Algebraic and Model Theoretic Techniques for Fusion Decidability in Modal Logics

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Abstract. We introduce a new method (derived from model theoretic general combination procedures in automated deduction) for proving fusion decidability in modal systems. We apply it to show fusion decidability in case not only the boolean connectives, but also a universal modality and nominals are shared symbols.

Introduction

The combination or fusion of two first order equational theories T_1 and T_2 – in the signatures Σ_1 and Σ_2, respectively – is the theory in the signature $\Sigma_1 \cup \Sigma_2$ having as axioms the set of equations $T_1 \cup T_2$. Among the transfer properties from the theories to their fusion, researchers have investigated when a positive answer to the decidability of uniform word problems for the two theories implies a positive answer to the decidability of uniform word problems for their fusion. In case Σ_1, Σ_2 are disjoint the positive answer has been known since [19], whereas in the general case the problem becomes undecidable (a simple undecidable example is supplied in [8]). Recently, a positive answer was independently obtained in [4, 8] in case the two theories share so-called 'constructors', however this rather natural hypothesis seems not to be applicable in the case of modal logics. On the other hand, rather strong fusion decidability transfer results for modal logics exist [32, 2], so a natural challenge arises: how to get them as instances of general combination methods in automated deduction?

In the area of combination problems in automated deduction another main technique consists on the so-called Nelson-Oppen combination procedure [18, 17, 27]. This procedure is also concerned with disjoint signatures, but it is not specifically tailored to uniform word problems and equational theories: it transfers in a general setting the decidability of the universal fragment to the combined

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theory, provided the input theories are only stably infinite, see below. Consequently, as pointed out in [3], the procedure can be useful for combined uniform word problems only in case the input theories have decidable universal fragments (which means, in case of modal logics, that global and not only local consequence relations should be decidable).

In [12, 11] a general algebraic and model theoretic framework is provided in order to extend the Nelson-Oppen combination procedure to non-disjoint signatures. The algorithm presented there relies on a skillful application of Robinson's joint consistency theorem in model theory [6]. In order to apply it the main ingredients are the following. (i) The shared theory $T_0 = T_1 \cap T_2$ should be locally finite: roughly speaking such a theory gives rise to a finite number of configurations to search for. (ii) The theory T_0 is completable, meaning that a model completion T_0^* of T_0 – see [6, 21, 31, 13] – exists. (iii) Every model of the theory T_i can be extended to a model whose T_0-reduct is a model of the completed theory T_0^*. This method is a real extension of the Nelson-Oppen combination procedure from disjoint to non disjoint signatures: indeed, for the Nelson-Oppen procedure to work, one needs that theories T_i are stably infinite, i.e. equivalently, that each model of T_i can be embedded into an infinite model of T_i. Considering that the theory of an infinite set is the model completion of the theory of pure equality, one realizes why the procedure of [12, 11] generalizes the Nelson-Oppen procedure. If, on the other hand, one considers that the theory of Boolean algebras – which is the theory shared by two distinct modal logics – is locally finite, that it has as model completion the theory of atomless Boolean algebras, and that every modal algebra in a given variety can be embedded into an atomless Boolean algebra in the same variety, one immediately recovers Wolter's result concerning fusion transfer of decidability of global consequence relation [32].

The main goal of this paper is to take advantage from the experience matured through [12, 11] to obtain an handy criterion on fusion decidability in modal logics with nominals and a universal modality. Nominals and the universal modality were introduced in modal logics in [20, 5] and further investigated in [14, 10]. Nominals can either be seen as (universally quantified) variables in axiomatization issues or as propositional constants, for instance in description logics (and in this paper too). In the latter sense, they provide a method to define atoms in a Boolean algebra and to code a small amount of predicative logic into the propositional setting. Many description logics [1] can be regarded as fragments of modal logics with nominals and a universal modality. An algorithm for fusion decidability in this setting can therefore be used to solve fusion decidability when dealing with knowledge representation systems.

With a similar aim of dealing with knowledge representation systems, fusion decidability results were partially extended from modal logics to description logics in [2]. Compared to this work, we emphasize here the algebraic perspective: we consider a logic as being determined by its syntax, its axioms, and by the standard inference rules; the logic is not determined by a class of frames and we avoid analyzing the completeness issue. The algebraic perspective makes it possible to specialize the general combined decidability algorithm of [12, 11] and to analyze with different tools the results obtained in [2]. A problem left open there was how to deal with description logics with non trivial use of the universal modality and of nominals. The setting we consider turns out to be quite comprehensive; for example, nominals and the universal modality are allowed in axioms: using a DL terminology, we allow an unrestricted use of individuals and of the universal modality in concept descriptions. We can also cover interesting examples of non trivial use of the universal modality – non trivial in that the universal modality is related to other modalities by specific axioms – that arise in computational logics. Among these logics are the logic of knowledge [16], the converse propositional dynamic logic [28], and the propositional μ-calculus with converse operators [29].

Our main result sounds as follows: we find a criterion on modal systems which ensures fusion decidability. More precisely, we call a modal system nominal closed if – roughly speaking – all the definable nominals are already explicitly named. The fusion of two decidable nominal closed modal systems is shown to be decidable.

The paper is structured as follows. In section 1 we formalize the algebraic setting of modal systems with nominals and present the main result about fusion decidability of modal systems. We discuss the concepts introduced with examples. In section 2 we present the model theoretic background and the extension of the Nelson-Oppen method. This method relies on the notion of model completion of the minimal modal system, which is the object of study of section 3. Finally, in section 4 we apply the method to fusion decidability of modal systems, therefore completing the proof of the main result.

