# From Free Agebras to Proof Bounds

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# 20 Years in Free Modal Algebras Constructions

*Step-by-step* constructions of free modal algebras have longstanding tradition:

- they can traced back to Fine normal forms (1975);
- Abramsky (1988, published much later in 2005);
- Ghilardi (1995; for Heyting case 1992, for S4 case 2010);
- coalgebraic literature (N. Bexhanishvili, Kurz, Kupke, Gehrke, Pattinson, Schröder, Venema, ... 2000–> );
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Coumans-van Gool emphasis is on partial algebras; a light reformulation of their point of view uses *two-sorted structures*.

These are called step-algebras and step-frames in (N. Bezhanishvili-Ghilardi-Jibladze, in press).

We review step-algebras and step-frames and then discuss ongoing work *concerning proof theoretic applications*

We anticipate the kind of proof theoretic aspects we want to investigate.

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# Logics and Inference Systems

A logic $L$ is a set of formulae containing tautologies, Aristotle's law and closed under necessitation, modus ponens and uniform substitution.

An inference system $Ax$ for $L$ is a set of inference rules

$$\frac{\phi_1(\underline{x}), \ldots, \phi_n(\underline{x})}{\psi(\underline{x})} \tag{1}$$

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# Logics and Inference Systems

- all rules in $Ax$ are derivable from $L$;
- from the rules in $Ax$ all formulae in $L$ are provable;
- all formulae occurring as premises or as conclusions of rules in $Ax$ have modal degree at most 1;
- if a propositional variable occurs in a rule $r \in Ax$, then it has an occurrence in $r$ which is located inside a modal operator.

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# Logics and Inference Systems

For every *L* one can find an *Ax* which is suitable for *L* (there are also canonical methods for that), but ...

... not all equivalent inference systems are proof-theoretically equally good ...

... for some of them proof search may become intricated ...

... we want at least the following: *to prove a formula $\phi$, only formulae up to the modal degree of $\phi$ are needed*.

Since the above property leads to decidability, it won't be possible to get it in general, but we want to have some criteria to recognize good systems.

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# Logics and Inference Systems

As an example, take GL-system. It can be axiomatized by the single axiom $\Box(\Box x \to x) \to \Box x$.
An inference system for $L$ consists of transitivity and Godel-Lob rules

$$\frac{\Box^+ y \to x}{\Box y \to \Box x} \qquad \frac{\Box x \to x}{x} \tag{2}$$

An alternative one (the good one!) consists of the Avron rule

$$\frac{\Box^+ x \wedge \Box y \to y}{\Box x \to \Box y} \tag{3}$$

# Outline

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1. Free Modal Algebras
   - The Local Method
   - Adding Equations
   - Step Correspondence
2. The Step Embedding Theorem
   - An Example
   - Some Case Studies
   - Conclusions

# One-Step Modal Algebras

## Definition

1. A *one-step modal algebra* is a quadruple $(A_0, A_1, i_0, \Diamond_0)$, where $A_0, A_1$ are Boolean algebras, $i_0 : A_0 \to A_1$ is a Boolean morphism, and $\Diamond_0 : A_0 \to A_1$ is a semilattice morphism. The algebras $A_0, A_1$ are called the *source* and the *target* Boolean algebras of the one-step modal algebra $(A_0, A_1, i_0, \Diamond_0)$.
2. A *one-step extension* of the one-step modal algebra $(A_0, A_1, i_0, \Diamond_0)$ is a one-step modal algebra $(A_1, A_2, i_1, \Diamond_1)$ satisfying $i_1 \Diamond_0 = \Diamond_1 i_0$.

