Path Rewriting and Combined Word Problems

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Abstract. We give an algorithm solving combined word problems (over non necessarily disjoint signatures) based on rewriting of equivalence classes of terms. The canonical rewriting system we introduce consists of few transparent rules and is obtained by applying Knuth-Bendix completion procedure to presentations of pushouts among categories with products. It applies to pairs of theories which are both constructible over their common reduct (on which we do not make any special assumption).

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1 Introduction

An essential problem in automated deduction consists in integrating theorem provers which are able to perform separated tasks. In the field of equational logic, this leads in particular to the following question: suppose you are able to solve word problems for theories T_1, T_2; can you solve word problem for $T_1 \cup T_2$? Better, can you design an algorithm taking as input two arbitrary algorithms for word problems for T_1 and T_2 and realizing a decision procedure for word problem for $T_1 \cup T_2$?

In case T_1, T_2 have disjoint signatures the positive answer was known from long time [12], although only more recently discovered within automated deduction community (see e.g. [11]). In the general case, combining decidable word problems may leads to undecidability, even if we suppose that T_1, T_2 are both conservative over their common reduct T_0. To this aim, consider the following example. Let T_0 be the theory of join-semilattices with zero (i.e. of commutative idempotent monoids) and let T_1 be the theory of Boolean algebras. As T_2 we take the theory of semilattice-monoids, which are algebras having both a monoid and a join-semilattice with zero structure and which satisfy the further equation:

$$(\bigvee_{i=1}^{n} x_i) \circ (\bigvee_{j=1}^{m} y_j) \;=\; \bigvee_{i=1}^{n} \bigvee_{j=1}^{m} (x_i \circ y_j).$$

T_2 clearly has decidable word problem (free algebras are finite sets of lists of the generators), as well as T_1. The union theory (which we better indicate with $T_1 +_{T_0} T_2$) corresponds to the 'distributive linear logic' of [8] and falls within the undecidability results of [1].

Clearly something must be assumed in order to have positive solution to combined word problems; in the literature it is usually assumed that T_1, T_2 share a set of constructors (we prefer the terminology 'they are both constructible over T_0'). There are various definitions of constructors and depending on such definitions there are variable strength results. Main papers on the subject are [5] and [3, 4]: the second has a weaker definition and consequently a stronger result. Our definition is again weaker (see Section 10 for details) and, more important, it covers natural mathematical examples and does not make any strong assumption on T_0 (in [5] T_0 is assumed to be free, in [3, 4] to be collapse-free).[1]

[5] and [3, 4] use quite different methods: in [3, 4] the combined decision algorithm is obtained through a refutation technique manipulating equations according to certain non-deterministic rules. As such it has the advantage of being more flexible, although it does not provide normal forms. On the contrary, [5] (and the similar method of [11] for the disjoint case) directly manipulate terms by abstracting and collapsing alien subterms and the suggested algorithm follows a complex and rigidly

[1] Recall that an equational theory is said to be collapse-free iff it cannot prove equations of the kind $x = t$, where x is a variable and t is a non-variable term.

preassigned procedure. Our method is more similar to that of [5] (in the sense that it manipulates terms), but has the same flexibility advantages as the method of [3, 4]. The idea is simple: we build a canonical rewriting system which is able to normalize paths of mixed pure terms.

The realization of such a plan looks very hard at a first glance: terms from combined signatures are quite unreliable datatypes, basically because they can compose, decompose and even collapse in many uncontrolled and overlapping ways. However, we shall put such complex combinatorics *under the control framework provided by the categorical approach to equational logic*: such approach goes back to the classical pioneering paper of F.W. Lawvere [9] in functorial semantics.[2] Basically, equational theories are identified with categories with products, so that in our situation we need to manipulate *presentations of pushouts* among such categories. We get a first general and simple presentation of these pushouts in Section 3 by means of two-sides rewrite rules. To this presentation we apply, in Section 5, *Knuth-Bendix completion procedure* and get the desired rewriting system, under some 'constructors' hypothesis for our theories.

This constructors hypothesis is formulated within a categorical framework in Section 5 by means of (weak) factorization systems and translated in symbolic terms in Section 10: roughly speaking, T_i is said to be constructible over T_0 iff there is a class E_i of terms (including variables and closed under renamings) in the signature Ω^i of T_i so that any Ω^i-term $t(x_1, \dots, x_n)$ decomposes uniquely (up to provable identity) as $u(v_1, \dots, v_k)$ where the $v_i(x_1, \dots, x_n)$ are (always up to provable identity) distinct terms from E_i and u is a k-minimized term in the signature Ω^0 of T_0 (a term $u(x_1, \dots, x_k)$ is said to be k-minimized iff it is not provably identical to any term in which only variables coming from a proper subset of $\{x_1, \dots, x_k\}$ occur). Examples are provided in Section 10 (a typical example is the case of commutative rings with unit which are constructible over abelian groups).

We briefly describe here the rewriting system $\mathcal{R}$ we obtain. $\mathcal{R}$ consists of only four rules (for technical reasons concerning 'colours' of terms, two of such rules are 'duplicated'). First rule (called *composition rule*) simply allows to compose equally coloured consecutive (equivalence classes of) terms. Second rule (called *ε-extraction rule*) minimizes terms by 'moving left' projections (i.e. n-tuples of distinct variables). Third rule (called *μ-extraction rule*) 'moves right' the second component of the above mentioned factorization of terms. The fourth rule (called *products rule*) is suggested by the completion procedure and has the following meaning: any projection (i.e. any tuple of distinct variable terms) appearing in an internal position of a path of pure terms *represents a 'hole' and the normalization process is supposed to fill such a hole by 'moving right' genuine terms* (i.e. terms which are not projections). The complete table of rules of $\mathcal{R}$ is given at the end of Section 5.

[2] We recall that there is another quite interesting category-theoretic approach to universal algebra, namely the *monads* approach (which has also been significantly used in questions related to rewriting, see e.g. [10]).

Although $\mathcal{R}$ is a quite simply described system, the confluence proof requires long work, because all critical pairs must be examined. This leads to a *large amount of details*, all consisting of elementary computations (in fact, once the technical tools are appropriately settled, single cases are treated in the most natural way).

The paper is organized as follows: in Section 2 we recall the necessary background from functorial semantics; in Section 3 we get a first presentation of pushouts among Lawvere categories. In Sections 4-5 we apply completion procedure and get the appropriate rewriting system $\mathcal{R}$. In Section 6 we provide local confluence and termination for a simple subsystem $\mathcal{R}_0$ of $\mathcal{R}$. In Section 7 a third rewriting system, called $\mathcal{R}^+$ is introduced ($\mathcal{R}^+$ is equivalent to $\mathcal{R}$, it normalizes slower but it is easier to manage); in addition useful technical facts are collected. In Section 8, $\mathcal{R}^+$ is proved to be locally confluent, whereas in Section 9 termination of both $\mathcal{R}$ and $\mathcal{R}^+$ is established. Finally, equivalence between $\mathcal{R}$ and $\mathcal{R}^+$ and canonicity of the former are obtained. Section 10 provides examples of constructible theories and of normalizations of paths of terms; a comparison with results of [3, 4] is done at the end of the paper.

Sections 6-7-8-9 can be skipped in a first reading by people mostly interested in our results (and less interested in their proofs).

This technical report is fully detailed and self-contained. We only assume a certain familiarity with rewriting (for some unexplained notions readers may consult [2]).

2 A short summary in functorial semantics

We recall that a category with *finite products* $\mathbf{C}$ is a category in which for every finite list of objects $X_1, \ldots, X_n$ ($n \geq 0$) there is an object $X_1 \times \cdots \times X_n$ and there are arrows

$$\pi^{X_1,\ldots,X_n}_{X_i} : X_1 \times \cdots \times X_n \longrightarrow X_i$$

(to be denoted simply as π_{X_i} or π_i) enjoying the following universal property:

- for every object Z, for every n-tuple of arrows $\alpha_i : Z \longrightarrow X_i$ ($i = 1, \ldots, n$) there is a unique arrow $\alpha : Z \longrightarrow X_1 \times \cdots \times X_n$ such that $\alpha \circ \pi_i = \alpha_i$ for all $i = 1, \ldots, n$[3] (such α is usually indicated by $\langle \alpha_1, \ldots, \alpha_n \rangle$).

The definition includes the case $n = 0$ and $n = 2$: in fact, such two cases are sufficient for the general case $n \geq 0$. We can so equivalently give the definition in the following way: a category $\mathbf{C}$ is said to have finite products iff

- there is a terminal object, namely an object $\mathbf{1}$ such that for every object X there is just one arrow $X \longrightarrow \mathbf{1}$ (such arrow is noted $\langle\ \rangle$ or $\langle\ \rangle_X$);

[3] Composition of arrows $\xrightarrow{\alpha}\xrightarrow{\beta}$ in a category is denoted as $\alpha \circ \beta$ in this paper (contrary to some more frequent notations like $\beta \circ \alpha$). We think that directly following 'arrow pictures' looks better for the purposes of this paper.

- for every pair of objects X_1, X_2 there are an object $X_1 \times X_2$ and arrows $\pi_1 : X_1 \times X_2 \longrightarrow X_1$, $\pi_2 : X_1 \times X_2 \longrightarrow X_2$, such that for every object Z and for every pair of arrows $\alpha_1 : Z \longrightarrow X_1$, $\alpha_2 : Z \longrightarrow X_2$, there is a unique arrow $\langle \alpha_1, \alpha_2 \rangle : Z \longrightarrow X_1 \times X_2$ such that $\langle \alpha_1, \alpha_2 \rangle \circ \pi_1 = \alpha_1$ and $\langle \alpha_1, \alpha_2 \rangle \circ \pi_2 = \alpha_2$.

In categories with finite products, we also use the standard abbreviation $\alpha_1 \times \cdots \times \alpha_n$ to denote (for $X_1 \xrightarrow{\alpha_1} Y_1, \ldots, X_n \xrightarrow{\alpha_n} Y_n$) the arrow $\langle \pi_1 \circ \alpha_1, \ldots, \pi_n \circ \alpha_n \rangle$.

In the paper *by 'category' we always mean 'category with finite products' and by 'functor' we always mean 'finite products preserving functor'.* Checking complex identities in categories with products might be a little painful if the above definition is used, however for general reasons it is sufficient to check such identities in the category **Set** of sets (basically, this is due to faithfulness of Yoneda embedding and to the fact that products are componentwise in presheaves). For instance, in order to check the identity

$$\langle \alpha \circ \beta, \gamma \rangle = \langle \alpha, \gamma \rangle \circ (\beta \times 1),$$

where $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z_1$, $X \xrightarrow{\gamma} Z_2$, [4] one can always assume that the data involved are just sets and functions and observe that "for every $x \in X$"

$$(\langle \alpha \circ \beta, \gamma \rangle)(x) = \langle \beta(\alpha(x)), \gamma(x) \rangle = (\beta \times 1) \circ (\langle \alpha(x), \gamma(x) \rangle) = (\langle \alpha, \gamma \rangle \circ (\beta \times 1))(x).$$

We shall meet many such identities in the paper, but we shall never justify them explicitly, we simply assume the reader realizes by himself that they are 'true in **Set**'.

An (equational) theory $T = \langle \Omega, Ax \rangle$ is just an ordinary signature Ω endowed with a set of pairs of terms ('the axioms' of T). We use letters $t, u, v, \ldots$ for terms and letters $x_1, x_2, \ldots$ for variables; $t(x_1, \ldots, x_n)$ means that the term t contains at most the variables $x_1, \ldots, x_n$. Notation $t(u_1/x_1, \ldots, t_n/x_n)$ (or simply $t(u_i/x_i)$ or again $t(u_1, \ldots, u_n)$) is used for substitutions; when we write $t(u/x_i)$ we mean $t(x_1/x_1, \ldots, u/x_i, \ldots, x_n/x_n)$. Notations like $\vdash_T t_1 = t_2$ refer to some sound and complete deduction system (e.g. equational logic). Deciding $\vdash_T t_1 = t_2$ is just the (uniform) *word problem* for T. In order to avoid irrelevant cases, *we shall always assume that our theories T match the following two requirements*:

- Ω always contains a constant symbol c_0 (this is harmless, because adding a free constant -if needed- does not change the nature of word problems);

- T is non-degenerate, namely $\not\vdash_T x_1 = x_2$.

Given a signature Ω and a category **C**, an Ω-interpretation $\mathcal{I}$ in **C** consists of the following data:

- an object A in **C** (called the support of $\mathcal{I}$);

[4] Identities are generically noted 1; in some cases, we may use a subscript for their domain if we want to evidentiate it (in the present case, we should have written 1_{Z_2}).

- for every $f \in \Omega_n$ (Ω_n is the set of function symbols having arity n), an arrow $\mathcal{I}(f) : A^n \longrightarrow A$ (notice that for $n = 0$, we have $\mathcal{I}(f) : \mathbf{1} \longrightarrow A$).

Given such an interpretation $\mathcal{I}$ and a term t, we can define $\mathcal{I}^n(t) : A^n \longrightarrow A$ (to be denoted simply as $\mathcal{I}(t)$ if confusion does not arise), for every n such that $x_1, \dots, x_n$ is a list containing all the variables occurring in t, as follows:

- $\mathcal{I}(x_i) = \pi_i$;

- $\mathcal{I}(f(t_1, \dots, t_m)) = \langle \mathcal{I}(t_1), \dots, \mathcal{I}(t_m) \rangle \circ \mathcal{I}(f)$.

An Ω-interpretation $\mathcal{I}$ is a *model* of $T = \langle \Omega, Ax \rangle$ (or is an internal T-algebra in $\mathbf{C}$) iff for every $(t_1, t_2) \in Ax$, we have $\mathcal{I}(t_1) = \mathcal{I}(t_2)$.[5]

Not only categories give models of theories, they can be used *as* theories. This is a basic point in categorical logic, which leads in our case to the notion of *Lawvere category.* Basically this is nothing but any one-sorted (finite products) category. Formally, a Lawvere category is a category having objects $\{X^n\}_{n \geq 0}$, in which X^n (endowed with specified projections $\pi_i : X^n \longrightarrow X$) is the product with itself n-times.[6] In our context (see below) π_i will be the (equivalence class of) the variable x_i. We fix the following convention about a Lawvere category: arrows $X^n \longrightarrow X^m$ of the kind $\langle \pi_{i_1}, \dots, \pi_{i_m} \rangle$ (where $i_1, \dots, i_m \leq n$) are called

- (pure) *projections* iff the $i_1, \dots, i_m$ are all distinct (in this case we must have $m \leq n$);
- *diagonals* iff $\{i_1, \dots, i_m\}$ include $\{1, \dots, n\}$ (in this case we must have $m \geq n$);
- *renamings* iff $i_1, \dots, i_m$ are just a permutation of $\{1, \dots, n\}$ (in this case we must have $n = m$).

In order to have a clearer picture, consider the category $\mathbf{Sf}$ having as objects the finite sets of the kind $\mathbf{n} = \{1, \dots, n\}$ and as arrows all functions (this is the skeleton category of finite sets); for every Lawvere category $\mathbf{T}$, we have a functor $S : \mathbf{Sf}^{\mathbf{op}} \longrightarrow \mathbf{T}$ associating X^n with $\mathbf{n}$ and $\langle \pi_{h(1)}, \dots, \pi_{h(m)} \rangle$ with every function $h : \mathbf{n} \longleftarrow \mathbf{m}$. Now an arrow in $\mathbf{T}$ is a (pure) projection iff it is the S-image of an injective function, it is a diagonal iff it is the S-image of a surjective function and it is a renaming iff it is the S-image of a bijection.

[5] Strictly speaking, one should show that this does not depend on the list $x_1, \dots, x_n$ (which includes all variables occurring in t_1, t_2) chosen in order to apply $\mathcal{I}$. Indeed it is so: in fact, a simple inductive argument shows that $\mathcal{I}^{n+1}(t_i)$ differs from $\mathcal{I}^n(t_i)$ by left composition with the n-tuple of projections $\langle \pi_1, \dots, \pi_n \rangle$. Now notice that any arrow of the kind $A^n \stackrel{\langle \pi_1, \dots, \pi_n, \alpha \rangle}{\longrightarrow} A^{n+1}$ gives the identity once composed on the right with such an n-tuple of projections (one can take as α anything, e.g. $A^n \stackrel{\mathcal{I}^n(t_1)}{\longrightarrow} A$). Thus $\mathcal{I}^{n+1}(t_1) = \mathcal{I}^{n+1}(t_2)$ holds iff $\mathcal{I}^n(t_1) = \mathcal{I}^n(t_2)$ holds (just take left composition with the above mentioned arrows).

[6] Of course, this implies that X^0 is equal to the terminal object $\mathbf{1}$ and that $X^{n_1+n_2}$ is the product of X^{n_1} and X^{n_2} with obvious tuples of π_i's as projections.

Lawvere categories are essentially in one-to-one correspondence with equational theories (we said 'essentially' because two equational theories differing only for the choice of the language and of the axioms are collapsed into the same 'invariant' Lawvere category). We need in this paper only one side of this correspondence, which we are going to explain. Let $T = (\Omega, Ax)$ be a theory; we build a Lawvere category $\mathbf{T}$ in the following way. We take as arrows $X^n \longrightarrow X^m$ the m-tuples of equivalence classes of terms containing at most the variables $x_1, \ldots, x_n$ (equivalence is intended through provable identity in T); equivalence classes of variables are the specified projections and *composition is substitution.* Explicitly, this means that composition of

$$\langle \{t_1\}, \ldots, \{t_m\} \rangle : X^n \longrightarrow X^m$$

and of

$$\langle \{u_1\}, \ldots, \{u_r\} \rangle : X^m \longrightarrow X^r$$

is the r-tuple of terms in the variables $x_1, \ldots, x_n$ given by:

$$\langle \{u_1(t_i/x_i)\}, \ldots, \{u_r(t_i/x_i)\} \rangle : X^n \longrightarrow X^r.$$

Let us now examine models; given a model $\mathcal{I}$ of a theory T in a category $\mathbf{C}$, we can associate with it a functor:

$$(1) \qquad F_{\mathcal{I}} : \mathbf{T} \longrightarrow \mathbf{C}$$

in the following way. If A is the support of $\mathcal{I}$, $F_{\mathcal{I}}(X^n) = A^n$; if $\langle \{t_1\}, \ldots, \{t_m\} \rangle$ is an arrow in $\mathbf{T}$,

$$F_{\mathcal{I}}(\langle \{t_1\}, \ldots, \{t_m\} \rangle) = \langle \mathcal{I}(t_1), \ldots, \mathcal{I}(t_m) \rangle.$$

Vice versa, given a functor $F : \mathbf{T} \longrightarrow \mathbf{C}$, we can associate with it the model $\mathcal{I}_F$ with support $F(X)$ given by

$$(2) \qquad \mathcal{I}_F(f) = F(\{f(x_1, \ldots, x_n)\})$$

for every $f \in \Omega_n$. The two correspondences (1) and (2) are inverse each other, thus we can *identify models with functors.*

Functors can be used also to deal with *syntactic interpretations*; we shall consider only special kinds of syntactic interpretations, those which matter for our purposes. Suppose we are given two theories $T_0 = (\Omega^0, Ax_0)$ and $T_1 = (\Omega^1, Ax_1)$, such that $\Omega^0 \subseteq \Omega^1$ and $Ax_0 \subseteq Ax_1$. Such data induce a functor

$$(3) \qquad I_1 : \mathbf{T_0} \longrightarrow \mathbf{T_1}$$

associating equivalence classes of terms with themselves (more precisely, equivalence class of t in $\mathbf{T_0}$ with equivalence class of t in $\mathbf{T_1}$). When T_1 is a conservative extension of T_0 (i.e. when Ω^0-terms are provably equal in T_0 iff they are provably equal in T_1) we write $T_0 \subseteq T_1$ for short. Notice that T_1 is a conservative extension of T_0 iff the functor I_1 is faithful (i.e. injective on arrows). Moreover, the restriction of a T_1-model to a T_0-model becomes composition on the left with I_1 (whenever models are seen as functors under the correspondence (1)-(2)).

3 Basic Equations

We now fix our main data for the paper: we have three theories

$$T_0 = (\Omega^0, Ax_0)$$
$$T_1 = (\Omega^1, Ax_1)$$
$$T_2 = (\Omega^2, Ax_2)$$

such that T_1 and T_2 are conservative extensions of T_0 and $\Omega^0 = \Omega^1 \cap \Omega^2$; taking (non disjoint) union of signatures and axioms we get a further theory which we call $T_1 +_{T_0} T_2$. *We suppose to be able to solve the word problem for* T_1, T_2; in general, as explained in the introduction, this is not enough for solving the word problem for $T_1 +_{T_0} T_2$ too,[7] however we may look for sufficient conditions yielding a positive solution.

The category $\mathbf{T_1} +_{\mathbf{T_0}} \mathbf{T_2}$ can be built as usual, by using terms; however we want to characterize it intrinsically in terms of $\mathbf{T_0}, \mathbf{T_1}, \mathbf{T_2}$. For this it is sufficient to look at its models. Let $\mathbf{C}$ be a category and let $\mathcal{I}_1, \mathcal{I}_2$ be models of T_1, T_2 in $\mathbf{C}$ restricting to the same model of T_0; from these data it is possible to build a unique model $\mathcal{I}$ of $T_1 +_{T_0} T_2$ in $\mathbf{C}$ restricting to $\mathcal{I}_1, \mathcal{I}_2$: the support is the same as the common support of $\mathcal{I}_1, \mathcal{I}_2$ and the interpretations of functions symbols can simply be joined (as they agree on Ω^0). Axioms $Ax_1 \cup Ax_2$ will be all true (as they involve only terms belonging to the same T_i). Translating everything in terms of functors, we have that $\mathbf{T_1} +_{\mathbf{T_0}} \mathbf{T_2}$ enjoys the following universal property: for every category $\mathbf{C}$, for every pair of functors $F_1 : \mathbf{T_1} \longrightarrow \mathbf{C}$ and $F_2 : \mathbf{T_2} \longrightarrow \mathbf{C}$ such that $I_1 \circ F_1 = I_2 \circ F_2$, there exists a unique functor $F : \mathbf{T_1} +_{\mathbf{T_0}} \mathbf{T_2} \longrightarrow \mathbf{C}$ such that $J_1 \circ F = F_1$ and $J_2 \circ F = F_2$ (here $I_1 : \mathbf{T_0} \longrightarrow \mathbf{T_1}$, $I_2 : \mathbf{T_0} \longrightarrow \mathbf{T_2}$, $J_1 : \mathbf{T_1} \longrightarrow \mathbf{T_1} +_{\mathbf{T_0}} \mathbf{T_2}$, $J_2 : \mathbf{T_2} \longrightarrow \mathbf{T_1} +_{\mathbf{T_0}} \mathbf{T_2}$ are functors coming from syntactic expansions as in (3)). Otherwise said, $\mathbf{T_1} +_{\mathbf{T_0}} \mathbf{T_2}$ is just the *pushout* of $\mathbf{T_1}, \mathbf{T_2}$ over $\mathbf{T_0}$.[8] This purely categorical property uniquely characterizes $\mathbf{T_1} +_{\mathbf{T_0}} \mathbf{T_2}$.

Next step consists in a direct description of a category (isomorphic to) $\mathbf{T_1} +_{\mathbf{T_0}} \mathbf{T_2}$, by using the above mentioned universal property: for this description we do not use terms anymore, but a more algebraic notion, namely mixed paths of arrows from $\mathbf{T_1}, \mathbf{T_2}$. To make the notation simpler, we act as functors I_1, I_2 (which are faithful) were just inclusions. Formally, a *path* $K : X^n \longrightarrow X^m$ is a non empty list of arrows coming from either $\mathbf{T_1}$ or $\mathbf{T_2}$ (or both)

$$K = \alpha_1, \ldots, \alpha_k$$

such that

[7]Notice that $T_1 +_{T_0} T_2$ might not be a conservative extension of T_1, T_2: for instance, both Boolean algebras and Heyting algebras are conservative over distributive lattices with 0 and 1, but putting together the two theories one gets again the theory of Boolean algebras which is obviously not conservative over Heyting algebras.

[8]In relevant contexts, a 2-dimensional pushout should be considered instead: it corresponds to the theory of pairs of models of T_1, T_2, endowed with an *isomorphism* among their respective reducts. 2-dimensional aspects could be conglobated with some further work in the approach of this paper.

(i) the domain of α_1 is X^n;

(ii) the codomain of α_k is X^m;

(iii) for every $i = 1, \ldots, k-1$, the codomain of α_i is equal to the domain of α_{i+1}.

Paths are just words (with 'typing' restrictions). Equivalence relations on paths (*stable* with right and left concatenation) can be introduced by two-side rewrite rules. The plan is quite simple: **identify such rules, orient and complete them into a canonical rewrite system** (after all, the situation is very similar to string-rewriting systems for monoid presentations).

In the remaining part of the paper, we make the following *conventions*:

- we shall use letters $\alpha, \beta, \ldots$ for arrows from $\mathbf{T_1} \cup \mathbf{T_2}$, letters $\alpha^1, \beta^1, \ldots$ for arrows from $\mathbf{T_1}$, letters $\alpha^2, \beta^2, \ldots$ for arrows from $\mathbf{T_2}$ and letters $\alpha^0, \beta^0, \ldots$ for arrows from $\mathbf{T_0}$; notice that any arrow like α^1 may happen to come from $\mathbf{T_0}$, the vice versa however cannot be;
- instead of indicating types (i.e. objects of Lawvere categories) with $X^n, X^m, \ldots$ we may use letters $Y, Z, U, \ldots$ if the knowledge of the exponent does not matter; letter X however can only indicate X^1;
- roman letters can be used to indicate arrows having codomain X, that is a^1 for instance, stands for an arrow in $\mathbf{T_1}$ (which might belong to $\mathbf{T_0}$ too) having domain some $Y = X^n$, but whose codomain can only be $X = X^1$.

Next, we give main definitions for path rewriting. Let $\mathcal{S}$ be a set of pairs of paths; we write

(i) $K \Rightarrow_{\mathcal{S}} K'$ (or simply $K \Rightarrow K'$, leaving $\mathcal{S}$ as understood from the context) iff $K = K_1, L, K_2$ and $K' = K_1, R, K_2$ for some pair $\langle L, R \rangle \in \mathcal{S}$;

(ii) $K \Leftrightarrow_{\mathcal{S}} K'$ (or simply $K \Leftrightarrow K'$) iff $K = K_1, L, K_2$ and $K' = K_1, R, K_2$ for some pair $\langle L, R \rangle$ such that either $\langle L, R \rangle \in \mathcal{S}$ or $\langle R, L \rangle \in \mathcal{S}$;

(iii) $K \Rightarrow^*_{\mathcal{S}} K'$ (or simply $K \Rightarrow^* K'$) for the reflexive-transitive closure of $\Rightarrow_{\mathcal{S}}$;

(iv) $K \Leftrightarrow^*_{\mathcal{S}} K'$ (or simply $K \Leftrightarrow^* K'$) for the least equivalence relation containing $\Rightarrow_{\mathcal{S}}$.

Clearly $\Leftrightarrow^*$ is the least stable equivalence relation extending $\mathcal{S}$. Pairs $\langle L, R \rangle \in \mathcal{S}$ will be directly written as $L \Rightarrow R$ and called *rules of* $\mathcal{S}$; alternatively, they might be written as $L \Leftrightarrow R$ (and called *basic equations of* $\mathcal{S}$), but in such a case we tacitly assume that $\mathcal{S}$ is symmetric, i.e. that $\mathcal{S}$ contains $\langle R, L \rangle$ in case it contains $\langle L, R \rangle$ (in such a case e.g. relations $\Rightarrow$ and $\Leftrightarrow$ obviously coincide).

Next theorem accomplishes our first goal ('finding appropriate basic equations'):

Theorem 3.1 *Let $\mathcal{P}$ be given by the following two kinds of pairs of paths:*

$$\alpha^i, \beta^i \Leftrightarrow \alpha^i \circ \beta^i \qquad (i = 1, 2)$$

$$1 \times \alpha_2, \alpha_1 \times 1 \Leftrightarrow \alpha_1 \times 1, 1 \times \alpha_2$$

(where in the last pair we have

$$\alpha_1 : Y_1 \longrightarrow Z_1 \qquad \alpha_2 : Y_2 \longrightarrow Z_2$$

and so

$$1 \times \alpha_2 : Y_1 \times Y_2 \longrightarrow Y_1 \times Z_2$$
$$\alpha_1 \times 1 : Y_1 \times Z_2 \longrightarrow Z_1 \times Z_2 \ ).$$

We have that $\mathbf{T_1} +_{\mathbf{T_0}} \mathbf{T_2}$ is isomorphic to the Lawvere category having as arrows the equivalence classes of paths under the relation $\Leftrightarrow^*_{\mathcal{P}}$.

Proof. Let $\mathbf{P}$ be the category having $\{X^n\}_{n\geq 0}$ as objects and as arrows $X^n \longrightarrow X^m$ the equivalence classes (wrt $\Leftrightarrow^*$) of paths of domain X^n and codomain X^m. Composition of $\{K\}$ and $\{L\}$ is $\{K, L\}$. Identity of X^n turns out to be just $\{1_{X^n}\}$.

We first show that $\mathbf{P}$ has finite products. $X^0 = \mathbf{1}$ is obviously terminal; in fact any path $K : Y \longrightarrow \mathbf{1}$ is equivalent to the singleton path $\langle\ \rangle_Y$ by iterated applications of the first basic equation of $\mathcal{P}$ (last member of K must be some $\langle\ \rangle_Z$, so it composes with the last-but-one member giving again something of the same kind, etc.).

Given objects $Y_1 = X^{n_1}, Y_2 = X^{n_2}$, we take $Y_1 \times Y_2$ (i.e. $X^{n_1+n_2}$) as binary product and $\{\pi_{Y_1}\}, \{\pi_{Y_2}\}$ as projections (here π_{Y_1}, π_{Y_2} are obviously the projections in $\mathbf{T_0}$). Let us now take two paths K_1, K_2 of domain Z and codomains Y_1, Y_2, respectively. Suppose for instance that

$$K_1 = \alpha_1, \ldots, \alpha_r \qquad K_2 = \beta_1, \ldots, \beta_s.$$

Let $\langle K_1, K_2\rangle$ be the path:

$$Z \stackrel{\langle 1_Z, 1_Z\rangle}{\longrightarrow} Z \times Z \stackrel{1_Z \times K_2}{\longrightarrow} Z \times Y_2 \stackrel{K_1 \times 1_{Y_2}}{\longrightarrow} Y_1 \times Y_2$$

where $1_Z \times K_2$ is $(1_Z \times \beta_1), \ldots, (1_Z \times \beta_s)$ ($K_1 \times 1_{Y_2}$ is defined analogously). We show that $\{\langle K_1, K_2\rangle\}$ enjoys the universal property for pairs. In fact

$$\langle 1_Z, 1_Z\rangle, (1_Z \times K_2), (K_1 \times 1_{Y_2}), \pi_{Y_1} \Leftrightarrow^* K_1$$

by successive applications of the first basic equation of $\mathcal{P}$ (we have $(\alpha_r \times 1_{Y_2}) \circ \pi_{Y_1} = \pi_{dom(\alpha_r)} \circ \alpha_r$, etc. so we finally get $\langle 1_Z, 1_Z\rangle, (1_Z \times K_2), \pi_{dom(\alpha_1)}, K_1 \Leftrightarrow^* K_1$, because $dom(\alpha_1) = Z$ and for every j, $(1_Z \times \beta_j) \circ \pi_Z = \pi_Z$). Similarly

$$\langle 1_Z, 1_Z\rangle, (1_Z \times K_2), (K_1 \times 1_{Y_2}), \pi_{Y_2} \Leftrightarrow^* K_2$$

(by the same passages in different order).

Let now K be another path from Z into $Y_1 \times Y_2$ such that $K, \pi_{Y_1} \Leftrightarrow^* K_1$ and $K, \pi_{Y_2} \Leftrightarrow^* K_2$. We must have $K = K', \langle\gamma_1, \gamma_2\rangle$, for some $\langle\gamma_1, \gamma_2\rangle : U \longrightarrow Y_1 \times Y_2$; so $K', \gamma_1 \Leftrightarrow^* K_1$ and $K', \gamma_2 \Leftrightarrow^* K_2$. From this, a glance to the shape of our basic equations[9] yields $(K' \times 1_{Y_2}), (\gamma_1 \times 1_{Y_2}) \Leftrightarrow^* (K_1 \times 1_{Y_2})$ and $(1_Z \times K'), (1_Z \times \gamma_2) \Leftrightarrow^* (1_Z \times K_2)$. Consequently

$$\langle K_1, K_2\rangle \;\Leftrightarrow^*\; \langle 1_Z, 1_Z\rangle, (1_Z \times K'), (1_Z \times \gamma_2), (K' \times 1_{Y_2}), (\gamma_1 \times 1_{Y_2}).$$

We only have to show that this last path is equivalent to $K = K', \langle\gamma_1, \gamma_2\rangle$. If $K' = \delta_1, \ldots, \delta_l$, by repeated applications of the second basic equation (first basic equation is also used e.g. in contracting $(1 \times \delta_j), (\delta_j \times 1)$ into $\delta_j \times \delta_j$), we have that

$$\langle 1_Z, 1_Z\rangle, (1_Z \times K'), (1_Z \times \gamma_2), (K' \times 1_{Y_2}), (\gamma_1 \times 1_{Y_2}) \;\Leftrightarrow^*\; \langle 1_Z, 1_Z\rangle, (K' \times K'), (\gamma_1 \times \gamma_2)$$

(where $K' \times K'$ is $(\delta_1 \times \delta_1), (\delta_2 \times \delta_2), \ldots, (\delta_l \times \delta_l)$). Finally, observe that $\langle 1_Z, 1_Z\rangle \circ (\delta_1 \times \delta_1) = \delta_1 \circ \langle 1_{cod(\delta_1)}, 1_{cod(\delta_1)}\rangle$, etc. hence repeated applications of the first basic equation yield

$$\langle K_1, K_2\rangle \;\Leftrightarrow^*\; K', \langle 1_U, 1_U\rangle, \gamma_1 \times \gamma_2 \;\Leftrightarrow\; K', \langle\gamma_1, \gamma_2\rangle,$$

as wanted.

