Does Continuity Matter to Modal Logicians?

Silvio GHILARDI

Dipartimento di Scienze dell'Informazione
Università degli Studi di Milano - Italy

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§1. Continuous Morphisms

Interior axioms for topology on a set X

$$Int\, S \subseteq S, \qquad Int\, Int\, S = Int\, S,$$
$$Int\, X = X, \qquad Int\,(S_1 \cap S_2) = Int\, S_1 \cap Int\, S_2$$

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But for morphisms $f : (X', Int') \longrightarrow (X, Int)$ the correspondence is broken ...

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The situation is similar in other mathematical models of modalities (see e.g. modalities arising from essential geometric morphisms among toposes).

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However, at the propositional level, the algebraic practice is dominating, to the point that continuity seems banned from papers and textbooks.

We are looking for exceptions ...

§1. Continuous Morphisms

We fix our framework. Let $\mathbb{V}$ be a variety of modal algebras (like $\mathbb{K}, \mathbb{S}4, \ldots$).
Arrows in $\mathbb{V}$ are *open* morphisms, i.e. Boolean morphisms μ such that

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Remark: *an iso in* $\mathbb{V}_c$ *is an iso in* $\mathbb{V}$ *too.*

§2. Basic Adjunction

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Consider the category $\mathbb{B}$ of Boolean algebras; we have a pair of contravariant adjoint functors

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This *basic adjunction* can be considered the natural background of Stone duality.

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We need continuous morphisms! In fact, in this way we have again a contravariant adjointness

$$(-)^* : \mathbb{K}_c \longrightarrow \mathbf{Grph}, \qquad (-)^* : \mathbf{Grph} \longrightarrow \mathbb{K}_c$$

where $\mathbf{Grph}$ is the category of graphs and relation-preserving maps (not p-morphisms!). Here both functors $(-)^*$ are extended from the Boolean case to the modal case in the well-known obvious way.

§3. Freeness for Presentations

Question: *is there anything like a free-continuous algebra* (value of an hypothetic left adjoint to the forgetful functor $\mathbb{V}_c \longrightarrow \mathbf{Set}$)? does anything like that make sense? is it useful?

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To get a more interesting notion, we shall introduce *presentations*. These give raise to initial objects in varieties expanded with finitely many constants and finitely many axioms.

§3. Freeness for Presentations

We call Σ the signature of modal algebras; a *(flat, finite) presentation* (in Σ) is a pair

$$P = (X_P, T_P)$$

where X_P is a finite set of variables and T_P is a set of equations of the kind $x = y, \Box x = y, \neg x = y, x_1 \wedge x_2 = y, x_1 \vee x_2 = y$.

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A modal algebra $(B, \Box)$ *satisfies the presentation* P iff there is an assignment $\alpha : X_P \longrightarrow B$ such that for every $(t, u) \in T_P$, we have $(B, \Box), \alpha \models P$.

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Definition 1. *Given a variety* $\mathbb{V}$ *and a presentation* P, *the* free $\mathbb{V}$-algebra over P *is any pair given by an algebra* $(\mathcal{F}_{\mathbb{V}}(P), \Box)$ *and an assignment* $\alpha_P : X_P \longrightarrow \mathcal{F}_{\mathbb{V}}(P)$ *such that:*

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(i) $(\mathcal{F}_{\mathbb{V}}(P), \Box) \in \mathbb{V}$ *;*

(ii) $(\mathcal{F}_{\mathbb{V}}(P), \Box), \alpha_P \models P$*;*

(iii) *for any other* $(B, \Box) \in \mathbb{V}$ *and any other* β *such that* $(B, \Box), \beta \models P$, *there exists a unique* open *morphism* $\mu : (\mathcal{F}_{\mathbb{V}}(P), \Box) \longrightarrow (B, \Box)$ *in* $\mathbb{V}$ *such that* $\mu \circ \alpha_P = \beta$*.*

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Definition 2. *Given a variety* $\mathbb{V}$ *and a presentation* P, *the* c-free $\mathbb{V}$-algebra over P *(provided it exists) is any pair given by an algebra* $(\mathcal{F}^c_{\mathbb{V}}(P), \Box)$ *and an assignment* $\alpha^c_P : X_P \longrightarrow \mathcal{F}^c_{\mathbb{V}}(P)$ *such that:*

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Fact 2. If $\mathbb{V}$ is axiomatized and c-locally finite (meaning that $(\mathcal{F}^c_{\mathbb{V}}(P), \Box)$ exists and is finite for every P), then conditional word problem (i.e. global consequence relation) is decidable in $\mathbb{V}$.

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Existence is still a major point ...

§3. Filtrations

Let us consider a modal algebra $(A, \Box_A)$ and a *finite* Boolean subalgebra B of A.

