k} \vdash_L g_{2k+1}$$

$$\cdots$$

and also such that:

$$\Gamma'_1(\underline{x}_0, \underline{x}_1), g_1, ..., g_n \vdash_L t_1 \Rightarrow u$$

$$\Gamma'_2(\underline{x}_0, \underline{x}_2), g_1, ..., g_n \vdash_L u \Rightarrow t_2$$

But, for $i = 1, 2$, by conservativity of Γ_i over Γ_0, we obtain that $\Gamma_i \vdash_L g_j$ (all $j = 1, \ldots, n$), hence $\Gamma_1 \vdash_L t_1 \Rightarrow u$ and $\Gamma_2 \vdash_L u \Rightarrow t_2$. Last two facts yield $[t_1] \leq [u] = \eta_1([u])$ in $F_{T_L}(X_0, X_1, \Gamma_1)$ and $[u] = \eta_2([u]) \leq [t_2]$ in $F_{T_L}(X_0, X_2, \Gamma_2)$, as required for the interpolation property to hold for the square $(\eta_1, \eta_2, \varepsilon_1, \varepsilon_2)$.

Conversely, suppose that L *has the superamalgamability property*. Let $\Gamma_1(\underline{x}, \underline{y})$, $\Gamma_2(\underline{x}, \underline{z})$ be finite sets of modal formulae and let $t_1(\underline{x}, \underline{y})$, $t_2(\underline{x}, \underline{z})$ be such that $\Gamma_1(\underline{x}, \underline{y}), \Gamma_2(\underline{x}, \underline{z}) \vdash_L t_1(\underline{x}, \underline{y}) \Rightarrow t_2(\underline{x}, \underline{z})$. We construct three algebras in T_L connected with (canonical) monomorphisms in order to apply the superamalgamability property. Let $\Gamma_0(\underline{x})$ be the set of the modal formulae $g(\underline{x})$ such that there is an $\underline{x}$-residue chain from $\Gamma_1(\underline{x}, \underline{y})$, $\Gamma_2(\underline{x}, \underline{z})$ ending up in $g(\underline{x})$. Then, we put $\Delta_1 := \Gamma_1 \cup \Gamma_0$ and $\Delta_2 := \Gamma_2 \cup \Gamma_0$; clearly Δ_1 and Δ_2 are conservative over Γ_0 by construction. Now, we are ready to take:

$$\mathcal{A}_1 := F_{T_L}(\underline{x}, \underline{y}, \Delta_1)$$

$$\mathcal{A}_2 := F_{T_L}(\underline{x}, \underline{z}, \Delta_2)$$

$$\mathcal{A}_0 := F_{T_L}(\underline{x}, \Gamma_0)$$

This is a triple of L-agebras connected by canonical embeddings. Then, by the superamalgamability property and by Proposition 6.6, the related pushout has the interpolation property. From (5), we know that such pushout is

$$F_{T_L}(\underline{x}, \underline{y}, \Delta_1) \xrightarrow{\varepsilon_1} F_{T_L}(\underline{x}, \underline{y}, \underline{z}, \Delta_1 \cup \Delta_2) \xleftarrow{\varepsilon_2} F_{T_L}(\underline{x}, \underline{z}, \Delta_2)$$

with canonical maps $\varepsilon_1, \varepsilon_2$.

From $\Gamma_1(\underline{x}, \underline{y}), \Gamma_2(\underline{x}, \underline{z}) \vdash_L t_1(\underline{x}, \underline{y}) \Rightarrow t_2(\underline{x}, \underline{z})$, it follows that we have

$$\varepsilon_1([t_1(\underline{x}, \underline{y})]) = [t_1(\underline{x}, \underline{y})] \leq [t_2(\underline{x}, \underline{z})] = \varepsilon_2([t_2(\underline{x}, \underline{z})])$$

in $F_{T_L}(\underline{x}, \underline{y}, \underline{z}, \Delta_1 \cup \Delta_2)$. By the interpolation property of the square, there exists a formula $u(\underline{x})$ such that we have $[t_1] \leq [u] = \eta_1[u]$ in $F_{T_L}(\underline{x}, \underline{y}, \Delta_1)$ and $\eta_2([u]) = [u] \leq [t_2]$ in $F_{T_L}(\underline{x}, \underline{z}, \Delta_2)$. This means that we have $\Gamma_1, \Gamma_0 \vdash_L t_1 \Rightarrow u$ and $\Gamma_2, \Gamma_0 \vdash_L u \Rightarrow t_2$. Since only finitely many modal formulae from Γ_0 are involved in these derivations and since all modal formulae in Γ_0 are obtained via $\underline{x}$-residue chains, the claim follows (we can obviously glue finitely many $\underline{x}$-residue chains into a single one). ⊣