In order to check that $\mathbf{P}$ is isomorphic to $\mathbf{T_1} +_{\mathbf{T_0}} \mathbf{T_2}$, we show it enjoys the related universal property. Functors

$$F_1 : \mathbf{T_1} \longrightarrow \mathbf{P} \qquad F_2 : \mathbf{T_2} \longrightarrow \mathbf{P}$$

associating with α^i the equivalence class $\{\alpha^i\}$ obviously commute with the inclusions $I_1 : \mathbf{T_0} \longrightarrow \mathbf{T_1}$ and $I_2 : \mathbf{T_0} \longrightarrow \mathbf{T_2}$. Now let $G_i : \mathbf{T_i} \longrightarrow \mathbf{C}$ $(i = 1, 2)$ be such that $I_1 \circ G_1 = I_2 \circ G_2$. There is in fact a unique functor $G : \mathbf{P} \longrightarrow \mathbf{C}$ such that $F_1 \circ G = G_1$ and $F_2 \circ G = G_2$: it is the functor associating with $\{\alpha_1^{i_1}, \ldots, \alpha_k^{i_k}\}$ the arrow $G_{i_1}(\alpha_1^{i_1}) \circ \cdots \circ G_{i_k}(\alpha_k^{i_k})$. This definition is forced by the conditions $F_1 \circ G = G_1$ and $F_2 \circ G = G_2$ and is good because basic equations of $\mathcal{P}$ express identities holding in any category with finite products. This completes the proof of the theorem. $\dashv$

[9] For the case of the second basic equation, you need identities like $1_Y \times (\delta \times 1_Z) = 1_Y \times \delta \times 1_Z = (1_Y \times \delta) \times 1_Z$, which hold in Lawvere categories (in fact, if e.g. $Y = X^n, Z = X^m$ and $\delta = \langle d_1, \ldots, d_{k_2}\rangle : X^{k_1} \longrightarrow X^{k_2}$, then unravelling the definitions the three members are all equal to

$$\langle \pi_1, \ldots, \pi_n, \pi \circ d_1, \ldots, \pi \circ d_{k_2}, \pi_{n+k_1+1}, \ldots, \pi_{n+k_1+m}\rangle,$$

where $\pi = \langle \pi_{n+1}, \ldots, \pi_{n+k_1}\rangle$). The point is that in Lawvere categories the finite product structure is freely generated (actually by one object); this is usual for categories coming from syntactic calculi, however in the general context of arbitrary category with products such identities hold only up to (coherent) isomorphisms.

In the applications, we should keep in mind that the isomorphism of categories among $\mathbf{T_1} +_{\mathbf{T_0}} \mathbf{T_2}$ and $\mathbf{P}$ is the unique expansion to the signature $\Omega^1 \cup \Omega^2$ of the models $F_1 : \mathbf{T_1} \longrightarrow \mathbf{P}$, $F_2 : \mathbf{T_2} \longrightarrow \mathbf{P}$ associating with α^i the equivalence class $\{\alpha^i\}$. This means the following: given an $\Omega^1 \cup \Omega^2$-term t, the universal model (isomorphism) $U : \mathbf{T_1} +_{\mathbf{T_0}} \mathbf{T_2} \longrightarrow \mathbf{P}$ *interpretes it as the equivalence class of any path obtained by expressing t as an iterated composition of terms which are pure*, i.e. which are either Ω^1 or Ω^2-terms. Such a path (called a *splitting path* for t) can be effectively computed from t in many ways (possibly yielding not the same path, but yielding in any case $\Leftrightarrow^*$-equivalent paths); one might for instance adopt usual abstraction of alien subterms, or alternatively make use of the following simply described inductive procedure (which applies to any tuple $t_1, \dots, t_n$ of terms having variables included in some fixed list $x_1, \dots, x_m$):

- if $t_1, \dots, t_n$ are all Ω^1 or Ω^2-terms, a splitting path is the singleton path

  $$\langle \{t_1\}, \dots, \{t_n\} \rangle$$

  having domain X^m and codomain X^n (recall that arrows in $\mathbf{T_1}, \mathbf{T_2}$ are equivalence classes of terms under provable identity in the corresponding theory);

- otherwise, we have e.g. that $t_i = f(u_1, \dots, u_k)$; a splitting path K of

  $$t_1, \dots, t_{i-1}, u_1, \dots, u_k, t_{i+1}, \dots, t_n$$

  is given (we apply multiset induction on term complexities) and it has codomain X^{n-1+k}, so we can take

  $$K, \langle \{x_1\}, \dots, \{f(x_i, \dots, x_{i+k})\}, \dots, \{x_{n-1+k}\} \rangle$$

  as a splitting path for $t_1, \dots, t_n$.

It is now clear how we can deal with word problems: to decide whether t and u are $T_1 +_{T_0} T_2$-equal, it is sufficient to split them into paths K and L according to one of the above mentioned procedures and then check whether $K \Leftrightarrow^* L$ holds or not. Of course, this will become convenient only after turning our basic equations into a canonical rewriting system. Let us see in any case an example.

Example Let us prove the well-known fact from elementary algebra saying that it is not possible to endow a given distributive lattice with 0 and 1 with two different Boolean algebra structures (the complement, in case it exists, is unique). Let T_0 be the theory of distributive lattices with 0 and 1 and let T_1, T_2 be the theory of Boolean algebras. We show that

$$\vdash_{T_1 +_{T_0} T_2} \neg_1 x_1 = \neg_1 x_1 \wedge \neg_2 x_1$$

(here $\neg_i$ is complement in T_i, what we prove is $\neg_1 x_1 \leq \neg_2 x_1$). We take $X \stackrel{\neg_1 x_1}{\longrightarrow} X$ as splitting path of $\neg_1 x_1$ (*we usually drop brackets in the examples*, to be precise we should write $X \stackrel{\{\neg_1 x_1\}}{\longrightarrow} X$); as splitting path of $\neg_1 x_1 \wedge \neg_2 x_1$, we take

$$X \stackrel{\langle \neg_1 x_1, x_1 \rangle}{\longrightarrow} X^2 \stackrel{x_1 \wedge \neg_2 x_2}{\longrightarrow} X.$$

Notice we could have used

$$X \stackrel{\langle x_1, \neg_2 x_1 \rangle}{\longrightarrow} X^2 \stackrel{\neg_1 x_1 \wedge x_2}{\longrightarrow} X$$

istead: indeed the two paths are $\Leftrightarrow^*$-equivalent (just expand $\langle \neg_1 x_1, x_1 \rangle, x_1 \wedge \neg_2 x_2$ to $\langle x_1, x_1 \rangle, \langle \neg_1 x_1, x_2 \rangle, \langle x_1, \neg_2 x_2 \rangle, x_1 \wedge x_2$, then apply second basic equation and contract once again). We need to show that

$$\neg_1 x_1 \ \Leftrightarrow^* \ \langle \neg_1 x_1, x_1 \rangle, x_1 \wedge \neg_2 x_2.$$

For this, let us consider the path

$$K \ = \ \langle \neg_1 x_1, \, x_1 \rangle, \langle x_1, x_2, x_1 \wedge x_2 \rangle, \, x_1 \wedge (\neg_2 x_2 \vee x_3);$$

We have

$$K \ \Leftrightarrow \ \langle \neg_1 x_1, x_1 \rangle, \, x_1 \wedge (\neg_2 x_2 \vee (x_1 \wedge x_2)) \ \Leftrightarrow \ \neg_1 x_1$$

because $\{x_1 \wedge (\neg_2 x_2 \vee (x_1 \wedge x_2))\} = \{x_1\}$. On the other hand

$$\begin{array}{rcl} K & \Leftrightarrow & \langle \neg_1 x_1, x_1, x_1 \wedge \neg_1 x_1 \rangle, \, x_1 \wedge (\neg_2 x_2 \vee x_3) \Leftrightarrow \\ & \Leftrightarrow & \langle \neg_1 x_1, x_1 \rangle, \, \langle x_1, x_2, 0 \rangle, \, x_1 \wedge (\neg_2 x_2 \vee x_3) \Leftrightarrow \\ & \Leftrightarrow & \langle \neg_1 x_1, x_1 \rangle, \, x_1 \wedge \neg_2 x_2. \end{array}$$

In conclusion, we have

$$\neg_1 x_1 \ \Leftrightarrow^* \ K \ \Leftrightarrow^* \ \langle \neg_1 x_1, x_1 \rangle, x_1 \wedge \neg_2 x_2$$

as wanted. $\dashv$

Notice that in the above example we sometimes replaced terms from Ω^i ($i = 1, 2$) with terms from Ω^0, when we realized this was possible. Such passages are indispensable in order to activate the first basic equation (which applies to consecutive equally coloured terms), but they might be non effective. The additional hypotheses we shall make on our data in order to be able to orient and complete basic equations into a canonical rewrite system will also be sufficient in order to make such passages effective.

4 Orientation

Before beginning orientation and completion, we make a modification to our 'datatypes', due to the fact that we do not want to bother distinguishing paths which are mere alphabetic variants each other. Formally, the involved definitions are the following (let K be the path $Y_1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_k} Y_{k+1}$ and let L be 'parallel' path $Y_1 \xrightarrow{\beta_1} \cdots \xrightarrow{\beta_k} Y_{k+1}$, with α_i, β_i equally coloured and having the same domain and codomain):

- K is said to be a *ρ-renaming* of L (where $\rho = \{\rho_i : Y_i \longrightarrow Y_i\}_{1 \le i \le k+1}$ is a list of renamings)[10] iff the following squares

$$\begin{array}{ccc} Y_i & \xrightarrow{\alpha_i} & Y_{i+1} \\ {\scriptstyle \rho_i}\downarrow & & \downarrow{\scriptstyle \rho_{i+1}} \\ Y_i & \xrightarrow[\beta_i]{} & Y_{i+1} \end{array}$$

  commute for $i = 1, \dots, k$ (otherwise said, we have $\beta_i = \rho_i^{-1} \circ \alpha_i \circ \rho_{i+1}$ for all i); we write $L = \rho(K)$ in order to express that L is (the) ρ-renaming of K;

- K is said to be the *ρ-alphabetic variant* of L (where $\rho = \{\rho_i : Y_i \longrightarrow Y_i\}_{1 \le i \le k+1}$ is a list of renamings) iff it is the ρ-renaming of L and moreover $\rho_1 = 1_{Y_1}$ and $\rho_{k+1} = 1_{Y_{k+1}}$ (the reason for this definition is that variables in internal equivalence classes of terms in a path are considered bounded).

Example. For every permutation σ on the n-elements set, we have that path

$$K_1, \langle a_1, \dots, a_n \rangle, \alpha, K_2$$

is an alphabetic variant of the path

$$K_1, \langle a_{\sigma(1)}, \dots, a_{\sigma(n)} \rangle, \langle \pi_{\sigma^{-1}(1)}, \dots, \pi_{\sigma^{-1}(n)} \rangle \circ \alpha, K_2$$

(here K_1, K_2 might be empty). Thus applying alphabetic variants allows permuting the components of an arrow in a path (provided such arrow is not in last position). $\dashv$

Example. Path

$$W \xrightarrow{K_1} Y \times Z \times U \xrightarrow{\langle \alpha, \pi_Z \rangle} V \times Z \xrightarrow{K_2} T$$

[10] Recall from Section 2 that a renaming $X^n \longrightarrow X^n$ in a Lawvere category is an n-tuple of projections of the kind $\langle \pi_{\sigma(1)}, \dots, \pi_{\sigma(n)} \rangle$, where σ is a permutation on $\{1, \dots, n\}$.

is an alphabetic variant of the path

$$W \stackrel{K_1\circ\langle\pi_Y,\pi_Z,\pi_U\rangle}{\longrightarrow} Y\times U\times Z \stackrel{\langle\langle\pi_Y,\pi_Z,\pi_U\rangle\circ\alpha,\pi_Z\rangle}{\longrightarrow} V\times Z \stackrel{K_2}{\longrightarrow} T$$

(here only K_2 might be empty and $K_1\circ\langle\pi_Y,\pi_Z,\pi_U\rangle$ is K_1 with last arrow composed with $\langle\pi_Y,\pi_Z,\pi_U\rangle$). Thus applying alphabetic variants allows assuming that certain projections (located not in first position) project, say, on last components of their domains. $\dashv$

The content of the last two examples will be frequently and tacitly used within the technical Sections of the paper.

We shall apply rewriting on *equivalence classes of paths modulo 'being an alphabetic variant of'*. This needs some additional conventions on our rules, because we want to have the following property (making the rewriting process easily manageable): *if K rewrites to L, then any alphabetic variant of K rewrites to some alphabetic variant of L.* In addition, notation of certain rules may be awkward in case we do not stipulate anything about their alphabetic variants. Consequently, we stipulate that *the renaming of any rule is always tacitly supposed to be available as a rule*: by this, we mean that if $K\Rightarrow K'$ is a rule, then $\rho(K)\Rightarrow\rho'(K')$ is also a rule, for any list ρ,ρ' of renamings such that first and last components of ρ,ρ' are respectively equal.[11]

A consequence of the above stipulation is that the normal forms we eventually obtain, will be unique only up to alphabetic variants. Checking whether two paths are alphabetic variants each other, in case we know they are both in normal forms, does not substantially affect efficiency, given the particular structure of normal forms (we shall turn on that in Section 10).

Before going on, we need another preliminary indispensable decision about our datatypes. As evidentiated also in the example at the end of Section 3, terms like $f(t_1,t_2)$, where $f\in\Omega^0$ and where $t_i(x_1)$ is a pure Ω^i-term, have (at least) two different splitting paths, namely

$$X \stackrel{\langle t_1(x_1),x_1\rangle}{\longrightarrow} X^2 \stackrel{f(x_1,t_2(x_2))}{\longrightarrow} X \quad\text{and}\quad X \stackrel{\langle x_1,t_2(x_1)\rangle}{\longrightarrow} X^2 \stackrel{f(t_1(x_1),x_2)}{\longrightarrow} X.$$

Our final aim is that of having (uniqueness of) normal forms for paths, so we must decide once for all which one has to be considered in normal form. This choice is clearly conventional, but has to be done one way or another: we choose the former path. This yields to the following notion: say that a path is *well-coloured* iff it has the form K,α^2 (where K is possibly empty). This means that the last arrow in a well-coloured path must come from $\mathbf{T_2}$ (which does not exclude it might come from $\mathbf{T_0}$ as well).

We modify our basic equations so that we need to consider only well-coloured paths. For a path $K:Y\longrightarrow Z$, let K^+ be the well-coloured path $K,1_Z$.

[11] We shall of course always deal with rules $K\Rightarrow K'$ such that K and K' agree on domains and codomains. Thus, our convention says that $\rho(K)\Rightarrow\rho(K')$ is a rule in case $K\Rightarrow K'$ is a rule, $\rho=\{\rho_1,\dots,\rho_n\}$, $\rho'=\{\rho'_1,\dots,\rho'_m\}$ and $\rho_1=\rho'_1$ and $\rho_n=\rho'_m$.

Let us reformulate our basic equations as follows:

$$\begin{array}{ll} (E1)^1 & \alpha^1, \beta^1, \gamma \ \Leftrightarrow\ \alpha^1 \circ \beta^1, \gamma \\ (E1)^2 & \alpha^2, \beta^2 \ \Leftrightarrow\ \alpha^2 \circ \beta^2 \\ (E2) & 1 \times \alpha_2,\ \alpha_1 \times 1,\ \beta \ \Leftrightarrow\ \alpha_1 \times 1,\ 1 \times \alpha_2,\ \beta. \end{array}$$

These new equations do not allow to rewrite a well-coloured path into a non well-coloured path; notice also that the 'interchange basic equation' $1 \times \alpha_2, \alpha_1 \times 1 \Leftrightarrow \alpha_1 \times 1, 1 \times \alpha_2$ now does not apply anymore in the last position of a path.

As we said, we shall only consider from now on *only well-coloured paths subject to the new basic equations* $(E1)^i, (E2)$.[12] There is no loss in that because for well-coloured paths K, L, we have $K \Leftrightarrow^* L$ (according to the old basic equations) iff $K \Leftrightarrow^* L$ (according to the new basic equations). In fact, one side is trivial; for the other side, let us consider a $\Leftrightarrow$-chain like

$$K = K_0 \Leftrightarrow K_1 \Leftrightarrow \cdots \Leftrightarrow K_n = L$$

obtained according to the old basic equations. We thus have

$$K^+ = K_0^+ \Leftrightarrow K_1^+ \Leftrightarrow \cdots \Leftrightarrow K_n^+ = L^+$$

according to the new basic equations; now two applications of $(E1)^2$ yields $K \Leftrightarrow K^+$ and $L \Leftrightarrow L^+$ because K, L are well-coloured. Thus $K \Leftrightarrow^* L$ holds by using the new equations too.
The obvious orientations of $(E1)^1, (E1)^2$ are

$$\begin{array}{ll} (R_c^1) & \alpha^1, \beta^1, \gamma \ \Rightarrow\ \alpha^1 \circ \beta^1, \gamma \\ (R_c^2) & \alpha^2, \beta^2 \ \Rightarrow\ \alpha^2 \circ \beta^2. \end{array}$$

Orientation of $(E2)$ depends on the colour of β. In case β has colour 2, we orient it as follows (supposing α_2 has colour 2 too):

$$(R_p^2)^* \qquad 1 \times \alpha_2^2,\ \alpha_1 \times 1,\ \beta^2 \ \Rightarrow\ \alpha_1 \times 1,\ (1 \times \alpha_2^2) \circ \beta^2$$

where second member has been reduced by a further (R_c^2)-rewrite step. In case β has colour 2, there are no other relevant cases. If α_1, α_2 have both colour 1, the two members are joinable by (R_c^1) and the equation can be deleted.[13] If α_2 has colour 1 and α_1 has colour 2, we do not need to add the rule

$$\alpha_1^2 \times 1,\ 1 \times \alpha_2^1,\ \beta^2 \ \Rightarrow 1\ \times \alpha_2^1,\ (\alpha_1^2 \times 1) \circ \beta^2$$

[12] Of course, this means also that, when computing the splitting path of a term, identity should be added at the end in case the top symbol of the term has wrong colour.

[13] We tolerate the use of $(R_p^2)^*$ in case α_1, α_2 both have colour 2. As a general philosophy, we prefer not to put provisoes on applications of rules, unless needed. So, for $(R_p^2)^*$ (and $(R_p^1)^*$ below), the only proviso is that first and third arrow in first member must be equally coloured.

because this is simply a renaming of $(R_p^2)^*$ and our convention about renamings automatically includes it. Notice that $(R_p^2)^*$ applies also in case $\beta \in \mathbf{T_0}$ (the fact that β has colour 2 does not prevent it from belonging to $\mathbf{T_0}$).

In case $\beta \in \mathbf{T_1} \backslash \mathbf{T_0}$, both members of $(E2)$ cannot occur in last position of a well-coloured path; taking into account this fact, the appropriate oriented rule is

$$(R_p^1)^* \qquad\qquad 1 \times \alpha_2^1,\ \alpha_1 \times 1,\ \beta^1,\ \gamma\ \Rightarrow\ \alpha_1 \times 1,\ (1 \times \alpha_2^1) \circ \beta^1,\ \gamma$$

Although, strictly speaking, we do not need such a rule in case $\beta \in \mathbf{T_0}$ (because orientation in this case is like in $(R_p^2)^*$), we allow its use in this case too.

During next section completion process, rules $(R_p^i)^*$ will be removed, whereas the rules (R_c^i) (called *composition rules*) are permanent (whenever a rule is deleted during completion, we always mark its name with a $*$).

Let us summarize the content of this section. We call $\mathcal{R}^*$ the rewriting system given by the rules

$$(R_c^1), (R_c^2), (R_p^1)^*, (R_p^2)^*.$$

$\mathcal{R}^*$ is our starting rewriting system: this system is sound and complete for our purposes (deciding path equivalence according to system $\mathcal{P}$ of Theorem 3.1), because the above discussion shows that

Lemma 4.1 *For well-coloured paths K_1, K_2, we have $K_1 \Leftrightarrow_{\mathcal{P}}^* K_2$ iff $K_1 \Leftrightarrow_{\mathcal{R}^*}^* K_2$.*

5 Completion

System $\mathcal{R}^*$ is clearly inadequate because it is far from being confluent, so we shall modify it by using Knuth-Bendix style completion as an *heuristic* guide.

Let us recall some general notions concerning a rewrite system $\mathcal{S}$ (these notions can be formulated within the context of abstract rewrite systems as in [2]). System $\mathcal{S}$ is said to be:

- *terminating* iff there are no infinite reduction sequences

$$K_1 \Rightarrow_{\mathcal{S}} K_2 \Rightarrow_{\mathcal{S}} \cdots K_i \Rightarrow_{\mathcal{S}} \cdots$$

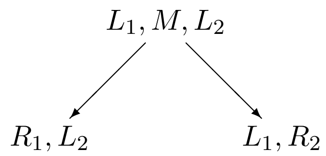
- *confluent* iff $K \Rightarrow_{\mathcal{S}}^* K_1$ and $K \Rightarrow_{\mathcal{S}}^* K_2$ imply that K_1, K_2 are *joinable* (i.e. that there exists K_0 such that $K_1 \Rightarrow_{\mathcal{S}}^* K_0$ and $K_2 \Rightarrow_{\mathcal{S}}^* K_0$);

- *locally confluent* iff $K \Rightarrow_{\mathcal{S}} K_1$ and $K \Rightarrow_{\mathcal{S}} K_2$ imply that K_1, K_2 are joinable;

- *canonical* iff it is terminating and confluent iff (by Newmann's Lemma) it is terminating and locally confluent.

It can be shown (see [2]) that in a canonical rewriting system $\mathcal{S}$ the relation $K_1 \Leftrightarrow^*_{\mathcal{S}} K_2$ holds iff K_1 and K_2 have the same normal form, which is moreover unique (a *normal form* for L is some L' such that $L \Rightarrow^*_{\mathcal{S}} L'$ and there is no L'' such that $L' \Rightarrow_{\mathcal{S}} L''$).

In order to prove canonicity of our path rewriting systems, we show that they are locally confluent and terminating; local confluence is, in its turn, easily reduced to the fact that critical pairs are all joinable. We recall that a *critical pair* is any pair of paths obtained as follows



where M is non empty and

$$L_1, M \Rightarrow R_1 \quad \text{and} \quad M, L_2 \Rightarrow R_2$$

are both rules of the system. In addition, there are also critical pairs induced by rules of the kind $L_1, M, L_2 \Rightarrow R_1$ and $M \Rightarrow R_2$ (where L_1, M, L_2 are all not empty). In all these cases, we say that such rules *superpose*.

In case some critical pairs are not joinable, the obvious thing to do is to enrich the system by adding it such oriented critical pairs as new rules. In addition, experience shows that it is better also to simplify - if possible - rules which are generated from the procedure. The following operations concerning a rewrite system $\mathcal{S}$ are in order:

(i) we can add to $\mathcal{S}$ a set of new rules $\{L_i \Rightarrow R_i\}_i$ such that (L_i, R_i) or (R_i, L_i) is a critical pair generated by rules in $\mathcal{S}$;

(ii) we can divide rules in $\mathcal{S}$ in two disjoint groups $\mathcal{S}' \cup \mathcal{S}''$ and remove all rules in $\mathcal{S}''$ in case we realize that left and right member of such rules are joinable in $\mathcal{S}'$;

(iii) we can divide rules in $\mathcal{S}$ in two disjoint groups $\mathcal{S}' \cup \mathcal{S}''$ and replace any rule $L \Rightarrow R$ in $\mathcal{S}''$ by some $L \Rightarrow R'$ such that $R \Rightarrow^*_{\mathcal{S}'} R'$.[14]

Clearly if $\mathcal{S}'$ results from $\mathcal{S}$ after a *finite* number of applications of (i)-(ii)-(iii), we have that $\mathcal{S}'$ is equivalent to $\mathcal{S}$ (in the sense that we have $K_1 \Leftrightarrow^*_{\mathcal{S}} K_2$ iff $K_1 \Leftrightarrow^*_{\mathcal{S}'} K_2$). If we are lucky, we can produce in this way a canonical rewrite system $\mathcal{S}'$ starting from a given $\mathcal{S}$. Notice that the above completion procedure - as it is formulated here - only has heuristic value (it cannot be fully mechanized as each step in (i)-(ii)-(iii) may consist in infinitely many operations).

Let us now apply completion to $\mathcal{R}^*$. The first obvious superposition we have in $\mathcal{R}^*$ is obtained by considering rules (R^1_c) and (R^2_c) as in the fork:

[14] We shall not need left member simplification steps.

$$\begin{array}{ccc} & \alpha^i, \beta^0, \alpha^j & \\ (R_c^i) \swarrow & & \searrow (R_c^j) \\ \alpha^i \circ \beta^0,\, \alpha^j & & \alpha^i,\, \beta^0 \circ \alpha^j \end{array}$$

(with $i \neq j$ and possibly with a γ appended everywhere to the right in case $j = 1$). Any naif global orientation of these critical pairs in one sense or in the other, would immediately cause infinite rewriting. Orientation from left to right $\alpha^i \circ \beta^0, \alpha^j \Rightarrow \alpha^i, \beta^0 \circ \alpha^j$ would produce for instance (let α have codomain Y and let β have domain Y):

$$\begin{array}{rcl} \alpha, \beta & = & \alpha \circ \langle 1_Y, 1_Y \rangle \circ \pi_1,\, \beta \\ & \Rightarrow & \alpha \circ \langle 1_Y,\, 1_Y \rangle,\, \pi_1 \circ \beta \;=\; \alpha \circ \langle 1_Y, 1_Y \rangle \circ \langle 1_{Y \times Y}, 1_{Y \times Y} \rangle \circ \pi_1',\, \pi_1 \circ \beta \\ & \Rightarrow & \cdots \end{array}$$

where $\pi_1 : Y \times Y \Rightarrow Y$ and $\pi_1' : (Y \times Y) \times (Y \times Y) \longrightarrow Y \times Y$ are first projections.

Critical pairs

$$(CP) \qquad \langle \alpha^i, \beta^0 \circ \alpha^j \quad ; \quad \alpha^i \circ \beta^0, \alpha^j \rangle$$

will be differently oriented, depending on the nature of the arrow β^0. In case the signature Ω^0 is empty, the solution is the following couple of rules:

$$\begin{array}{ll} (Rpr)^* & \alpha^i,\, \pi \circ \alpha^j \;\Rightarrow\; \alpha^i \circ \pi,\, \alpha^j \\ (Rdi)^* & \alpha^i \circ \delta,\, \alpha^j \;\Rightarrow\; \alpha^i,\, \delta \circ \alpha^j \end{array}$$

where π is any strict projection and δ is any strict diagonal [15] (a projection - resp. diagonal - is strict iff it is not a renaming). Given that any β^0 factors as $\pi \circ \delta$, where π is a projection and δ is a diagonal, this pair of rules is sufficient to make all critical pairs (CP) joinable, if Ω^0 is empty.

In our case, we cannot suppose Ω^0 to be empty, however we can try to identify two different classes of arrows in $\mathbf{T_0}$ forming a factorization system; arrows in the first class will be associated 'to the left' and arrows in the second class will be associated 'to the right', as in the case in which Ω^0 is empty. There is a standard notion of factorization system in category theory (see [6]), however such a notion is too strong in the present context, so that we weaken it.

Let $\mathbf{C}$ be any category; by a *weak factorization system* in $\mathbf{C}$, we mean a pair of classes of arrows $(\mathcal{E}, \mathcal{M})$ from $\mathbf{C}$ such that:

(i) both $\mathcal{E}$ and $\mathcal{M}$ contain identities and are closed with respect to left and right composition with arrows in $\mathcal{E} \cap \mathcal{M}$;

[15] Recall from Section 2 that we call projections (resp. diagonals) arrows $X^n \longrightarrow X^m$ which are m-tuples of distinct π_i (resp. m-tuples of π_i including all the $\pi_1, \ldots, \pi_n$).

(ii) for every $\alpha \in \mathbf{C}$, there are $\varepsilon \in \mathcal{E}$, $\mu \in \mathcal{M}$ such that $\alpha = \varepsilon \circ \mu$;

(iii) whenever we have a commutative square

$$\begin{array}{ccc} Y_0 & \xrightarrow{\varepsilon_1} & Y_1 \\ {\scriptstyle \varepsilon_2}\downarrow & & \downarrow{\scriptstyle \mu_1} \\ Y_2 & \xrightarrow[\mu_2]{} & Y \end{array}$$

with $\varepsilon_1, \varepsilon_2 \in \mathcal{E}$, $\mu_1, \mu_2 \in \mathcal{M}$, there is a unique $\rho \in \mathcal{E} \cap \mathcal{M}$ such that $\varepsilon_2 \circ \rho = \varepsilon_1$ and $\rho \circ \mu_1 = \mu_2$ (this condition says that the factorization given by (ii) is essentially unique).

¿From the above axioms it follows that arrows $\rho \in \mathcal{E} \cap \mathcal{M}$ are invertible (because they have two trivial factorizations, namely $\rho \circ 1$ and $1 \circ \rho$, hence...); such arrows will be just renamings in our cases. Notice that we do not ask for $\mathcal{E}$, $\mathcal{M}$ to be closed under composition, not even that they contain isomorphisms and are 'orthogonal' each other (like in standard factorization systems).

Main Example. For any equational theory $T = (\Omega, Ax)$, the corresponding Lawvere category $\mathbf{T}$ always has a weak factorization system $(\mathcal{E}, \mathcal{M})$ (which we call the *standard weak factorization system* for $\mathbf{T}$):

- arrows in $\mathcal{E}$ are just projections;
- arrows in $\mathcal{M}$ are those α such that in case it happens that $\alpha = \varepsilon \circ \alpha'$ (with $\varepsilon \in \mathcal{E}$), we must have that ε is just a renaming.

The factorizations $\alpha = \alpha_\varepsilon \circ \alpha_\mu$ (with $\alpha_\varepsilon \in \mathcal{E}$, $\alpha_\mu \in \mathcal{M}$) are obtained as follows. Let $\vec{t}(x_1, \dots, x_n)$ be a tuple of terms containing at most the variables $x_1, \dots, x_n$; say that this tuple is *n-minimized* iff for no $i = 1, \dots, n$ we have $\vdash_T \vec{t} = \vec{t}(c_0/x_i)$.[16] Now we have that the m-tuple of terms $\vec{t}$ *is n-minimized iff the arrow $\alpha : X^n \longrightarrow X^m$ belongs to $\mathcal{M}$*, where α is the vector of the equivalence classes of terms represented by the m components of $\vec{t}$ (if, say, $\vec{t} = \langle t_1, \dots, t_m \rangle$, then α is $\langle \{t_1\}, \dots, \{t_m\} \rangle$). Suppose in fact on one side that we have $\alpha = \varepsilon \circ \alpha'$, where $\varepsilon : X^n \longrightarrow X^k$ is the tuple $\langle \pi_{i_1}, \dots, \pi_{i_k} \rangle$; as such a tuple is a strict projection, the i_j are all distinct and some $s = 1, \dots, n$ is missed. Let α' be formed by the equivalence classes represented by the terms $\vec{t'}(x_1, \dots, x_k)$; the relation $\alpha = \varepsilon \circ \alpha'$ means that $\vdash_T \vec{t}(x_1, \dots, x_n) = \vec{t'}(x_{i_1}/x_1, \dots, x_{i_k}/x_k)$. Replacing x_s by the constant c_0, we get

$$\vdash_T \vec{t}(c_0/x_s) = \vec{t'}(x_{i_1}, \dots, x_{i_k}), \quad \text{hence} \quad \vdash_T \vec{t}(c_0/x_s) = \vec{t}$$

[16] Notations like $\vdash_T \vec{u} = \vec{v}$, for $\vec{u} = \langle u_1, \dots, u_m \rangle$ and $\vec{v} = \langle v_1, \dots, v_m \rangle$, mean that $\vdash_T \bigwedge_{j=1}^m u_j = v_j$. Recall that in Section 2 we assumed that there is at least one ground term c_0 in our signatures.

contrary to the fact that $\vec{t}$ is n-minimized. Conversely, if $\vec{t}$ is not n-minimized, we have $\vdash_T \vec{t}(c_0/x_s) = \vec{t}$ for some s, hence α admits a factorization through the proper projection $\langle \pi_1, \dots, \pi_{s-1}, \pi_{s+1}, \dots, \pi_n \rangle$.

We so established that $\alpha : X^n \longrightarrow X^m$ belongs to $\mathcal{M}$ iff it is represented by some n-minimized vector of terms. Let now α be arbitrary; how can we get the factorization $\alpha = \alpha_\varepsilon \circ \alpha_\mu$, where $\alpha_\varepsilon \in \mathcal{E}$ and $\alpha_\mu \in \mathcal{M}$? This is easy: take any vector of terms in the equivalence classes of α containing a minimal set of variables: if such a vector is $\vec{t}(x_{i_1}, \dots, x_{i_k})$, then the factorization is $\alpha = \langle \pi_{i_1}, \dots, \pi_{i_k} \rangle \circ \beta$, where β represents the vector of terms $\vec{t}(x_1, \dots, x_k)$. Notice that this process is effective in case word problem for T is solvable: one takes any $\vec{t}$ representing α and then go on by replacing variables in it by c_0; the procedure stops when only terms not provably equal to $\vec{t}$ can be obtained.