1 Modal Systems with Nominals

Fix $k \geq 0$; by a *k-nominals modal signature* (or simply a *modal signature*, if k is understood) we mean a tuple $\Sigma M = \langle \mathcal{O}, \mu, N_1, \dots, N_k \rangle$, where $\mathcal{O}$ is a set (called the set of of *modal operators*) and μ is an arity function associating with every $O \in \mathcal{O}$ a natural number $\mu(O) \geq 0$. We always assume that $\mathcal{O}$ contains a special operator $\Box_U$ (the *universal modality*) of arity 1. ΣM-formulas are built up in the usual way using countably many propositional variables $x, y, z, \dots$, the propositional constants $N_1, \dots, N_k$ (called the *nominals*), the Boolean connectives $\neg, \wedge, \vee, \top, \bot, \rightarrow, \leftrightarrow$, and the operators in $\mathcal{O}$. We use $\alpha, \beta \dots$ as metavariables for formulas; $\Diamond_U \alpha$ abbreviates $\neg \Box_U \neg \alpha$. A *k-nominals modal system* L (or simply a *modal system*) based on ΣM is a set of formulas closed under uniform substitution and containing (at least) the basic axiom schemata below. We write $L \vdash \alpha$ in order to say that there is an L-deduction of α, where an L-deduction is a deduction using the axiom schemata for L and two inference rules: modus ponens and necessitation (from α infer $\Box_U \alpha$). The *basic axiom schemata* are all the propositional tautologies, the congruential axioms

$$\bigwedge_{i=1}^{n} \Box_U(\alpha_i \leftrightarrow \beta_i) \rightarrow (O(\alpha_1, \dots, \alpha_n) \leftrightarrow O(\beta_1, \dots, \beta_n)) \quad (1)$$

(we have one such axiom for every $O \in \mathcal{O}$ of arity n), the S5 axioms for $\Box_U$

$$\Box_U(\alpha \to \beta) \to (\Box_U \alpha \to \Box_U \beta) \qquad \Box_U \alpha \to \alpha$$
$$\Box_U \alpha \to \Box_U \Box_U \alpha \qquad \Diamond_U \alpha \to \Box_U \Diamond_U \alpha\,,$$

and the following axioms for nominals

$$\Diamond_U N_i \qquad \Diamond_U (N_i \wedge \alpha) \to \Box_U (N_i \to \alpha)\,. \tag{2}$$

We say that a unary modal operator $\Box \in \mathcal{O}$ is normal if L contains the axiom schemata $\Box\top$ and $\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$; if $\Box \in \mathcal{O}$ is normal, its necessitation rule follows from necessitation rule for $\Box_U$ and the remaining axiom schemata; also, the corresponding congruential axioms (1) can equivalently be replaced by the simpler axiom $\Box_U \alpha \to \Box\alpha$.

It is standard practice in propositional logic to associate a variety of algebras with a logic, see for example [9, §1.5]. In the case of a modal system L based on the modal signature ΣM, we can define *a first order equational theory* T_L as follows. The first order signature of T_L is obtained by adding to ΣM the Boolean operators $\neg, \wedge, \vee, \top, \bot, \to$.[1] The axioms of T_L contain for every $\alpha \in L$ a corresponding equation $\alpha = \top$ (here and in the following, we may treat ambiguously the metavariables $\alpha, \beta, \ldots$ as denoting propositional modal formulas and first order terms in the signature of T_L). Models of T_L form the variety V_L, called the variety of L-algebras. It should be clear (for general reasons) that $L \vdash \alpha$ *is equivalent to the fact that* $T_L \models \alpha = \top$. The problem $L \vdash \alpha$ is known as the *decidability problem* for the modal system L, whereas the problem $T_L \models \alpha = \top$ is known as the (uniform) *word problem* for V_L. Given the above mentioned equivalence, in this paper *we shall treat decidability problems for modal logics as word problems in universal algebra*, thus applying general combination methods from automated reasoning.

A main result of Universal Algebra – see for example [15] – is that every algebra can be embedded in a product of subdirectly irreducible algebras, i.e. of algebras having a minimal non trivial congruence. Consequently an equation fails in an arbitrary algebra if and only if it fails in a subdirectly irreducible one. By means of the congruential axioms (1) we can ensure a nice behaviour of congruences of an L-algebra $\mathcal{A}$: these bijectively correspond with $\Box_U$-closed filters of the algebra.[2] Using this information it is easily argued that: the finitely generated congruences are all principal and form a Boolean algebra; in the variety V_L subdirectly irreducible algebras coincide with *simple* algebras, i.e. the algebras with exactly one non trivial congruence; simple algebras are the models of the first order theory T_L^s obtained by adding to T_L the axioms $\bot \neq \top$ and $\forall x(x \neq \top \Rightarrow \Box_U x = \bot)$. Recall that an atom in a Boolean algebra is a minimal

[1] In order to avoid confusion, we shall use different symbols, like $\sim$ for negations, & for conjunctions, $\Rightarrow$ for implications, etc. when we denote connectives in first order logic (this is because $\neg, \wedge, \to, \ldots$ are used to form first order terms containing the operations of the signature of Boolean algebras).

[2] A filter $\Phi \subseteq \mathcal{A}$ is $\Box_U$-closed iff $a \in \Phi$ implies $\Box_U a \in \Phi$.

non-zero element. Observe then that if N is an element of a simple L-algebra $\mathcal{A}$, then the relations

$$\Diamond_U N = \top \qquad\qquad \Diamond_U(N \wedge x) \leq \Box_U(N \to x)$$

– that correspond to the axioms (2) – hold for N and for all $x \in \mathcal{A}$ iff N is an atom of $\mathcal{A}$.

1.1 Main Result

A modal system L is *nominal closed* if and only if, for every formula α, whenever

$$L \vdash \Diamond_U(\alpha \wedge x) \to \Box_U(\alpha \to x)$$

for a propositional variable x not occurring in α, then

$$L \vdash \Box_U(\alpha \to \bot) \vee \bigvee_{i=1}^{k} \Box_U(\alpha \to N_i)\,.$$

Roughly speaking, in a nominal-closed system, there are no hidden nominals, apart from $N_1, \dots, N_k$ which are explicitly mentioned.