The following Corollary is well-known ([12],[7],[28]):

Corollary 6.8. *If L is normal, then L has the comprehensive interpolation property iff it has the local interpolation property. Consequently T_L has the superamalgamability property iff L satisfies the local interpolation property.*

Proof. Suppose first that L has *the local interpolation property.* Consider the deduction relation $\Gamma_1(\underline{x}, \underline{y}), \Gamma_2(\underline{x}, \underline{z}) \vdash_L t_1(\underline{x}, \underline{y}) \Rightarrow t_2(\underline{x}, \underline{z})$. Since L is normal, applying the 'deduction theorem' it is clear that the latter is equivalent to[13] $\vdash_L (o\Gamma_1 \cap o\Gamma_2) \Rightarrow (t_1 \Rightarrow t_2)$, which implies (using propositional tautologies) that $\vdash_L (o\Gamma_1 \cap t_1) \Rightarrow (o\Gamma_2 \Rightarrow t_2)$ holds. By the local interpolation property, there exists a formula $u(\underline{x})$ such that $\vdash_L (o\Gamma_1 \cap t_1) \Rightarrow u$ and $\vdash_L u \Rightarrow (o\Gamma_2 \Rightarrow t_2)$. Applying again propositional tautologies and the 'deduction theorem', the previous statement is equivalent to $\Gamma_1 \vdash_L t_1 \Rightarrow u$ and $\Gamma_2 \vdash_L u \Rightarrow t_2$, which means that L has the comprehenesive interpolation property (here, the $\underline{x}$-residue chain is empty).

For the other implication[14], suppose that L has *the comprehensive interpolation property* and that $t_2(\underline{x}, \underline{z})$ is locally deducible from $t_1(\underline{x}, \underline{z})$, i.e. that $\vdash_L t_1(\underline{x}, \underline{y}) \Rightarrow t_2(\underline{x}, \underline{z})$ holds. This fact implies, by comprehensive interpolation property, that there exist a $\underline{x}$-residue chain $C = \{g_1, \dots, g_n\}$ and a formula $u(\underline{x})$ such that $C \vdash_L t_1 \Rightarrow u$ and $C \vdash_L u \Rightarrow t_2$. Hence, in order to achieve the aim it is sufficient to show that $\vdash_L C$ holds. In fact, since $\Gamma_i = \emptyset$, we have $\vdash_L g_1$; moreover, reasoning by induction, $\vdash_L g_1, \dots, \vdash_L g_{n-1}$ and $g_1, ..., g_{n-1} \vdash_L g_n$ imply, replacing the g_i $(i = 1, ..., n-1)$ with their proofs, $\vdash_L g_n$. Therefore, we conclude $\vdash_L C$, as wanted. ⊣

Classical propositional logic can be seen as the modal logic over the empty signature (as such, it is clearly also normal and Corollary 6.8 applies to it). From ordinary Craig interpolation theorem for classical propositional logic and Corollary 6.8, we can get the following very well-known fact:

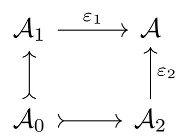
Lemma 6.9. *BA has the superamalgamation property.*

Pushouts in BA can be better described:

[13] Recall from Subsection 4.1, that we use the notation $o\Gamma$ to indicate the conjunction of a finite set of modal formulae of the kind $o_1(o_2 \cdots o_n(g) \cdots)$, for $g \in \Gamma$ and $n \geq 0$, $o_1, \dots, o_n \in \Sigma_M$.

[14] Notice that the following argument does not require the normality of L. Thus, the comprehensive interpolation property implies the local interpolation property even in the non-normal case.