Next we show that the above factorization is unique up to a renaming. Suppose we have a commutative square in $\mathbf{T}$

$$\begin{array}{ccc} X^m & \xrightarrow{\varepsilon_2} & X^l \\ {\scriptstyle \varepsilon_1}\downarrow & & \downarrow{\scriptstyle \mu_2} \\ X^k & \xrightarrow[\mu_1]{} & X^n \end{array}$$

with $\varepsilon_1, \varepsilon_2 \in \mathcal{E}$ and $\mu_1, \mu_2 \in \mathcal{M}$. For the sake of simplicity, we can apply a suitable renaming to X^m so that we have $\varepsilon_1 = \langle \pi_1, \dots, \pi_k \rangle$ (i.e. ε_1 projects on first k components) and $\varepsilon_2 = \langle \pi_{j_1}, \dots, \pi_{j_l} \rangle$; now μ_1, μ_2 must be represented by k, l-minimized vectors of terms $\vec{t_1}, \vec{t_2}$. The commutativity of the square says that we have

$$\vdash_T \vec{t_1}(x_1, \dots, x_k) = \vec{t_2}(x_{j_1}, \dots, x_{j_l}).$$

By minimization, we must have $\{1, \dots, k\} = \{j_1, \dots, j_l\}$ (otherwise, one can 'reduce' variables in t_1 or t_2 by replacing them with c_0); this means also that $k = l$. Now the renaming $\langle \pi_{j_1}, \dots, \pi_{j_l} \rangle : X^k \longrightarrow X^k$ fills the 'bottom-top' diagonal of the square

$$\begin{array}{ccc} X^{k+l} & \xrightarrow{\varepsilon_2} & X^k \\ {\scriptstyle \varepsilon_1}\downarrow & & \downarrow{\scriptstyle \mu_2} \\ X^k & \xrightarrow[\mu_1]{} & X^n \end{array}$$

(and is the unique such), as wanted. $\dashv$

Using the above described standard weak factorization system (which we conveniently call $(\mathcal{E}_0, \mathcal{M}_0)$) available in $\mathbf{T_0}$, we can replace rule $(Rdi)^*$ by the following one

$(R_\mu)^*$ $\qquad \alpha^i \circ \mu,\, \alpha^j \;\Rightarrow\; \alpha^i,\, \mu \circ \alpha^j \quad (\mu \in \mathcal{M}_0)$

which is stronger than $(Rdi)^*$ because diagonals always are in $\mathcal{M}_0$ (they cannot be factored through a strict projection or, to say it differently, they always are represented by minimized vectors of terms). As rule $(Rpr)^*$ is kept, the effect of rules $(Rpr)^*$ and $(R_\mu)^*$ is that all critical pairs (CP) are now joinable (because β^0 can be factored as $\beta^0_\varepsilon \circ \beta^0_\mu$, with $\beta^0_\varepsilon \in \mathcal{E}_0$ and $\beta^0_\mu \in \mathcal{M}_0$). Apart from evident problems concerning the effectiveness of applicability of rule $(R_\mu)^*$, this is still bad because it may once again produce infinite rewriting. This is especially evident in case T_0 is not collapse-free. Suppose for instance we have a collapsing equation like $f(x_1, x_1) = x_1$ in T_0; then if we start with the path $x_1, t(x_1)$ (where $t(x_1)$ is any term), we can decompose x_1 as $\langle x_1, x_1\rangle \circ f(x_1, x_2)$, thus producing the following rewrite steps:

$$\langle x_1, x_1\rangle \circ f(x_1, x_2),\, t(x_1) \;\Rightarrow\; \langle x_1, x_1\rangle,\, t(f(x_1, x_2)) \;\Rightarrow\; x_1,\, \langle x_1, x_1\rangle \circ t(f(x_1, x_2))$$

yielding again $x_1, t(x_1)$. We cannot overcome this problem without postulating anything (after all, combined solvable word problems might be unsolvable...). We shall postulate that *there is a canonical way of extracting $\mathcal{M}_0$-components* from terms in Ω^i (of course, we shall also have to assume that such extraction can be done in an effective way, see Section 10).[17] This extra assumption will restrict rule $(R_\mu)^*$ (or, to put it in a different form, will allow the completion/simplification process to get rid of undesired instances of rule $(R_\mu)^*$). We need first to come back once again to the abstract categorical framework.

Let $\mathbf{C}$ to be a subcategory of $\mathbf{C}'$ and let $(\mathcal{E}, \mathcal{M})$ be a weak factorization system in $\mathbf{C}$. A weak factorization system $(\mathcal{E}', \mathcal{M})$ in $\mathbf{C}'$ (notice that $\mathcal{M}$ is the same!) is said to be a *left extension* of $(\mathcal{E}, \mathcal{M})$ iff the following hold:

- $\mathcal{E}' \cap \mathbf{C} = \mathcal{E}$;
- if $\varepsilon_1, \varepsilon_2 \in \mathcal{E}$ and if $\varepsilon \in \mathcal{E}'$, then $\varepsilon_1 \circ \varepsilon \in \mathcal{E}'$ and $\varepsilon \circ \varepsilon_2 \in \mathcal{E}'$ (whenever compositions make sense).

Notice that this implies that $\mathcal{E}$ - not necessarily $\mathcal{E}'$ - is closed under composition. Let us say that T_i *is constructible over* T_0 iff in $\mathbf{T_i}$ there is a left extension $(\mathcal{E}_i, \mathcal{M}_0)$ of the standard weak factorization system $(\mathcal{E}_0, \mathcal{M}_0)$ of $\mathbf{T_0}$.

Assumption. *We assume that T_1, T_2 are both constructible over T_0.*

We postpone to Section 10 a symbolic translation of this assumption as well as the analysis of some examples (and counterexamples). For the moment, let us underline that, as an effect of the above assumption, we now have that any arrow α^i admits two factorizations, namely:

[17] The assumption of [3, 4] may be seen as the stronger requirement that there is a maximal way of extracting $\mathcal{M}_0$-components (such stronger requirement is incompatible with existence of collapsing equations in T_0).

- it can be factored as $\alpha^i_\varepsilon \circ \alpha^i_m$ according to the standard weak factorization system $(\mathcal{E}_0, \mathcal{M}_i)$ of $\mathbf{T_i}$ (we recall that here $\mathcal{E}_0$ is formed by arrows which are projections, whereas $\mathcal{M}_i$ is formed by arrows represented by minimized -in the sense of the theory T_i- vectors of terms);
- it can be factored as $\alpha^i_e \circ \alpha^i_\mu$ according to the left extension $(\mathcal{E}_i, \mathcal{M}_0)$ of the standard weak factorization system of $\mathbf{T_0}$ (here the class $\mathcal{E}_i$ is axiomatically given by the above Assumption, whereas $\mathcal{M}_0$ is the class of arrows from $\mathbf{T_0}$ represented by minimized vectors of terms -in the sense of the theory T_0).[18]

Rules $(Rpr)^*$, $(R_\mu)^*$ are so restricted:

$$\begin{array}{lll} (R_\varepsilon) & \alpha^i, \alpha^j \;\Rightarrow\; \alpha^i \circ \alpha^j_\varepsilon,\, \alpha^j_m \\ (R_\mu) & \alpha^i, \alpha^j \;\Rightarrow\; \alpha^i_e,\, \alpha^i_\mu \circ \alpha^j \end{array}$$

and called *ε-extraction and μ-extraction rules*, respectively.[19] Let us call $\mathcal{R}_0$ the rewriting system formed by rules $(R^i_c), (R_\varepsilon), (R_\mu)$; in Section 6 we shall prove that

Theorem 5.1 *$\mathcal{R}_0$ is canonical.*

As an effect of the above Theorem, rules $(Rpr)^*, (Rdi)^*, (R_\mu)^*$ are all canceled during completion/simplification process (because their members are joinable in $\mathcal{R}_0$); we shall nevertheless artificially postpone cancellation of rules $(Rpr)^*$ and $(Rdi)^*$ at the end of the completion, because we shall make further use of them in order to identify the good superpositions/simplifications steps needed to treat the remaining rule $(R^i_p)^*$ (which causes some further confluence problems).

In fact, in order to finish our completion process we need only to identify a couple of very specific superpositions yielding the right modification of the rule $(R^i_p)^*$

$$(R^i_p)^* \qquad\qquad 1 \times \alpha^i_2,\, \alpha^j_1 \times 1,\, \beta^i \;\Rightarrow\; \alpha^j_1 \times 1,\, (1 \times \alpha^i_2) \circ \beta^i$$

(recall that in case $i = 1$, there is an extra arrow to the right of both members).[20] Let us consider the path

$$Y_2 \xrightarrow{\langle \gamma^i, 1_{Y_2}\rangle} Y_1 \times Y_2 \xrightarrow{1_{Y_1} \times \alpha^i_2} Y_1 \times Z_2 \xrightarrow{\alpha^j_1 \times 1_{Z_2}} Z_1 \times Z_2 \xrightarrow{\beta^i} U$$

giving rise to the superposition (among rules (R^i_c) and $(R^i_p)^*$)

[18] These vectors of terms are also n-minimized in the sense of T_i, given that T_i is conservative over T_0.

[19] It goes without saying that such rules do not apply in case of trivial factorizations (i.e. when α^j_ε -resp. α^i_μ- are, up to a renaming, just identities). We allow applying the rule also in case $i = j$ (although in principle this is not needed).

[20] Recall from Section 4 that we allow j to be different or equal to i.

$$
\begin{array}{ccc}
 & \langle \gamma^i, 1\rangle,\ 1\times\alpha_2^i,\ \alpha_1^j\times 1,\ \beta^i & \\
 \swarrow{\scriptstyle (R_c^i)} & & \searrow{\scriptstyle (R_p^i)^*} \\
\langle \gamma^i, \alpha_2^i\rangle,\ \alpha_1^j\times 1,\ \beta^i & & \langle \gamma^i, 1\rangle,\ \alpha_1^j\times 1,\ (1\times\alpha_2^i)\circ\beta^i
\end{array}
$$

The related critical pair is oriented as follows:

$(R_p')^*$ $\qquad \langle \gamma^i, \alpha_2^i\rangle,\ \alpha_1^j\times 1,\ \beta^i \ \Rightarrow\ \langle \gamma^i, 1\rangle,\ \alpha_1^j\times 1,\ (1\times\alpha_2^i)\circ\beta^i$

We need another final superposition (among $(R_p')^*$ and $(Rdi)^*$): consider the path (here $\alpha^j : Y_1\times Z \longrightarrow Y_2$)

$$Y \stackrel{\langle\gamma^i,\delta^i,\delta^i\rangle}{\longrightarrow} Y_1\times Z\times Z \stackrel{\alpha^j\times 1_Z}{\longrightarrow} Y_2\times Z \stackrel{\beta^i}{\longrightarrow} U$$

and the superposition

$$
\begin{array}{ccc}
 & \langle \gamma^i, \delta^i, \delta^i\rangle,\ \alpha^j\times 1_Z,\ \beta^i & \\
 \swarrow{\scriptstyle (Rdi)^*} & & \searrow{\scriptstyle (R_p^i)^*} \\
\langle \gamma^i, \delta^i\rangle,\ \langle\alpha^j,\pi_Z\rangle,\ \beta^i & & \langle \gamma^i, \delta^i, 1_Y\rangle,\ \alpha^j\times 1_Y,\ (1_{Y_2}\times\delta^i)\circ\beta^i
\end{array}
$$

where we used the fact that $\langle\gamma^i,\delta^i,\delta^i\rangle = \langle\gamma^i,\delta^i\rangle\circ(1_{Y_1}\times\Delta_Z)$ (Δ_Z is the diagonal $\langle 1_Z, 1_Z\rangle$) and the fact that $(1_{Y_1}\times\Delta_Z)\circ(\alpha^j\times 1_Z) = \langle\alpha^j,\pi_Z\rangle$ (π_Z is the projection $Y_1\times Z\longrightarrow Z$). We are near to the end of the completion process; we first reduce the second component of the above critical pair by using two $(Rpr)^*$-reduction steps:[21] suppose that $\delta^i : Y\longrightarrow Z$ has factorization

$$Y \stackrel{\delta^i_\varepsilon}{\longrightarrow} Y' \stackrel{\delta^i_m}{\longrightarrow} Z,$$

then we have

$$
\begin{array}{c}
\langle\gamma^i,\delta^i,1_Y\rangle, \alpha^j\times 1_Y, (1_{Y_2}\times\delta^i)\circ\beta^i \\
\Downarrow \\
\langle\gamma^i,\delta^i,1_Y\rangle, \alpha^j\times \delta^i_\varepsilon, (1_{Y_2}\times\delta^i_m)\circ\beta^i \\
\Downarrow \\
\langle\gamma^i,\delta^i,\delta^i_\varepsilon\rangle, \alpha^j\times 1_{Y'}, (1_{Y_2}\times\delta^i_m)\circ\beta^i
\end{array}
$$

[21] This reduction is important: without it, we may have problems in the termination proof.

So our final *products rule* is

$$(R_p^i) \qquad \langle\gamma^i,\delta^i\rangle,\ \langle\alpha^j,\ \pi_Z\rangle,\beta^i\ \Rightarrow\ \langle\gamma^i,\delta^i,\delta^i_\varepsilon\rangle,\ \alpha^j\times 1_{Y'},\ (1_{Y_2}\times\delta^i_m)\circ\beta^i$$

(recall we have an extra arrow to the right in both members in case $i=1$). Putting types, first member is

$$(I) \qquad Y \stackrel{\langle\gamma^i,\delta^i\rangle}{\longrightarrow} Y_1\times Z \stackrel{\langle\alpha^j,\pi_Z\rangle}{\longrightarrow} Y_2\times Z \stackrel{\beta^i}{\longrightarrow} U$$

whereas second member is

$$(II) \qquad Y \stackrel{\langle\gamma^i,\delta^i,\delta^i_\varepsilon\rangle}{\longrightarrow} Y_1\times Z\times Y' \stackrel{\alpha^j\times 1_{Y'}}{\longrightarrow} Y_2\times Y' \stackrel{(1_{Y_2}\times\delta^i_m)\circ\beta^i}{\longrightarrow} U$$

We add a proviso for this rule: $\delta^i\notin\mathcal{E}_0$ *(that is, δ^i cannot be a projection).* The reason for this last proviso is that, in case δ^i is a projection, then the second member of (R_p^i) can be re-written to the first by using rule $(Rdi)^*$ (thus causing termination problems). In fact, in case δ^i is a projection, we have that $\delta^i=\delta^i_\varepsilon$ and $\delta^i_m=1_Z$, hence the second member is $\langle\gamma^i,\delta^i,\delta^i\rangle,\alpha^j\times 1_Z,\beta^i$ and we can rewrite it as follows

$$\langle\gamma^i,\delta^i,\delta^i\rangle,\alpha^j\times 1_Z,\beta^i = \langle\gamma^i,\delta^i\rangle\circ(1_{Y_1}\times\Delta_Z),\alpha^j\times 1_Z,\beta^i\ \Rightarrow\ \langle\gamma^i,\delta^i\rangle,\langle\alpha^j,\pi_Z\rangle,\beta^i$$

thus getting the first member. To finish, we observe that rules $(R_p^i)^*$ and $(R'_p)^*$ can be removed, because their members become joinable in the rewrite system $\mathcal{R}$ obtained by adding (R_p^i) to $\mathcal{R}_0$. We check it for the former rule, leaving the latter (which is treated in a very similar way) to the reader.

First member of $(R_p^i)^*$ is

$$Y_1\times Y_2 \stackrel{\langle\pi_{Y_1},\pi_{Y_2}\circ\alpha^i\rangle}{\longrightarrow} Y_1\times Z_2 \stackrel{\langle\pi_{Y_1}\circ\alpha^j_1,\pi_{Z_2}\rangle}{\longrightarrow} Z_1\times Z_2 \stackrel{\beta^i}{\longrightarrow} U$$

whereas the second member is

$$Y_1\times Y_2 \stackrel{\alpha^j_1\times 1_{Y_2}}{\longrightarrow} Z_1\times Y_2 \stackrel{(1_{Z_1}\times\alpha^i)\circ\beta^i}{\longrightarrow} U.$$

Applying a $(Rpr)^*$-rewrite step to the second member, we get (suppose that $Y_2\stackrel{\alpha^i_\varepsilon}{\longrightarrow} Y'_2\stackrel{\alpha^i_m}{\longrightarrow} Z_2$):

$$(*) \qquad \alpha^j_1\times 1_{Y_2},(1_{Z_1}\times\alpha^i)\circ\beta^i\ \Rightarrow\ \alpha^j_1\times\alpha^i_\varepsilon,(1_{Z_1}\times\alpha^i_m)\circ\beta^i.$$

Let us now operate on first member by successively using rules $(R_p^i),(Rpr)^*,(R_c^j)$ as follows (to apply (R_p^i), notice that $\pi_{Y_2}\circ\alpha^i=(\pi_{Y_2}\circ\alpha^i_\varepsilon)\circ\alpha^i_m$, so by uniqueness this is the factorization of $\pi_{Y_2} \circ \alpha^i$ in the standard weak factorization system of $\mathbf{T_i}$):[22]

$$\begin{array}{c}
\langle \pi_{Y_1}, \pi_{Y_2} \circ \alpha^i \rangle, \langle \pi_{Y_1} \circ \alpha_1^j, \pi_{Z_2} \rangle, \beta^i \\
\Downarrow_{(R_p^i)} \\
\langle \pi_{Y_1}, \pi_{Y_2} \circ \alpha^i, \pi_{Y_2} \circ \alpha_\varepsilon^i \rangle, \langle \pi_{Y_1} \circ \alpha_1^j \rangle \times 1_{Y_2'}, (1_{Z_1} \times \alpha_m^i) \circ \beta^i \\
= \\
\langle \pi_{Y_1}, \pi_{Y_2} \circ \alpha^i, \pi_{Y_2} \circ \alpha_\varepsilon^i \rangle, \langle \pi_{Y_1}, \pi_{Y_2'} \rangle \circ (\alpha_1^j \times 1_{Y_2'}), (1_{Z_1} \times \alpha_m^i) \circ \beta^i \\
\Downarrow_{(Rpr)^*} \\
\langle \pi_{Y_1}, \pi_{Y_2} \circ \alpha^i, \pi_{Y_2} \circ \alpha_\varepsilon^i \rangle \circ \langle \pi_{Y_1}, \pi_{Y_2'} \rangle, \alpha_1^j \times 1_{Y_2'}, (1_{Z_1} \times \alpha_m^i) \circ \beta^i \\
= \\
\langle \pi_{Y_1}, \pi_{Y_2} \circ \alpha_\varepsilon^i \rangle, \alpha_1^j \times 1_{Y_2'}, (1_{Z_1} \times \alpha_m^i) \circ \beta^i \\
\Downarrow_{(R_c^j)} \\
\alpha_1^j \times \alpha_\varepsilon^i, (1_{Z_1} \times \alpha_m^i) \circ \beta^i
\end{array}$$

as in $(*)$. In conclusion, we obtained the rewriting system $\mathcal{R}$ which is described by Table 1 (in the last two rules of the Table, Z, Y' and Y_2 are the codomains of $\delta^i, \delta_\varepsilon^i$ and α^j, respectively, as in the fully displayed paths (I) and (II) above).[23]

Recall that renamings of rules of $\mathcal{R}$ are available as rules of $\mathcal{R}$. However, such a convention can be slightly simplified, given that the above rules are all closed under the operation of composing first (or last) arrow in each member by the same single renaming. Thus we can merely stipulate that *if $L \Rightarrow R$ is a rule, then $L' \Rightarrow R'$ is also a rule, where L' is any alphabetic variant of L and R' is any alphabetic variant of R.*

The content of the present section can be so summarized (recall that $\mathcal{R}$ is obtained from $\mathcal{R}^*$ by few completion steps):

Lemma 5.2 *For well-coloured paths K_1, K_2, we have $K_1 \Leftrightarrow_{\mathcal{R}}^* K_2$ iff $K_1 \Leftrightarrow_{\mathcal{R}^*}^* K_2$.*

---

[22] If α_i is a projection, rule (R_p^i) does not apply, however in this case $1_{Z_1} \times \alpha_m^i$ is the identity, $\alpha_\varepsilon^i = \alpha^i$ and a single (R_c^j)-rewrite step reduces the first member as in $(*)$.

[23] Notice the following subtility concerning rule (R_c^1) (a similar observation applies to rule (R_p^1) too): paths α^1, β^0 and $\alpha^1 \circ \beta^0$ are well-coloured in case the composed arrow $\alpha^1 \circ \beta^0$ collapses to an arrow from $\mathbf{T_0}$. In this case, rule (R_c^1) does not apply, but the two paths are nevertheless joinable by eliminating peaks from the following $\Leftrightarrow_{\mathcal{R}}^*$-chain (according to the instructions given in the confluence proof):

$$\alpha^1 \circ \beta^0 \Leftarrow \alpha^1 \circ \beta^0, 1 \Leftarrow \alpha^1, \beta^0, 1 \Rightarrow \alpha^1, \beta^0.$$

The point behind that lies in the above mentioned properties of left extension of factorization systems: if $\alpha^1 \circ \beta^0$ collapses, then $(\alpha_e^1 \circ (\alpha_\mu^1 \circ \beta^0)_\varepsilon) \circ (\alpha_\mu^1 \circ \beta^0)_\mu$ is, by uniqueness, the factorization, *in $\mathbf{T_0}$ as well as in $\mathbf{T_1}$*, of the arrow $\alpha^1 \circ \beta^0$, hence we have

$$\alpha^1, \beta^0 \Rightarrow \alpha_e^1, \alpha_\mu^1 \circ \beta \Rightarrow \alpha_e^1 \circ (\alpha_\mu^1 \circ \beta^0)_\varepsilon, (\alpha_\mu^1 \circ \beta^0)_\mu \Rightarrow (\alpha_e^1 \circ (\alpha_\mu^1 \circ \beta^0)_\varepsilon) \circ (\alpha_\mu^1 \circ \beta^0)_\mu = \alpha^1 \circ \beta^0$$

where last passage is now correct (it is an (R_c^2)-step applied to arrows from $\mathbf{T_0}$).

(R_c^1)	$\alpha^1, \beta^1, \gamma \Rightarrow \alpha^1 \circ \beta^1, \gamma$
(R_c^2)	$\alpha^2, \beta^2 \Rightarrow \alpha^2 \circ \beta^2$
(R_ε)	$\alpha, \beta \Rightarrow \alpha \circ \beta_\varepsilon, \beta_m$
(R_μ)	$\alpha, \beta \Rightarrow \alpha_e, \alpha_\mu \circ \beta$
(R_p^1)	$\langle \gamma^1, \delta^1 \rangle, \langle \alpha, \pi_Z \rangle, \beta^1, \theta \Rightarrow \langle \gamma^1, \delta^1, \delta_\varepsilon^1 \rangle, \alpha \times 1_{Y'}, (1_{Y_2} \times \delta_m^1) \circ \beta^1, \theta$ where $\delta^1 \notin \mathcal{E}_0$
(R_p^2)	$\langle \gamma^2, \delta^2 \rangle, \langle \alpha, \pi_Z \rangle, \beta^2 \Rightarrow \langle \gamma^2, \delta^2, \delta_\varepsilon^2 \rangle, \alpha \times 1_{Y'}, (1_{Y_2} \times \delta_m^2) \circ \beta^2$ where $\delta^2 \notin \mathcal{E}_0$

Table 1: The system $\mathcal{R}$

In Section 9 we shall prove our main result, namely that

Theorem 5.3 *$\mathcal{R}$ is canonical.*

6 Local confluence, I

In this section we will prove the canonicity of the system $\mathcal{R}_0$ which, we recall, is the system described by Table 2.

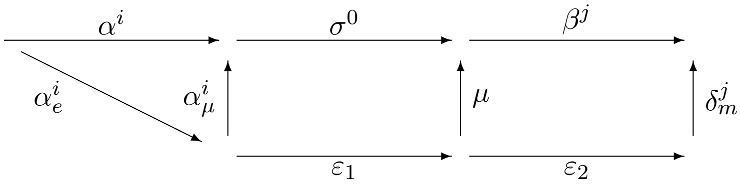
(R_c^1)	$\alpha^1, \beta^1, \gamma \;\Rightarrow\; \alpha^1 \circ \beta^1, \gamma$
(R_c^2)	$\alpha^2, \beta^2 \;\Rightarrow\; \alpha^2 \circ \beta^2$
(R_ε)	$\alpha, \beta \;\Rightarrow\; \alpha \circ \beta_\varepsilon, \beta_m$
(R_μ)	$\alpha, \beta \;\Rightarrow\; \alpha_e, \alpha_\mu \circ \beta$

Table 2: The system $\mathcal{R}_0$

We begin by showing that $\mathcal{R}_0$ is locally confluent: we single out all critical pairs arising from superpositions between the rules of $\mathcal{R}_0$ and we prove that they are joinable. Most of the cases can be reduced to the critical pairs treated in the following lemma.

Lemma 6.1 *The paths $\alpha^i \circ \sigma^0, \beta^j$ and $\alpha^i, \sigma^0 \circ \beta^j$ are joinable in $\mathcal{R}_0$.*

Proof. Let α^i_e and α^i_μ be the e/μ components of α^i and let us consider the following commutative diagram, where $\varepsilon_1 \circ \mu$ corresponds to the ε/μ factorization in $\mathbf{T_0}$ of $\alpha^i_\mu \circ \sigma^0$ and $\varepsilon_2 \circ \delta^j_m$ is the ε/m factorization of $\mu \circ \beta^j$ in $\mathbf{T_j}$.



Since $\alpha^i_e \circ \varepsilon_1$ belongs to $\mathcal{E}_i$ (recall the definition of left extensions of factorization systems) and δ^j_m belongs to $\mathcal{M}_j$, we have (up to a renaming):

$$(\alpha^i \circ \sigma^0)_e = \alpha^i_e \circ \varepsilon_1 \qquad (\alpha^i \circ \sigma^0)_\mu = \mu$$
$$(\alpha^i_\mu \circ \sigma^0 \circ \beta^j)_\varepsilon = \varepsilon_1 \circ \varepsilon_2 \qquad (\alpha^i_\mu \circ \sigma^0 \circ \beta^j)_m = \delta^j_m$$

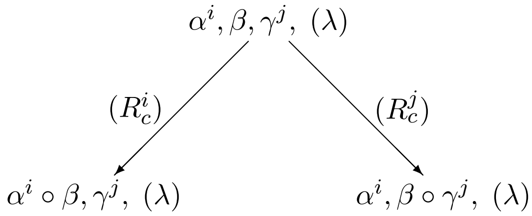
We can do the following rewriting steps:

$$\alpha^i \circ \sigma^0, \beta^j \quad \Rightarrow_{R_\mu} \quad \alpha^i_e \circ \varepsilon_1,\ \mu \circ \beta^j \quad \Rightarrow_{R_\varepsilon} \quad \alpha^i_e \circ \varepsilon_1 \circ \varepsilon_2,\ \delta^j_m$$
$$\alpha^i, \sigma^0 \circ \beta^j \quad \Rightarrow_{R_\mu} \quad \alpha^i_e,\ \alpha^i_\mu \circ \sigma^0 \circ \beta^j \quad \Rightarrow_{R_\varepsilon} \quad \alpha^i_e \circ \varepsilon_1 \circ \varepsilon_2,\ \delta^j_m$$

and this proves the lemma. ⊣

In subsequent sections we proceed with a systematic analyses of the cases. To simplify the exposition, we treat (R^1_c) and (R^2_c) together; however some applications of (R^1_c) may require an additional arrow λ to the right (we put it within round brackets).

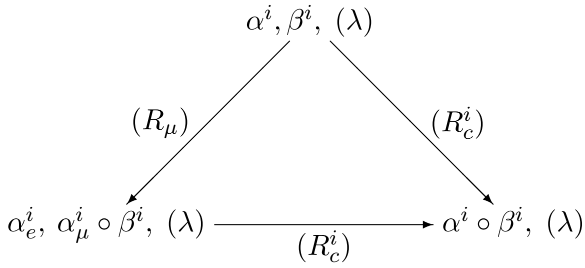
6.1 Superposition between (R^i_c) and (R^j_c)



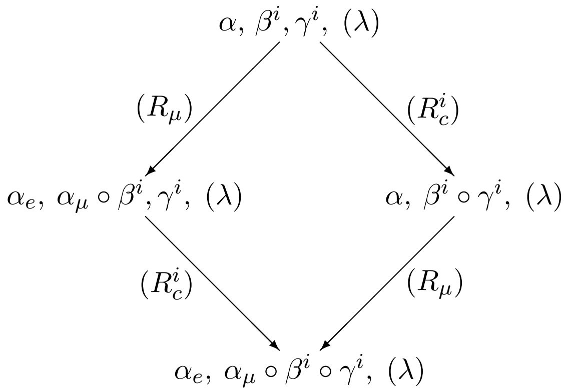
If $i = j$, we can rewrite both members to $\alpha^i \circ \beta \circ \gamma^i$. Otherwise, necessarily β belongs to $\mathbf{T_0}$ and we can apply Lemma 6.1.

6.2 Superpositions between (R_c^i) and (R_μ)

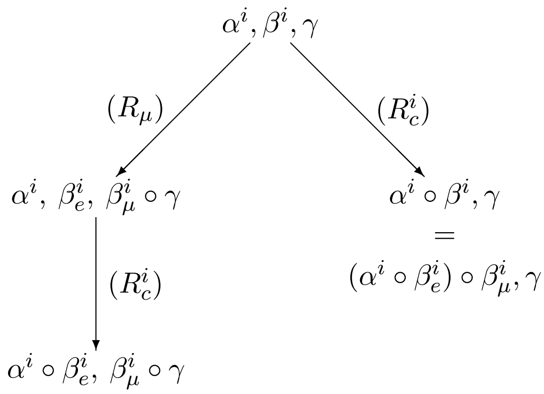
<u>CASE 1</u>



<u>CASE 2</u>



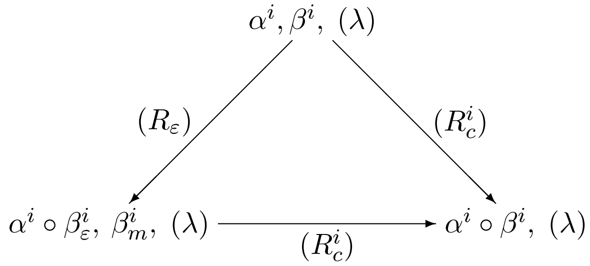
<u>CASE 3</u>



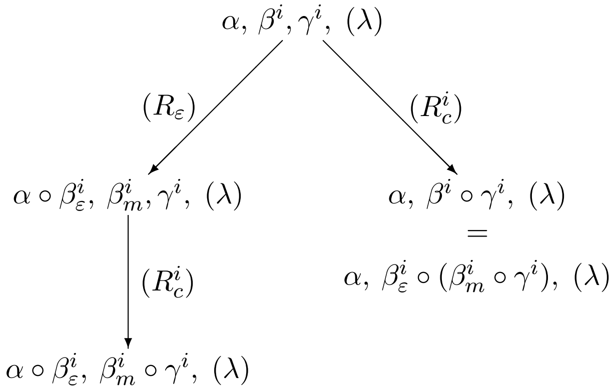
By Lemma 6.1, with $\sigma^0 = \beta_\mu^i$.

6.3 Superpositions between (R_c^i) and (R_ε)

CASE 1

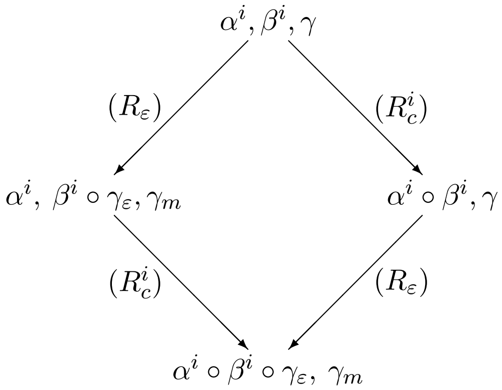


CASE 2

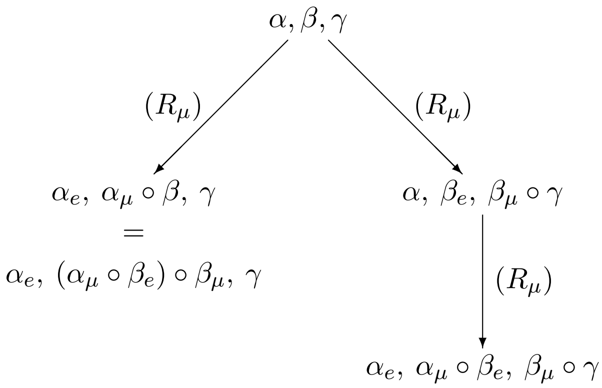


By Lemma 6.1, with $\sigma^0 = \beta_\varepsilon^i$.

CASE 3



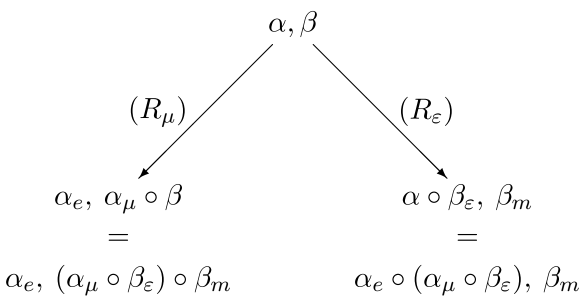
6.4 Superposition between (R_μ) and itself



By Lemma 6.1, with $\sigma^0 = \beta_\mu$.

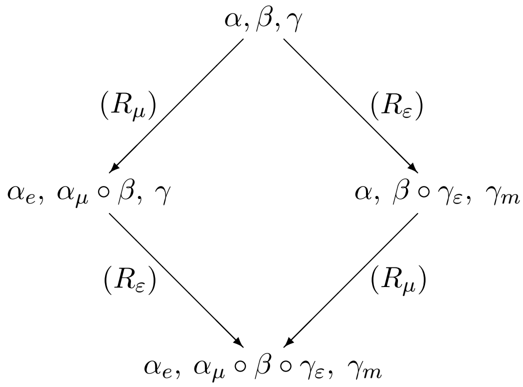
6.5 Superpositions between (R_μ) and (R_ε)

<u>CASE 1</u>

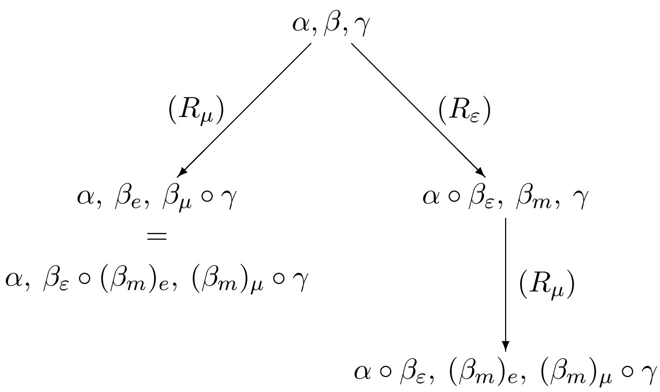


By Lemma 6.1, with $\sigma^0 = \alpha_\mu \circ \beta_\varepsilon$.