Let now L_1, L_2 be k-nominals modal systems over the signatures ΣM_1, ΣM_2 (notice that k is the same in both cases); assume also that $\Sigma M_1 \cap \Sigma M_2$ contains just the universal modality $\Box_U$ and the k-nominals $N_1, \dots, N_k$. The *fusion* of L_1, L_2 is the modal system $L_1 \cup L_2$ over the signature $\Sigma M_1 \cup \Sigma M_2$. Our main result is a decidability transfer (to be proved in sections 2-4):

Theorem *If L_1, L_2 are both decidable and nominal-closed, their fusion $L_1 \cup L_2$ is decidable too.*

We summarize here the variant of the Nelson-Oppen [18, 17] combination schema suggested by the proofs of Theorem 2.3, Lemma 4.1 and Theorem 4.4 of this paper:

Input A $(\Sigma M_1 \cup \Sigma M_2)$- formula α.

Step 1. Apply successive variable abstractions of alien subterms to the literal $\alpha \neq \top$ in order to produce a set Γ_1 of pure T_{L_1}-literals and a set Γ_2 of pure T_{L_2}-literals, so that $\Gamma_1 \cup \Gamma_2$ is equisatisfiable with $\{\alpha \neq \top\}$.

Step 2. Guess a Boolean arrangement[3] Δ of the shared variables appearing both in Γ_1 and in Γ_2 and let $\Gamma_1' = \Gamma_1 \cup \Delta$, $\Gamma_2' = \Gamma_2 \cup \Delta$.

Step 3. Using the decision procedures for L_1, L_2, check whether

$$L_i \not\vdash \Big(\bigwedge_{\beta = \top \in \Gamma_i'} \Box_U \beta\Big) \to \Big(\bigvee_{\gamma \neq \top \in \Gamma_i'} \Box_U \gamma\Big).$$

If this is the case for both $i = 1$ and $i = 2$, **return** '$L_1 \cup L_2 \not\vdash \alpha$'. Otherwise, go back to Step 2.

[3] By a Boolean arrangement on a finite set $\{x_1, \dots, x_n\}$ we mean a set of unit literals representing the diagram of a (necessarily finite) Boolean algebra generated by $x_1, \dots, x_n$. A Boolean arrangement can be guessed by specifying which terms of the kind $\bigwedge_{i=1}^{n} \epsilon_i\, x_i$ are equal to $\top$ and which ones are not (here $\epsilon_i\, x_i$ is either x_i or $\neg x_i$).

Step 4. If this step is reached (namely if all Boolean arrangements in Step 2 have been unsuccessfully tried), **return** '$L_1 \cup L_2 \vdash \alpha$'.

As for complexity of this combined procedure, the same observations of [2] apply: since the purified problem can be produced in linear time and since the guess takes exponential space, the combined procedure may raise the complexity from polynomial to exponential space and from exponential time to double exponential time.[4]

1.2 Examples

Description Logics. Inclusion of nominals into modal systems makes it possible to translate terminologies and assertions of Description Logics into our framework – see also [24] for earlier considerations on description and modal logics. The following is a simple example: consider the terminology and assertion

$$\texttt{Mother} \equiv \texttt{Female} \sqcap \exists \texttt{hasChild}.\top \qquad\qquad \texttt{Mother(MARY)}\,.$$

It can be translated into a modal system (in our sense) with given propositional constants $P_{\texttt{Mother}}$, $P_{\texttt{Female}}$, one normal modal operator $\Diamond_{\texttt{hasChild}}$, and one nominal $N_{\texttt{MARY}}$. On the top of the axioms for nominals and modal operators, the modal system contains the axioms

$$P_{\texttt{Mother}} \leftrightarrow (P_{\texttt{Female}} \wedge \Diamond_{\texttt{hasChild}} \top) \qquad\qquad N_{\texttt{MARY}} \to P_{\texttt{Mother}}\,.$$

Many extensions of the core description logic $\mathcal{ALC}$ can also be algebraized within our modal systems. We list, among them, extensions with transitive roles [25], least or greatest fixed point semantics of cyclic terminologies [23], number restrictions [7], functional and inverse roles – which we are going to exemplify within example (e).

The Universal Modality in Computational Logics. The universal modality is also worth studying in its own since it arises in many contexts. For a finite set of modalities $\Box_\sigma$, $\sigma \in \Sigma$, define $\Box_U x$ as the greatest fixed point

$$\Box_U x = \nu_y.(x \wedge \bigwedge_{\sigma\in\Sigma} \Box_\sigma y)\,. \tag{3}$$

That is, $\Box_U x$ is a fixed point of $f(y) = x \wedge \bigwedge_{\sigma\in\Sigma} \Box_\sigma y$ which is greater of any z such that $z \leq f(z)$. Usually, this modality satisfies the S4 axioms but not the S5 axiom, but there are interesting logics where the latter axiom holds too. In the logic of knowledge [16] the modality $\Box_\sigma$ represents the knowledge of the individual agent σ; usually this is an S5 modality. The common knowledge modality, which is defined as in (3), becomes an S5 modality too.

[4] Notice that the addition of the universal modality usually leads by itself to EXPTIME-complete decision problems (see [26]).

Another interesting example is the converse (test-free) PDL and its extension to the full propositional modal μ-calculus with converse modalities [28, 29]. In these logics there is a converse action $\overline{\sigma} \in \Sigma$ for each $\sigma \in \Sigma$, and the formulas

$$\Diamond_\sigma \Box_{\overline{\sigma}} x \to x \qquad\qquad x \to \Box_{\overline{\sigma}} \Diamond_\sigma x \,, \tag{4}$$

expressing that $\Diamond_\sigma$ and $\Box_{\overline{\sigma}}$ is a residuated pair, are axioms. Again $\Box_U x$, defined as in (3), is easily seen to be a universal modality.