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# One-Step Modal Algebras

The ‘universal’ one-step extension of a given $(A_0, A_1, i_0, \Diamond_0)$ is built by using pushouts ($A_2 \simeq A_1 +_{V(A_0)} V(A_1)$) as follows

$$
\begin{array}{ccc}
V(A_0) & \xrightarrow{V(i_0)} & V(A_1) \\
\downarrow{\scriptstyle \Diamond_0^T} & & \downarrow{\scriptstyle \Diamond_1^T} \\
A_1 & \xrightarrow{i_1} & A_2
\end{array}
$$

where we used the bijective correspondence given by Vietoris functor

$$
\frac{\Diamond : C \longrightarrow D \qquad \text{(in } \textbf{SemiL}\text{)}}{\Diamond^T : V(C) \longrightarrow D \quad \text{(in } \textbf{Bool}\text{)}}
$$

# One-Step Modal Algebras

We can use the above construction to built a chain; we start with a finite Boolean algebra $B_0$ and build the one step modal algebra $i_0, \Diamond_0 : B_0 \rightrightarrows B_0 + V(B_0)$, where $i_0, \Diamond_0^T$ are the coproduct injections.

Then we go on with universal one-step extensions. We have a chain

$$B_0 \rightrightarrows B_1 \rightrightarrows B_2 \rightrightarrows \cdots \quad (4)$$

The upper morphisms $i_k$ are Boolean morphisms and the lower morphisms $\Diamond_k$ are Semilattice morphisms. Since we have $\Diamond_{k+1} \circ i_k = i_{k+1} \circ \Diamond_k$, we can define

$$\Diamond : \varinjlim B_k \longrightarrow \varinjlim B_k$$

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The point is now to add equations axiomatizing modal algebras for a given logic. Following a suggestions by Coumans & Van Gool (in press), we transform axioms into inferences of modal degree 1.

For instance, in case of **K4**, the axiom

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Equation (5) can be interpreted in a one-step modal algebra $(A_0, A_1, i, \Diamond)$, where is says

$$\forall x, y \in A_0 \ (\Box x \leq i(y) \Rightarrow \Box x \leq \Box y). \tag{6}$$

Thus it is sufficient, whenever we build one-step extensions, *to divide the newly built Boolean algebra by the minimum congruence leading* (6) *to hold*.

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There is a point to care indeed. In the quotient chain

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we are not guaranteed that the upper Boolean morphisms $i_k$ are injective.

Thus, we can claim that the colimit algebra is the free algebra (modulo the extra conditions), but we cannot claim that the quotient algebra $\tilde{B}_k$ is the algebra of formulae of modal complexity less or equal to $k$.

To claim it, we need an extra argument showing injectivity of the $i_k$. This amounts to show that provable formulae of a certain modal complexity have proofs of that complexity. The required argument cannot be general, because it implies decidability of the logic!

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# Applying Duality

To get a better understanding of the situation, it is nice to apply duality. Since we are interested in finite algebras, duality needs not topological machinery and is easy.

## Definition

1. A *one-step frame* is a quadruple $(W_1, W_0, f, R)$, where $W_0, W_1$ are sets, $f : W_1 \longrightarrow W_0$ is a function and $R \subseteq W_1 \times W_0$ is a relation.
2. A *one-step extension* of the one-step frame $(W_1, W_0, f, R)$ is a one-step frame $(W_2, W_1, g, S)$ satisfying $f \circ S = g \circ R$.

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All constructions we used have duals: coproducts are turned into products, pushouts into pullbacks, Vietoris functor is turned into covariant power set functor, etc.

Thus we can easily build, starting from a finite set $B_0^*$ the dual of the Boolean algebras chain (4)

$$B_0^* \leftleftarrows B_1^* \leftleftarrows B_2^* \leftleftarrows \cdots$$

and (in presence of further axioms) the dual of the quotient chain (7)

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Our problem is turned into the problem of showing that the upper functions are *surjective*.

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# Step Correspondence Theory

To clarify our goal we need to understand better what it means for (the dual of) a one-step frame $(W_1, W_0, f, R)$ to validate an inference rule like the transitivity rule we got for **K4**.