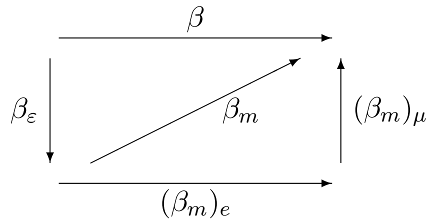
<u>CASE 2</u>



<u>CASE 3</u>

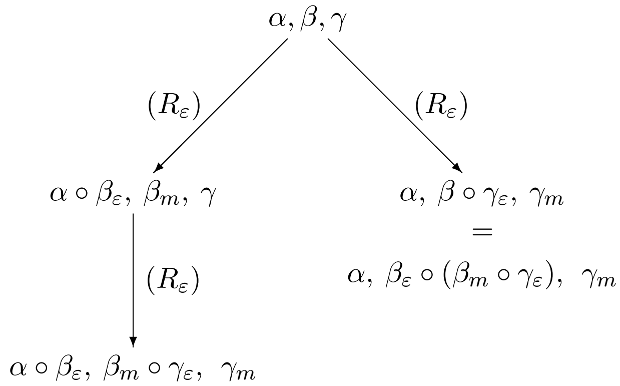


In first member we use the fact that the following diagram is commutative



Thus, reasoning as usual (by uniqueness of factorizations - up to a renaming), we can state that $\beta_e = \beta_\varepsilon \circ (\beta_m)_e$ and $\beta_\mu = (\beta_m)_\mu$. We can apply Lemma 6.1, with $\sigma^0 = \beta_\varepsilon$.

6.6 Superposition between (R_ε) and itself



By Lemma 6.1, with $\sigma^0 = \beta_\varepsilon$.

We can conclude:

Theorem 6.2 *$\mathcal{R}_0$ is locally confluent.*

It remains to show the termination of $\mathcal{R}_0$. This result is a consequence of Theorem 9.8, however here we give a direct proof, which uses less machinery. We need a complexity measure for paths which decreases with application of our rules. At this aim, we define:

$$\mu(\alpha^i) = \begin{cases} 0 & \text{if } \alpha^i \in \mathcal{E}_i \\ 1 & \text{otherwise} \end{cases} \qquad \varepsilon(\alpha^i) = \begin{cases} 0 & \text{if } \alpha^i \in \mathcal{M}_i \\ 1 & \text{otherwise} \end{cases}$$

Let K be the path $\alpha_1, \ldots, \alpha_n$. We define:

$$\mu(K) = \langle \mu(\alpha_1), \ldots, \mu(\alpha_n) \rangle \qquad \varepsilon(K) = \langle \varepsilon(\alpha_n), \ldots, \varepsilon(\alpha_1) \rangle$$

(notice that $\mu(K) = \mu(K')$ and $\varepsilon(K) = \varepsilon(K')$ hold in case K and K' are alphabetic variants each other).

Finally, we introduce the following order relation $\succ$ between paths K, L:

- $K \succ L$ if and only if either (i) or (ii) hold:
  - (i) $|K| > |L|$ (where $|K|$ denotes the length of K);
  - (ii) $|K| = |L|$ and $\langle \mu(K), \varepsilon(K) \rangle >_l \langle \mu(L), \varepsilon(L) \rangle$ (where $>_l$ denotes the lexicographic order between n-ple of integers).

It is a standard fact that

Lemma 6.3 *$\succ$ is a terminating transitive relation.*

Now we prove that $\succ$ is a stable relation.

Lemma 6.4 *$K \succ K'$ implies $L, K, R \succ L, K', R$ (for all L, R).*

Proof. If $|K| > |K'|$ we immediately have $L, K, R \succ L, K', R$. Suppose now $|K| = |K'|$; we have to show that

$$\langle \mu(L), \mu(K), \mu(R), \varepsilon(R), \varepsilon(K), \varepsilon(L) \rangle \ >_l \ \langle \mu(L), \mu(K'), \mu(R), \varepsilon(R), \varepsilon(K'), \varepsilon(L) \rangle.$$

This follows from the fact that either $\mu(K) >_l \mu(K')$ or $\mu(K) = \mu(K')$ and $\varepsilon(K) >_l \varepsilon(K')$. ⊣

Lemma 6.5 *Let $L \Rightarrow_{\mathcal{R}_0} L'$, then $L \succ L'$.*

Proof. Since $\succ$ is stable, it is sufficient to analyze the following three cases.

(1) $\alpha^i, \beta^i \Rightarrow_{(R_c^i)} \alpha^i \circ \beta^i$.

We have $|\alpha^i, \beta^i| > |\alpha^i \circ \beta^i|$, hence $\alpha^i, \beta^i \succ \alpha^i \circ \beta^i$.

(2) $\alpha, \beta \Rightarrow_{(R_\mu)} \alpha_e, \alpha_\mu \circ \beta$.

The two paths have the same length, moreover $\mu(\alpha) = 1$ and $\mu(\alpha_e) = 0$ (otherwise there is no way to apply (R_μ)). This implies that

$$\langle \mu(\alpha), \mu(\beta), \varepsilon(\beta), \varepsilon(\alpha) \rangle >_l \langle \mu(\alpha_e), \mu(\alpha_\mu \circ \beta), \varepsilon(\alpha_\mu \circ \beta), \varepsilon(\alpha_e) \rangle$$

from which $\alpha, \beta \succ \alpha_e, \alpha_\mu \circ \beta$ follows.

(3) $\alpha, \beta \Rightarrow_{(R_\varepsilon)} \alpha \circ \beta_\varepsilon, \beta_m$.

Clearly $|\alpha, \beta| = |\alpha \circ \beta_\varepsilon, \beta_m|$. Moreover:

- $\mu(\alpha) \geq \mu(\alpha \circ \beta_\varepsilon)$.

In fact, if $\mu(\alpha) = 0$, then $\alpha \in \mathcal{E}_i$ $(i = 1, 2)$, which implies $\alpha \circ \beta_\varepsilon \in \mathcal{E}_i$, hence $\mu(\alpha \circ \beta_\varepsilon) = 0$.

- $\mu(\beta) \geq \mu(\beta_m)$.

Suppose that $\mu(\beta_m) = 1$, that is $\beta_m = (\beta_m)_e \circ \mu$, with μ different from identity. Since $\beta = (\beta_\varepsilon \circ (\beta_m)_e) \circ \mu$ and $\beta_\varepsilon \circ (\beta_m)_e$ belongs to $\mathcal{E}_k$, μ is also the μ-component of β, and this means that $\mu(\beta) = 1$.

- $\varepsilon(\beta) > \varepsilon(\beta_m)$.

Otherwise we cannot apply (R_ε). We get:

$$\langle \mu(\alpha), \mu(\beta), \varepsilon(\beta), \varepsilon(\alpha) \rangle >_l \langle \mu(\alpha \circ \beta_\varepsilon), \mu(\beta_m), \varepsilon(\beta_m), \varepsilon(\alpha \circ \beta_\varepsilon) \rangle$$

and this proves the lemma. ⊣

Since $\succ$ is a terminating transitive relation, the following theorem is proved.

Theorem 6.6 $\mathcal{R}_0$ *is terminating.*

This concludes the proof of Theorem 5.1.[24]

7 The system $\mathcal{R}^+$

Proving directly local confluence of $\mathcal{R}$ leads to unnecessary complications, this is why we prefer to introduce another system (which we call $\mathcal{R}^+$) and prove local confluence of the latter. In Section 9 we shall prove termination of both $\mathcal{R}$ and $\mathcal{R}^+$ and then we shall make a more precise comparison between $\mathcal{R}$ and $\mathcal{R}^+$: from this comparison, canonicity of $\mathcal{R}$ follows immediately.

In order to introduce $\mathcal{R}^+$ we first consider slight modifications of rules (R_μ) and (R^i_p).

Rule (R_μ) is enlarged as follows:

$$(R_\mu)^+ \qquad\qquad \langle \alpha,\, \beta\rangle, \gamma \;\Rightarrow\; \langle \alpha_e, \beta\rangle,\, (\alpha_\mu \times 1)\circ\gamma$$

(notice that in case vector β is empty, we get ordinary (R_μ)-rule).

Rules (R^i_p) are on the other hand restricted so that only 1-component arrows are 'moved to the right' (let us call $(R^i_p)^+$ the rules obtained by this restriction). In conclusion, we let $\mathcal{R}^+$ be the rewriting system of Table 3.

It should be noticed that (as for $\mathcal{R}$) also in $\mathcal{R}^+$ alphabetic variants of the above rules are available as rules. For instance, rule $(R^i_p)^+$ has the following alphabetic variant

$$Y \overset{\langle\gamma^i_1, d^i, \gamma^i_2\rangle}{\longrightarrow} Y_1\times X\times Y_2 \overset{\langle\alpha^j_1, \pi_X, \alpha^j_2\rangle}{\longrightarrow} Z_1\times X\times Z_2 \overset{\beta^i}{\longrightarrow} U$$
$$\Downarrow$$
$$Y \overset{\langle\gamma^i_1, d^i, d^i_\varepsilon, \gamma^i_2\rangle}{\longrightarrow} Y_1\times X\times Y'\times Y_2 \overset{\langle\pi\circ\alpha^j_1, \pi_{Y'}, \pi\circ\alpha^j_2\rangle}{\longrightarrow} Z_1\times Y'\times Z_2 \overset{(1_{Z_1}\times d^i_m\times 1_{Z_2})\circ\beta^i}{\longrightarrow} U$$

(where a further arrow must be inserted to the right in case $i = 1$, where Y' is the codomain of d^i_ε and where π is the projection $Y_1\times X\times Y'\times Y_2 \longrightarrow Y_1\times X\times Y_2$). Other alphabetic variants are possible, e.g. by permuting the components of $\langle\gamma^i_1, d^i, d^i_\varepsilon, \gamma^i_2\rangle$. Such alphabetic variants will be sometimes used during confluence proofs.

In the remaining part of this section we collect useful technical facts. We first analyze the relationship between old and new μ-extraction rules.

[24]Notice that all results in this Section depends only on the definition of left extensions of weak factorization systems. As such, they can be used to handle pushouts (for faithful and bijective on objects functors) in **Cat**, the category of all small categories. Recalling that monoids are just one-object categories, Theorem 5.1 specializes to a little result in pure string-rewriting.

(R_c^1)	$\alpha^1, \beta^1, \gamma \;\Rightarrow\; \alpha^1 \circ \beta^1, \gamma$
(R_c^2)	$\alpha^2, \beta^2 \;\Rightarrow\; \alpha^2 \circ \beta^2$
(R_ε)	$\alpha, \beta \;\Rightarrow\; \alpha \circ \beta_\varepsilon, \beta_m$
$(R_\mu)^+$	$\langle \alpha, \beta \rangle, \gamma \;\Rightarrow\; \langle \alpha_e, \beta \rangle, (\alpha_\mu \times 1) \circ \gamma$
$(R_p^1)^+$	$\langle \gamma^1, d^1 \rangle, \langle \alpha, \pi_X \rangle, \beta^1, \theta \;\Rightarrow\; \langle \gamma^1, d^1, d_\varepsilon^1 \rangle, \alpha \times 1, (1 \times d_m^1) \circ \beta^1, \theta$ where $d^1 \notin \mathcal{E}_0$
$(R_p^2)^+$	$\langle \gamma^2, d^2 \rangle, \langle \alpha, \pi_X \rangle, \beta^2 \;\Rightarrow\; \langle \gamma^2, d^2, d_\varepsilon^2 \rangle, \alpha \times 1, (1 \times d_m^2) \circ \beta^2$ where $d^2 \notin \mathcal{E}_0$

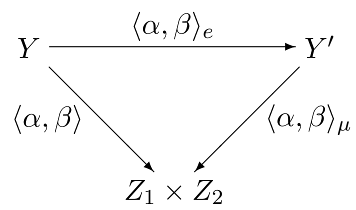
Table 3: The system $\mathcal{R}^+$

Lemma 7.1 *If $K \;\Rightarrow\; K'$ by a single $(R_\mu)^+$-step, then there is K'' such that K' rewrites to K'' by (at most) 2 $(R_\mu)^+$-rewrite steps and K rewrites to K'' by a single (R_μ)-rewrite step.*

Proof. We have the following three $(R_\mu)^+$-rewrite steps:

$$\begin{aligned} \langle \alpha, \beta \rangle, \gamma \;&\Rightarrow \langle \alpha_e, \beta \rangle,\, (\alpha_\mu \times 1) \circ \gamma \;\Rightarrow\; \langle \alpha_e, \beta_e \rangle,\, (\alpha_\mu \times \beta_\mu) \circ \gamma \;\Rightarrow \\ &\Rightarrow\; \langle \alpha_e, \beta_e \rangle_e, \langle \alpha_e, \beta_e \rangle_\mu \circ (\alpha_\mu \times \beta_\mu) \circ \gamma \end{aligned}$$

We need only to show that $\langle \alpha_e, \beta_e \rangle_e = \langle \alpha, \beta \rangle_e$ and $\langle \alpha_e, \beta_e \rangle_\mu \circ (\alpha_\mu \times \beta_\mu) = \langle \alpha, \beta \rangle_\mu$. Let us consider the factorization



and let us put $\langle \alpha, \beta \rangle_\mu = \langle \sigma, \tau \rangle$. We have $(\langle \alpha, \beta \rangle_e \circ \sigma_\varepsilon) \circ \sigma_\mu = \alpha_e \circ \alpha_\mu$, hence (by uniqueness of factorization)

$$\langle \alpha, \beta \rangle_e \circ \sigma_\varepsilon = \alpha_e \quad \text{and} \quad \alpha_\mu = \sigma_\mu$$

and similarly

$$\langle \alpha, \beta \rangle_e \circ \tau_\varepsilon = \beta_e \quad \text{and} \quad \beta_\mu = \tau_\mu.$$

Thus

$$(*) \qquad \langle \alpha, \beta \rangle = \langle \alpha, \beta \rangle_e \circ (\langle \sigma_\varepsilon, \tau_\varepsilon \rangle \circ (\alpha_\mu \times \beta_\mu)).$$

The arrow $\langle \sigma_\varepsilon, \tau_\varepsilon \rangle \circ (\alpha_\mu \times \beta_\mu)$ belongs to $\mathcal{M}_0$ as it is equal to $\langle \sigma, \tau \rangle = \langle \alpha, \beta \rangle_\mu$; so if we factorize $\langle \sigma_\varepsilon, \tau_\varepsilon \rangle$ as $\varepsilon \circ \mu$ and then $\mu \circ (\alpha_\mu \times \beta_\mu)$ as $\varepsilon' \circ \mu'$, we get that $\varepsilon \circ \varepsilon'$ is the identity (being equal to the first component of the ε/μ-factorization of an arrow in $\mathcal{M}_0$, namely $\langle \sigma_\varepsilon, \tau_\varepsilon \rangle \circ (\alpha_\mu \times \beta_\mu)$). This can happen only if ε itself (which is a projection) is in fact identity (up to a renaming); we thus established that $\langle \sigma_\varepsilon, \tau_\varepsilon \rangle$ belongs to $\mathcal{M}_0$ - which means that

$$(*)' \qquad \langle \sigma_\varepsilon, \tau_\varepsilon \rangle \quad \text{is a diagonal.}$$

(this is clear as $\sigma_\varepsilon, \tau_\varepsilon$ are both projections). From $\langle \alpha, \beta \rangle_e \circ \sigma_\varepsilon = \alpha_e$ and $\langle \alpha, \beta \rangle_e \circ \tau_\varepsilon = \beta_e$, we get

$$\langle \alpha_e, \beta_e \rangle = \langle \alpha, \beta \rangle_e \circ \langle \sigma_\varepsilon, \tau_\varepsilon \rangle$$

As first component is in $\mathcal{E}^i$ and second component is in $\mathcal{M}_0$, we get by uniqueness of factorization,

$$\langle \alpha_e, \beta_e \rangle_e = \langle \alpha, \beta \rangle_e$$

and

$$(*)'' \qquad \langle \alpha_e, \beta_e \rangle_\mu = \langle \sigma_\varepsilon, \tau_\varepsilon \rangle$$

which gives the claim (combined with $\langle \alpha, \beta \rangle_\mu = \langle \sigma_\varepsilon, \tau_\varepsilon \rangle \circ (\alpha_\mu \times \beta_\mu)$ coming from $(*)$). $\dashv$

The above Lemma guarantees that there is no need in the local confluence proof to compute superpositions between rule $(R_\mu)^+$ and other rules ($(R_\mu)^+$ itself included): it is sufficient to compute superpositions between (R_μ) and other rules.[25] Using $(R_\mu)^+$ instead of (R_μ) allows us to apply a less restrictive rule during confluence proofs; this makes some passages shorter (the only little price we pay for that is that we shall need to prove termination of $(R_\mu)^+$ too). Next Corollary will be used in Section 9 and is a slightly more accurate reformulation of what comes from the proof of Lemma 7.1 (recall that, according to $(*)'$ and $(*)''$ the third step was in fact a $(Rdi)^*$-step, where $(Rdi)^*$ is the diagonalization rule we met in Section 5):

Lemma 7.2 *Let $(R_\mu)^{+1}$ be the following special case of rule $(R_\mu)^+$:*

$$(R_\mu)^{+1} \qquad \langle a, \alpha \rangle, \beta \;\Rightarrow\; \langle a_e, \alpha \rangle, (a_\mu \times 1) \circ \beta.$$

If $K \Rightarrow K'$ by a single (R_μ) or $(R_\mu)^+$-rewrite step, then K rewrites to K' by using a finite number of $(R_\mu)^{+1}$-rewrite steps followed by a single $(Rdi)^*$-rewrite step.

[25]If $K \Rightarrow K'$ and $K \Rightarrow K''$ give rise to the critical pair (K', K'') and, say, $K \Rightarrow K'$ is a $(R_\mu)^+$-step, we can find K_0 such that $K' \Rightarrow^*_{\mathcal{R}^+} K_0$ and the pair (K_0, K'') is a critical pair generated by rule (R_μ) (instead of rule $(R_\mu)^+$).

In words: the e/μ factorization of $\langle a_1, \ldots, a_n\rangle$ is obtained by taking the componentwise e/μ factorizations and then by applying a diagonalization step. The following fact is very useful:

Lemma 7.3 *If $\langle e^i_1, \ldots, e^i_n\rangle \in \mathcal{E}^i$, the e^i_j are pairwise distinct.*

Proof. As $\mathcal{E}^i$ is closed under composition with projections, all e^i_j are in $\mathcal{E}^i$. Let $\langle e^i_{j_1}, \ldots, e^i_{j_m}\rangle$ be a list of distinct arrows containing exactly all the arrows among $e^i_1, \ldots, e^i_n$. By the previous Lemma, $\langle e^i_{j_1}, \ldots, e^i_{j_m}\rangle \in \mathcal{E}^i$. According to the definition of $\langle e^i_{j_1}, \ldots, e^i_{j_m}\rangle$, there is a diagonal δ such that

$$\langle e^i_{j_1}, \ldots, e^i_{j_m}\rangle \circ \delta = \langle e^i_1, \ldots, e^i_n\rangle.$$

As $\delta \in \mathcal{M}_0$, by uniqueness of e/μ-factorizations, δ is a renaming (thus showing the claim). $\dashv$

Corollary 7.4 *$\alpha^i \in \mathcal{E}^i$ iff the components of α^i are pairwise distinct and all belong to $\mathcal{E}^i$.*

A consequence of the above results is that e/μ-factorizations are stable under certain pullbacks, in the sense of the following:

Lemma 7.5 *If $\alpha : Y_1 \longrightarrow Y_2$ has factorization $\alpha_e \circ \alpha_\mu$, then for every Z, $\alpha \times 1_Z$ has factorization $(\alpha_e \times 1_Z) \circ (\alpha_\mu \times 1_Z)$.*

Proof. It is sufficient to show that the components of $\pi_{Y_1} \circ \alpha_e$ cannot be equal to the components of π_Z. This is clear, otherwise we would have in our theories provable equations of the kind $t = x_i$, where t is a term not containing the variable x_i: this cannot be, otherwise (after making term t a ground term by a substitution, if you like) we would obtain degeneration, i.e. that all terms are provably equal. $\dashv$

We now show that also rule (R^i_p) can be roughly achieved by finitely many $(R^i_p)^+$-rewrite steps. Let us use *the notation $K \searrow L$ in order to express that there is K' such that $K \Rightarrow^*_{\mathcal{R}^+} K'$ and $K' \Leftrightarrow^*_{\mathcal{R}_0} L$.*

Lemma 7.6 *Let L be the path*

$$L \;=\; Y \xrightarrow{\langle\gamma^i,\delta^i\rangle} Y_1 \times Z \xrightarrow{\langle\alpha,\pi_Z\rangle} Y_2 \times Z \xrightarrow{\beta^i} U \xrightarrow{(\lambda)} V$$

(where the arrow λ is missed in case $i = 2$) and let R, R' be the following two paths:

$$R = \langle\gamma^i, \delta^i, \delta^i_\varepsilon\rangle, \alpha \times 1_{Y'}, (1_{Y_2} \times \delta^i_m) \circ \beta^i, (\lambda)$$
$$R' = \langle\gamma^i, \delta^i, 1_Y\rangle, \alpha \times 1_Y, (1_{Y_2} \times \delta^i) \circ \beta^i, (\lambda)$$

(where we supposed that Y' is the codomain of δ_ε). We have:

(i) $R \Leftrightarrow^*_{\mathcal{R}_0} R'$;

(ii) *if* $\delta^i \in \mathbf{T_0}$, *then* $L \Leftrightarrow^*_{\mathcal{R}_0} R$;

(iii) *in the general case,* $L \searrow R$ *(and consequently* $L \searrow R'$*).*

Proof. (i) was proved at the end of Section 5 (it was actually used as simplification step during completion). (ii) is easy, because we can move to the left $(1 \times \delta^i_m)$ in R by $\Leftrightarrow^*_{\mathcal{R}_0}$-equivalence:

$$\langle \gamma^i, \delta^i, \delta^i_\varepsilon \rangle,\ \alpha \times 1,\ (1 \times \delta^i_m) \circ \beta^i, (\lambda)\ \Leftrightarrow^*_{\mathcal{R}_0}\ \langle \gamma^i, \delta^i, \delta^i \rangle,\ \alpha \times 1,\ \beta^i, (\lambda) \Leftrightarrow^*_{\mathcal{R}_0} L.$$

(iii) is proved by induction on the number of components of δ. If such number is 1, there is nothing to prove (because either $(R^i_p)^+$ or (ii) applies). So suppose it is bigger than 1. If $\delta \in \mathbf{T_0}$, we just proved a stronger claim; otherwise L and R (up to an alphabetic variant) are

$$(1) \qquad Y \xrightarrow{\langle \gamma, \delta, d \rangle} Y_1 \times Z \times X \xrightarrow{\langle \alpha, \pi_Z, \pi_X \rangle} Y_2 \times Z \times X \xrightarrow{\beta} U \xrightarrow{(\lambda)} V$$

and

$$(2) \qquad Y \xrightarrow{\langle \gamma, \delta, d, \langle \delta, d \rangle_\varepsilon \rangle} Y_1 \times Z \times X \times Y' \xrightarrow{\alpha \times 1_{Y'}} Y_2 \times Y' \xrightarrow{(1_{Y_2} \times \langle \delta, d \rangle_m) \circ \beta} U \xrightarrow{(\lambda)} V$$

respectively (with $d \notin \mathbf{T_0}$). To the former, we can apply a $(R^i_p)^+$-rewrite step thus getting

$$(3) \qquad Y \xrightarrow{\langle \gamma, \delta, d, d_\varepsilon \rangle} Y_1 \times Z \times X \times Y''_0 \xrightarrow{\langle \alpha, \pi_Z \rangle \times 1_{Y''_0}} Y_2 \times Z \times Y''_0 \xrightarrow{(1_{Y_2} \times 1_Z \times d_m) \circ \beta} U \xrightarrow{(\lambda)} V$$

(where we called Y''_0 the codomain of d_ε). By induction hypothesis, there is path K'' such that (3) $\Rightarrow^*_{\mathcal{R}^+} K''$ and $K'' \Leftrightarrow^*_{\mathcal{R}_0}$ (4), where (4) is the path (let Y'_0 be the codomain of δ_ε):

$$(4) \quad Y \xrightarrow{\langle \gamma, \delta, d, \delta_\varepsilon, d_\varepsilon \rangle} Y_1 \times Z \times X \times Y'_0 \times Y''_0 \xrightarrow{\alpha \times 1_{Y'_0} \times 1_{Y''_0}} Y_2 \times Y'_0 \times Y''_0 \xrightarrow{(1_{Y_2} \times \delta_m \times d_m) \circ \beta} U \xrightarrow{(\lambda)} V$$

As $\langle \gamma, \delta, d, \delta_\varepsilon, d_\varepsilon \rangle$ is equal to $\langle \gamma, \delta, d, 1_Y \rangle \circ (1_{Y_1} \times 1_Z \times 1_X \times \langle \delta_\varepsilon, d_\varepsilon \rangle)$, we can move right $1_{Y_1} \times 1_Z \times \langle \delta_\varepsilon, d_\varepsilon \rangle$ by $\Leftrightarrow^*_{\mathcal{R}_0}$-equivalence, thus getting the path

$$(5) \qquad Y \xrightarrow{\langle \gamma, \delta, d, 1_Y \rangle} Y_1 \times Z \times X \times Y \xrightarrow{\alpha \times 1_Y} Y_2 \times Y \xrightarrow{(1_{Y_2} \times \langle \delta, d \rangle) \circ \beta} U \xrightarrow{(\lambda)} V$$

which we know from (i) it is $\Leftrightarrow^*_{\mathcal{R}_0}$-equivalent to (2). In conclusion, we have

$$(1) \Rightarrow_{\mathcal{R}^+} (3) \Rightarrow^*_{\mathcal{R}^+} K'' \Leftrightarrow^*_{\mathcal{R}_0} (4) \Leftrightarrow^*_{\mathcal{R}_0} (5) \Leftrightarrow^*_{\mathcal{R}_0} (2)$$

thus showing the claim $\dashv$

We need a final Lemma for next Section:

Lemma 7.7 *We have $R_1 \searrow R_2$, where R_1, R_2 are the paths*

$$R_1 = Y \times Z_1 \overset{\langle\alpha_1^i,\alpha_2^i\rangle\times 1}{\longrightarrow} Y_1 \times Y_2 \times Z_1 \overset{\pi_{Y_1}\times\beta^j}{\longrightarrow} Y_1 \times Z_2 \overset{\gamma^i}{\longrightarrow} W \overset{(\lambda)}{\longrightarrow} V$$
$$R_2 = \quad Y \times Z_1 \overset{1\times\beta^j}{\longrightarrow} Y \times Z_2 \overset{(\alpha_1^i\times 1)\circ\gamma^i}{\longrightarrow} W \overset{(\lambda)}{\longrightarrow} V$$

(λ is missed in case $i = 2$).

Proof. Applying (an alphabetic variant of) Lemma 7.6 (iii), we have that

$$R_1 = \langle \pi_Y \circ \alpha_1^i, \pi_Y \circ \alpha_2^i, \pi_{Z_1}\rangle,\ \langle \pi_{Y_1}, \pi_{Z_1} \circ \beta^j\rangle,\ \gamma^i, (\lambda)$$
$$\searrow$$
$$\langle 1_{Y\times Z_1}, \pi_Y \circ \alpha_1^i, \pi_Y \circ \alpha_2^i, \pi_{Z_1}\rangle,\ \langle 1_{Y\times Z_1} \times (\pi_{Z_1} \circ \beta^j)\rangle,\ ((\pi_Y \circ \alpha_1^i) \times 1_{Z_2}) \circ \gamma^i, (\lambda)$$
$$=$$
$$\langle \pi_Y, \pi_{Z_1}, \pi_Y \circ \alpha_1^i, \pi_Y \circ \alpha_2^i, \pi_{Z_1}\rangle,$$
$$\langle 1_{Y\times Z_1} \times (\pi_{Z_1} \circ \beta^j)\rangle,\ \langle \pi_Y, \pi_{Z_2}\rangle \circ (\alpha_1^i \times 1_{Z_2}) \circ \gamma^i,\ (\lambda)$$
$$\Leftrightarrow^*_{\mathcal{R}_0} \quad \text{(see Figure 1)}$$
$$\langle \pi_Y, \pi_{Z_1}, \pi_Y \circ \alpha_1^i, \pi_Y \circ \alpha_2^i, \pi_{Z_1}\rangle,\ \langle \pi_Y, \pi^2_{Z_1}\rangle \circ (1_Y \times \beta^j),\ (\alpha_1^i \times 1_{Z_2}) \circ \gamma^i, (\lambda)$$
$$\Leftrightarrow^*_{\mathcal{R}_0}$$
$$\langle \pi_Y, \pi_{Z_1}, \pi_Y \circ \alpha_1^i, \pi_Y \circ \alpha_2^i, \pi_{Z_1}\rangle \circ \langle \pi_Y, \pi^2_{Z_1}\rangle,\ 1_Y \times \beta^j,\ (\alpha_1^i \times 1_{Z_2}) \circ \gamma^i, (\lambda)$$
$$=$$
$$\langle \pi_Y, \pi_{Z_1}\rangle,\ 1_Y \times \beta^j,\ (\alpha_1^i \times 1_{Z_2}) \circ \gamma^i, (\lambda)$$
$$=$$
$$1_{Y\times Z_1},\ 1_Y \times \beta^j,\ (\alpha_1^i \times 1_{Z_2}) \circ \gamma^i, (\lambda)$$
$$\Leftrightarrow^*_{\mathcal{R}_0}$$
$$1_Y \times \beta^j,\ (\alpha_1^i \times 1_{Z_2}) \circ \gamma^i, (\lambda)\ =\ R_2$$

as wanted. ⊣

8 Local confluence, II

In this section we prove that $\mathcal{R}^+$ is locally confluent. In order to show confluence of a pair of paths (R_1, R_2), *we shall use the following schema: we find L_1, L_2 such that $R_1 \searrow L_1$ and $R_2 \searrow L_2$ and $L_1 \Leftrightarrow^*_{\mathcal{R}_0} L_2$.* Canonicity of $\mathcal{R}_0$ (which was proved in Section 6) guarantees that in such a condition K_1, K_2 are joinable.

Throughout this section we shall mention arrows $\gamma, d, \alpha, \beta, \theta, \lambda$ whose domains and codomains are fixed as follows:

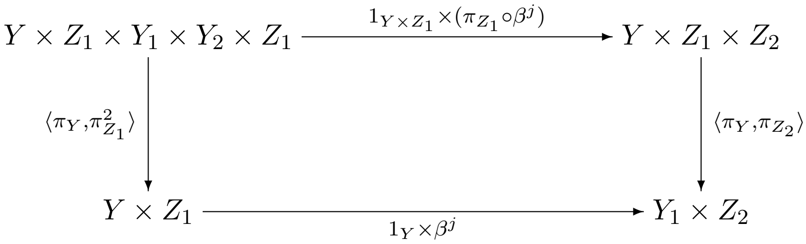
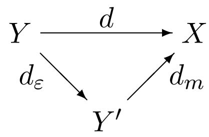


Figure 1: Commutative diagram

$$Y \xrightarrow{\langle \gamma, d \rangle} Y_1 \times X \xrightarrow{\langle \alpha, \pi_X \rangle} Y_2 \times X \xrightarrow{\beta} U \xrightarrow{\theta} V \xrightarrow{\lambda} T$$

We also assume that d factorizes in ε/m-components as follows



We first analyze some situations which are very frequent during local confluence proof.

Lemma 8.1 *Let K_i $(i \in \{1, 2\})$ be the following path:*

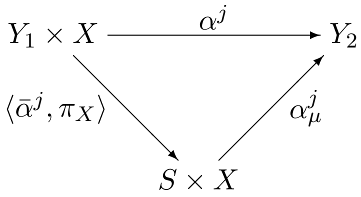
$$K_i = \langle \gamma^i, d^i, d^i_\varepsilon \rangle, \alpha^j \times 1_{Y'}, (1_{Y_2} \times d^i_m) \circ \beta^0 \circ \theta^i, (\lambda)$$

(where λ lacks in case $i = 2$). Then:
(i) The path $K'_i = \langle \gamma^i, d^i \rangle, (\langle \alpha^j, \pi_X \rangle \circ \beta^0)_e, (\langle \alpha^j, \pi_X \rangle \circ \beta^0)_\mu \circ \theta^i, (\lambda)$ is joinable with K_i in $\mathcal{R}^+$.
(ii) The path $K''_i = \langle \gamma^i, d^i \rangle, \langle \alpha^j, \pi_X \rangle \circ \beta^0, \theta^i, (\lambda)$ is joinable with K_i in $\mathcal{R}^+$.

Proof. (ii) is trivially reduced to (i) (just apply (R_μ) in K''_i to decompose $\langle \alpha^j, \pi_X \rangle \circ \beta^0$).

To prove (i), we have to factorize the arrow $\langle \alpha^j, \pi_X \rangle \circ \beta^0$ in components e/μ. We first factorize $\langle \alpha^j, \pi_X \rangle$: by Lemmas 7.2, 7.3, such factorization is obtained by first factorizing α_j in e/μ components and then diagonalizing with π_X in case π_X appears among the components of α^j_e. We have to distinguish whether π_X is among the components of α^j_e or not.