Considering the μ-calculus with converse modalities is appropriate in this context since this logic can be fully algebraized, see [23]. With the μ-calculus we can also exemplify the way non-normal operators arise: the interpretation of an arbitrary inductively defined μ-term gives rise to a non-normal modal operator satisfying the congruential axiom (1), see [22]; moreover such operator is uniquely determined by the original modalities $\Box_\sigma$ and therefore algebraically related to the universal modality $\Box_U$.

How to Check that a Modal System is Nominal Closed. Apparently the property of a modal system L to be nominal closed is difficult to decide: one would expect to need a nice presentation of L such as a sequent calculus with good proof theoretic properties. This is not the case; the following Proposition gives an equivalent algebraic criterion which turns out to be very useful:

Proposition 1.1. *A system L is nominal closed if and only if for any simple L-algebra $\mathcal{A}$ and any atom $a \in \mathcal{A}$ distinct from the N_i, there is a simple L-algebra $\mathcal{B}$ and an L-algebra monomorphism $h : \mathcal{A} \longrightarrow \mathcal{B}$ such that $h(a)$ is not an atom.*

Later on, after Proposition 4.3, it will be evident why the Proposition holds.

(a) We apply Proposition 1.1 to a trivial modal system L. That is, we assume that L has no nominals and that the universal modality interacts with other operators (including itself) only through the congruential axioms (1). It is enough to consider the diagonal $\mathcal{A} \longrightarrow \mathcal{A} \times \mathcal{A}$ where the universal modality is defined on $\mathcal{A} \times \mathcal{A}$ in the unique possible way to obtain a simple algebra: $\Box_U z = \top$ if $z = \top$ and $\Box_U z = \bot$ otherwise. The operations $O \in \mathcal{O}$ are defined as usual in a product algebra. It is an exercise to verify that the congruential axioms hold. Finally, the image of an atom a is the pair (a, a) which is not anymore an atom since $(\bot, \bot) < (a, \bot) < (a, a)$. Thus we have proved: *a trivial modal system L is nominal closed*. This statement is analogous to [2, §25]. $\Box$

(b) We consider now the modal system L – introduced above – with one nominal $N_{\texttt{MARY}}$, propositional constants $P_{\texttt{Mother}}$, $P_{\texttt{Female}}$, and one normal modal operator $\Diamond_{\texttt{hasChild}}$. We use Proposition 1.1 to show that this system is nominal closed.

Suppose that $\mathcal{A}$ is a simple L-algebra and that $a \neq N_{\texttt{MARY}}$ is an atom of $\mathcal{A}$. Recall that the function $\phi : \mathcal{A} \longrightarrow 2$ defined by $\phi(x) = \top$ if and only if $a \leq x$ is morphism of Boolean algebras. Thus we define $h_a = \langle \texttt{id}, \phi \rangle : \mathcal{A} \longrightarrow \mathcal{A} \times 2$ and put a structure of a simple L-algebra on $\mathcal{A} \times 2$ as follows: the universal modality is defined as in (a) while we define

$$P_{\texttt{Female}} = (P_{\texttt{Female}}, \phi(P_{\texttt{Female}})) \qquad P_{\texttt{Mother}} = (P_{\texttt{Mother}}, \phi(P_{\texttt{Mother}}))$$
$$N_{\texttt{MARY}} = (N_{\texttt{MARY}}, \phi(N_{\texttt{MARY}})) \qquad \Diamond_{\texttt{hasChild}}(x, y) = (\Diamond_{\texttt{hasChild}} x, \phi(\Diamond_{\texttt{hasChild}} x)) \,.$$

By definition the extension h_a preserves the L-structure and therefore the relations on constants axiomatizing the system. It is readily seen that the operator $\Diamond_{\texttt{hasChild}}$ on $\mathcal{A} \times 2$ is normal and that the constant $N_{\texttt{MARY}}$ is an atom of $\mathcal{A} \times 2$. This follows since $\phi(N_{\texttt{MARY}}) = \bot$, as a is an atom of $\mathcal{A}$ distinct from $N_{\texttt{MARY}}$. $\Box$

(c) There is an easy sufficient semantic criterion for modal systems L which are Kripke complete w.r.t. a class of frames – see a standard textbook on modal logic such as [9] for frame semantics. Say that a Kripke frame is simple iff the universal modality is interpreted by means of the total relation. In this case, *a modal system L which is complete w.r.t. a class $\mathcal{C}$ of simple Kripke frames is nominal closed provided for every simple frame F in the class $\mathcal{C}$ and each element $w \in F$ distinct from the interpretation of a nominal, there are a simple frame F' in the class $\mathcal{C}$ and a surjective p-morphism $f : F' \longrightarrow F$ such that the fiber $f^{-1}(w)$ contains at least two elements.* The criterion is easily seen to be sufficient by considering the contrapositive of the definition of a nominal closed system.

Here is an application of this semantic criterion. Let L be a modal system containing nominals (and the universal modality) and normal modal operators $\Box_i$, $i = 1, \dots, n$. The axioms of L are the ones for nominals and the universal modality and any combination of the axioms K, T, K4, B for the normal modal operators $\Box_i$. As a Kripke completeness theorem is available, we can apply the above semantic criterion. Argue as follows: let F be a frame for L and $w \in F$ be distinct from the interpretation of nominals; define F' to be $F + \{*\}$ and the function f by $f(v) = v$, for $v \in F$, $f(*) = w$. Define the relation R_i (corresponding to the normal operator $\Box_i$) as follows: $v_1 R_i v_2$ iff $f(v_1) R_i f(v_2)$. This proves that L is nominal closed.

(d) The same technique used in the previous example can be applied to show that the systems we met in the previous subsection are nominal closed, whenever a Kripke completeness theorem is available. For example, the standard filtration technique can be adapted to prove that the converse PDL with k nominals has the finite model property, hence it is complete with respect to the related class of simple Kripke frames. Using this technique is then easy to see that converse PDL with at least one nominal is nominal closed.