Lemma 6.10. *Let* $(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_0)$ *be a* BA*-fork and let*



be the Boolean pushout of the fork; then all elements of $|\mathcal{A}|$ *can be written as finite meets of elements of the kind* $\varepsilon_1(a_1) \Rightarrow \varepsilon_2(a_2)$*, for* $a_1 \in |\mathcal{A}_1|$ *and* $a_2 \in |\mathcal{A}_2|$.

Proof. We replace our BA-fork with a canonical presentation

$$F_{BA}(X_0, X_1, \Gamma_1) \xleftarrow{\eta_1} F_{BA}(X_0, \Gamma_0) \xrightarrow{\eta_2} F_{BA}(X_0, X_2, \Gamma_2)$$

of it; then the pushout can be presented as

$$F_{BA}(X_0, X_1, \Gamma_1) \xrightarrow{\varepsilon_1} F_{BA}(X_0, X_1, X_2, \Gamma_1 \cup \Gamma_2) \xleftarrow{\varepsilon_2} F_{BA}(X_0, X_2, \Gamma_2)) \qquad (9)$$

Elements of $F_{BA}(X_0, X_1, X_2, \Gamma_1 \cup \Gamma_2)$ are equivalence classes of classical propositional formulae built up from the variables X_0, X_1, X_2. By conjunctive normal forms, they are conjunctions of clauses $l^1_1 \vee \cdots \vee l^1_n \vee l^2_1 \vee \cdots \vee l^2_m$, where the literals l^1_j are built from $X_0 \cup X_1$ and the literals l^2_k are built up from $X_0 \cup X_2$ (this representation is of course not unique). These clauses in turn can be written as

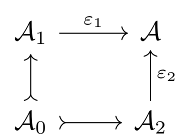
$$(\neg l^1_1 \wedge \cdots \wedge \neg l^1_n) \Rightarrow (l^2_1 \vee \cdots \vee l^2_m) \qquad (10)$$

and the equivalence class of (10) in $F_{BA}(X_0, X_1, X_2, \Gamma_1 \cup \Gamma_2)$ is of the desired shape. ⊣

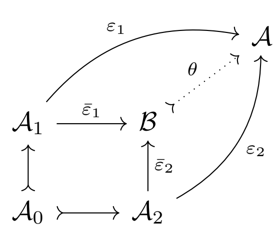
Theorem 4.2 *A* BAO*-theory* T *has the superamalgamation property iff it is* BA*-strongly amalgamable.*

Proof. Let T be equal to T_L for a modal logic L.

Suppose first that T_L *has the superamalgamation property* and let $(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_0)$ be a T_L-fork. Since it is superamalgamable, we can fill the inlusions $\mathcal{A}_0 \subseteq \mathcal{A}_1$ and $\mathcal{A}_0 \subseteq \mathcal{A}_2$ into a commutative square



having the interpolation property. Considering the BA-reducts, we can build the Boolean pushout square and the Boolean comparison morphism θ as in the following diagram:

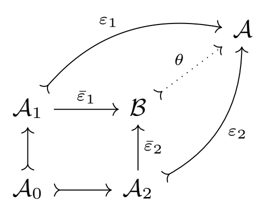


We want to show that θ is injective (i.e. a Boolean embedding), because this is precisely what it is required by BA-strong amalgamability. Using Lemma 6.10 (and recalling that in a Boolean algebra the meet of a finite set is equal to 1 iff all elements from the set are equal to 1), it is sufficient to prove that if $\theta(\bar{\varepsilon}_1(a_1) \Rightarrow \bar{\varepsilon}_2(a_2)) = 1$, then $\bar{\varepsilon}_1(a_1) \leq \bar{\varepsilon}_2(a_2)$ (i.e. $\bar{\varepsilon}_1(a_1) \Rightarrow \bar{\varepsilon}_2(a_2) = 1$). But

$$1 = \theta(\bar{\varepsilon}_1(a_1) \Rightarrow \bar{\varepsilon}_2(a_2)) = \theta(\bar{\varepsilon}_1(a_1)) \Rightarrow \theta(\bar{\varepsilon}_2(a_2)) = \varepsilon_1(a_1) \Rightarrow \varepsilon_2(a_2)$$