Case 1: π_X is among the components of α^j_e, hence α^j has the following factorization in e/μ-components:

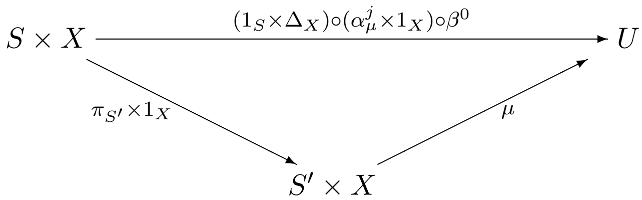


Then:

$$\langle \alpha^j, \pi_X \rangle = \langle \bar{\alpha}^j, \pi_X, \pi_X \rangle \circ (\alpha^j_\mu \times 1_X) = \langle \bar{\alpha}^j, \pi_X \rangle \circ [(1_S \times \Delta_X) \circ (\alpha^j_\mu \times 1_X)]$$

where $X \xrightarrow{\Delta_X} X \times X$ is a diagonal. We have two subcases, depending whether π_X appears in the ε-component of $(1_S \times \Delta_X) \circ (\alpha^j_\mu \times 1_X) \circ \beta^0$ or not.

Subcase 1.1: let us assume that $(1_S \times \Delta_X) \circ (\alpha^j_\mu \times 1_X) \circ \beta^0$ has the following factorization in $\mathbf{T_0}$



It follows that $\langle \alpha^j, \pi_X \rangle \circ \beta^0 = \langle \bar{\alpha}^j, \pi_X \rangle \circ (\pi_{S'} \times 1_X) \circ \mu$; by the fact that $\langle \bar{\alpha}^j, \pi_X \rangle \circ (\pi_{S'} \times 1_X)$ belongs to $\mathcal{E}_j$ and by the uniqueness of decomposition we have:

$$(\langle \alpha^j, \pi_X \rangle \circ \beta^0)_e = \langle \bar{\alpha}^j, \pi_X \rangle \circ (\pi_{S'} \times 1_X) = \langle \bar{\alpha}^j \circ \pi_{S'}, \pi_X \rangle$$
$$(\langle \alpha^j, \pi_X \rangle \circ \beta^0)_\mu = \mu$$

It follows that $K'_i = \langle \gamma^i, d^i \rangle, \langle \bar{\alpha}^j \circ \pi_{S'}, \pi_X \rangle, \mu \circ \theta^i, (\lambda)$. We can apply Lemma 7.6(iii) (in fact, if $i = 1$ the arrow λ belongs to the path) and we obtain $K'_i \searrow L_1$, where

$$L_1 = \ \langle \gamma^i, d^i, 1_Y \rangle, (\bar{\alpha}^j \circ \pi_{S'}) \times 1_Y, (1_{S'} \times d^i) \circ \mu \circ \theta^i, (\lambda)$$

Let us consider K_i. We first observe that $\alpha^j \times 1_{Y'}$ can be decomposed in e/μ components as $(\alpha^j_e \times 1_{Y'}) \circ (\alpha^j_\mu \times 1_{Y'})$ by Lemma 7.5; therefore an application of (R_μ) yields to

$$K_i \Rightarrow_{\mathcal{R}^+} \langle \gamma^i, d^i, d^i_\varepsilon \rangle, \alpha^j_e \times 1_{Y'}, (\alpha^j_\mu \times 1_{Y'}) \circ (1_{Y_2} \times d^i_m) \circ \beta^0 \circ \theta^i, (\lambda)$$
$$=$$
$$\langle \gamma^i, d^i, d^i_\varepsilon \rangle, \langle \pi \circ \bar{\alpha}^j, \pi_X, \pi_{Y'} \rangle, (\alpha^j_\mu \times 1_{Y'}) \circ (1_{Y_2} \times d^i_m) \circ \beta^0 \circ \theta^i, (\lambda)$$

where $Y_1 \times X \times Y' \xrightarrow{\pi} Y_1 \times X$. We can apply Lemma 7.6(iii) on $\langle d^i, d^i_\varepsilon \rangle$ and we get:

$$K_i$$

$$\searrow$$

$$\langle \gamma^i, d^i, d^i_\varepsilon, 1_Y \rangle, (\pi \circ \bar{\alpha}^j) \times 1_Y,\ (1_S \times \langle d^i, d^i_\varepsilon \rangle) \circ (\alpha^j_\mu \times 1_{Y'}) \circ (1_{Y_2} \times d^i_m) \circ \beta^0 \circ \theta^i,\ (\lambda)$$

$$=$$

$$\langle \gamma^i, d^i, d^i_\varepsilon, 1_Y \rangle,\ (\pi \circ \bar{\alpha}^j) \times 1_Y,$$

$$(1_S \times \langle d^i, d^i_\varepsilon \rangle) \circ (1_S \times 1_X \times d^i_m) \circ (\alpha^j_\mu \times 1_X) \circ \beta^0 \circ \theta^i,\ (\lambda)$$

$$= \quad \text{(see Figure 2)}$$

$$\langle \gamma^i, d^i, d^i_\varepsilon, 1_Y \rangle,\ (\pi \circ \bar{\alpha}^j) \times 1_Y,\ (1_S \times d^i) \circ (1_S \times \Delta_X) \circ (\alpha^j_\mu \times 1_X) \circ \beta^0 \circ \theta^i,\ (\lambda)$$

$$=$$

$$\langle \gamma^i, d^i, d^i_\varepsilon, 1_Y \rangle,\ (\pi \circ \bar{\alpha}^j) \times 1_Y,\ (1_S \times d^i) \circ (\pi_{S'} \times 1_X) \circ \mu \circ \theta^i,\ (\lambda)$$

$$=$$

$$\langle \gamma^i, d^i, d^i_\varepsilon, 1_Y \rangle,\ (\pi \times 1_Y) \circ (\bar{\alpha}^j \times 1_Y),\ (\pi_{S'} \times 1_Y) \circ (1_{S'} \times d^i) \circ \mu \circ \theta^i,\ (\lambda)$$

$$\Leftrightarrow^*_{\mathcal{R}_0}$$

$$\langle \gamma^i, d^i, d^i_\varepsilon, 1_Y \rangle \circ (\pi \times 1_Y),\ (\bar{\alpha}^j \times 1_Y) \circ (\pi_{S'} \times 1_Y),\ (1_{S'} \times d^i) \circ \mu \circ \theta^i,\ (\lambda)$$

$$=$$

$$\langle \gamma^i, d^i, 1_Y \rangle,\ (\bar{\alpha}^j \circ \pi_{S'}) \times 1_Y,\ (1_{S'} \times d^i) \circ \mu \circ \theta^i,\ (\lambda)$$

which coincides with L_1, and this prove (i).

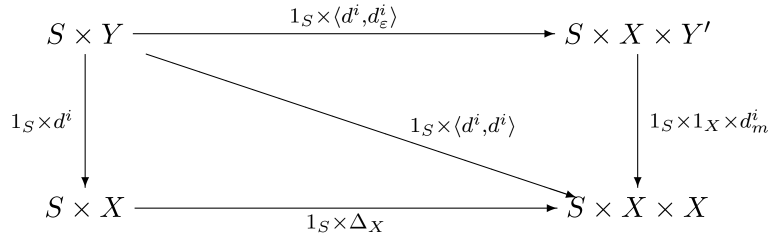
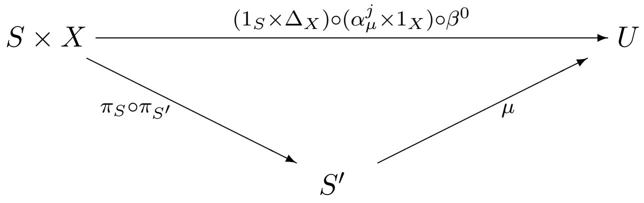


Figure 2: Commutative diagram

Subcase 1.2: now $(1_S \times \Delta_X) \circ (\alpha^j_\mu \times 1_X) \circ \beta^0$ has the following factorization in $\mathbf{T_0}$:



We consequently have

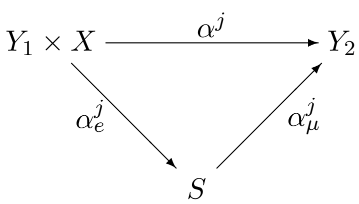
$$((\langle \alpha^j, \pi_X \rangle \circ \beta^0)_e = \langle \bar{\alpha}^j, \pi_X \rangle \circ \pi_S \circ \pi_{S'} = \bar{\alpha}^j \circ \pi_{S'}$$
$$((\langle \alpha^j, \pi_X \rangle \circ \beta^0)_\mu = \mu$$

In the present subcase we do not need manipulating K'_i; moreover by manipulating K_i as in the previous case, we get

$$K_i$$
$$\searrow$$
$$\langle \gamma^i, d^i, d^i_\varepsilon, 1_Y \rangle, (\pi \circ \bar{\alpha}^j) \times 1_Y, (1_S \times d^i) \circ (1_S \times \Delta_X) \circ (\alpha^j_\mu \times 1_X) \circ \beta^0 \circ \theta^i, (\lambda)$$
$$=$$
$$\langle \gamma^i, d^i, d^i_\varepsilon, 1_Y \rangle, (\pi \circ \bar{\alpha}^j) \times 1_Y, (1_S \times d^i) \circ \pi_S \circ \pi_{S'} \circ \mu \circ \theta^i, (\lambda)$$
$$=$$
$$\langle \gamma^i, d^i, d^i_\varepsilon, 1_Y \rangle, (\pi \circ \bar{\alpha}^j) \times 1_Y, \pi_S \circ \pi_{S'} \circ \mu \circ \theta^i, (\lambda)$$
$$\Leftrightarrow^*_{\mathcal{R}_0}$$
$$\langle \gamma^i, d^i, d^i_\varepsilon, 1_Y \rangle, ((\pi \circ \bar{\alpha}^j) \times 1_Y) \circ \pi_S \circ \pi_{S'}, \mu \circ \theta^i, (\lambda)$$
$$=$$
$$\langle \gamma^i, d^i, d^i_\varepsilon, 1_Y \rangle, \langle \pi_{Y_1}, \pi_X \rangle \circ \bar{\alpha}^j \circ \pi_{S'}, \mu \circ \theta^i, (\lambda)$$
$$\Leftrightarrow^*_{\mathcal{R}_0}$$
$$\langle \gamma^i, d^i \rangle, \bar{\alpha}^j \circ \pi_{S'}, \mu \circ \theta^i, (\lambda)$$

which coincides with K'_i, and this prove (i).

Case 2: suppose now that π_X does not belong to α^j_e, namely α_j has the following factorization in e/μ-components:

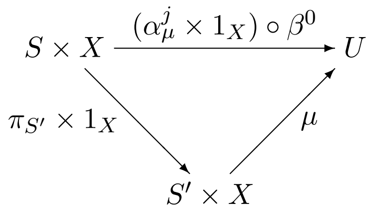


This implies that:

$$\langle \alpha^j, \pi_X \rangle_e = \langle \alpha^j_e, \pi_X \rangle \qquad \langle \alpha^j, \pi_X \rangle_\mu = \alpha^j_\mu \times 1_X$$

We need to factorize $(\alpha^j_\mu \times 1_X) \circ \beta^0$ in $\mathbf{T_0}$: again, we have two subcases, depending whether π_X appears or not in the ε-component.

Subcase 2.1: let $(\alpha^j_\mu \times 1_X) \circ \beta^0$ factorize as follows:



Reasoning as in Case 1, it follows that:

$$(\langle \alpha^j, \pi_X \rangle \circ \beta^0)_e = \langle \alpha^j_e \circ \pi_{S'}, \pi_X \rangle \qquad (\langle \alpha^j, \pi_X \rangle \circ \beta^0)_\mu = \mu$$

We have

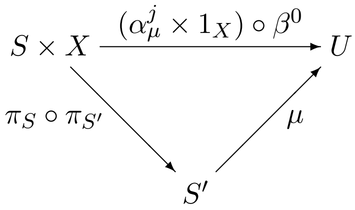
$$K'_i = \langle \gamma^i, d^i \rangle, \langle \alpha^j_e \circ \pi_{S'}, \pi_X \rangle, \mu \circ \theta^i, (\lambda)$$
$$\searrow$$
$$\langle \gamma^i, d^i, 1_Y \rangle, (\alpha^j_e \circ \pi_{S'}) \times 1_Y, (1_{S'} \times d^i) \circ \mu \circ \theta^i, (\lambda) \qquad (L'_1)$$

The arrow $\alpha^j \times 1_{Y'}$ decomposes in e/μ components as $(\alpha^j_e \times 1_{Y'}) \circ (\alpha^j_\mu \times 1_{Y'})$ (see Lemma 7.5); thus, by (R_μ), K_i rewrites to

$$\langle \gamma^i, d^i, d^i_\varepsilon \rangle, \alpha^j_e \times 1_{Y'}, (\alpha^j_\mu \times 1_{Y'}) \circ (1_{Y_2} \times d^i_m) \circ \beta^0 \circ \theta^i, (\lambda)$$
$$\Leftrightarrow^*_{\mathcal{R}_0} \quad \text{(by Lemma 7.6(i))}$$
$$\langle \gamma^i, d^i, 1_Y \rangle, \alpha^j_e \times 1_Y, (1_S \times d^i_\varepsilon) \circ (\alpha^j_\mu \times 1_{Y'}) \circ (1_{Y_2} \times d^i_m) \circ \beta^0 \circ \theta^i, (\lambda)$$
$$=$$
$$\langle \gamma^i, d^i, 1_Y \rangle, \alpha^j_e \times 1_Y, (1_S \times d^i_\varepsilon) \circ (1_S \times d^i_m) \circ (\alpha^j_\mu \times 1_X), \beta^0 \circ \theta^i, (\lambda)$$
$$\Leftrightarrow^*_{\mathcal{R}_0}$$
$$\langle \gamma^i, d^i, 1_Y \rangle, \alpha^j_e \times 1_Y, (1_S \times d^i) \circ (\alpha^j_\mu \times 1_X) \circ \beta^0 \circ \theta^i, (\lambda)$$
$$=$$
$$\langle \gamma^i, d^i, 1_Y \rangle, \alpha^j_e \times 1_Y, (1_S \times d^i) \circ (\pi_{S'} \times 1_X) \circ \mu \circ \theta^i, (\lambda)$$
$$\Leftrightarrow^*_{\mathcal{R}_0}$$
$$\langle \gamma^i, d^i, 1_Y \rangle, (\alpha^j_e \times 1_Y) \circ (\pi_{S'} \times 1_Y), (1_{S'} \times d^i) \circ \mu \circ \theta^i, (\lambda)$$

which coincides with L_1', and this concludes Subcase 2.1.

Subcase 2.2.: let $(\alpha^j_\mu \times 1_X) \circ \beta^0$ factorize as follows:



We have

$$(\langle \alpha^j, \pi_X \rangle \circ \beta^0)_e = \alpha^j_e \circ \pi_{S'} \qquad (\langle \alpha^j, \pi_X \rangle \circ \beta^0)_\mu = \mu$$

Reasoning as in Subcase 2.1, we have

$$\begin{gathered} K_i \;\Leftrightarrow^*_{\mathcal{R}_0}\; \langle \gamma^i, d^i, 1_Y \rangle,\ \alpha^j_e \times 1_Y,\ (1_S \times d^i) \circ (\alpha^j_\mu \times 1_X) \circ \beta^0 \circ \theta^i, (\lambda) \\ = \\ \langle \gamma^i, d^i, 1_Y \rangle,\ \alpha^j_e \times 1_Y,\ (1_S \times d^i) \circ \pi_S \circ \pi_{S'} \circ \mu \circ \theta^i, (\lambda) \\ \Leftrightarrow^*_{\mathcal{R}_0} \\ \langle \gamma^i, d^i, 1_Y \rangle,\ \langle \pi_{Y_1}, \pi_X \rangle \circ \alpha^j_e \circ \pi_{S'},\ \mu \circ \theta^i, (\lambda) \\ \Leftrightarrow^*_{\mathcal{R}_0} \\ \langle \gamma^i, d^i \rangle,\ \alpha^j_e \circ \pi_{S'},\ \mu \circ \theta^i, (\lambda) \end{gathered}$$

which coincides with K_i'. ⊣

Lemma 8.2 *Let K_j ($j \in \{1,2\}$) be the following path:*

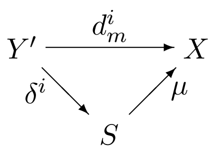
$$K_j = \langle \gamma^i, d^i, d^i_\varepsilon \rangle, \alpha^j \times 1_{Y'}, (1_{Y_2} \times d^i_m) \circ \beta^0, \theta^j, (\lambda)$$

(where λ lacks in case $j = 2$). Then the path

$$K_i''' = \langle \gamma^i, d^i \rangle, \langle \alpha^j, \pi_X \rangle \circ \beta^0 \circ \theta^j, (\lambda)$$

is joinable with K_j in $\mathcal{R}^+$.

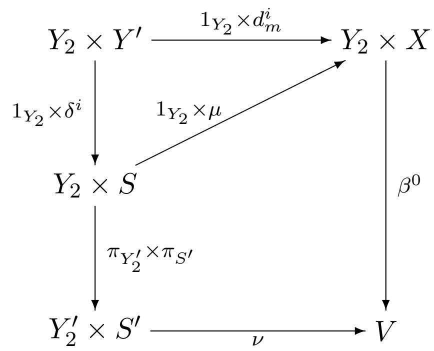
Proof. Here we cannot apply the products rule on K_i''', therefore we have to act on K_j; thus we have to decompose $(1_{Y_2} \times d^i_m) \circ \beta^0$ in e/μ components. Suppose that the e/μ-components of d^i_m are



Then by Lemma 7.5:

$$(1_{Y_2} \times d^i_m)_e = 1_{Y_2} \times \delta^i \qquad (1_{Y_2} \times d^i_m)_\mu = 1_{Y_2} \times \mu$$

We decompose $(1_{Y_2} \times d^i_m) \circ \beta^0$ as follows:



Since $1_{Y_2} \times \delta^i$ belongs to $\mathcal{E}_i$, we can state that

$$((1_{Y_2} \times d^i_m) \circ \beta^0)_e = (1_{Y_2} \times \delta^i) \circ (\pi_{Y_2'} \times \pi_{S'}) = \pi_{Y_2'} \times (\delta^i \circ \pi_{S'})$$
$$((1_{Y_2} \times d^i_m) \circ \beta^0)_\mu = \nu$$

By (R_μ), we have

$$K_j \Rightarrow_{\mathcal{R}^+} \langle \gamma^i, d^i, d^i_\varepsilon \rangle, \alpha^j \times 1_{Y'}, \pi_{Y_2'} \times (\delta^i \circ \pi_{S'}), \nu \circ \theta^j, (\lambda)$$

Lemma 7.7 yields (by a $\searrow$-step):[26]

$$\langle \gamma^i, d^i, d^i_\varepsilon \rangle,\ 1_{Y_1} \times 1_X \times (\delta^i \circ \pi_{S'}),\ ((\alpha^j \circ \pi_{Y_2'}) \times 1_{S'}) \circ \nu \circ \theta^j,\ (\lambda)$$
$$\Leftrightarrow^*_{\mathcal{R}_0}$$
$$\langle \gamma^i, d^i, d^i_\varepsilon \rangle,\ 1_{Y_1} \times 1_X \times \delta^i,\ (1_{Y_1} \times 1_X \times \pi_{S'}) \circ ((\alpha^j \circ \pi_{Y_2'}) \times 1_{S'}) \circ \nu \circ \theta^j,\ (\lambda)$$
$$\Leftrightarrow^*_{\mathcal{R}_0}$$
$$\langle \gamma^i, d^i, d^i_\varepsilon \circ \delta^i \rangle,\ (\alpha^j \times 1_S) \circ (\pi_{Y_2'} \times \pi_{S'}) \circ \nu \circ \theta^j,\ (\lambda)$$
$$=$$
$$\langle \gamma^i, d^i, d^i_\varepsilon \circ \delta^i \rangle,\ (\alpha^j \times 1_S) \circ (1_{Y_2} \times \mu) \circ \beta^0 \circ \theta^j,\ (\lambda)$$
$$=$$
$$\langle \gamma^i, d^i, d^i_\varepsilon \circ \delta^i \rangle,\ (1_{Y_1} \times 1_X \times \mu) \circ (\alpha^j \times 1_X) \circ \beta^0 \circ \theta^j,\ (\lambda)$$
$$\Leftrightarrow^*_{\mathcal{R}_0}$$
$$\langle \gamma^i, d^i, d^i_\varepsilon \circ \delta^i \rangle \circ (1_{Y_1} \times 1_X \times \mu),\ (\alpha^j \times 1_X) \circ \beta^0 \circ \theta^j,\ (\lambda)$$
$$=$$
$$\langle \gamma^i, d^i, d^i \rangle,\ (\alpha^j \times 1_X) \circ \beta^0 \circ \theta^j,\ (\lambda)$$
$$\Leftrightarrow^*_{\mathcal{R}_0}$$
$$\langle \gamma^i, d^i \rangle,\ \langle \alpha^j, \pi_X \rangle \circ \beta^0 \circ \theta^j,\ (\lambda)$$

which coincides with K_i'''. $\dashv$

Let us now prove local confluence of $\mathcal{R}^+$. To this aim, by Section 6 results, it suffices to study the superpositions between the rule $(R^i_p)^+$ and the other rules, itself included (see also the observation following the proof of Lemma 7.1).

8.1 Superpositions between $(R^j_p)^+$ and (R^i_c)

CASE 1

We have a path of four arrows $\theta_1, \theta_2, \theta_3, \theta_4$ and we apply (R^i_c) on θ_1, θ_2 and $(R^j_p)^+$ on $\theta_2, \theta_3, \theta_4$ (clearly, if the rule applied is $(R^1_p)^+$, we have to add an arrow λ to the path).

Let us first suppose $i \neq j$; in such a case θ_2 must belong to $\mathbf{T_0}$:

[26] We have a projection $\pi_{Y_2'} : Y_2 \longrightarrow Y_2'$, hence α^j must be a pair (of vectors), whose component having codomain Y_2' is obviously $\alpha^j \circ \pi_{Y_2'}$.

$$\zeta^i, \langle \gamma^0, d^0 \rangle, \langle \alpha, \pi_X \rangle, \beta^j, (\lambda)$$

$$\swarrow (R_c^i) \qquad\qquad (R_p^j)^+ \searrow$$

$$\zeta^i \circ \langle \gamma^0, d^0 \rangle, \langle \alpha, \pi_X \rangle, \beta^j, (\lambda) \qquad \zeta^i, \langle \gamma^0, d^0, d^0_\varepsilon \rangle, \alpha \times 1_{Y'}, (1_{Y_2} \times d^0_m) \circ \beta^j, (\lambda)$$

In this case the two members of the critical pair are $\Leftrightarrow^*_{\mathcal{R}_0}$-equivalent by Lemma 7.6(ii).

If $i = j$, then we have:

$$\zeta^i, \langle \gamma^i, d^i \rangle, \langle \alpha, \pi_X \rangle, \beta^i, (\lambda)$$

$$\swarrow (R_c^i) \qquad\qquad (R_p^i)^+ \searrow$$

$$\zeta^i \circ \langle \gamma^i, d^i \rangle, \langle \alpha, \pi_X \rangle, \beta^i, (\lambda) \qquad \zeta^i, \langle \gamma^i, d^i, d^i_\varepsilon \rangle, \alpha \times 1_{Y'}, (1_{Y_2} \times d^i_m) \circ \beta^i, (\lambda)$$

By applying Lemma 7.6(iii) to first member (where $\zeta^i \circ \langle \gamma^i, d^i \rangle = \langle \zeta^i \circ \gamma^i, \zeta^i \circ d^i \rangle$) we have a $\searrow$-step to the path (let W be the domain of ζ^i):

$$\langle \zeta^i \circ \gamma^i, \zeta^i \circ d^i, 1_W \rangle, \alpha \times 1_W, (1_{Y_2} \times \zeta^i \circ d^i) \circ \beta^i, (\lambda) \qquad (L_1)$$

By applying (R_c^i) to the second member, where $\alpha \times 1_{Y'}$ is $\langle \pi \circ \alpha, \pi_{Y'} \rangle$ (with $Y_1 \times X \times Y' \xrightarrow{\pi} Y_1 \times X$), we get

$$\langle \zeta^i \circ \gamma^i, \zeta^i \circ d^i, \zeta^i \circ d^i_\varepsilon \rangle, \langle \pi \circ \alpha, \pi_{Y'} \rangle, (1_{Y_2} \times d^i_m) \circ \beta^i, (\lambda)$$

$$\searrow \text{ Lemma 7.6(iii)}$$

$$\langle \zeta^i \circ \gamma^i, \zeta^i \circ d^i, \zeta^i \circ d^i_\varepsilon, 1_W \rangle, (\pi \circ \alpha) \times 1_W, (1_{Y_2} \times \zeta^i \circ d^i_\varepsilon) \circ (1_{Y_2} \times d^i_m) \circ \beta^i, (\lambda)$$

$$=$$

$$\langle \zeta^i \circ \gamma^i, \zeta^i \circ d^i, \zeta^i \circ d^i_\varepsilon, 1_W \rangle, (\pi \times 1_W) \circ (\alpha \times 1_W), (1_{Y_2} \times \zeta^i \circ d^i_\varepsilon \circ d^i_m) \circ \beta^i, (\lambda)$$

$$\Leftrightarrow^*_{\mathcal{R}_0}$$

$$\langle \zeta^i \circ \gamma^i, \zeta^i \circ d^i, \zeta^i \circ d^i_\varepsilon, 1_W \rangle \circ (\pi \times 1_W), \alpha \times 1_W, (1_{Y_2} \times \zeta^i \circ d^i) \circ \beta^i, (\lambda)$$

$$=$$

$$\langle \zeta^i \circ \gamma^i, \zeta^i \circ d^i, 1_W \rangle, \alpha \times 1_W, (1_{Y_2} \times \zeta^i \circ d^i) \circ \beta^i, (\lambda)$$

which coincides with (L_1).

For future reference let us mark the following fact we established during the above proof:[27]

Lemma 8.3 *Paths*

$$\zeta^i \circ \langle \gamma^i, d^i \rangle, \langle \alpha, \pi_X \rangle, \beta^i, (\lambda)$$

$$\zeta^i \circ \langle \gamma^i, d^i, d^i_\varepsilon \rangle, \alpha \times 1_{Y'}, (1_{Y_2} \times d^i_m) \circ \beta^i, (\lambda)$$

are joinable in $\mathcal{R}^+$.

Let us go on by examining other superpositions.

<u>CASE 2</u>

We have a path of three arrows $\theta_1, \theta_2, \theta_3$, we apply (R^j_c) on θ_1, θ_2 and $(R^i_p)^+$ on the whole path. If $i = j$ everything trivially compose; otherwise θ_1 must belong to $\mathbf{T_0}$. Therefore we have:

$$\langle \gamma^0, d^0 \rangle, \langle \alpha^j, \pi_X \rangle, \beta^i, (\lambda)$$

$$\swarrow (R^j_c) \qquad\qquad (R^i_p)^+ \searrow$$

$$\langle \gamma^0, d^0 \rangle \circ \langle \alpha^j, \pi_X \rangle, \beta^i, (\lambda) \qquad\qquad \langle \gamma^0, d^0, d^0_\varepsilon \rangle, \alpha^j \times 1_{Y'}, (1_{Y_2} \times d^0_m) \circ \beta^i, (\lambda)$$

The two members are $\Leftrightarrow^*_{\mathcal{R}_0}$-equivalent by Lemma7.6(ii).

<u>CASE 3</u>

We have a path of three arrows $\theta_1, \theta_2, \theta_3$ and we apply (R^j_c) on θ_2, θ_3 and $(R^i_p)^+$ on the whole path. Again everything compose if $i = j$; otherwise θ_3 must belong to $\mathbf{T_0}$. Moreover as $i = 1$ or $j = 1$, we need a fourth arrow θ_4 (θ_4, in its turn, must be followed *in a well-coloured path*[28] by a further arrow λ in case θ_4 belongs to $\mathbf{T_1} \backslash \mathbf{T_0}$). We have:

[27] The Lemma comes from the fact that the first step we applied to second member was a (R^i_c)-step.

[28] Of course, only well-coloured paths occur in our rewriting, so we are justified in limiting ourselves to such paths.

$$\begin{array}{ccc} & \langle \gamma^i, d^i \rangle, \langle \alpha^j, \pi_X \rangle, \beta^0, \theta, (\lambda) & \\ \swarrow (R_c^j) & & (R_p^i)^+ \searrow \\ \langle \gamma^i, d^i \rangle, \langle \alpha^j, \pi_X \rangle \circ \beta^0, \theta, (\lambda) & & \langle \gamma^i, d^i, d^i_\varepsilon \rangle, \alpha^j \times 1_{Y'}, (1_{Y_2} \times d^i_m) \circ \beta^0, \theta, (\lambda) \end{array}$$

If $\theta \in \mathbf{T_j} \backslash \mathbf{T_0}$, we compose $\langle \alpha^j, \pi_X \rangle \circ \beta^0$ with θ and then apply Lemma 8.2. If $\theta \in \mathbf{T_i} \backslash \mathbf{T_0}$, we compose $(1_{Y_2} \times d^i_m) \circ \beta^0$ with θ and the confluence immediately follows by Lemma 8.1(ii). If $\theta \in \mathbf{T_0}$, we can in any case apply one of the two previous solutions (because either i or j must be 2, hence lack of λ does not matter).

CASE 4

We have a path of four arrows $\theta_1, \theta_2, \theta_3, \theta_4$ and we apply (R_c^j) on θ_3, θ_4 and $(R_p^i)^+$ on $\theta_1, \theta_2, \theta_3$. Suppose $j = i$; that is:

$$\begin{array}{ccc} & \langle \gamma^i, d^i \rangle, \langle \alpha, \pi_X \rangle, \beta^i, \theta^i, (\lambda) & \\ \swarrow (R_c^i) & & (R_p^i)^+ \searrow \\ \langle \gamma^i, d^i \rangle, \langle \alpha, \pi_X \rangle, \beta^i \circ \theta^i, (\lambda) & & \langle \gamma^i, d^i, d^i_\varepsilon \rangle, \alpha \times 1_{Y'}, (1_{Y_2} \times d^i_m) \circ \beta^i, \theta^i, (\lambda) \end{array}$$

Then we can reduce both first member (by $(R_p^i)^+$) and second member (by $(R_c^i)^+$) to the path

$$\langle \gamma^i, d^i, d^i_\varepsilon \rangle, \alpha \times 1_{Y'}, (1_{Y_2} \times d^i_m) \circ \beta^i \circ \theta^i, (\lambda).$$

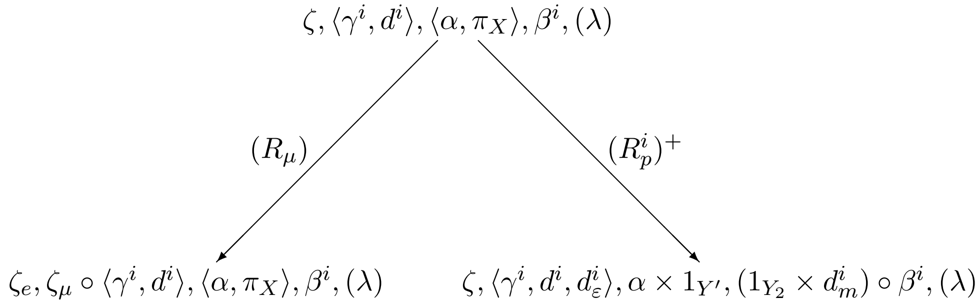
Suppose that $i \neq j$; in this case $\theta_3 \in \mathbf{T_0}$ and we have

$$\begin{array}{ccc} & \langle \gamma^i, d^i \rangle, \langle \alpha, \pi_X \rangle, \beta^0, \theta^j, (\lambda) & \\ \swarrow (R_c^j) & & (R_p^i)^+ \searrow \\ \langle \gamma^i, d^i \rangle, \langle \alpha, \pi_X \rangle, \beta^0 \circ \theta^j, (\lambda) & & \langle \gamma^i, d^i, d^i_\varepsilon \rangle, \alpha \times 1_{Y'}, (1_{Y_2} \times d^i_m) \circ \beta^0, \theta^j, (\lambda) \end{array}$$

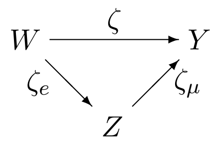
If $\alpha \in \mathbf{T_i}$, we can trivially apply (R_c^i) to the second member and then get first member by $\Leftrightarrow^*_{\mathcal{R}_0}$-steps. The relevant case is when $\alpha \in \mathbf{T_j}$: here we can rewrite first member by (R_c^j) to $\langle \gamma^i, d^i \rangle, \langle \alpha, \pi_X \rangle \circ \beta^0 \circ \theta^j, (\lambda)$ and then we apply Lemma 8.2.