(e) Finally, we exhibit a modal system L that is not nominal closed. The modal system L has one nominal N_1 and contains a normal operator $\overrightarrow{\Box}$ as well as its converse modality $\overleftarrow{\Box}$: formulas analogous to those in (4) are axioms. The formula $\overleftarrow{\Diamond} x \to \overleftarrow{\Box} x$ is an axiom of L too. We first observe that: *if $\mathcal{A}$ is a simple L-algebra and a is an atom of $\mathcal{A}$, then either $\overrightarrow{\Diamond} a = \bot$ or $\overrightarrow{\Diamond} a$ is an atom.* For this, it is enough to show that the condition $\bot \le x \le \overrightarrow{\Diamond} a$ implies either $\overrightarrow{\Diamond} a \le x$ or $x = \bot$, so let us assume this condition. If $a \wedge \overleftarrow{\Box} x \neq \bot$, then $a \le \overleftarrow{\Box} x$, and therefore $\overrightarrow{\Diamond} a \le x$. If $a \wedge \overleftarrow{\Box} x = \bot$, then $a \le \neg \overleftarrow{\Box} x = \overleftarrow{\Diamond} \neg x \le \overleftarrow{\Box} \neg x$. It follows that $x \le \overrightarrow{\Diamond} a \le \overrightarrow{\Diamond} \overleftarrow{\Box} \neg x \le \neg x$, and therefore $x \le \bot$. It is now easy to construct a simple L-algebra – by means of a Kripke frame – with the property that $\overrightarrow{\Diamond}^n N_1$ is distinct from $\overrightarrow{\Diamond}^m N_1$ if $n \neq m$. In particular we have a definable infinity of atoms: if $\overrightarrow{\Diamond}^l N_1$ is distinct from the N_i, then it is not possible to embed such an algebra into an algebra where $\Diamond^l N_1$ is not an atom. This shows that the system is not nominal closed.[5] □

2 Model Theory and Combination Problems

We are planning to adapt the extension of the Nelson-Oppen combined decision procedure outlined in [12, 11] to our fusion decidability transfer problem. To this aim, we need to recall some classical model-theoretic ingredients.

A *first order signature* Σ is a set of functions and predicate symbols (each of them endowed with the corresponding arity). We assume the binary equality predicate symbol $=$ to be always present in Σ. The signature obtained from Σ by the addition of a set of new constants ($=$ 0-ary function symbols) X is denoted by $\Sigma \cup X$ or by Σ^X. We have the usual notions of Σ-*term*, (full first order) -*formula*, -*atom*, -*literal*, -*clause*, etc.: e.g. atoms are just atomic formulas, literals are atoms and their negations, clauses are disjunctions of literals, etc. We use letters $\alpha, \beta, \ldots$ for terms and letters $\phi, \psi, \ldots$ for formulas. Terms, literals and clauses are called *ground* whenever free variables do not appear in them. *Sentences* are formulas without free variables. A Σ-*theory* T is a set of sentences (called the axioms of T) in the signature Σ; however when we write $T \subseteq T'$ for theories, we may mean not just set-theoretic inclusion but the fact that all the axioms for T are logical consequences of the axioms for T'.

From the semantic side, we have the standard notion of a Σ-*structure* $\mathcal{A}$: this is nothing but a support set endowed with an arity-matching interpretation of the predicate and function symbols from Σ. We shall notationally confuse, for the sake of simplicity, a structure with its support set. Truth of a Σ-formula in $\mathcal{A}$ is defined in any one of the standard ways; a Σ-structure $\mathcal{A}$ is a *model* of a Σ-theory T (in symbols $\mathcal{A} \models T$) iff all axioms of T are true in $\mathcal{A}$ (for models of a Σ-theory T we shall sometimes use the letters $\mathcal{M}, \mathcal{N}, \ldots$ to distinguish them from arbitrary Σ-structures). If ϕ is a formula, $T \models \phi$ ('ϕ *is a logical consequence of* T') means that ϕ is true in any model of T. The *word problem* for T is the problem of deciding whether the universal closure of a Σ-atom is a logical consequence of T; similarly, the *clausal word problem* for T is the problem of deciding whether the universal closure of a Σ-clause is a logical consequence of T. A Σ-theory T is *complete* iff for every Σ-sentence ϕ, either ϕ or $\neg\phi$ is a logical consequence of T; T is *consistent* iff it has a model.

An *embedding* between two Σ-structures $\mathcal{A}$ and $\mathcal{B}$ is any map $f : \mathcal{A} \longrightarrow \mathcal{B}$ among the corresponding support sets satisfying the condition

$$(*) \qquad \mathcal{A} \models A \quad \text{iff} \quad \mathcal{B} \models A$$

for all $\Sigma^{\mathcal{A}}$ atoms A (here $\mathcal{A}$ is regarded as a $\Sigma^{\mathcal{A}}$-structure by interpreting each $a \in \mathcal{A}$ into itself and $\mathcal{B}$ is regarded as a $\Sigma^{\mathcal{A}}$-structure by interpreting each $a \in \mathcal{A}$

[5] A little warning: there might be 0-nominals modal systems which are not nominal-closed. Notice however that this may happen only in case the universal modality has a non-trivial interaction with the remaining modal operators, because of example (a).

into $f(a)$). In case (*) holds for all first order formulas, the embedding is said to be *elementary*.

The *diagram* $\Delta(\mathcal{A})$ of a Σ-structure $\mathcal{A}$ is the set of ground $\Sigma^{\mathcal{A}}$-literals which are true in $\mathcal{A}$; the elementary diagram $\Delta^e(\mathcal{A})$ of a Σ-structure $\mathcal{A}$ is the set of $\Sigma^{\mathcal{A}}$-sentences which are true in $\mathcal{A}$. Robinson (elementary) diagram theorem [6] says that there is an (elementary) embedding between the Σ-structures $\mathcal{A}$ and $\mathcal{B}$ iff it is possible to expand $\mathcal{B}$ to a $\Sigma^{\mathcal{A}}$-structure in such a way that it becomes a model of the (elementary) diagram of $\mathcal{A}$. This theorem will be repeatedly used without explicit mention in the paper.