This happens iff

$$\forall x, y \subseteq W_0\ (\Box_R x \subseteq f^*(y) \Rightarrow \Box_R x \subseteq \Box_R y). \tag{8}$$

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# Step Correspondence Theory

Applying adjunction $\exists_f \dashv f^*$, Ackermann rule, set-theoretic definition of $\subseteq$, definition of $\Box_R$ and Ackermann rule again we get:

$$\begin{array}{rc}
\forall x, y \subseteq W_0. & \Box_R x \subseteq f^*(y) \Rightarrow \Box_R x \subseteq \Box_R y \\
\forall x, y \subseteq W_0. & \exists_f(\Box_R x) \subseteq y \Rightarrow \Box_R x \subseteq \Box_R y \\
\forall x \subseteq W_0. & \Box_R x \subseteq \Box_R \exists_f (\Box_R x) \\
\forall x \subseteq W_0 \forall w \in W_0. & w \in \Box_R x \Rightarrow w \in \Box_R \exists_f (\Box_R x). \\
\forall x \subseteq W_0 \; \forall w \in W_0. & R(w) \subseteq x \Rightarrow w \in \Box_R \exists_f (\Box_R x) \\
\forall w \in W_0. & w \in \Box_R \exists_f (\Box_R R(w)).
\end{array}$$

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Last condition reads as:

$$\forall w \forall v \ (R(w,v) \to \exists w_1 \ (f(w_1) = v \ \& \ R(w_1) \subseteq R(w))) \ . \qquad (9)$$

The possibility of getting a first order condition is subject to sufficient syntactic conditions very similar to those of Sahlqvist theorem. Using the above first order characterization it is possible to give a direct description of the dual quotient chain

$$\tilde{B}_0^* \leftleftarrows \tilde{B}_1^* \leftleftarrows \tilde{B}_2^* \leftleftarrows \cdots$$

and to prove that the upper fuctions are surjective: details are in our paper (N. Bezhanishvili & Ghilardi & Jibladze, in press). We do not insist on such details here, because we want to show that the whole machinery becomes much more manageable if we let finite model property (for global consequence relation) to enter in the picture.

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and to prove that the upper fuctions are surjective: details are in our paper (N. Bezhanishvili & Ghilardi & Jibladze, in press).

We do not insist on such details here, because we want to show that the whole machinery becomes much more manageable if we let finite model property (for global consequence relation) to enter in the picture.

# Step Correspondence Theory

Last condition reads as:

$$\forall w \forall v \; (R(w,v) \to \exists w_1 \; (f(w_1) = v \;\&\; R(w_1) \subseteq R(w)))\;. \qquad (9)$$

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We do not insist on such details here, because we want to show that the whole machinery becomes much more manageable if we let finite model property (for global consequence relation) to enter in the picture.

# The Step Embedding Theorem

Given a logic $L$, for a finite set of formulae $\Gamma$ and for a formula $\phi$, we write

$$\Gamma \vdash_L \phi \tag{10}$$

for the *global* consequence relation: this means that $\phi$ has a proof from axioms and rules of $L$ with premises $\Gamma$ (uniform substitution applies only to formulae in $L$, not to formulae in $\Gamma$). For transitive systems, $\Gamma \vdash_L \phi$ is the same as $\Box^+(\bigwedge \Gamma) \to \phi \in L$.
For axioms systems $Ax$, we have a similar notion

$$\Gamma \vdash_{Ax} \phi \tag{11}$$

Notice that if $Ax$ is an inference system for $L$, the relations (10) and (11) are equivalent.

# The Step Embedding Theorem

## Definition

An inference system *Ax* has the *bounded proof property* (bpp) iff whenever $\Gamma \vdash_{Ax} \phi$ holds, there is a proof in *Ax* of $\phi$ with premises $\Gamma$ in which formulae not exceeding the modal degree of formulae in $\Gamma, \phi$ occur.

## Definition

A logic *L* has the *finite model property* (fmp) iff whenever $\Gamma \vdash_{Ax} \phi$ does not hold, then there is a Kripke model based on a finite frame for *L* where the $\Gamma$ are everywhere true and $\phi$ is not.