8.2 Superpositions between $(R_p^i)^+$ and (R_μ)

CASE 1



where ζ factorizes as follows



By applying Lemma 7.6(iii) to first member (where $\zeta_\mu \circ \langle \gamma^i, d^i \rangle = \langle \zeta_\mu \circ \gamma^i, \zeta_\mu \circ d^i \rangle$), we obtain, through a $\searrow$-step:

$$\zeta_e, \langle \zeta_\mu \circ \gamma^i, \zeta_\mu \circ d^i, 1_Z \rangle, \alpha \times 1_Z, (1_{Y_2} \times \zeta_\mu \circ d^i) \circ \beta^i, (\lambda) \qquad (L_1)$$

Let us apply (R_μ) to the second member; we get

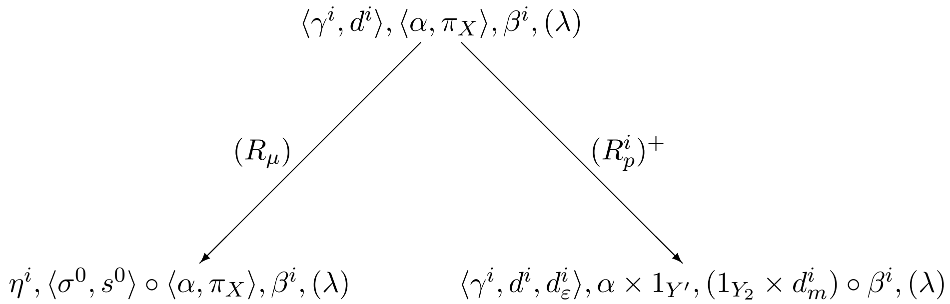
$$\zeta_e, \langle \zeta_\mu \circ \gamma^i, \zeta_\mu \circ d^i, \zeta_\mu \circ d^i_\varepsilon \rangle, \alpha \times 1_{Y'}, (1_{Y_2} \times d^i_m) \circ \beta^i, (\lambda)$$
$$=$$
$$\zeta_e, \langle \zeta_\mu \circ \gamma^i, \zeta_\mu \circ d^i, \zeta_\mu \circ d^i_\varepsilon \rangle, \langle \pi \circ \alpha, \pi_{Y'} \rangle, (1_{Y_2} \times d^i_m) \circ \beta^i, (\lambda)$$

with $Y_1 \times X \times Y' \xrightarrow{\pi} Y_1 \times X$. We can apply again Lemma 7.6(iii) and get, by a $\searrow$-step

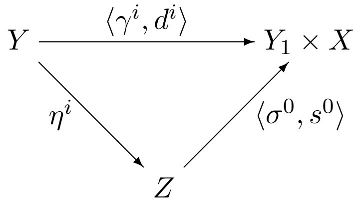
$$\zeta_e, \langle \zeta_\mu \circ \gamma^i, \zeta_\mu \circ d^i, \zeta_\mu \circ d^i_\varepsilon, 1_Z \rangle, (\pi \circ \alpha) \times 1_Z, (1_{Y_2} \times \zeta_\mu \circ d^i_\varepsilon) \circ (1_{Y_2} \times d^i_m) \circ \beta^i, (\lambda)$$
$$=$$
$$\zeta_e, \langle \zeta_\mu \circ \gamma^i, \zeta_\mu \circ d^i, \zeta_\mu \circ d^i_\varepsilon, 1_Z \rangle, (\pi \circ \alpha) \times 1_Z, (1_{Y_2} \times \zeta_\mu \circ d^i) \circ \beta^i, (\lambda)$$
$$\Leftrightarrow^*_{\mathcal{R}_0}$$
$$\zeta_e, \langle \zeta_\mu \circ \gamma^i, \zeta_\mu \circ d^i, \zeta_\mu \circ d^i_\varepsilon, 1_Z \rangle \circ (\pi \times 1_Z), \alpha \times 1_Z, (1_{Y_2} \times \zeta_\mu \circ d^i) \circ \beta^i, (\lambda)$$
$$=$$
$$\zeta_e, \langle \zeta_\mu \circ \gamma^i, \zeta_\mu \circ d^i, 1_Z \rangle, \alpha \times 1_Z, (1_{Y_2} \times \zeta_\mu \circ d^i) \circ \beta^i, (\lambda)$$

which coincides with (L_1).

<u>CASE 2</u>



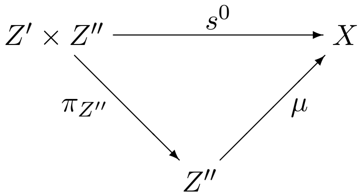
where we suppose $\langle \gamma^i, d^i \rangle$ to have the following factorization in components e/μ



We apply $(R_\mu)^+$ on the second member to the component $\langle \gamma^i, d^i \rangle$ of $\langle \gamma^i, d^i, d^i_\varepsilon \rangle$ and we obtain

$$\langle \eta^i, d^i_\varepsilon \rangle, (\langle \sigma^0, s^0 \rangle \times 1_{Y'}) \circ (\alpha \times 1_{Y'}), (1_{Y_2} \times d^i_m) \circ \beta^i, (\lambda)$$
$$=$$
$$\langle \eta^i, d^i_\varepsilon \rangle, (\langle \sigma^0, s^0 \rangle \circ \alpha) \times 1_{Y'}, (1_{Y_2} \times d^i_m) \circ \beta^i, (\lambda)$$
$$\Leftrightarrow^*_{\mathcal{R}_0}$$
$$\langle \eta^i, 1_Y \rangle, (\langle \sigma^0, s^0 \rangle \circ \alpha) \times 1_Y, (1_{Y_2} \times d^i_\varepsilon) \circ (1_{Y_2} \times d^i_m) \circ \beta^i, (\lambda)$$
$$=$$
$$\langle \eta^i, 1_Y \rangle, (\langle \sigma^0, s^0 \rangle \circ \alpha) \times 1_Y, (1_{Y_2} \times d^i) \circ \beta^i, (\lambda) \qquad (L_2)$$

We need to factorize s^0 in components ε/μ in $\mathbf{T_0}$.



where $Z' \times Z'' = Z$. This implies that η^i has the form $\langle \eta_1^i, \eta_2^i \rangle$, where $Y \xrightarrow{\eta_1^i} Z'$ and $Y \xrightarrow{\eta_2^i} Z''$. By applying $(R_\mu)^+$ on the first member to the arrow $\langle \sigma^0, s^0 \rangle \circ \langle \alpha, \pi_X \rangle = \langle \langle \sigma^0, s^0 \rangle \circ \alpha, s^0 \rangle$, in order to decompose s^0, we obtain:

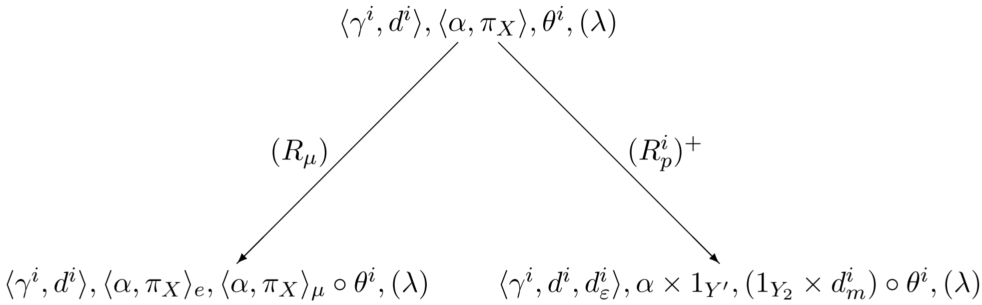
$$\langle \eta_1^i, \eta_2^i \rangle,\ \langle \langle \sigma^0, s^0 \rangle \circ \alpha, \pi_{Z''} \rangle,\ (1_{Y_2} \times \mu) \circ \beta^i,\ (\lambda)$$

which, by Lemma 7.6(iii), becomes (through a $\searrow$-step)

$$\langle \eta_1^i, \eta_2^i, 1_Y \rangle,\ (\langle \sigma^0, s^0 \rangle \circ \alpha) \times 1_Y,\ (1_{Y_2} \times \eta_2^i) \circ (1_{Y_2} \times \mu) \circ \beta^i,\ (\lambda)$$
$$=$$
$$\langle \eta^i, 1_Y \rangle,\ (\langle \sigma^0, s^0 \rangle \circ \alpha) \times 1_Y,\ (1_{Y_2} \times \eta_2^i \circ \mu) \circ \beta^i,\ (\lambda) \qquad (L_1)$$

Since $\eta_2^i \circ \mu = \eta^i \circ \pi_{Z''} \circ \mu = \eta^i \circ s^0 = d^i$, we can conclude that (L_1) coincides with (L_2).

CASE 3



Confluence is an immediate application of Lemma 8.1(i) (taking as β^0 the identity).

CASE 4

$$\langle\gamma^i,d^i\rangle,\langle\alpha,\pi_X\rangle,\beta^i,\theta \xrightarrow{(R_\mu)} \langle\gamma^i,d^i\rangle,\langle\alpha,\pi_X\rangle,\beta^i_e,\beta^i_\mu\circ\theta$$

$$\langle\gamma^i,d^i\rangle,\langle\alpha,\pi_X\rangle,\beta^i,\theta \xrightarrow{(R^i_p)^+} \langle\gamma^i,d^i,d^i_\varepsilon\rangle,\alpha\times 1_{Y'},(1_{Y_2}\times d^i_m)\circ\beta^i,\theta$$

It suffices to apply $(R^i_p)^+$ to first member and the confluence immediately follows by Lemma 6.1 (with $\sigma^0=\beta^i_\mu$).

8.3 Superpositions between $(R^i_p)^+$ and (R_ε)

CASE 1

$$\zeta,\langle\gamma^i,d^i\rangle,\langle\alpha,\pi_X\rangle,\beta^i,(\lambda) \xrightarrow{(R_\varepsilon)} \zeta\circ\varepsilon,\langle\bar{\gamma}^i,\bar{d}^i\rangle,\langle\alpha,\pi_X\rangle,\beta^i,(\lambda)$$

$$\zeta,\langle\gamma^i,d^i\rangle,\langle\alpha,\pi_X\rangle,\beta^i,(\lambda) \xrightarrow{(R^i_p)^+} \zeta,\langle\gamma^i,d^i,d^i_\varepsilon\rangle,\alpha\times 1_{Y'},(1_{Y_2}\times d^i_m)\circ\beta^i,(\lambda)$$

where we suppose $\langle\gamma^i,d^i\rangle$ to have the following factorization in ε/m-components

$$Y \xrightarrow{\langle\gamma^i,d^i\rangle} Y_1\times X,\qquad Y\xrightarrow{\varepsilon}\tilde{Y}\xrightarrow{\langle\bar{\gamma}^i,\bar{d}^i\rangle} Y_1\times X$$

Let us suppose that $\bar{d}^i$ has the following ε/m-factorization

$$\tilde{Y}\xrightarrow{\bar{d}^i} X,\qquad \tilde{Y}\xrightarrow{\bar{d}^i_\varepsilon}\tilde{Y}'\xrightarrow{\bar{d}^i_m} X$$

Then, by the uniqueness of decompositions, since $\varepsilon \circ \bar{d}^i_\varepsilon \circ \bar{d}^i_m = d^i$, we have:

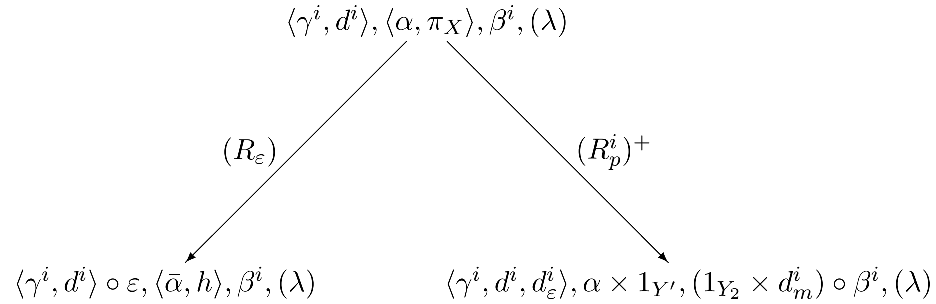
$$\varepsilon \circ \bar{d}^i_\varepsilon = d^i_\varepsilon \qquad \bar{d}^i_m = d^i_m \qquad \tilde{Y}' = Y'$$

We apply $(R^i_p)^+$ to the first member and we obtain:

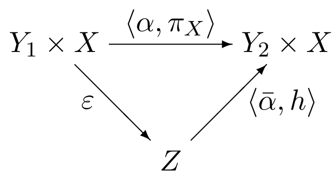
$$\zeta \circ \varepsilon, \langle \bar{\gamma}^i, \bar{d}^i, \bar{d}^i_\varepsilon \rangle, \alpha \times 1_{Y'}, (1_{Y_2} \times d^i_m) \circ \beta^i, (\lambda)$$
$$\Leftrightarrow^*_{\mathcal{R}_0}$$
$$\zeta, \varepsilon \circ \langle \bar{\gamma}^i, \bar{d}^i, \bar{d}^i_\varepsilon \rangle, \alpha \times 1_{Y'}, (1_{Y_2} \times d^i_m) \circ \beta^i, (\lambda)$$
$$=$$
$$\zeta, \langle \varepsilon \circ \bar{\gamma}^i, \varepsilon \circ \bar{d}^i, \varepsilon \circ \bar{d}^i_\varepsilon \rangle, \alpha \times 1_{Y'}, (1_{Y_2} \times d^i_m) \circ \beta^i, (\lambda)$$
$$=$$
$$\zeta, \langle \gamma^i, d^i, d^i_\varepsilon \rangle, \alpha \times 1_{Y'}, (1_{Y_2} \times d^i_m) \circ \beta^i, (\lambda)$$

which coincides with the second member.

<u>CASE 2</u>



where we have factorized $\langle \alpha, \pi_X \rangle$ in components ε/m as follows



On the other hand, let $h = h_\varepsilon \circ h_m$. Since $\varepsilon \circ h_\varepsilon \circ h_m = \pi_X$, by uniqueness of factorizations, h_m must coincide with 1_X. Therefore $h = h_\varepsilon$ is the projection[29] on X, hence (up to renaming) we have:

$$\varepsilon = \pi_{Y'} \times 1_X \qquad Z = Y'_1 \times X$$

[29] As $\varepsilon \circ h_\varepsilon = \pi_X$, we have that h_ε composed on the left with a projection is π_X: it follows that h_ε itself must be the projection into X (with domain Z).

Thus, first member coincides with

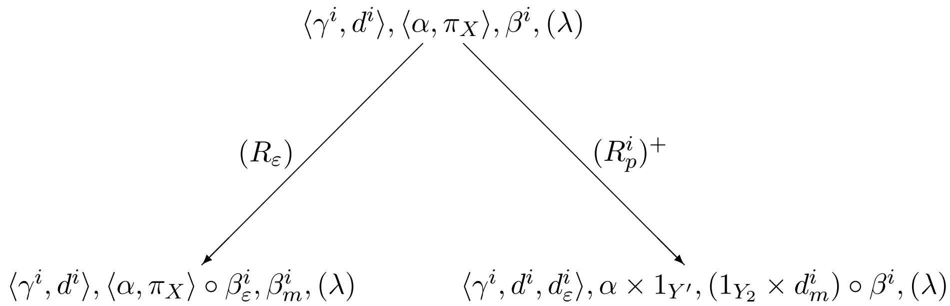
$$\langle \gamma^i \circ \pi_{Y'}, d^i \rangle, \langle \bar{\alpha}, \pi_X \rangle, \beta^i, (\lambda)$$

which, by $(R_p^i)^+$, becomes

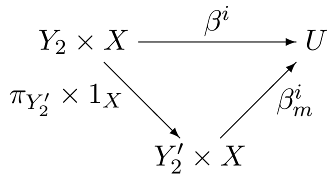
$$\langle \gamma^i \circ \pi_{Y'}, d^i, d_\varepsilon^i \rangle, \bar{\alpha} \times 1_{Y'}, (1_{Y_2} \times d_m^i) \circ \beta^i, (\lambda)$$
$$=$$
$$\langle \gamma^i, d^i, d_\varepsilon^i \rangle \circ (\varepsilon \times 1_{Y'}), \bar{\alpha} \times 1_{Y'}, (1_{Y_2} \times d_m^i) \circ \beta^i, (\lambda)$$
$$\Leftrightarrow^*_{\mathcal{R}_0}$$
$$\langle \gamma^i, d^i, d_\varepsilon^i \rangle, (\varepsilon \times 1_{Y'}) \circ (\bar{\alpha} \times 1_{Y'}), (1_{Y_2} \times d_m^i) \circ \beta^i, (\lambda)$$
$$=$$
$$\langle \gamma^i, d^i, d_\varepsilon^i \rangle, \varepsilon \circ \bar{\alpha} \times 1_{Y'}, (1_{Y_2} \times d_m^i) \circ \beta^i, (\lambda)$$

and, since $\varepsilon \circ \bar{\alpha} = \alpha$, the last path coincides with the second member.

<u>CASE 3</u>



We have to factorize β^i. Suppose that 'X belongs to the codomain of β_ε^i', that is



Then the first member coincides with

$$\langle \gamma^i, d^i \rangle, \langle \alpha \circ \pi_{Y_2'}, \pi_X \rangle, \beta_m^i, (\lambda)$$

which is rewritten by $(R_p^i)^+$ as

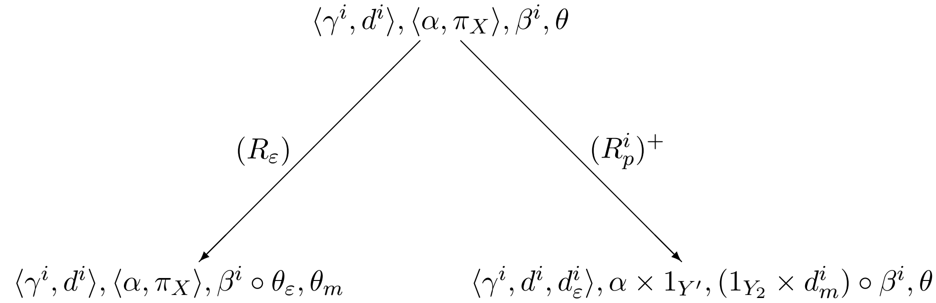
$$\langle\gamma^i, d^i, d^i_\varepsilon\rangle, (\alpha \circ \pi_{Y_2'}) \times 1_{Y'}, (1_{Y_2'} \times d^i_m) \circ \beta^i_m, (\lambda)$$
$$\Leftrightarrow^*_{\mathcal{R}_0}$$
$$\langle\gamma^i, d^i, d^i_\varepsilon\rangle, \alpha \times 1_{Y'}, (\pi_{Y_2'} \times 1_{Y'}) \circ (1_{Y_2'} \times d^i_m) \circ \beta^i_m, (\lambda)$$
$$=$$
$$\langle\gamma^i, d^i, d^i_\varepsilon\rangle, \alpha \times 1_{Y'}, (1_{Y_2} \times d^i_m) \circ (\pi_{Y_2'} \times 1_X) \circ \beta^i_m, (\lambda)$$
$$=$$
$$\langle\gamma^i, d^i, d^i_\varepsilon\rangle, \alpha \times 1_{Y'}, (1_{Y_2} \times d^i_m) \circ \beta^i, (\lambda)$$

which coincides with the second member.
If 'X does not belong to the codomain of β^i_ε', that is $\beta^i_\varepsilon = \pi_{Y_2} \circ \pi_{Y_2'}$, then the second member coincides with

$$\langle\gamma^i, d^i, d^i_\varepsilon\rangle, \alpha \times 1_{Y'}, (1_{Y_2} \times d^i_m) \circ \pi_{Y_2} \circ \pi_{Y_2'} \circ \beta^i_m, (\lambda)$$
$$=$$
$$\langle\gamma^i, d^i, d^i_\varepsilon\rangle, \alpha \times 1_{Y'}, \pi_{Y_2} \circ \pi_{Y_2'} \circ \beta^i_m, (\lambda)$$
$$\Leftrightarrow^*_{\mathcal{R}_0}$$
$$\langle\gamma^i, d^i, d^i_\varepsilon\rangle, (\alpha \times 1_{Y'}) \circ \pi_{Y_2} \circ \pi_{Y_2'}, \beta^i_m, (\lambda)$$
$$=$$
$$\langle\gamma^i, d^i, d^i_\varepsilon\rangle, \langle\pi_{Y_1}, \pi_X\rangle \circ \alpha \circ \pi_{Y_2'}, \beta^i_m, (\lambda)$$
$$\Leftrightarrow^*_{\mathcal{R}_0}$$
$$\langle\gamma^i, d^i, d^i_\varepsilon\rangle \circ \langle\pi_{Y_1}, \pi_X\rangle, \alpha \circ \pi_{Y_2'}, \beta^i_m, (\lambda)$$
$$=$$
$$\langle\gamma^i, d^i\rangle, \alpha \circ \pi_{Y_2'}, \beta^i_m, (\lambda)$$

which, by the fact that $\alpha \circ \pi_{Y_2'}$ is the same as $\langle\alpha, \pi_X\rangle \circ \pi_{Y_2} \circ \pi_{Y_2'}$, coincides with the first member.

<u>CASE 4</u>



It suffices to apply $(R_p^i)^+$ to first member and the confluence immediately follows by Lemma 6.1 (with $\sigma^0 = \theta_\varepsilon$).

8.4 Superpositions between $(R_p^i)^+$ and $(R_p^j)^+$

CASE 1

Here we have three arrows $\theta_1, \theta_2, \theta_3$ and we apply both rules to the whole path; the case $i \neq j$ is trivial (it implies that θ_1 and θ_3 belongs to $\mathbf{T_0}$, so that everything compose). Let us suppose $i = j$. Arrow θ_2 (up to an alphabetic variant) must be of the form $\langle \alpha, \pi_X^1, \pi_X^2 \rangle$. We have in principle two cases (to be treated in a very similar way) depending on whether π_X^1 and π_X^2 are the same projection or not.

SUBCASE 1.1

$$\begin{array}{ccc} & \langle \gamma^i, d^i \rangle, \langle \alpha, \pi_X, \pi_X \rangle, \beta^i, (\lambda) & \\ {\scriptstyle (R_p^i)^+} \swarrow & & \searrow {\scriptstyle (R_p^i)^+} \\ K_1 & & K_2 \end{array}$$

where

$$Y \xrightarrow{\langle \gamma^i, d^i \rangle} Y_1 \times X \xrightarrow{\langle \alpha, \pi_X, \pi_X \rangle} Y_2 \times X \times X \xrightarrow{\beta^i} V$$

and $Y \xrightarrow{d_\varepsilon^i} Y' \xrightarrow{d_m^i} X$ corresponds to the factorization ε/m of d^i. The two members are:

$$\begin{aligned} K_1 &= \langle \gamma^i, d^i, d_\varepsilon^i \rangle \ , \ \langle \alpha, \pi_X \rangle \times 1_{Y'} \ , \ (1_{Y_2} \times 1_X \times d_m^i) \circ \beta^i, (\lambda) \\ &= \langle \gamma^i, d^i, d_\varepsilon^i \rangle \ , \ \langle \langle \pi_{Y_1}, \pi_X \rangle \circ \alpha, \pi_X, \pi_{Y'} \rangle \ , \ (1_{Y_2} \times 1_X \times d_m^i) \circ \beta^i, (\lambda) \\ K_2 &= \langle \gamma^i, d^i, d_\varepsilon^i \rangle \ , \ \langle \langle \pi_{Y_1}, \pi_X \rangle \circ \alpha, \pi_{Y'}, \pi_X \rangle \ , \ (1_{Y_2} \times d_m^i \times 1_X) \circ \beta^i, (\lambda) \end{aligned}$$

By applying $(R_p^i)^+$ on both members with respect to π_X, we get the path

$$\langle \gamma^i, d^i, d_\varepsilon^i, d_\varepsilon^i \rangle \ , \ \langle \langle \pi_{Y_1}, \pi_X \rangle \circ \alpha, \pi_{Y'}^1, \pi_{Y'}^2 \rangle \ , \ (1_{Y_2} \times d_m^i \times d_m^i) \circ \beta^i, (\lambda)$$

where $\pi_{Y'}^1$ and $\pi_{Y'}^2$ project on the first and on the second Y' respectively.

SUBCASE 1.2

$$\begin{array}{ccc} & \langle \gamma^i, d^i, c^i \rangle, \langle \alpha, \pi^1_X, \pi^2_X \rangle, \beta^i, (\lambda) & \\ {\scriptstyle (R^i_p)^+} \swarrow & & \searrow {\scriptstyle (R^i_p)^+} \\ K_1 & & K_2 \end{array}$$

where

$$Y \xrightarrow{\langle \gamma^i, d^i, c^i \rangle} Y_1 \times X \times X \xrightarrow{\langle \alpha, \pi^1_X, \pi^2_X \rangle} Y_2 \times X \times X \xrightarrow{\beta^i} V$$

and $Y \xrightarrow{d^i_\varepsilon} Y' \xrightarrow{d^i_m} X$, $Y \xrightarrow{c^i_\varepsilon} Y'' \xrightarrow{c^i_m} X$ correspond to the factorization ε/m of d^i, c^i. The two members are:

$$\begin{aligned} K_1 &= \langle \gamma^i, d^i, c^i, c^i_\varepsilon \rangle \ , \ \langle \pi \circ \alpha, \pi^1_X, \pi_{Y''} \rangle \ , \ (1_{Y_2} \times 1_X \times c^i_m) \circ \beta^i, (\lambda) \\ K_2 &= \langle \gamma^i, d^i, d^i_\varepsilon, c^i \rangle \ , \ \langle \pi \circ \alpha, \pi_{Y'}, \pi^2_X \rangle \ , \ (1_{Y_2} \times d^i_m \times 1_X) \circ \beta^i, (\lambda) \end{aligned}$$

where π denotes in both cases the projection from the corresponding domains onto $Y_1 \times X \times X$. By applying $(R^i_p)^+$ on both members with respect to the suitable projection on X, we get the same path, namely

$$\langle \gamma^i, d^i, d^i_\varepsilon, c^i, c^i_\varepsilon \rangle \ , \ \langle \pi \circ \alpha, \pi_{Y'}, \pi_{Y''} \rangle \ , \ (1_{Y_2} \times d^i_m \times c^i_m) \circ \beta^i, (\lambda)$$

<u>CASE 2</u>

Here we have a four-arrows path, $(R^i_p)^+$ is applied to the first three arrows and $(R^j_p)^+$ to the last three. We have

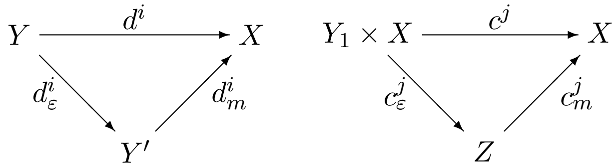
$$\begin{array}{ccc} & \langle \gamma^i, d^i \rangle, \langle \alpha^j, c^j, \pi_X \rangle, \langle \beta^i, \pi^1_X \rangle, \theta^j, (\lambda) & \\ {\scriptstyle (R^i_p)^+} \swarrow & & \searrow {\scriptstyle (R^j_p)^+} \\ K_1 & & K_2 \end{array}$$

where

$$Y \xrightarrow{\langle \gamma^i, d^i \rangle} Y_1 \times X \xrightarrow{\langle \alpha^j, c^j, \pi_X \rangle} Y_2 \times X \times X \xrightarrow{\langle \beta^i, \pi^1_X \rangle} U \times X \xrightarrow{\theta^j} V$$

and $Y_2 \times X \times X \xrightarrow{\pi^1_X} X$ is the projection on the first X.[30] We also assume that d^i and c^j have the following factorizations:

[30] It cannot be the projection on second X, otherwise the proviso for rule $(R^j_p)^+$ is violated.



It follows that

$$\begin{aligned} K_1 = &\quad \langle \gamma^i, d^i, d^i_\varepsilon \rangle \ , \ \langle \alpha^j, c^j \rangle \times 1_{Y'} \ , \ (1_{Y_2} \times 1_X \times d^i_m) \circ \langle \beta^i, \pi^1_X \rangle, \ \theta^j, (\lambda) \\ K_2 = &\quad \langle \gamma^i, d^i \rangle, \langle \alpha^j, c^j, \pi_X, c^j_\varepsilon \rangle \ , \ \beta^i \times 1_Z \ , \ (1_U \times c^j_m) \circ \theta^j, (\lambda) \end{aligned}$$

First member can be written as follows

$$\langle \gamma^i, d^i, d^i_\varepsilon \rangle \ , \ \langle \langle \pi_{Y_1}, \pi_X \rangle \circ \alpha^j, \langle \pi_{Y_1}, \pi_X \rangle \circ c^j, \pi_{Y'} \rangle \ , \ \langle (1_{Y_2} \times 1_X \times d^i_m) \circ \beta^i, \pi_X \rangle \ , \ \theta^j \ , \ (\lambda)$$

We can apply $(R^j_p)^+$ with respect to the projection π_X (we point out that X is the codomain of $\langle \pi_{Y_1}, \pi_X \rangle \circ c^j$, which factorizes, in ε/m components, as $(\langle \pi_{Y_1}, \pi_X \rangle \circ c^j_\varepsilon) \circ c^j_m$). This yields to

$$\begin{aligned} &\langle \gamma^i, d^i, d^i_\varepsilon \rangle \ , \ \langle \langle \pi_{Y_1}, \pi_X \rangle \circ \alpha^j, \langle \pi_{Y_1}, \pi_X \rangle \circ c^j, \pi_{Y'}, \langle \pi_{Y_1}, \pi_X \rangle \circ c^j_\varepsilon \rangle \ , \\ &((1_{Y_2} \times 1_X \times d^i_m) \circ \beta^i) \times 1_Z \ , \ (1_U \times c^j_m) \circ \theta^j, (\lambda) \end{aligned} \qquad (L_1)$$

By applying $(R^i_p)^+$ to second member with respect to the projection π_X, we get

$$\begin{aligned} &\langle \gamma^i, d^i, d^i_\varepsilon \rangle \ , \ \langle \langle \pi_{Y_1}, \pi_X \rangle \circ \alpha^j, \langle \pi_{Y_1}, \pi_X \rangle \circ c^j, \pi_{Y'}, \langle \pi_{Y_1}, \pi_X \rangle \circ c^j_\varepsilon \rangle \ , \\ &(1_{Y_2} \times 1_X \times d^i_m \times 1_Z) \circ (\beta_i \times 1_Z) \ , \ (1_U \times c^j_m) \circ \theta^j, (\lambda) \qquad (L_2) \end{aligned}$$

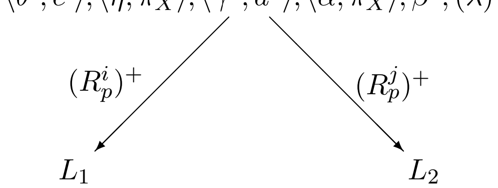
Since $((1_{Y_2} \times 1_X \times d^i_m) \circ \beta^i) \times 1_Z$ coincides with $(1_{Y_2} \times 1_X \times d^i_m \times 1_Z) \circ (\beta_i \times 1_Z)$, paths L_1 and L_2 coincide.

<u>CASE 3</u>

Here we have a five-arrows path, $(R^i_p)^+$ is applied to the first three components and $(R^j_p)^+$ to the last three components. We distinguish the subcases $i \neq j$ and $i = j$.

<u>SUBCASE 3.1</u>

The third arrow must belong to $\mathbf{T_0}$, hence we have



where

$$W \xrightarrow{\langle\theta^i, c^i\rangle} W_1 \times X \xrightarrow{\langle\eta, \pi_X\rangle} W_2 \times X \xrightarrow{\langle\gamma^0, d^0\rangle} Y_1 \times X \xrightarrow{\langle\alpha, \pi_X\rangle} Y_2 \times X \xrightarrow{\beta^j} U$$

Therefore L_1 and L_2 are as follows (let W', Y' be the codomains of $c^i_\varepsilon, d^0_\varepsilon$):

$$\begin{aligned} L_1 &= \langle\theta^i, c^i, c^i_\varepsilon\rangle\ ,\ \eta \times 1_{W'}\ ,\ (1_{W_2} \times c^i_m) \circ \langle\gamma^0, d^0\rangle, \langle\alpha, \pi_X\rangle, \beta^j, (\lambda) \\ L_2 &= \langle\theta^i, c^i\rangle, \langle\eta, \pi_X\rangle, \langle\gamma^0, d^0, d^0_\varepsilon\rangle\ ,\ \alpha \times 1_{Y'}\ ,\ (1_{Y_2} \times d^0_m) \circ \beta^j, (\lambda) \end{aligned}$$

Applying $(R^i_p)^+$ to L_2, one gets

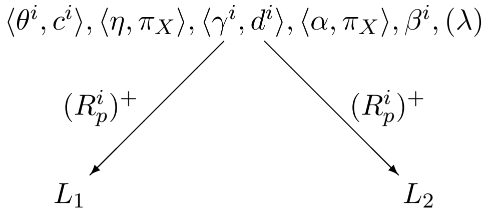
$$\langle\theta^i, c^i, c^i_\varepsilon\rangle\ ,\ \eta \times 1_{W'}\ ,\ (1_{W_2} \times c^i_m) \circ \langle\gamma^0, d^0, d^0_\varepsilon\rangle\ ,\ \alpha \times 1_{Y'}\ ,\ (1_{Y_2} \times d^0_m) \circ \beta^j, (\lambda)$$

By an $\Leftrightarrow^*_{\mathcal{R}_0}$-step, we get

$$\langle\theta^i, c^i, c^i_\varepsilon\rangle\ ,\ \eta \times 1_{W'}\ ,\ (1_{W_2} \times c^i_m), \langle\gamma^0, d^0, d^0_\varepsilon\rangle\ ,\ \alpha \times 1_{Y'}\ ,\ (1_{Y_2} \times d^0_m) \circ \beta^j, (\lambda)$$

which is $\Leftrightarrow^*_{\mathcal{R}_0}$-equivalent to L_1 by Lemma 7.6(ii).

SUBCASE 3.2



where

$$W \xrightarrow{\langle\theta^i, c^i\rangle} W_1 \times X \xrightarrow{\langle\eta, \pi_X\rangle} W_2 \times X \xrightarrow{\langle\gamma^i, d^i\rangle} Y_1 \times X \xrightarrow{\langle\alpha, \pi_X\rangle} Y_2 \times X \xrightarrow{\beta^i} U$$

Therefore L_1 and L_2 are as follows (let W', Y' be the codomains of $c^i_\varepsilon, d^i_\varepsilon$):

$$\begin{aligned} L_1 &= \langle\theta^i, c^i, c^i_\varepsilon\rangle\ ,\ \eta \times 1_{W'}\ ,\ (1_{W_2} \times c^i_m) \circ \langle\gamma^i, d^i\rangle, \langle\alpha, \pi_X\rangle, \beta^i, (\lambda) \\ L_2 &= \langle\theta^i, c^i\rangle, \langle\eta, \pi_X\rangle, \langle\gamma^i, d^i, d^i_\varepsilon\rangle\ ,\ \alpha \times 1_{Y'}\ ,\ (1_{Y_2} \times d^i_m) \circ \beta^i, (\lambda) \end{aligned}$$

Applying $(R^i_p)^+$, L_2 rewrites to

$$\langle\theta^i, c^i, c^i_\varepsilon\rangle\ ,\ \eta \times 1_{W'}\ ,\ (1_{W_2} \times c^i_m) \circ \langle\gamma^i, d^i, d^i_\varepsilon\rangle\ ,\ \alpha \times 1_{Y'}\ ,\ (1_{Y_2} \times d^i_m) \circ \beta^i, (\lambda)$$

which is confluent with L_1 by Lemma 8.3 (take ζ^i to be $1_{W_2} \times c^i_m$)).