We shall need the well-known notion of a *model completion* of a theory; we take the definition from e.g. [30]. Let T be a Σ-theory and let $T^* \supseteq T$ be a further Σ-theory; we say that T^* is a model completion of T iff (i) every model of T has an embedding into a model of T^* and (ii) for every model $\mathcal{M}$ of T, we have that $T^* \cup \Delta(\mathcal{M})$ is a complete theory. The following Proposition gives an equivalent formulation in case T is universal and the signature is at most countable:

Proposition 2.1. *Let T be a universal Σ-theory (where Σ is at most countable) and let $T^* \supseteq T$ be a further Σ-theory such that every model of T has an embedding into a model of T^*. Then T^* is a model completion of T iff whenever $\mathcal{M}_1$, $\mathcal{M}_2$ are both at most countable models of T^* extending a common finitely generated substructure $\mathcal{A}$, then $\mathcal{M}_1$ and $\mathcal{M}_2$ are elementarily equivalent as $\Sigma \cup \mathcal{A}$-structures.*

Proof. One side is trivial (because $\mathcal{M}_1, \mathcal{M}_2$ are both models of $T^* \cup \Delta(\mathcal{A})$ and $\mathcal{A} \models T$, being T universal). For the other side, let $\mathcal{M}$ be a model of T and let $\phi(\underline{a})$ be a formula with parameters $\underline{a}$ from $\mathcal{M}$. Suppose that there are models $\mathcal{M}_1, \mathcal{M}_2$ of $T^* \cup \Delta(\mathcal{M})$ such that $\mathcal{M}_1 \models \phi(\underline{a})$ and $\mathcal{M}_2 \models \neg\phi(\underline{a})$. Let $\mathcal{A}$ be the substructure of $\mathcal{M}$ generated by $\underline{a}$; by downward Löwenheim-Skolem theorem, $\mathcal{M}_1, \mathcal{M}_2$ are elementarily equivalent as $\Sigma^{\underline{a}}$-structures to at most countable models $\mathcal{M}'_1, \mathcal{M}'_2$, respectively: this gives a contradiction. ☐

It can be shown that a model completion T^* of a theory T is unique, in case it exists, see [6]. There are many classical examples of model completions from algebra [6]: the theory of algebraically closed fields is the model completion of the theory of fields, the theory of divisible torsion free abelian groups is the model completion of the theory of torsion free abelian groups, etc. An example which is more relevant for this paper is the following: the theory of atomless Boolean algebras is the model completion of the theory of Boolean algebras (for model completions arising in the algebra of logic, see the book [13]). Next, we give the definition of Σ_0-compatibility [12, 11]:

Definition 2.2. *Let T be a theory in the signature Σ and let T_0 be a universal theory in a subsignature $\Sigma_0 \subseteq \Sigma$. We say that T is T_0-*compatible *iff* (i) $T_0 \subseteq T$*;* (ii) *T_0 has a model-completion T_0^*;* (iii) *every model of T embeds into a model of $T \cup T_0^*$.*

We say that a Σ_0-universal theory T_0 is *locally finite* iff Σ_0 is finite and for every finite set $\underline{a}$ of new free constants, there are only finitely many $\Sigma^{\underline{a}}$-ground terms up to T_0-identity (let their number be $k_{T_0}(\underline{a})$). As we are mainly dealing with computational aspects, we consider part of the definition the further request that $k_{T_0}(\underline{a})$ is effectively computable from $\underline{a}$. Examples of locally finite theories important for this paper are the theory of Boolean algebras and of $S5$-(uni)modal algebras.

For our fusion decidability results, the main ingredient is the following theorem [12, 11]. The decision algorithm presented in the previous section is suggested from its proof which therefore is included.

Theorem 2.3. *Assume that T_1 is a Σ_1-theory and that T_2 is a Σ_2-theory which are both compatible with respect to a locally finite universal Σ_0-theory T_0 (where Σ_0 is $\Sigma_1 \cap \Sigma_2$). If the clausal word problem in both T_1, T_2 is decidable, so is the clausal word problem in $T_1 \cup T_2$.*

Proof. Let $\forall \underline{x}\ (\alpha_1 \neq \beta_1 \vee \cdots \vee \alpha_n \neq \beta_n \vee \alpha'_1 = \beta'_1 \vee \cdots \vee \alpha'_m = \beta'_m)$ be the $\Sigma_1 \cup \Sigma_2$-clause we want to decide. Taking negation and skolemization, we need to test for $T_1 \cup T_2$-consistency the set of ground literals (here $\underline{b}$ are new constants)

$$\{\alpha_1(\underline{b}) = \beta_1(\underline{b}), \ldots, \alpha_n(\underline{b}) = \beta_n(\underline{b}), \alpha'_1(\underline{b}) \neq \beta'_1(\underline{b}), \ldots, \alpha'_m(\underline{b}) \neq \beta'_m(\underline{b})\}.$$

This set, call it Γ, can be purified: in fact we get equiconsistency if we replace in it a subterm γ with a new constant c and add the further equation $c = \gamma$. Doing that repeatedly, we finally get two sets of literals Γ_1, Γ_2 such that for a certain finite set $\underline{a}$ of new constants (including the $\underline{b}$'s) we have that: (a) Γ_1 is a set of $\Sigma_1^{\underline{a}}$-ground literals; (b) Γ_2 is a set of $\Sigma_2^{\underline{a}}$-ground literals; (c) $\Gamma_1 \cup \Gamma_2$ is $T_1 \cup T_2$-equiconsistent with Γ.