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## Definition

A one-step frame $(W_1, W_0, f, R)$ is *conservative* iff (i) $f$ is surjective and (ii) for every $w_1, w_2 \in W_1$ we have that

$$f(w_1) = f(w_2) \ \& \ R(w_1) = R(w_2) \ \Rightarrow \ w_1 = w_2. \tag{12}$$

Dually, a one-step modal algebra $(A_0, A_1, i, \Diamond)$ is conservative iff (i) $i$ is injective and (ii) the set

$$\{i(a) \mid a \in A_0\} \cup \{\Diamond a \mid a \in A_0\}$$

generate $A_1$ as a Boolean algebra.

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# The Step Embedding Theorem

**Theorem**

*Let Ax be an inference system for a logic L. Then every finite conservative step-frame validating Ax is a p-morphic image of a finite Kripke frame for L iff Ax has the bpp and L has the fmp.*

Kripke frames are here seen as step-frames with identical step transition function $f$; the notion of a p-morphism between step-frames is the obvious one.

# The Step Embedding Theorem

A *p-morphism* between step frames $\mathcal{F}' = (W_1', W_0', f', R')$ and $\mathcal{F} = (W_1, W_0, f, R)$ is a pair of surjective maps $\mu : W_1' \longrightarrow W_1, \quad \nu : W_0' \longrightarrow W_0$ such that

$$f \circ \mu = \nu \circ f' \quad \text{and} \quad R \circ \mu = \nu \circ R' \; . \tag{13}$$

$$\begin{array}{ccc} W_1' & \xrightarrow{\ \mu\ } & W_1 \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ W_0' & \xrightarrow{\ \nu\ } & W_0 \end{array} \qquad\qquad \begin{array}{ccc} W_1' & \xrightarrow{\ \mu\ } & W_1 \\ {\scriptstyle R'}\downarrow & & \downarrow{\scriptstyle R} \\ W_0' & \xrightarrow{\ \nu\ } & W_0 \end{array}$$

# An example

Using the above theorem, it is easy to prove fmp and bpp for simple logics like $K, K4, T, S4, \ldots$. As an example, let us consider the density axiom $\Box\Box x \to \Box x$ (we add this axiom to $K$).

*First*, we turn the axiom into a rule; there is a default naif method for that giving

$$\frac{y \to \Box x}{\Box y \to \Box x.} \tag{14}$$

*Second*, applying step correspondence, we get the following first-order characterization for validation of (14) in step frames:

$$\forall w \forall v \ (wRv \Rightarrow \exists k \ (wRf(k) \ \& \ kRv)) \tag{15}$$

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*Third*, we fix a finite conservative step-frame $\mathcal{S} = (W_1, W_0, f, R)$ satisfying (15); we must find a finite frame $\mathfrak{F} = (V, S)$ which dense in the standard sense

$$\forall w \forall v\ (wSv \Rightarrow \exists k\ (kSv\ \&\ wSk)). \tag{16}$$

and a surjective map $\mu : V \longrightarrow W_1$ such that $R \circ \mu = f \circ \mu \circ S$.

The idea is to take $V := W_1$ and $\mu := id_{W_1}$, so that we need to check

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Some ingenuity is needed in the general case to find the appropriate $S$ but there are templates. Our template is

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Thus, taking into consideration that $f$ is also surjective (because $S$ is conservative)

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can be shown to be valid manually or by a prover (SPASS solves the problem in less than half a second; the superposition proof it finds takes 47 lines).

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Let's summarize the three steps:

- First: produce the inference rules (there are automatic methods, not always they give the good rules).
- Second: apply correspondence theory (this is automatic).
- Third: produce p-morphic extensions to standard frames (not automatic, but there are templates); provers can discharge the final proof obbligation.

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# Some case studies

We wonder to which extent the above mechanization of the metatheory can be pushed.

We analyzed some more significant cases. The first case is **GL** system axiomatized by the single axiom $\Box(\Box x \to x) \to \Box x$.