We so completed the proof of the following:

Theorem 8.4 $\mathcal{R}^+$ *is locally confluent.*

9 Termination

In order to show termination of $\mathcal{R}$ and of $\mathcal{R}^+$, we shall associate with our paths certain commutative labelled trees. Such trees are represented as terms built up from the countable set of variables $\{x_i\}_{i\geq 1}$ by using four[31] constructors f_i ($i \in \{0,1\}^2$) of type $TermMultiset \longrightarrow Term$.

$\mathcal{R}$-*trees* (or, briefly, trees) are inductively defined as follows:

- x_i is an $\mathcal{R}$-tree for every $i \geq 1$;
- if $\{T_1, \ldots, T_n\}$ is a multiset of $\mathcal{R}$-trees and $i \in \{0,1\}^2$, then $f_i(T_1, \ldots, T_n)$ is an $\mathcal{R}$-tree.

As a next step, we introduce a relation $>$ among our trees; we have $T_1 > T_2$ iff one of the following two conditions is satisfied:

- T_1 is $f_i(T'_1, \ldots, T'_n)$ and T_2 is $f_j(T''_1, \ldots, T''_k)$ and $\{T'_1, \ldots, T'_n\} >_m \{T''_1, \ldots, T''_k\}$;
- T_1 is $f_i(T'_1, \ldots, T'_n)$ and T_2 is $f_j(T'_1, \ldots, T'_n)$ and $i > j$ (in the lexicographic sense).

Some comments are in order. First $>_m$ is the multiset extension of $>$; secondly the definition is by induction on the height $h(T_1)$ of the tree T_1. It is easily seen that $T_1 > T_2$ implies $h(T_1) \geq h(T_2)$.[32] In the following, we use $\geq$ for the reflexive closure of $>$.

We have the following easy

Lemma 9.1 *$>$ is a transitive and terminating relation.*

Proof. For transitivity, let us show that

$$T_1 > T_2 > T_3 \quad \text{implies} \quad T_1 > T_3$$

by induction on $h(T_1) + h(T_2) + h(T_3)$. We have two cases:

(i) suppose that $T_1 > T_2$ holds by the first clause, so that T_1 is $f_i(T'_1, \ldots, T'_n)$, T_2 is $f_j(T''_1, \ldots, T''_k)$ and $\{T'_1, \ldots, T'_n\} >_m \{T''_1, \ldots, T''_k\}$; $T_1 > T_3$ follows from the fact that $>_m$ is transitive (as $>$ is transitive on lower height trees by induction hypothesis);

(ii) suppose that $T_1 > T_2$ holds by the second clause, so that T_1 is $f_i(T'_1, \ldots, T'_n)$, T_2 is $f_j(T'_1, \ldots, T'_n)$ and $i > j$; if $T_2 > T_3$ holds by the first clause, then $T_1 > T_3$ holds by the same clause, if it holds by the second clause, then $T_1 > T_3$ holds by transitivity of lexicographic orders.

[31] Actually only three such constructors will be really used ($f_{\langle 0,1\rangle}$ is useless).

[32] $h(T)$ is obviously defined as follows: variables have height 1, $f_i(T_1, \ldots, T_n)$ has height $1 + max(h(T_1), \ldots, h(T_n))$.

For termination, suppose we have a chain

$$T_1 > T_2 > \cdots T_i > \cdots$$

We show that this cannot be by induction on $h(T_1)$. As $>_m$ is terminating (by inductive hypothesis on $>$), first clause cannot be used infinitely many times; so starting from a certain T_k on, only the second clause applies, which is absurd as such clause can be consecutively applied only at most 3 times. ⊣

As our trees are represented as terms, it makes sense to speak about substitutions. Substitutions are compatible with $>$ in the following sense:

Lemma 9.2 *Let a succession $\{T_i\}_{i\geq 1}$ of trees be given and let T', T'' be such that $T' > T''$; we then have $T'(T_i/x_i) > T''(T_i/x_i)$.*

Proof. Immediate. ⊣

Let us now turn to our paths. First, we need a definition. For an arrow α^i, let us put

$$e(\alpha^i) = \begin{cases} 0 & \text{if } \alpha^i \in \mathcal{E}_0 \\ 1 & \text{otherwise} \end{cases} \qquad m(\alpha^i) = \begin{cases} 0 & \text{if } \alpha^i \in \mathcal{E}_i \\ 1 & \text{otherwise} \end{cases}$$

$$\chi(\alpha^i) = \langle m(\alpha^i), e(a^i) \rangle.$$

Lemma 9.3 *For every arrow α and for every $\varepsilon \in \mathcal{E}_0$, we have $\chi(\varepsilon \circ \alpha) = \chi(\alpha)$ (whenever composition makes sense).*

Proof. If $e(\alpha) = 0$ then clearly $e(\varepsilon \circ \alpha) = 0$ too; vice versa, if $e(\varepsilon \circ \alpha) = 0$, then the two ε/m factorizations $(\varepsilon \circ \alpha) \circ 1 = (\varepsilon \circ \alpha_\varepsilon) \circ \alpha_m$ of $\varepsilon \circ \alpha$ must be equal, so that α_m is the identity; hence $\alpha = \alpha_\varepsilon$, that is $\alpha \in \mathcal{E}_0$. The proof of $m(\alpha) = 0$ iff $m(\varepsilon \circ \alpha) = 0$ is similar. ⊣

For a path $K : Y \longrightarrow Z$ and for $\beta^0 : Z \longrightarrow V$, let $K \circ \beta^0$ be the path obtained by composing the last arrow of K with β^0 (that is, if $K = K', \alpha$, then $K \circ \beta^0$ is $K', \alpha \circ \beta^0$).

With a path $K : X^n \longrightarrow X$ (resp. $L : X^n \longrightarrow X^m$), we now associate an $\mathcal{R}$-tree $T(K)$ (resp. a multiset of $\mathcal{R}$-trees $T(L)$) as follows (definition is by induction on the lengths $|K|$, $|L|$ of K and L):

$$\begin{aligned} T(a) &= f_{\chi(a)}(x_{i_1}, \ldots, x_{i_k}), \quad \text{if } a_\varepsilon = \langle \pi_{i_1}, \ldots, \pi_{i_k} \rangle; \\ T(\langle a_1, \ldots, a_m \rangle) &= \{T(a_1), \ldots, T(a_m)\}; \\ T(K', a) &= f_{\chi(a)}(T(K' \circ a_\varepsilon)); \\ T(L', \langle a_1, \ldots, a_m \rangle) &= \{T(L', a_1), \ldots, T(L', a_m)\}. \end{aligned}$$

Lemma 9.4 *Let $L : Y \longrightarrow X^n$ and $K : X^n \longrightarrow X^m$. We have that*

$$T(L, K) \;=\; T(K)(T(L_1)/x_1, \ldots, T(L_n)/x_n),$$

where $L_1 = L \circ \pi_1, \ldots, L_n = L \circ \pi_n$.

Proof. The claim is shown by induction on the length $|K|$ of K. If length is 1, then K is just $\alpha = \langle a_1, \ldots, a_m\rangle$; if $(a_j)_\varepsilon$ is $\langle \pi_{i_{j(1)}}, \ldots, \pi_{i_{j(k_j)}}\rangle$, we have

$$\begin{aligned} T(L, \alpha) &= \{f_{\chi(a_j)}(T(L_{i_{j(1)}}), \ldots, T(L_{i_{j(k_j)}}))\}_{j=1,\ldots,m} \\ &= T(\alpha)(T(L_1)/x_1, \ldots, T(L_n)/x_n). \end{aligned}$$

If length is greater than 1, then K is K', α (for $\alpha = \langle a_1, \ldots, a_m\rangle$), so that

$$\begin{aligned} T(L, K', \alpha) &= \{f_{\chi(a_j)}(T(L, K' \circ (a_j)_\varepsilon)\}_{j=1,\ldots,m} \\ &= \{f_{\chi(a_j)}(T(K' \circ (a_j)_\varepsilon)(T(L_i)/x_i))\}_{j=1,\ldots,m} \end{aligned}$$

by inductive hypothesis; on the other hand

$$\begin{aligned} T(K', \alpha)(T(L_i)/x_i) &= \{f_{\chi(a_j)}(T(K' \circ (a_j)_\varepsilon)\}_{j=1,\ldots,m}(T(L_i)/x_i) \\ &= \{f_{\chi(a_j)}(T(K' \circ (a_j)_\varepsilon)(T(L_i)/x_i)\}_{j=1,\ldots,m} \end{aligned}$$

and the two members are equal by the inductive definition of substitution. ⊣

Lemma 9.5 *Let $K' = \rho(K)$ for a list of renamings whose first component is identity;*[33] *we have $T(K) = T(K')$.*

Proof. We first collect some easy facts. Fix any path $L : Y \longrightarrow Z$ and a renaming $\rho : Z \longrightarrow Z$. We have:

(i) $T(L) = T(L \circ \rho)$;

(ii) for every $\alpha = \langle a_1, \ldots, a_n\rangle : Z \longrightarrow X^n$ and for every $i = 1, \ldots, n$, $T(L, \rho \circ a_i) = T(L \circ \rho, a_i)$: in fact,

$$T(L, \rho \circ a_i) = f_{\chi(\rho \circ a_i)}(T(L \circ (\rho \circ a_i)_\varepsilon) = f_{\chi(a_i)}(T(L \circ \rho \circ (a_i)_\varepsilon) = T(L \circ \rho, a_i)$$

by uniqueness of factorization and Lemma 9.3;

(iii) for every L', $T(L, \rho \circ \alpha, L') = T(L \circ \rho, \alpha, L')$, by (ii) and Lemma 9.4.

Now let $K = \alpha_1, \ldots, \alpha_k$, $K' = \alpha'_1, \ldots, \alpha'_k$ and let $\rho = \{1 = \rho_0, \rho_1, \ldots, \rho_k\}$ (recall we have $\rho_{i-1} \circ \alpha'_i = \alpha_i \circ \rho_i$ for all i). We have

$$T(K) = T(K \circ \rho_k) = T(\alpha_1, \ldots, \rho_{k-1} \circ \alpha'_k) = T(\alpha_1, \ldots, \alpha_{k-1} \circ \rho_{k-1}, \alpha'_k) = \cdots = T(K')$$

as wanted. ⊣

[33] See Section 4 for these concepts.

Notice that the above Lemma yields in particular that $T(K) = T(K')$ in case K' is an alphabetic variant of K: this is important, as we rewrite on equivalence classes of paths modulo alphabetic variants. Moreover the above Lemma (which will be tacitly used many times during the termination proof) yields the possibility of replacing K with any $\rho(K)$ (where ρ has identity as first component) when computing $T(K)$: this allows moving certain arrows to last position in a tuple of arrows, assuming that certain projections located in an internal position of a path project on last components, etc (see the Examples of Section 4).

Lemma 9.6 *Let $\delta = \langle d_1, \ldots, d_n \rangle : X^m \longrightarrow X^n$ be an arrow which is not in $\mathcal{E}_0$ (i.e. it is not a projection); suppose that $\delta_\varepsilon = \langle \pi_{i_1}, \ldots, \pi_{i_k} \rangle : X^m \longrightarrow X^k$. We have that $T(\delta, 1_{X^n}) > T(\delta_\varepsilon, 1_{X^k})$.*

Proof. We have

$$T(\delta_\varepsilon, 1_{X^k}) = \{f_{\langle 0,0 \rangle}(f_{\langle 0,0 \rangle}(x_s))\}_{s=i_1,\ldots,i_k}$$

and

$$T(\delta, 1_{X^n}) = \{f_{\langle 0,0 \rangle}(f_{\chi(d_j)}(x_{i_{j(1)}}, \ldots, x_{i_{j(l_j)}}))\}_{j=1,\ldots,n},$$

where we supposed that $(d_j)_\varepsilon = \langle \pi_{i_{j(1)}}, \ldots, \pi_{i_{j(l_j)}} \rangle$. Now elements of the former multiset are all distinct and for every $s = i_1, \ldots, i_k$, there is j such that s is among $j(1), \ldots, j(l_j)$ (otherwise π_s would be missed in δ_ε). This means in particular that for such s, j we have $f_{\langle 0,0 \rangle}(x_s) \leq f_{\chi(d_j)}(x_{i_{j(1)}}, \ldots, x_{i_{j(l_j)}})$ (where this inequality is strict in case the same j corresponds to different s). Consequently the former multiset is less or equal than the latter. It is strictly less indeed; in fact δ cannot be in $\mathcal{E}_0$ for two independent reasons: some of the $\chi(d_j)$ is not $\langle 0, 0 \rangle$ or some projection among $\langle \pi_{i_1}, \ldots, \pi_{i_k} \rangle$ appears at least twice in δ. In both cases, this is a sufficient reason for the latter multiset to be bigger. ⊣

For a path $K = \alpha_1, \ldots, \alpha_k$, we define $c(K)$ to be the vector

$$\langle T(\alpha_1, \ldots, \alpha_k), T(\alpha_1, \ldots, \alpha_{k-1}), \ldots, T(\alpha_1) \rangle$$

and for paths K, L, we put

$$K > L \quad \text{iff} \quad c(K) > c(L)$$

where second member refers to the lexicographic extension of $>_m$. Next Lemma says that c is 'almost stable by concatenation' as a complexity measure:

Lemma 9.7 *Let $K : X^m \longrightarrow X^n$ and $K' : X^m \longrightarrow X^n$ be two paths such that $K > K'$ (notice that they agree on domains and codomains); then*

(i) *for every path L having codomain X^m, we have $L, K > L, K'$;*

(ii) *suppose that* $K = K_0, \langle a_1, \ldots, a_n \rangle$, $K' = K'_0, \langle a'_1, \ldots, a'_n \rangle$ *and that* $T(K_0, a_i) \geq T(K'_0, a'_i)$ *holds for all* $i = 1, \ldots, n$; *then for every path* R *having domain* X^n, *we have* $K, R > K', R$.

Proof. Claim (i) directly follows from Lemmas 9.4, 9.2. Let us show (ii). Here it is sufficient to prove that if $T(K_0, a_i) \geq T(K'_0, a'_i)$ holds for every $i = 1, \ldots, n$, then $T(K, b) \geq T(K', b)$ holds for every $b : X^n \longrightarrow X$ (this yields the claim, by induction on the length of R, because the complexity measure of a path is the vector of trees associated to left segments of the path itself). Supposing that b_ε is $\langle \pi_{i_1}, \ldots, \pi_{i_k} \rangle$, we have

$$T(K, b) = f_{\chi(b)}(T(K_0, a_{i_1}), \ldots, T(K_0, a_{i_k}))$$

$$T(K', b) = f_{\chi(b)}(T(K'_0, a'_{i_1}), \ldots, T(K'_0, a'_{i_k}))$$

hence $T(K, b) \geq T(K', b)$ as wanted. ⊣

Theorem 9.8 $\mathcal{R}$ *and* $\mathcal{R}^+$ *are terminating.*

Proof. If we have $K \Rightarrow K'$ by rules (R^i_c), then $K > K'$ always holds because such rules are length-reducing (recall that in lexicographic orders for variable length vectors, length is principal parameter).

According to the above Lemma, it is sufficient to show that for every other rule $L \Rightarrow R$ of $\mathcal{R}^+ \cup \mathcal{R}$, we have both

$$(1) \qquad T(L \circ \pi_i) \geq T(R \circ \pi_i),$$

for every $i = 1, \ldots, n$ (here X^n is the common codomain of L, R) and

$$(2) \qquad c(L) > c(R).$$

Notice that any (R_ε)-rewrite step is a special case of a $(Rpr)^*$-rewrite step, where $(Rpr)^*$ is the rewrite rule

$$(Rpr)^* \qquad \alpha, \varepsilon \circ \beta \;\Rightarrow\; \alpha \circ \varepsilon, \beta$$

(here ε is any strict projection). Moreover, we know from Lemma 7.2 that any (R_μ) or $(R_\mu)^+$-rewrite step is a composition of a finite number of $(R_\mu)^{+1}$ and of $(Rdi^{+1})^*$-rewrite steps, where $(R_\mu)^{+1}$ is (any alphabetic variant of)

$$(R_\mu)^{+1} \qquad \langle \alpha, a \rangle, \beta \;\Rightarrow\; \langle \alpha, a_e \rangle, (1 \times a_\mu) \circ \beta$$

and $(Rdi^{+1})^*$ is (any alphabetic variant of)

$$(Rdi^{+1})^* \qquad \langle \alpha, a, a \rangle, \beta \;\Rightarrow\; \langle \alpha, a \rangle, (1 \times \Delta_X) \circ \beta$$

Consequently, it is sufficient to prove (1) and (2) for rules $(Rpr)^*$, $(R_\mu)^{+1}$, $(Rdi^{+1})^*$ and (R_p^i).

Proof of (1) *for rule* $(Rpr)^*$:

$$\alpha, \varepsilon \circ \beta \Rightarrow \alpha \circ \varepsilon, \beta.$$

Let b any component of β; as $(\varepsilon \circ b_\varepsilon) \circ b_m$ is the factorization of $\varepsilon \circ b$, we have (taking into account Lemma 9.3):

$$T(\alpha, \varepsilon \circ b) = f_{\chi(b)}(T(\alpha \circ \varepsilon \circ b_\varepsilon)) = T(\alpha \circ \varepsilon, b),$$

as wanted.

Proof of (2) *for rule* $(Rpr)^*$: by the previous point, we have $T(\alpha, \varepsilon \circ \beta) = T(\alpha \circ \varepsilon, \beta)$; however $T(\alpha) > T(\alpha \circ \varepsilon)$ because the projection is strict.

Notice that the above established fact that $T(\alpha, \varepsilon \circ \beta)$ and $T(\alpha \circ \varepsilon, \beta)$ are componentwise equal (together with Lemma 9.4), yields the following important information to be used in the sequel: let us write $K \Rightarrow_\varepsilon^* K'$ in order to express that K' is obtained from K by a sequence of $(Rpr)^*$-rewrite steps; we have that[34]

$$(*) \qquad K \Rightarrow_\varepsilon^* K' \quad \text{implies} \quad T(K) = T(K').$$

Proof of (1) *for rule* $(R_\mu)^{+1}$: first member of the rule is

$$X^n \xrightarrow{\langle \alpha, a \rangle} Z \times X \xrightarrow{\beta} U$$

whereas second member is (let $a_e = \langle e_1, \ldots, e_k \rangle$)

$$X^n \xrightarrow{\langle \alpha, e_1, \ldots, e_k \rangle} Z \times X^k \xrightarrow{(1 \times a_\mu) \circ \beta} U.$$

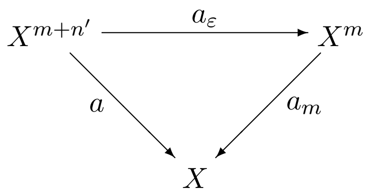
Let b be any component of the vector β; we first suppose that b_ε is the identity and then reduce to this case. If b_ε is identity, we have

$$T(\langle \alpha, a \rangle, b) = f_{\chi(b)}(T(\alpha) \cup \{T(a)\})$$

$$T(\langle \alpha, e_1, \ldots, e_k \rangle, (1 \times a_\mu) \circ b) \leq f_{\chi((1 \times a_\mu) \circ b)}(T(\alpha) \cup \{T(e_j)\}_{j=1,\ldots,k}).$$

(we put $\leq$ here, because we do not know what $((1 \times a_\mu) \circ b)_\varepsilon$ is, so we supposed - worst case - it is identity). We need to prove that $T(a) > T(e_j)$ for all $j = 1, \ldots, k$ (then first clause of the definition of orders among our trees applies). Suppose that a factors as follows

[34] $(*)$ is shown as follows: suppose that $K \Rightarrow_\varepsilon K'$ (in one step); then $K = S', L, S''$ and $K' = S', R, S''$, where L and R are the left and right side of a $(Rpr)^*$-rule. We have that $T(L, S'') = T(R, S'')$ because such multisets of trees are obtained by replacing variables in the same multiset of trees by equal trees (notice that $T(L)$ and $T(R)$ are not only equal, but also componentwise equal); the same happens to $T(S', L, S'')$ and $T(S', R, S'')$ (only a further substitution is operated to get them).



where $n = m + n'$ and α_ε is $\langle \pi_1, \ldots, \pi_m \rangle$. We have

$$T(a) = f_{\chi(a)}(x_1, \ldots, x_m).$$

Now observe that each e_j factors through a_ε (in fact, we have $a = a_\varepsilon \circ ((a_m)_e \circ (a_m)_\mu)$, hence, by uniqueness of factorizations, $\langle e_1, \ldots, e_k \rangle = a_e = a_\varepsilon \circ (a_m)_e$), so that

$$T(e_j) \le f_{\chi(e_j)}(x_1, \ldots, x_m).$$

As $\chi(a) = \langle 1, 1 \rangle$ and $\chi(e_j) = \langle 0, - \rangle$, [35] we have $T(a) > f_{\chi(e_j)}(x_1, \ldots, x_m)$ by second clause in our definition of order among trees.

Let us now consider the case in which b_ε is not identity; we apply $\Rightarrow_\varepsilon^*$-rewriting to both members (being sure that the corresponding trees do not change by $(*)$). We have two subcases. In the first subcase $Z = Z' \times Z''$ (consequently α is split as α', α'') and b_ε is the projection $Z' \times Z'' \times X \longrightarrow Z'' \times X$. We have for first member

$$\langle \alpha', \alpha'', a \rangle, b \;\Rightarrow_\varepsilon^*\; \langle \alpha'', a \rangle, b_m$$

and

$$\langle \alpha', \alpha'', a_e \rangle, (1 \times a_\mu) \circ b \;\Rightarrow_\varepsilon^*\; \langle \alpha'', a_e \rangle, (1 \times a_\mu) \circ b_m$$

for second member, thus reducing to the above special case (now $(b_m)_\varepsilon$ is identity). In the second subcase, b_ε is the projection $Z' \times Z'' \times X \longrightarrow Z''$. In this case, both members $\Rightarrow_\varepsilon^*$-rewrite to the path $Y \stackrel{\alpha''}{\longrightarrow} Z'' \stackrel{b_m}{\longrightarrow} X$.

Proof of (2) *for rule* $(R_\mu)^{+1}$: by the previous point, we have that the multiset of trees corresponding to the first member of the rule is greater or equal to the multiset of trees corresponding to the second member. In case they are equal, we need to compare $T(\alpha, a)$ and $T(\alpha, a_e)$; as we saw above, the former is greater as a multiset, because for every component e_j of a_e, we have $T(a) > T(e_j)$.

Proof of (1) *for rule* $(Rdi^{+1})^*$: first member of the rule is

$$Y \stackrel{\langle \alpha, a, a \rangle}{\longrightarrow} Z \times X \times X \stackrel{\beta}{\longrightarrow} U$$

whereas second member is

$$Y \stackrel{\langle \alpha, a \rangle}{\longrightarrow} Z \times X \stackrel{(1 \times \Delta_X) \circ \beta}{\longrightarrow} U.$$

[35] Of course rule does not apply in case first component of $\chi(a)$ is 0, because in such a case a would have trivial e/μ factorization as $a \circ 1_X$.

Let b be any component of the vector β; we first suppose that b_ε is the identity and then reduce to this case. If b_ε is identity, we have

$$T(\langle \alpha, a, a\rangle, b) = f_{\chi(b)}(T(\alpha) \cup \{T(a), T(a)\})$$

which is trivially bigger than

$$f_{\chi((1\times\Delta_X)\circ b)}(T(\alpha) \cup \{T(a)\}) \geq T(\langle \alpha, a\rangle, (1 \times \Delta_X) \circ b),$$

by first clause in definition of order for trees.

If b_ε is not identity, let $Z = Z' \times Z''$ (consequently α splits as α', a'') and let b_ε be either i) $\langle \pi_{Z''}, \pi^1_X, \pi^2_X\rangle$, or ii) $\langle \pi_{Z''}, \pi^1_X\rangle$, or iii) $\langle \pi_{Z''}, \pi^2_X\rangle$ or finally iv) $\pi_{Z''}$. In the last three cases both members have a $\Rightarrow^*_\varepsilon$-rewriting to the same path (which is $\langle \alpha'', a\rangle, b_m$ for ii)-iii) and α'', b_m for iv)), so the corresponding trees are equal by $(*)$. In the first case, first member $\Rightarrow^*_\varepsilon$-rewrites to $\langle \alpha'', a, a\rangle, b_m$, whereas second member $\Rightarrow^*_\varepsilon$-rewrites to $\langle \alpha'', a\rangle, (1 \times \Delta_X) \circ b_m$, thus reducing to the above considered special case.

Proof of (2) *for rule* $(Rdi^{+1})^*$: by the previous point, we have that the multiset of trees corresponding to the first member of the rule is greater or equal to the multiset of trees corresponding to the second member. In case they are equal, we need to compare $T(\alpha, a, a)$ and $T(\alpha, a)$: the former is clearly bigger.

Proof of (1) *for rule* (R^i_p): we recall that first member of (R^i_p) is

$$Y \xrightarrow{\langle\gamma,\delta\rangle} Y_1 \times Z \xrightarrow{\langle\alpha,\pi_Z\rangle} Y_2 \times Z \xrightarrow{\beta} U$$

whereas second member is

$$Y \xrightarrow{\langle\gamma,\delta,\delta_\varepsilon\rangle} Y_1 \times Z \times Y' \xrightarrow{\alpha\times 1_{Y'}} Y_2 \times Y' \xrightarrow{(1_{Y_2}\times\delta_m)\circ\beta} U$$

(with an extra arrow to the right in case $i = 1$). This rule is subject to the proviso that δ cannot be a projection. Let b be any component of β; we first assume that b_ε is the identity (and then reduce to this case). We have that

$$T(\langle\gamma,\delta\rangle, \langle\alpha, \pi_Z\rangle, b) = f_{\chi(b)}(T(\langle\gamma,\delta\rangle, \langle\alpha,\pi_Z\rangle)) = f_{\chi(b)}(T(\langle\gamma,\delta\rangle, \alpha) \cup T(\delta, 1_Z))$$

where $\cup$ refers to multiset union (notice that we used $(*)$ above in missed intermediate passages). We do not know what is $((1\times\delta_m)\circ b)_\varepsilon$: let so take worst case (it is identity) and proceed as follows by using $(*)$ again:

$$\begin{aligned} T(\langle\gamma,\delta,\delta_\varepsilon\rangle, \alpha\times 1, (1\times\delta_m)\circ b) &\leq f_{\chi((1\times\delta_m)\circ b)}(T(\langle\gamma,\delta,\delta_\varepsilon\rangle, \alpha\times 1)) = \\ &= f_{\chi((1\times\delta_m)\circ b)}(T(\langle\gamma,\delta\rangle,\alpha) \cup T(\delta_\varepsilon, 1_{Y'})). \end{aligned}$$

This tree is indeed smaller than $f_{\chi(b)}(T(\langle\gamma,\delta\rangle,\alpha) \cup T(\delta, 1_Z))$ (by the first clause of the definition of trees order): in fact, by Lemma 9.6 we have $T(\delta, 1_Z) > T(\delta_\varepsilon, 1_{Y'})$.

Let us now turn to the general case (b_ε may not be identity). In such a case, let us transform both

$$Y \stackrel{\langle\gamma,\delta\rangle}{\longrightarrow} Y_1 \times Z \stackrel{\langle\alpha,\pi_Z\rangle}{\longrightarrow} Y_2 \times Z \stackrel{b}{\longrightarrow} X$$

and

$$Y \stackrel{\langle\gamma,\delta,\delta_\varepsilon\rangle}{\longrightarrow} Y_1 \times Z \times Y' \stackrel{\alpha\times 1_{Y'}}{\longrightarrow} Y_2 \times Y' \stackrel{(1_{Y_2}\times\delta_m)\circ b}{\longrightarrow} X$$

by $\Rightarrow^*_\varepsilon$-rewriting and then apply $(*)$. Suppose we have $Y_2 = Y_2' \times Y_2''$ and $Z = Z' \times Z''$ (consequently δ and α are also splitted as δ', δ'' and α', α'', respectively); let b factors as follows:

$$\begin{array}{ccc}
Y_2' \times Y_2'' \times Z' \times Z'' & \stackrel{b_\varepsilon}{\longrightarrow} & Y_2'' \times Z'' \\
& {\scriptstyle b} \searrow \quad \swarrow {\scriptstyle b_m} & \\
& X &
\end{array}$$

where b_ε is the obvious projection. We then have for the first member

$$\langle\gamma,\delta\rangle, \langle\alpha,\pi_Z\rangle, b \;\Rightarrow^*_\varepsilon\; \langle\gamma,\delta',\delta''\rangle, \langle\alpha'',\pi_{Z''}\rangle, b_m.$$

Let us also split $\delta_m : Y' \longrightarrow Z' \times Z''$ as θ', θ'' (as a consequence, from $\langle\delta',\delta''\rangle = \delta = \delta_\varepsilon \circ \delta_m$, we have in particular $\delta_\varepsilon \circ \theta'' = \delta''$); an analogous transformation on the second member gives

$$\langle\gamma,\delta,\delta_\varepsilon\rangle, \alpha \times 1, (1 \times \delta_m) \circ b \;\Rightarrow^*_\varepsilon\; \langle\gamma,\delta',\delta'',\delta_\varepsilon\rangle, \alpha'' \times 1, (1 \times \theta'') \circ b_m.$$

Let us now factorize $\theta'' = \theta''_\varepsilon \circ \theta''_m$; from $\delta_\varepsilon \circ \theta'' = \delta''$, by uniqueness of factorizations, we get $\delta''_\varepsilon = \delta_\varepsilon \circ \theta''_\varepsilon$ and $\delta''_m = \theta''_m$; thus, by further $\Rightarrow^*_\varepsilon$-rewrite steps, we get

$$\langle\gamma,\delta',\delta'',\delta_\varepsilon\rangle, \alpha'' \times 1, (1 \times \theta'') \circ b_m \;\Rightarrow^*_\varepsilon\; \langle\gamma,\delta',\delta'',\delta''_\varepsilon\rangle, \alpha'' \times 1, (1 \times \delta''_m) \circ b_m.$$

Now

$$\langle\gamma,\delta',\delta''\rangle, \langle\alpha'',\pi_{Z''}\rangle, b_m$$

and

$$\langle\gamma,\delta',\delta'',\delta''_\varepsilon\rangle, \alpha'' \times 1, (1 \times \delta''_m) \circ b_m$$

are first and second member of a (R^i_p)-rewrite rule and $(b_m)_\varepsilon$ is the identity. We can so reduce to the above particular case, except that now there is no guarantee that δ'' is not a projection: this further case has to be considered separately. However in such a case, $1 \times \delta''_m$ is the identity, $\delta''_\varepsilon = \delta''$ and all what we need is to prove that trees corresponding to the paths

$$Y \stackrel{\langle\gamma,\delta',\delta''\rangle}{\longrightarrow} Y_1 \times Z' \times Z'' \stackrel{\langle\alpha'',\pi_{Z''}\rangle}{\longrightarrow} Y_2'' \times Z''$$

$$Y \stackrel{\langle\gamma,\delta',\delta'',\delta''\rangle}{\longrightarrow} Y_1 \times Z' \times Z'' \times Z'' \stackrel{\alpha''\times 1}{\longrightarrow} Y_2'' \times Z''$$

are the same. Indeed they are both equal to $T(\langle\gamma,\delta',\delta''\rangle,\alpha'') \cup T(\delta'', 1_{Z''})$ (again by $(*)$).

Proof of (2) *for rule* (R_p^i): by the previous point, we have that the multiset of trees corresponding to the first member of the rule is greater or equal to the multiset of trees corresponding to the second member. This does not prevent from them to be equal, in some cases; hence, we compare trees corresponding to

$$Y \stackrel{\langle\gamma,\delta\rangle}{\longrightarrow} Y_1 \times Z \stackrel{\langle\alpha,\pi_Z\rangle}{\longrightarrow} Y_2 \times Z$$

and to

$$Y \stackrel{\langle\gamma,\delta,\delta_\varepsilon\rangle}{\longrightarrow} Y_1 \times Z \times Y' \stackrel{\alpha\times 1_{Y'}}{\longrightarrow} Y_2 \times Y'.$$

The former is $T(\langle\gamma,\delta\rangle,\alpha) \cup T(\delta, 1_Z)$ whereas the latter is $T(\langle\gamma,\delta\rangle,\alpha) \cup T(\delta_\varepsilon, 1_{Y'})$: as δ cannot be a projection, Lemma 9.6 applies, showing that the former is greater. $\dashv$

From the previous section results, we immediately get:

Corollary 9.9 *$\mathcal{R}^+$ is canonical.* $\dashv$

We now compare rewrite systems $\mathcal{R}^+$ and $\mathcal{R}$: it will turn out that they are essentially the same, hence in particular canonicity of $\mathcal{R}$ will follow.