We shall show that $\Gamma_1 \cup \Gamma_2$ is $T_1 \cup T_2$-consistent iff there is a Σ_0-structure $\mathcal{A}$ such that: (i) $\mathcal{A}$ is generated by $\underline{a}$; (ii) $\Gamma_1 \cup \Delta(\mathcal{A})$ is T_1-consistent and (iii) $\Gamma_2 \cup \Delta(\mathcal{A})$ is T_2-consistent. This will prove the theorem since the input problem has been reduced to finitely many pairs of pure problems, which are solvable by hypothesis.[6]

One side is trivial; so suppose that there is $\mathcal{A}$ satisfying (i)-(ii)-(iii). This means that $T_1 \cup \Gamma_1 \cup \Delta(\mathcal{A})$ has a Σ_1-model $\mathcal{M}_1$ and that $T_2 \cup \Gamma_2 \cup \Delta(\mathcal{A})$ has a Σ_2-model $\mathcal{M}_2$: by T_0-compatibility, we can freely suppose that $\mathcal{M}_i \models T_0^*$ for $i = 1, 2$ (recall that truth of Γ_i, which is a set of ground literals, is preserved by superstructures). Also, we can rename elements in the supports so that $\mathcal{M}_1 \cap \mathcal{M}_2 = \mathcal{A}$. Now $T_0^* \cup \Delta(\mathcal{A})$ is a $\Sigma_0 \cup \mathcal{A}$-complete theory and $\Delta^e(\mathcal{M}_1)$, $\Delta^e(\mathcal{M}_2)$ are both consistent extensions of its (in signatures $\Sigma_1 \cup \mathcal{M}_1$ and $\Sigma_2 \cup \mathcal{M}_2$ such that $(\Sigma_1 \cup \mathcal{M}_1) \cap (\Sigma_2 \cup \mathcal{M}_2) = \Sigma_0 \cup \mathcal{A}$). By Robinson joint consistency theorem [6], $\Delta^e(\mathcal{M}_1) \cup \Delta^e(\mathcal{M}_2)$ has a model which in particular is a $(\Sigma_1 \cup \Sigma_2)^{\underline{a}}$-model of $T_1 \cup \Gamma_1 \cup T_2 \cup \Gamma_2$, as desired. □

[6] Notice that Σ_0-structures satisfying (i)-(ii)-(iii) are finitely many and effectively computable: since T_0 is universal, they must be models of T_0, hence they cannot have more than $k_{T_0}(\underline{a})$ elements, because T_0 is locally finite.

3 The Model Completion of $T^s_{L_0}$

We call L_0 the minimum k-nominal modal system: it just contains the universal modality and the nominals $N_1, \ldots, N_k$; the basic axiom schemata of every k-nominal modal system (see section 1) are the only axiom schemata in L_0. In view of applying Theorem 2.3 to the fusion of modal systems, we need a better grasp on the model completion of $T^s_{L_0}$, the theory of simple L_0-algebras.

Definition 3.1. *A simple L_0-algebra $\mathcal{A}$ is said to be* quasi-atomless *if the only atoms of $\mathcal{A}$ are $N_1, \ldots, N_k$.*

Clearly, there are many first order formulas that axiomatize quasi-atomless L_0-algebras; among them

$$\forall y \ (\ y \neq \bot \ \& \ y \neq N_1 \ \& \cdots \& \ y \neq N_k \ \Rightarrow \ \exists x \ (\bot < x < y) \ ) \, . \qquad \text{(QA)}$$

We have that:

Theorem 3.2. *The theory $(T^s_{L_0})^*$ of quasi-atomless simple L_0-algebras is the model completion of the theory $T^s_{L_0}$ of simple L_0-algebras.*

Using Proposition 2.1, the theorem is proved observing that:

Proposition 3.3. *(i) Every simple L_0-algebra can be embedded into a quasi-atomless simple L_0-algebra. (ii) Given a finite L_0-algebra $\mathcal{A}$ and two at most countable simple quasi-atomless extensions $\mathcal{B}, \mathcal{C}$ of $\mathcal{A}$, there is an isomorphism from $\mathcal{B}$ to $\mathcal{C}$ fixing $\mathcal{A}$.*

For lack of space, we only sketch the proof. The first statement (i) is proved using a tool introduced in section 1: given a simple L_0-algebra and an atom $a \in \mathcal{A}$ distinct from the N_i one produces a simple L_0-algebra on $\mathcal{A} \times 2$ and shows that the embedding h_a defined in example (b) is an homomorphism of L_0-algebras; starting from a simple L_0-algebra, one carefully iterates this construction producing an infinite chain: the inductive limit of this chain is the desired quasi-atomless simple L_0-algebra. The second statement (ii) is a consequence of:

Lemma 3.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be two finite simple L_0-algebras, and let $\mathcal{C}$ be an infinite quasi-atomless L_0-algebras. Given monomorphisms $i : \mathcal{A} \longrightarrow \mathcal{C}$ and $j : \mathcal{A} \longrightarrow \mathcal{B}$, there exists a monomorphism $k : \mathcal{B} \longrightarrow \mathcal{C}$ such that $k \circ j = i$.*

This lemma – easily verified using Stone duality – allows to progressively construct the isomorphism of (ii) using back and forth.

4 Fusion Decidability for Modal Systems

We are now ready is to apply Theorem 2.3 to fusion decidability of modal systems.

Lemma 4.1. *If the modal system L is decidable, then so is the clausal word problem in T^s_L.*

Proof. A finite set $\{\,\alpha_1 = \top, \ldots, \alpha_n = \top, \beta_1 \neq \top, \ldots, \beta_m \neq \top\,\}$ of T^s_L-literals (containing additional free constants induced by the Skolemization of the universal closure of a T_L-clause) is satisfiable iff there is a simple algebra $\mathcal{A}$ such that $\mathcal{A} \models \Box_U \alpha_1 = \top, \ldots \mathcal{A} \models \Box_U \alpha_n = \top, \mathcal{A} \models \Box_U \beta_1 = \bot, \ldots, \Box_U \beta_m = \bot$ i.e. iff there is a simple algebra $\mathcal{A}$ such that

$$\mathcal{A} \models \Box_U \alpha_1 \wedge \cdots \wedge \Box_U \alpha_n \wedge \neg \Box_U \beta_1 \wedge \cdots \wedge \neg \Box_U \beta_m = \top.$$