First Step can be driven so that to obtain a rule which is equivalent (for our purposes) to Avron's rule

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Second Step (via Ackermann rule applied to fixpoint logic) gives

$$\forall w\ R(w) \subseteq \mu(Y, w)\exists_f(f^*(R(w)) \cap \Box_R R(w) \cap \Box_R Y). \qquad (20)$$

In *finite* one-step frames this simplifies to

$$\forall w\ (R(w) \subseteq \{f(w') \mid R(w') \subset R(w)\})\ . \qquad (21)$$

Notice that $\subset$ is strict inclusion, so the above condition is a ‘step’ irreflexivity.

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Third Step is not difficult, but is not fully automatic. We can use the template $S$ for transitive systems, but then the resulting Kripke frame is not irreflexive, so one needs to take the dijoint union of the irreflexive subframes satisfying (17).

It should be noticed that, if we do the same analysis for the system axiomatized by transitivity and Löb rule, we get a weaker condition than (20). Using the fact that the condition is too weak, it is possible to prove formally that bpp fails.

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Our second case study is the system **S4.3** axiomatized via **S4** reflexivity and transitivity axioms plus $\Box(\Box x \to y) \vee \Box(\Box y \to x)$.

First step: the inference rule extracted automatically from the axiom is not good (bpp fails). Instead, we use Goré infinitely many rules:

$$\frac{\cdots \Box y \to x_j \vee \bigvee_{j \neq i} \Box x_i \cdots}{\Box y \to \bigvee_{i=1}^{n} \Box x_i} \tag{22}$$

The rules are indexed by $n$ and the $n$-th rule has $n$ premises, according to the values $j = 1, \ldots, n$.

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Second Step: correspondence theory applies to these rules. Interpreting the results in finite frames one gets

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Third Step: the same method used in **GL** case shows that one-step frames satisfying (23) are p-morphic images of Kripke frames for **S4.3**. This establishes bpp and fmp.

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As a further case study let us consider **S5**.

First Step The following rule has been proposed in the literature:

$$\frac{\Box\Gamma \Rightarrow y, \Box\Delta}{\Box\Gamma \Rightarrow \Box y, \Box\Delta\ .} \tag{24}$$

In the resulting system, cuts cannot be completely eliminated, but can be limited to subformulae of the sequent to be proved. This 'analytic' cut-elimination property is sufficient to imply the bpp, and thus we should be able to get the bpp directly by our methods.

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# Some case studies

Second Step Correspondence theory gives

$$\forall w \forall v \ \ (wRv \to \exists \tilde{w} \ (f(\tilde{w}) = v \ \& \ R(w) = R(\tilde{w}))). \tag{25}$$

Third Step Step frames satisfying the above property are easily seen to be p-morphic images of reflexive, transitive, symmetric Kripke frames; this establishes bpp and fmp.

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As a final case study consider the system obtained by adding to **K** the axiom $\Box\Box x \leftrightarrow \Box x$. This is density+transitivity; we can join the rules we already used for density and transitivity. This is not a good idea: bpp fails!

Instead, we use the following couple of rules suggested to us by G. Mints:

$$\frac{\Box^{+}\Gamma \rightarrow \alpha}{\Box\Gamma \rightarrow \Box\alpha} \qquad \frac{\Gamma, \Box\Delta \Rightarrow \Box\alpha}{\Box\Gamma, \Box\Delta \Rightarrow \Box\alpha} \tag{26}$$

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Second Step Correspondence theory gives, besides step transitivity (9), the condition

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Third Step Step frames satisfying the above property are easily seen to be p-morphic images of transitive and dense Kripke frames; this establishes bpp and fmp.

Notice that the fact that (15)+ (9) do not imply (27) is a formal argument proving that bpp fails is we adopt the old rule (14) in a transitive context. Thus, at least in principle, *model finders* can be used as automatic supports for showing that bpp fails.

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# Conclusions

- step methods (via step embedding theorem and step correspondence) seem to be quite effective in jointly proving bpp and fmp;
- in simple cases the application of the methods is fully automatic (in a sense we are *mechanizing the metatheory* of modal logic!);
- in more complex cases some ingenuity is needed, still uniform arguments often work;
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