Lemma 9.10 *If $K \Rightarrow^*_{\mathcal{R}^+} K'$, then there exists K'' such that $K' \Rightarrow^*_{\mathcal{R}^+} K''$ and $K \Rightarrow^*_{\mathcal{R}} K''$.*

Proof. Statement is proved by noetherian induction on K (with respect to the order $>$ among paths which has been used in the termination proof). If $K = K'$, the statement is trivial; otherwise we have, for some K_0,

$$K \Rightarrow_{\mathcal{R}^+} K_0 \Rightarrow^*_{\mathcal{R}^+} K'.$$

Now there is K_0' such that

$$K_0 \Rightarrow^*_{\mathcal{R}^+} K_0' \quad \text{and} \quad K \Rightarrow_{\mathcal{R}} K_0'$$

(if the $\Rightarrow_{\mathcal{R}^+}$-step is done by a rule different than $(R_\mu)^+$ this is trivial, otherwise apply Lemma 7.1). As $\mathcal{R}^+$ is confluent, there exists K_0'' such that

$$K_0' \Rightarrow^*_{\mathcal{R}^+} K_0'' \quad \text{and} \quad K' \Rightarrow^*_{\mathcal{R}^+} K_0''.$$

As $K_0' < K$ (any kind of rewrite step decreases complexity, as we saw in the termination proof), we can apply inductive hypothesis to K_0', yielding K'' such that

$$K' \Rightarrow^*_{\mathcal{R}^+} K_0'' \Rightarrow^*_{\mathcal{R}^+} K'' \quad \text{and} \quad K \Rightarrow_{\mathcal{R}} K_0' \Rightarrow^*_{\mathcal{R}} K''$$

as wanted. $\dashv$

Lemma 9.11 *If* $K \Rightarrow^*_{\mathcal{R}} K'$*, then* $K \Leftrightarrow^*_{\mathcal{R}^+} K'$*.*

Proof. Statement is again proved by noetherian induction on K. The only relevant case is when we have $K \Rightarrow_{\mathcal{R}} K'$ by a single (R^i_p)-rewrite step, which is covered by Lemma 7.6 (iii). $\dashv$

We can finally complete the

Proof of Theorem 5.3. As we know from Proposition 9.8 that $\mathcal{R}$ is terminating, we only have to prove its confluence. Suppose we have that

$$K \Rightarrow^*_{\mathcal{R}} K' \quad \text{and} \quad K \Rightarrow^*_{\mathcal{R}} K''.$$

Then $K' \Leftrightarrow^*_{\mathcal{R}^+} K''$ by Lemma 9.11; as $\mathcal{R}^+$ is canonical, K' and K'' both $\Rightarrow^*_{\mathcal{R}^+}$-rewrite to their common normal form N. By Lemma 9.10, there are N', N'' such that

$$N \Rightarrow^*_{\mathcal{R}^+} N', \quad K' \Rightarrow^*_{\mathcal{R}} N'$$

$$N \Rightarrow^*_{\mathcal{R}^+} N'', \quad K'' \Rightarrow^*_{\mathcal{R}} N''$$

However N is in $\mathcal{R}^+$-normal form, hence $N' = N = N''$ is a path to which K', K'' both $\Rightarrow^*_{\mathcal{R}}$-reduce. $\dashv$

10 Examples and Related Work

In this Section we illustrate our results in concrete cases. First, we gave in Section 5 a definition of constructibility for theories referring to their associated Lawvere categories. Now we give a useful equivalent purely symbolic definition:

Proposition 10.1 *A theory* $T' = \langle \Omega', Ax' \rangle$ *is constructible over a theory* $T = \langle \Omega, Ax \rangle$ *iff* T' *is a conservative extension of* T *and there exists a class* E' *of* Ω'*-terms such that:*

(i) *E' contains the variables and is closed under renamings of terms;*

(ii) *for every* Ω'*-term* $t(x_1, \dots, x_n)$ *there exist a k-minimized* Ω*-term* $u(x_1, \dots, x_k)$ *and pairwise distinct (with respect to provable identity in* T'*)* Ω'*-terms*

$$v_1(x_1, \dots, x_n), \dots, v_k(x_1, \dots, x_n)$$

belonging to E' *such that*

$$\vdash_{T'} t = u(v_1, \dots, v_k);$$

(iii) *whenever u, u' are k (resp. k')-minimized Ω-terms and we have*

$$\vdash_{T'} u(v_1, \dots, v_k) = u'(v'_1, \dots, v'_{k'})$$

for pairwise distinct (wrt T'-provability) terms $v_1, \dots, v_k \in E'$ and pairwise distinct (wrt T'-provability) terms $v'_1, \dots, v'_{k'} \in E'$, then $k = k'$ and there is a permutation σ acting on the k-elements set,[36] *such that*

$$\vdash_{T'} v'_{\sigma(i)} = v_i \ (i = 1, \dots, k) \quad \text{and} \quad \vdash_T u' = u(x_{\sigma(i)}/x_i).$$

Proof. If T' is constructible over T, in $\mathbf{T}'$ there is a left extension $(\mathcal{E}', \mathcal{M})$ of the standard weak factorization system $(\mathcal{E}, \mathcal{M})$ of $\mathbf{T}$. In order to find E' fulfilling the above requirements it is sufficient to take the set of terms $t(x_1, \dots, x_n)$ such that the equivalence class of t (seen as an arrow $X^n \longrightarrow X$ in $\mathbf{T}'$) belongs to $\mathcal{E}'$. To see that (i)-(iii) hold, we only have to show that if an arrow from $\mathbf{T}$ like $\langle e_1, \dots, e_m \rangle : X^n \longrightarrow X^m$ belongs to $\mathcal{E}'$, then the e_i are pairwise distinct and, vice versa, that if all e_i belong to $\mathcal{E}'$ and are pairwise distinct, then $\langle e_1, \dots, e_n \rangle$ belongs to $\mathcal{E}'$; these facts follow from Corollary 7.4.

Vice versa, suppose that a class E' of Ω'-terms fulfilling the above requirements is given. We define a left extension $(\mathcal{E}', \mathcal{M})$ of the standard weak factorization system $(\mathcal{E}, \mathcal{M})$ of $\mathbf{T}$ by taking as $\mathcal{E}'$ the set of arrows $\langle e_1, \dots, e_m \rangle : X^n \longrightarrow X^m$ such that the e_i are represented by distinct (up to provable identity in T') terms in E'.

First notice that, if $\alpha = \langle a_1, \dots, a_n \rangle \in \mathcal{E}' \cap \mathbf{T}$, then α ia an n-tuple of distinct projections by an immediate application of (iii) to (the symbolic meaning of) the commutativity of the squares

$$\begin{array}{ccc} Y & \xrightarrow{a_i} & X \\ {\scriptstyle (a_i)_\varepsilon}\downarrow & & \downarrow{\scriptstyle 1_X} \\ Z = X & \xrightarrow[(a_i)_\mu]{} & X \end{array}$$

We can easily factorize arrows a having codomain X (just apply (ii) to find a_e and a_μ). To factorize arrows $\langle a_1, \dots, a_m \rangle : X^n \longrightarrow X^m$, it is sufficient to factorize each a_i as $\langle e_{i1}, \dots, e_{ik_i} \rangle \circ \mu_i$ and then 'diagonalize' as follows: let $\langle e_1, \dots, e_s \rangle$ be any list of the distincts elements of $\{e_{ij}\}$ and let δ be a diagonal such that $\langle e_1, \dots, e_s \rangle \circ \delta = \langle e_{11}, \dots, e_{mk_m} \rangle$. We factorize $\langle a_1, \dots, a_m \rangle$ as $\langle e_1, \dots, e_s \rangle \circ (\delta \circ (\mu_1 \times \cdots \times \mu_m))$. Notice that $\delta \circ (\mu_1 \times \cdots \times \mu_m)$ is still represented by a minimized *vector* of terms: in fact, for every i, as $e_{i1}, \dots, e_{ik_i}$ are all distinct, δ composed with the projection from $X^{\sum k_i}$ onto the domain of μ_i is a projection $\pi_i : X^s \longrightarrow X^{k_i}$ (hence the i-th component of

[36] Such σ is clearly unique given that the v_i are distinct.

$\delta \circ (\mu_1 \times \cdots \times \mu_m)$ is $\pi_i \circ \mu_i$); moreover projections $\{\pi_i\}$ altogether cannot miss any component of X^s (by the very definition of the list $\langle e_1, \ldots, e_s \rangle$).

To show uniqueness of factorizations suppose you have a commutative square

$$\begin{array}{ccc} Z & \xrightarrow{\alpha_2} & Y_1 \\ {\scriptstyle \alpha_1}\downarrow & & \downarrow{\scriptstyle \mu_2} \\ Y_2 & \xrightarrow[\mu_1]{} & X^n \end{array}$$

with $\alpha_1, \alpha_2 \in \mathcal{E}'$ and $\mu_1, \mu_2 \in \mathcal{M}$. The α_1, α_2 are lists formed by distinct components (by the definition of the class $\mathcal{E}'$); let us first show that each component a of α_1 appears as a component of α_2 too (and vice versa, so that α_1 and α_2 differ only by a renaming). As μ_1 is represented by a minimized vector of terms, there is a component s of μ_1 such that $\alpha_1 \circ s_\varepsilon$ contains a; if s is the i-th component of μ_1, let r be the corresponding i-th component of μ_2. By commutativity of the square, we have $(\alpha_1 \circ s_\varepsilon) \circ s_\mu = (\alpha_2 \circ r_\varepsilon) \circ r_\mu$; by (iii), there is a renaming ρ such that $\alpha_1 \circ s_\varepsilon \circ \rho = \alpha_2 \circ r_\varepsilon$ and $\rho \circ r_\mu = s_\mu$. The former shows that a is a component of α_2.

We so established that α_1, α_2 differ only for a renaming, i.e. that there is a renaming ρ such that $\alpha_1 \circ \rho = \alpha_2$. Now $\rho \circ \mu_2 = \mu_1$ immediately follows from the commutativity of the above square and from the following

Claim. If $\alpha \in \mathcal{E}'$ and $\tau, \sigma \in \mathbf{T_0}$, then $\alpha \circ \tau = \alpha \circ \sigma$ implies $\tau = \sigma$.

The claim is obvious in case σ, τ are projections, because the components of α are distinct. In the general case, it is sufficient to prove the Claim for σ, τ having codomain X; if codomain is X, from $(\alpha \circ \sigma_\varepsilon) \circ \sigma_\mu = (\alpha \circ \tau_\varepsilon) \circ \tau_\mu$, we have (by (iii)) $(\alpha \circ \sigma_\varepsilon) \circ \rho = \alpha \circ \tau_\varepsilon$ and $\rho \circ \tau_\mu = \sigma_\mu$ for a renaming ρ. As $\sigma_\varepsilon \circ \rho$ and τ_ε are projections, we just saw (this is the above mentioned obvious case) that $\sigma_\varepsilon \circ \rho = \tau_\varepsilon$, hence

$$\sigma = \sigma_\varepsilon \circ \sigma_\mu = \sigma_\varepsilon \circ \rho \circ \tau_\mu = \tau_\varepsilon \circ \tau_\mu = \tau$$

as required. $\dashv$

We say that T' is *effectively constructible* over T iff it is constructible over T and moreover for every term t, terms $u, v_1, \ldots, v_k$ satisfying (ii) above are provided by a total recursive function. As an immediate corollary to our main Theorem 5.3, we have:

Theorem 10.2 *Suppose that T_1, T_2 are both effectively constructible over T_0 and that word problems for T_1, T_2 are solvable; then word problem for $T_1 +_{T_0} T_2$ is solvable too.*

Proof. By Theorem 3.1, Lemma 4.1, 5.2 and Theorem 5.3, it is sufficient to show that applicability of rules of $\mathcal{R}$ is effective whenever a path is given as a list of terms, representing their respective equivalence classes (in order to be able to compare normal forms, we need also to check that it is effectively recognizable whether two paths are alphabetic variants each other).

For rules (R^i_c) we need to be able to recognize whether a certain arrow α^i comes from $\mathbf{T_0}$: this happens iff $\alpha_e \in \mathcal{E}_0$ (by uniqueness of e/μ factorization and by the fact that $\mathcal{E}_0 \subseteq \mathcal{E}_i$), a fact which is effective by appealing to the solvability of word problem for T_i.[37] For rule (R_ε) we already observed in Section 5 that ε-extraction is effective in case word problem is decidable. For rule (R_μ), one just use effective constructibility, together with the fact that the e/μ factorization of $\langle a_1, \dots, a_n \rangle$ can be reduced to the e/μ factorization of components, see Lemma 7.2. Finally, in order to apply rules (R^i_p) (and checking the relative proviso) it is sufficient to be able to recognize projections, a fact which is reduced once again to solvability of the input word problems.

Last, we show that it is effectively recognizable whether two paths are alphabetic variants each other. In case they are both in normal form (which is the relevant case), there is a quick procedure for that. First, for $\alpha_1, \dots, \alpha_k$ to be an alphabetic variant of $\beta_1, \dots, \beta_{k'}$ we need $k = k'$; secondly, as the components of α_1 and β_1 are distinct (because paths are in normal form and (R_μ) does not apply), it is easily computed - provided it exists - the renaming ρ_1 such that $\alpha_1 \circ \rho_1 = \beta_1$; at this point, we recursively need to check whether $\rho_1^{-1} \circ \alpha_2, \dots, \alpha_k$ is an alphabetic variant of $\beta_2, \dots, \beta_k$ and so on. $\dashv$

Example. Commutative rings with unit are constructible over abelian groups. In fact terms $t(x_1, \dots, x_n)$ in the theory of abelian groups can be represented as homogeneous linear polynomials in the indeterminates $x_1, \dots, x_n$ with integer coefficients (they are minimized iff no coefficient is zero); terms in the theory of commutative rings with unit can be represented as arbitrary polynomials with integer coefficients. Class E' needed for constructibility is formed by monic monomials (1 included): in fact, every integer polynomial can be uniquely expressed as a linear combination (with integer non-zero coefficients) of distinct monic monomials. $\dashv$

Example. Let T be the theory of join-semilattices with zero and let T' be the theory of semilattice-monoids we met in the Introduction. T' is constructible over T: class E' is given by terms of the form $x_{i_1} \circ \cdots \circ x_{i_k}$ (for $k \geq 0$). $\dashv$

Example The theory of abelian groups endowed with an endomorphism f is constructible over the theory of abelian groups: class E' is given by terms of the form $f^n(x_i)$ (for $n \geq 0$). $\dashv$

Example Differential rings (i.e. of rings endowed with a differentiation operator ∂ satisfying usual laws for derivatives of sums and products) are constructible over commutative rings with unit: class E' is given by terms of the form $\{\partial^k x_i\}$ (for $k \geq 0$).

[37] Clearly if term t represents $a : X^n \longrightarrow X$, then a is a projection iff t *collapses to* (i.e. it is provably equal to) a variable x_i (for $i = 1, \dots, n$); a similar observation applies to vector of terms.

$\dashv$

Notice that in the above examples the smaller theory *is not collapse-free*. Additional examples of different nature can be found in [3, 4]. In order to build counterexamples, a useful tool is the following Proposition (clearly inspired from [3, 4]):

Proposition 10.3 *If T' is constructible over T, then the T-reduct of any free T'-algebra is a free T-algebra (on a bigger set of generators).*

Proof. Let $F_{T'}(G)$ be the free T'-algebra on the set G of generators; we show that its T-reduct is free over the set of elements of the form $u(g_1, \dots, g_n)$ where $u(x_1, \dots, x_n) \in E'$ and $g_1, \dots, g_n$ are distinct elements from G. Clearly the claim follows from the case in which G is finite. To have a quick proof we translate everything in the terminology of functorial semantics.

Let $(\mathcal{E}, \mathcal{M})$ be the standard weak factorization system of $\mathbf{T}$ and let $(\mathcal{E}', \mathcal{M})$ be its left extension to $\mathbf{T}'$. For any functor F having domain $\mathbf{T}'$ let us call $|F|$ its restriction to $\mathbf{T}$; for any type Y let $\mathcal{E}'(Y,X)$ be $\mathbf{T}'(Y,X) \cap \mathcal{E}'$. Fix a type Y and a T-algebra $A : \mathbf{T} \longrightarrow \mathbf{Set}$; we need to find a bijective natural correspondence between set-theoretic functions

$$\bar{N} : \mathcal{E}'(Y,X) \longrightarrow A(X)$$

and natural transformations

$$N : |\mathbf{T}'(Y,-)| \longrightarrow A.$$

Given N, let $\bar{N}$ be the restriction of N_X to $\mathcal{E}'(Y,X)$ in the domain. Conversely, if $\bar{N}$ is given, we define for every Z and $\alpha : Y \longrightarrow Z$

$$N_Z(\alpha) = A(\alpha_\mu)(\bar{N}(e_1), \dots, \bar{N}(e_k))$$

where $\alpha_e = \langle e_1, \dots, e_k \rangle$. In order to prove naturality of N so defined, take $\nu : Z \longrightarrow Z'$ in $\mathbf{T}$; we need to show the commutativity of the square

$$\begin{array}{ccc} |\mathbf{T}(Y,Z)| & \xrightarrow{N_Z} & A(Z) \\ {\scriptstyle |\mathbf{T}(Y,\nu)|}\downarrow & & \downarrow{\scriptstyle A(\nu)} \\ |\mathbf{T}(Y,Z')| & \xrightarrow[N_{Z'}]{} & A(Z') \end{array}$$

We have

$$A(\nu)(N_Z(\alpha)) = A(\nu)(A(\alpha_\mu)(\bar{N}(e_1), \dots, \bar{N}(e_k))) = A(\alpha_\mu \circ \nu)(\bar{N}(e_1), \dots, \bar{N}(e_k)).$$

On the other hand, let $\alpha_\mu \circ \nu$ factorize in ε/μ-components as follows:

$$\begin{array}{ccc} X^k & \xrightarrow{\alpha_\mu} & Z \\ \pi \downarrow & & \downarrow \nu \\ X^m & \xrightarrow[\mu]{} & Z' \end{array}$$

where $\pi = \langle \pi_{i_1}, \ldots, \pi_{i_m} \rangle$. We have

$$\begin{aligned} N_{Z'}(|\mathbf{T}(Y,\nu)(\alpha)|) &= N_{Z'}(\alpha \circ \nu) \\ &= N_{Z'}(\alpha_e \circ \pi \circ \mu) \\ &= A(\mu)(\bar{N}(e_{i_1}), \ldots, \bar{N}(e_{i_m})) \\ &= A(\mu)(A(\pi)(\bar{N}(e_1), \ldots, \bar{N}(e_k))) \\ &= A(\pi \circ \mu)(\bar{N}(e_1), \ldots, \bar{N}(e_k)) \\ &= A(\alpha_\mu \circ \nu)(\bar{N}(e_1), \ldots, \bar{N}(e_k)). \end{aligned}$$

Bijectivity and naturality of the correspondence $N \longleftrightarrow \bar{N}$ are immediate. $\dashv$

Counterexample. Boolean algebras are not constructible over join-semilattices with zero. In fact the free join-semilattice with zero over an infinite set G of generators is just the set of finite subsets of G; in this algebra, clearly the strict part of the partial order relation associated with the join is terminating. It is not so however in the countably generated free Boolean algebra, which is atomless. $\dashv$

Counterexample. Modal algebras (also $K4$-modal algebras, interior algebras, diagonalizable algebras, etc.) are not constructible over Boolean algebras: in fact, in such varieties, finitely generated free algebras are atomic and infinite,[38] whereas free Boolean algebras are either finite or atomless. $\dashv$

Let us now give examples of normalization through our rewriting system $\mathcal{R}$. In order to apply normalization to paths of equivalence classes of terms, algebraic notation for rules must be converted into ordinary symbolic notation. This is not difficult (all needed information is contained in Section 2 above), however some care is needed. Suppose e.g. you want to apply products rule to the path

$$X^3 \xrightarrow{\langle t,\, u \rangle} X^2 \xrightarrow{\langle v,\, x_2 \rangle} X^2 \xrightarrow{w} X$$

First $u(x_1, x_2, x_3)$ has to be minimized (this is the factorization $\delta = \delta_\varepsilon \circ \delta_m$ in the Table of rules of Section 5). Suppose it minimizes as $u'(x_1, x_3)$; the pair of projections $\langle x_1, x_3 \rangle$ stays in first position, whereas $u'(x_1, x_2)$ is moved in third position. However, the term moved to last position for composition with $w(x_1, x_2)$ (the arrow $1 \times \delta_m$ of

[38] These are well-known results. For a proof making use of normal forms, see [7].

the Table of rules), requires a renaming away from x_1 and consequently it is the pair $\langle x_1,\, u'(x_2, x_3)\rangle$. Thus the products rule rewrite step produces

$$X^3 \xrightarrow{\langle t,\, u,\, x_1, x_3\rangle} X^4 \xrightarrow{\langle v,\, x_3,\; x_4\rangle} X^3 \xrightarrow{w(x_1,\, u'(x_2,x_3))} X$$

In the examples below we consider the following theories:

$T_0 =$ Abelian groups with period 2

$T_1 =$ Boolean rings

$T_2 =$ T_0 + an idempotent endomorphism f (i.e., such that $f(f(x_1)) = f(x_1)$)

We leave to the reader to check that T_1, T_2 are both constructible over T_0.

Example. Let us consider the following instance of word problem in the theory $T_1 +_{T_0} T_2$:

$$f(x_1 \cdot x_2 + x_2 + f(x_2)) \stackrel{?}{=} f(x_1 \cdot x_2)$$

Let us rewrite a splitting path of first member in $\mathcal{R}$.

$$X^2 \xrightarrow{\langle x_1,\, x_2,\, f(x_2)\rangle} X^3 \xrightarrow{x_1 \cdot x_2 + x_2 + x_3} X \xrightarrow{f(x_1)} X$$

$$\Downarrow_{R_\mu}$$

$$X^2 \xrightarrow{\langle x_1,\, x_2,\, f(x_2)\rangle} X^3 \xrightarrow{\langle x_1 \cdot x_2,\, x_2,\, x_3\rangle} X^3 \xrightarrow{f(x_1 + x_2 + x_3)} X$$

$$\Downarrow_{(R_p^2)}$$

$$X^2 \xrightarrow{\langle x_1,\, x_2,\, f(x_2),\, x_2\rangle} X^4 \xrightarrow{\langle x_1 \cdot x_2,\, x_2, x_4\rangle} X^3 \xrightarrow{f(x_1 + x_2 + f(x_3))} X$$

$$\Downarrow_{R_\varepsilon}$$

$$X^2 \xrightarrow{\langle x_1,\, x_2,\, x_2\rangle} X^3 \xrightarrow{\langle x_1 \cdot x_2,\, x_2,\, x_3\rangle} X^3 \xrightarrow{f(x_1 + x_2 + f(x_3))} X$$

$$\Downarrow_{(R_c^1)}$$

$$X^2 \xrightarrow{\langle x_1 \cdot x_2,\, x_2,\, x_2\rangle} X^3 \xrightarrow{f(x_1 + x_2 + f(x_3))} X$$

$$\Downarrow_{R_\mu}$$

$$X^2 \xrightarrow{\langle x_1 \cdot x_2,\, x_2\rangle} X^2 \xrightarrow{f(x_1 + x_2 + f(x_2))} X$$

$$\Downarrow_{R_\varepsilon}$$

$$X^2 \xrightarrow{x_1 \cdot x_2} X \xrightarrow{f(x_1)} X$$

where the last path corresponds to the splitting path of the term $f(x_1 \cdot x_2)$. $\dashv$

Example. Let us consider the following instance of word problem for $T_1 +_{T_0} T_2$:

$$f(x_1) \cdot f(x_2) + f(x_1) \cdot (f(x_1) + f(x_2)) \stackrel{?}{=} f(x_1)$$

We rewrite first member as follows.

$$X^2 \xrightarrow{\langle f(x_1),\, f(x_2),\, f(x_1)+f(x_2)\rangle} X^3 \xrightarrow{\langle x_1 \cdot x_2,\, x_1 \cdot x_3\rangle} X^2 \xrightarrow{x_1 + x_2} X$$

$$\Downarrow_{R_\mu}$$

$$X^2 \xrightarrow{\langle f(x_1),\, f(x_2)\rangle} X^2 \xrightarrow{\langle x_1,\, x_2,\, x_1+x_2\rangle \circ \langle x_1 \cdot x_2,\, x_1 \cdot x_3\rangle} X^2 \xrightarrow{x_1 + x_2} X$$

$$=$$

$$X^2 \xrightarrow{\langle f(x_1),\, f(x_2)\rangle} X^2 \xrightarrow{\langle x_1 \cdot x_2,\, x_1 \cdot (x_1+x_2)\rangle} X^2 \xrightarrow{x_1 + x_2} X$$

$$\Downarrow_{R_\mu}$$

$$X^2 \xrightarrow{\langle f(x_1),\, f(x_2)\rangle} X^2 \xrightarrow{\langle x_1 \cdot x_2,\, x_1\rangle} X^2 \xrightarrow{\langle x_1,\, x_1+x_2\rangle \circ (x_1 + x_2)} X$$

$$=$$

$$X^2 \xrightarrow{\langle f(x_1),\, f(x_2)\rangle} X^2 \xrightarrow{\langle x_1 \cdot x_2,\, x_1\rangle} X^2 \xrightarrow{x_1 + x_1 + x_2} X$$

$$\Downarrow_{R_\varepsilon}$$

$$X^2 \xrightarrow{\langle f(x_1),\, f(x_2)\rangle} X^2 \xrightarrow{\langle x_1 \cdot x_2,\, x_1\rangle \circ x_2} X \xrightarrow{x_1} X$$

$$=$$

$$X^2 \xrightarrow{\langle f(x_1),\, f(x_2)\rangle} X^2 \xrightarrow{x_1} X \xrightarrow{x_1} X$$

$$\Downarrow_{(R_c^2)}$$

$$X^2 \xrightarrow{f(x_1)} X$$

where the last path coincides with the second term of the problem. $\dashv$

We now make a comparison with results from [3, 4]. Let $T = (\Omega, Ax)$ and $T' = (\Omega', Ax')$ be equational theories such that T' is a conservative extension of T. Let G be the set of Ω'-terms r such that $\not\vdash_{T'} r = t$ for all Ω'-terms t with top symbol in Ω. Notice that $G \neq \emptyset$ iff $V \subseteq G$, where V is the set of variables. Moreover, *G is empty in case T is not collapse-free.* We say that T' is BT-constructible over T iff the following hold:

(I) $V \subseteq G$ (hence T is collapse-free);

(II) for all Ω'-term t, there are an Ω-term s and a vector $\vec{r}$ of terms in G such that $\vdash_{T'} t = s(\vec{r})$;

(III) for every pair s_1, s_2 of Ω-terms and for every pair of vectors $\vec{r_1}, \vec{r_2}$ of terms from G, we have

$$\vdash_{T'} s_1(\vec{r_1}) = s_2(\vec{r_2}) \quad \text{iff} \quad \vdash_T s_1(\vec{z_1}) = s_2(\vec{z_2})$$

where $\vec{z_1}, \vec{z_2}$ are fresh vectors of variables abstracting $\vec{r_1}, \vec{r_2}$ so that two terms in $\vec{r_1}, \vec{r_2}$ are abstracted by the same variable iff they are provably equal in T'.

We now show that *if T' is BT-constructible over T, then T' is constructible over T* (in our sense). We use Proposition 10.1 above taking $E' = G$. Let us first show uniqueness of factorizations. Suppose that we have k (resp. k')-minimized Ω-terms u, u' and that we have $\vdash_{T'} u(v_1, \dots, v_k) = u'(v'_1, \dots, v'_{k'})$ for pairwise distinct (wrt T'-provability) terms $v_1, \dots, v_k \in G$ and pairwise distinct (wrt T'-provability) terms $v'_1, \dots, v'_{k'} \in G$. Let $w_1, \dots, w_s$ be the terms which are common to the lists $v_1, \dots, v_k$ and $v'_1, \dots, v'_{k'}$. For simplicity, let us also rearrange such lists as

$$v_1, \dots, v_k = w_1, \dots, w_s, r_1, \dots r_l \quad \text{and} \quad v'_1, \dots, v'_{k'} = w_1, \dots, w_s, r'_1, \dots, r'_{l'}.$$

Then, applying (III), we get

$$\vdash_T u(x_1, \dots, x_s, y_1, \dots, y_l) = u'(x_1, \dots, x_s, z_1, \dots, z_{l'})$$

which cannot be (unless $l = l' = 0$, yielding what we need) because u and u' are minimized: in fact, replacing e.g. all the y_i by a ground term c we would get

$$\vdash_T u(x_1, \dots, x_s, c, \dots, c) = u'(x_1, \dots, x_s, z_1, \dots, z_{l'}) = u(x_1, \dots, x_s, y_1, \dots, y_l)$$

contrary to the fact that u is minimized.

Showing the existence of factorization is a little more tricky because the requirements in (II) above look more liberal than those in Proposition 10.1(ii) (it is not asked for s to be minimized, not for the $\vec{r}$ to be distinct (up to probability) and to contain only at most the variables of the original t). We progressively refine the factorization in (II). First, if we have (let $\vec{r} = r_1, \dots, r_k$) $\vdash_{T'} t = s(r_1, \dots, r_k)$ for non distinct r_i, then we can identify variables in $s(x_1, \dots, x_k)$ and reduce correspondingly the list $r_1, \dots, r_k$ to a list formed by distinct elements. If furthermore $s(x_1, \dots, x_k)$ is not minimized, then we can minimize it and remove the corresponding r_i from the list. Thus we obtained a factorization of $t(x_1, \dots, x_n)$ as $s(r_1, \dots, r_k)$ where s is k-minimized and the r_i are pairwise distinct (up to provability in T') terms of G. Suppose that the r_i contain additional variables, say that they contain variables from $x_1, \dots, x_k, \vec{y}$; we have (let $\vec{x} = x_1, \dots, x_k$)

$$\vdash_{T'} t(\vec{x}) = s(r_1(\vec{x}, \vec{y}), \dots, r_k(\vec{x}, \vec{y}))$$

Let $\vec{z}$ be renamings of the $\vec{y}$; we get

$$\vdash_{T'} t(\vec{x}) = s(r_1(\vec{x}, \vec{z}), \ldots, r_k(\vec{x}, \vec{z}))$$

hence

$$\vdash_{T'} s(r_1(\vec{x}, \vec{y}), \ldots, r_k(\vec{x}, \vec{y})) = s(r_1(\vec{x}, \vec{z}), \ldots, r_k(\vec{x}, \vec{z})).$$

We can now apply (III) to this situation; as s is minimized we must have

$$\vdash_{T'} r_i(\vec{x}, \vec{y}) = r_i(\vec{x}, \vec{z})$$

for all $i = 1, \ldots, k$ (eventually up to a permutation). Replacing all the $\vec{y}$ by ground terms $\vec{c}$ (we may use the same ground term for all of them), we get

$$\vdash_{T'} r_i(\vec{x}, \vec{c}) = r_i(\vec{x}, \vec{z}) = r_i(\vec{x}, \vec{y}).$$

Now $r_i(\vec{x}, \vec{c})$ is provably equal to $r_i(\vec{x}, \vec{y})$, hence as the latter is in G so is the former (G is closed under provably identical terms according to its definition). For the same reason, all the $r_i(\vec{x}, \vec{c})$ are pairwise distinct (with respect to provable identity in T') because so are the $r_i(\vec{x}, \vec{y})$. We finally get

$$\vdash_{T'} t(\vec{x}) = s(r_1(\vec{x}, \vec{c}), \ldots, r_k(\vec{x}, \vec{c}))$$

which is a factorization matching all the requirements from Proposition 10.1(ii).

Summing up, the difference between the definition of constructibility of [3, 4] and ours, lies in the fact that we do not need any *specific* definition for the class E' of terms used in factorizations.

The refinement factorization technique we used above for comparison with BT-constructibility is interesting by itself. Combining it with the proof of Proposition 10.3, it is not difficult to get the following third characterization of constructibility:

Proposition 10.4 *Let T' be a conservative extension of T. We have that T' is constructible over T iff the T-reduct of any T'-free algebra $F_{T'}(G')$ is a free T-algebra over a set of generators G such that*

- *$G' \subseteq G$;*
- *G is invariant under the T'-isomorphisms of $F_{T'}(G')$ which are the extension of a bijection on the set of free generators G'.*

To finish, let us mention some possible *directions for future research.* Of course, there is the problem of extending our results to combined unification. Secondly, one may try to generalize combined word problems to the case in which the definition of constructibility is related to a weak factorization system of the smaller theory which may not be the standard one (that is, class $\mathcal{E}_0$ is supposed to be larger than the class of projections). Results from Section 6 are still valid, however it is not clear what happens with critical pairs arising from superpositions with products rule. Such enlargements of the definition of constructibility are important because they could cover additional mathematically relevant examples. Finally, although quite difficult, it would be essential to be able to deal with theories extending $T_1 +_{T_0} T_2$ with further axioms. In principle, as our combination algorithm is obtained through rewriting, one may try to apply some form of Knuth-Bendix completion to get decision procedures in such situations too.

References

[1] H. Andréka, Á. Kurucz, I. Németi, I. Sain, and A. Simon. Causes and remedies for undecidability in arrow logics and in multi-modal logics. In *Arrow logic and multi-modal logic*, pages 63–99. CSLI Publ., Stanford, CA, 1996.

[2] F. Baader and T. Nipkow. *Term rewriting and all that.* Cambridge University Press, Cambridge, 1998.

[3] F. Baader and C. Tinelli. Deciding the word problem in the union of equational theories. Technical Report UIUCDCS 98-2073, Department of Computer Science, University of Illinois at Urbana-Champaign, 1998.

[4] F. Baader and C. Tinelli. Deciding the word problem in the union of equational theories sharing constructors. In *Rewriting techniques and applications (Trento, 1999)*, pages 175–189. Springer, Berlin, 1999.

[5] E. Domenjoud, F. Klay, and C. Ringeissen. Combination techniques for nondisjoint equational theories. In *Proc. CADE-12*, volume 814 of *Springer LNAI*, pages 48–64, 1996.

[6] P. J. Freyd and G. M. Kelly. Categories of continuous functors. I. *Journal of Pure and Applied Algebra*, 2:169–191, 1972.

[7] S. Ghilardi. An algebraic theory of normal forms. *Annals of Pure and Applied Logic*, 71(3):189–245, 1991.

[8] S. Ghilardi and G. C. Meloni. Modal logics with n-ary connectives. *Z. Math. Logik Grundlag. Math.*, 36(3):193–215, 1990.

[9] F. William Lawvere. Functorial semantics of algebraic theories. *Proc. Nat. Acad. Sci. U.S.A.*, 50:869–872, 1963.

[10] Christoph Lüth and Neil Ghani. Monads and modular term rewriting. In *Category theory and computer science (Santa Margherita Ligure, 1997)*, pages 69–86. Springer, Berlin, 1997.

[11] T. Nipkow. Combining matching algorithms: the regular case. *J. Symbolic Comput.*, 12(6):633–653, 1991.

[12] Don Pigozzi. The join of equational theories. *Colloq. Math.*, 30:15–25, 1974.