and in a Boolean algebra, this is equivalent to $\varepsilon_1(a_1) \leq \varepsilon_2(a_2)$. Since the outer square has the interpolation property, we conclude that there exists an element $a \in \mathcal{A}_0$ such that $a_1 \leq \eta_1(a)$ and $\eta_2(a) \leq a_2$, where η_1, η_2 are the inclusions of $\mathcal{A}_0$ into $\mathcal{A}_1, \mathcal{A}_2$. Therefore, we compute $\bar{\varepsilon}_1(a_1) \leq \bar{\varepsilon}_1(\eta_1(a)) = \bar{\varepsilon}_2(\eta_2(a)) \leq \bar{\varepsilon}_2(a_2)$, as wanted.

Conversely, *suppose that T_L is BA-strongly amalgamable.* Thus, by Proposition 6.6, it is sufficient to show that every T_L-fork $(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_0)$ is superamalgamable. By BA-strong amalgamability, the minimal BA-amalgam $(\mathcal{B}, \bar{\varepsilon}_1, \bar{\varepsilon}_2)$, which is the pushout in BA, can be embedded via a θ (unique by the universal property of the pushout) into a T_L-amalgam $(\mathcal{A}, \varepsilon_1, \varepsilon_2)$.



Now Lemma 6.9 states that BA has the superamalgamation property; so, by Proposition 6.6, the Boolean pushout

$$\begin{array}{ccc} \mathcal{A}_1 & \overset{\bar{\varepsilon}_1}{\rightarrowtail} & \mathcal{B} \\ \uparrow & & \uparrow {\scriptstyle \bar{\varepsilon}_2} \\ \mathcal{A}_0 & \rightarrowtail & \mathcal{A}_2 \end{array}$$

has the interpolation property. We show that, since θ is injective, the commutative square

$$\begin{array}{ccc} \mathcal{A}_1 & \overset{\varepsilon_1}{\rightarrowtail} & \mathcal{A} \\ \uparrow & & \uparrow {\scriptstyle \varepsilon_2} \\ \mathcal{A}_0 & \rightarrowtail & \mathcal{A}_2 \end{array}$$

has the interpolation property. In fact, if for $a_1 \in |\mathcal{A}_1|$ and $a_2 \in |\mathcal{A}_2|$, we have $\varepsilon_1(a_1) \leq \varepsilon_2(a_2)$ in $|\mathcal{A}|$, then we get $\theta(\bar{\varepsilon}_1(a_1)) \leq \theta(\bar{\varepsilon}_2(a_2))$, and also $\bar{\varepsilon}_1(a_1) \leq$

$\bar{\varepsilon}_2(a_2)$, because θ is injective. Thus, by the fact that the Boolean pushout has the interpolation property, we conclude that:

$$\exists a_0 \in |\mathcal{A}_0| \; (a_1 \leq \eta_1(a_0) \wedge \eta_2(a_0) \leq a_2)$$

where η_1, η_2 are the inclusions of $\mathcal{A}_0$ into $\mathcal{A}_1, \mathcal{A}_2$, as wanted. $\dashv$

Theorem 4.7 *If the modal logics L_1 and L_2 both have the comprehensive interpolation property, so does their fusion $L_1 \oplus L_2$.*

Proof. Immediate, from Proposition 6.7 and Corollary 4.3. $\dashv$

Finally, we state Wolter's theorem [25], which is an immediate consequence of Theorem 4.7 and of Corollary 6.8:

Theorem 6.11. *If the normal modal logics L_1 and L_2 both have the local interpolation property, so does their fusion $L_1 \oplus L_2$.*

References

27. J. Adámek and J. Rosický. *Locally presentable and accessible categories*, volume 189 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge, 1994.
28. M. Kracht. *8. Modal consequence relations.* In Johan Van Benthem Patrick Blackburn and Frank Wolter, editors, *Handbook of Modal Logic*, volume 3, pages 491 – 545. Elsevier, 2007.
29. H. Rasiowa. *An algebraic approach to non-classical logics.* North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1974. Studies in Logic and the Foundations of Mathematics, Vol. 78.