As simple algebras coincide with subdirectly irreducible algebras in our variety, this simply means that the formula

$$\neg(\Box_U \alpha_1 \wedge \cdots \wedge \Box_U \alpha_n \wedge \neg \Box_U \beta_1 \wedge \cdots \wedge \neg \Box_U \beta_m)$$

is not a theorem of the system L (notice that this formula can denote either $\bot$ or $\top$ in any simple algebra). $\Box$

Lemma 4.2. *Suppose that L is nominal closed. Let $\mathcal{A}$ be a model of T^s_L and let a be an atom of $\mathcal{A}$ which is different from $N_1, \ldots, N_k$. Then it is possible to embed $\mathcal{A}$ into a model of T^s_L in which a is not an atom anymore.*

Proof. Let $\mathcal{F}(X)/\Phi$ be a presentation of $\mathcal{A}$ as a quotient of the free algebra $\mathcal{F}(X)$ divided by the $\Box_U$-closed filter Φ; suppose that a is (in this presentation) the equivalence class $[\alpha]$ of the term α. Take $y \notin X$ and consider the algebra $\mathcal{B}$ having $\mathcal{F}(X \cup \{y\})/\Psi$ as a presentation, where Ψ is the $\Box_U$-closed filter generated by the set of formulas $\Phi \cup \{\Diamond_U(\alpha \wedge y), \Diamond_U(\alpha \wedge \neg y)\}$. If we are able to show that Ψ does not contain $\bot$, then we are done, since if $\mathcal{C}$ is a simple quotient of $\mathcal{B}$ then the composite map $\mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C}$ is injective – because $\mathcal{A}$ is simple – and the relations $\bot < [\alpha \wedge y] < [\alpha]$ hold in $\mathcal{C}$ by construction. Suppose therefore that Ψ contains $\bot$, then for some X-term β such that $[\Box_U \beta] \in \Phi$, we have $L \vdash \Box_U \beta \wedge \Diamond_U(\alpha \wedge y) \wedge \Diamond_U(\alpha \wedge \neg y) \to \bot$ and also, by the S5 axioms, $L \vdash \Diamond_U(\Box_U \beta \wedge \alpha \wedge y) \wedge \Diamond_U(\Box_U \beta \wedge \alpha \wedge \neg y) \to \bot$. By Boolean transformations, we get $L \vdash \Diamond_U(\Box_U \beta \wedge \alpha \wedge y) \to \Box_U(\Box_U \beta \wedge \alpha \to y)$. As L is nominal closed, we have $L \vdash \bigvee_{i=0}^k \Box_U(\Box_U \beta \wedge \alpha \to N_i)$ (where we take $N_0 = \bot$). If we read this relation within the simple algebra $\mathcal{A}$, we get $a = [\alpha] \leq N_i$ for some i, a contradiction. $\Box$

Proposition 4.3. *T^s_L is $T^s_{L_0}$-compatible iff L is nominal-closed.*

Proof. Suppose that L is nominal-closed. It is sufficient to repeatedly apply the previous Lemma: given a simple algebra $\mathcal{A}_0$, let $\{a_i\}_i$ be a well-ordering of the atoms of $\mathcal{A}_0$ different from $N_1, \ldots, N_k$. Define simple algebras $\mathcal{A}^i_0$ by transfinite induction by inserting new elements $\bot < b_i < a_i$; then take the union $\mathcal{A}_1 = \bigcup_i \mathcal{A}^i_0$ and repeat this construction ω-times (in order to eliminate also newly introduced atoms).

Suppose, vice-versa, that T^s_L is $T^s_{L_0}$-compatible and let α be such that $L \vdash \Diamond_U(\alpha \wedge x) \to \Box_U(\alpha \to x)$ holds for x not occurring in α. If $L \nvdash \bigvee_{i=0}^k \Box_U(\alpha \to N_i)$, then there is a subdirectly irreducible (hence simple) algebra $\mathcal{A}$ such that $\mathcal{A} \not\models \bigvee_{i=0}^k \Box_U(\alpha(\underline{a}) \to N_i) \neq \top$, for some $\underline{a} \in \mathcal{A}$ (replacing the variables appearing in α). This means that $\alpha(\underline{a}) \neq \bot$ and that for all $i = 1, \ldots, k$, in $\mathcal{A}$ we have $\alpha(\underline{a}) \not\leq N_i$. By compatibility, we can embed $\mathcal{A}$ into a simple algebra $\mathcal{B}$ in which $\alpha(\underline{a})$ is not an atom. This means that in $\mathcal{B}$ there is b such that $\bot < b < \alpha(\underline{a})$: such b contradicts the fact that $L \vdash \Diamond_U(\alpha \wedge x) \to \Box_U(\alpha \to x)$ (just replace the propositional variables in α by $\underline{a}$ and x by b). ☐

We can now prove our main result:

Theorem 4.4. *Let L_1, L_2 be both nominal-closed. If L_1, L_2 are decidable, so is their fusion $L_1 \cup L_2$.*

Proof. By general reasons, $L_1 \cup L_2 \not\vdash \alpha$ iff there is a subdirectly irreducible (hence simple) algebra $\mathcal{A} \models T_{L_1} \cup T_{L_2}$ in which the equation $\alpha = \top$ fails. Hence it is sufficient to be able to solve word problems in $(T_{L_1} \cup T_{L_2})^s = T^s_{L_1} \cup T^s_{L_2}$: this is in fact the case, by Theorem 2.3, Lemma 4.1, Theorem 3.2 and Proposition 4.3. ☐

It is easily seen that T_0-compatibility is a modular property (see in any case Proposition 3.3 of [12]), hence it is an immediate corollary of Proposition 4.3 that if L_1, L_2 are both nominal closed, so is their fusion. We can consequently generalize the above results to:

Corollary 4.5. *If $L_1, \ldots, L_n$ are nominal closed and decidable, then their iterated fusion $L_1 \cup \ldots \cup L_n$ is also decidable.*

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