Definition 3. A *filtration* of $(A, \Box_A)$ over $B \overset{i}{\hookrightarrow} A$ is a hemimorphism $\Box_B : B \longrightarrow B$ such that:

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(i) i is continuous, i.e. we have $i(\Box_B b) \leq \Box_A i(b)$, for all $b \in B$;

(ii) for every $b, c \in B$, it happens that

$$\Box_A i(b) = i(c) \quad \Rightarrow \quad c \leq \Box_B b.$$

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Filtration Lemma. *Let* $\Box_B$ *be a filtration of* $(A, \Box_A)$ *over* $B \overset{i}{\hookrightarrow} A$*; then for every* $b, c \in B$*, it holds that*

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We say that $\mathbb{V}$ *admits filtrations* iff for every $B \overset{i}{\hookrightarrow} (A, \Box) \in \mathbb{V}$, there exists a $\mathbb{V}$*-filtration* of A over B, (i.e. a filtration $\Box_B$ such that $(B, \Box_B) \in \mathbb{V}$).

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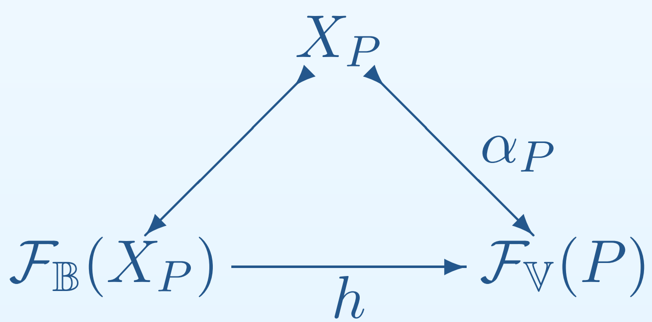
Proof idea. Given a presentation $P = (X_P, T_P)$, to build $(\mathcal{F}^c_{\mathbb{V}}(P), \Box)$ just filter $(\mathcal{F}_{\mathbb{V}}(P), \Box)$ over the image of the universal Boolean morphism h



(here $\mathcal{F}_{\mathbb{B}}(X_P)$ is the free Boolean algebra over the set X_P).

§3. Filtrations

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Fix $B \overset{i}{\hookrightarrow} (A, \Box)$; define for $b \in B$:

$$\begin{aligned} \Box_1 b &:= i_* \Box_A i(b); \\ \Box_0 b &:= \bigvee \{c \in B \mid \exists a \in B \ (a \leq b \ \& \ i(c) = \Box_A i(a))\}. \end{aligned}$$

§3. Filtrations

Proposition 1. *For every* $b \in B$*, we have that* $\Box_0 b \leq \Box_1 b$*. Moreover, a hemimorphism* $\Box_B : B \longrightarrow B$ *is a filtration iff we have* $\Box_0 b \leq \Box_B b \leq \Box_1 b$*, for every* $b \in B$*.*

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Let first analyze $\mathbb{K}$-case:

Proposition 2. $\Box_0$ *and* $\Box_1$ *are* $\mathbb{K}$*-filtrations, indeed they are the smallest and the biggest* $\mathbb{K}$*-filtrations.*

§3. Filtrations

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Proposition 3. *$\Box_T$ and $\Box_t$ are $\mathbb{S}4$-filtrations, where*

$$\Box_T\, b \;:=\; \bigwedge_{n\geq 0} \Box_1^n(b)$$

$$\Box_t\, b \;:=\; \bigvee\{c \in B \mid \exists a \in B\ (a \wedge c \leq b\ \&\ i(c) = \Box_A i(a))\}.$$

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$\Box_T$ is the 'reflexive-transitive closure' of $\Box_1$, whereas $\Box_t$ is the 'Lemmon' filtration (see below why we call it so).

§3. Filtrations

Since the filtered modal algebra $(B, \Box)$ is finite, it is dual to a finite frame $(atoms(B), R)$. For instance, it turns out that the relations dual to $\Box_0$ and $\Box_t$ are:

$$pR_0q \quad \Leftrightarrow \quad (\forall a, c \in B)\ [i(c) = \Box_A i(a) \Rightarrow (p \leq c \Rightarrow q \leq a)]$$

$$pR_tq \quad \Leftrightarrow \quad (\forall a, c \in B)\ [i(c) = \Box_A i(a) \Rightarrow (p \leq c \Rightarrow q \leq a \ \&\ q \leq c)]$$

This can be useful to recognize classical formulations for filtrations (read '$i(c) = \Box_A i(a)$' as 'a represents a formula ϕ in a filtering set Γ such that $\Box\phi$ is also in Γ and is represented by c').

§3. Filtrations

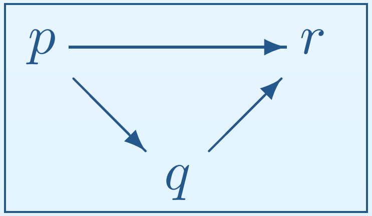
Nevertheless, we have much less filtrations than in the classical case. Take $(A, \Box)$ to be the finite modal algebra dual to the the reflexive graph

$$p \longrightarrow q \qquad q' \longrightarrow r$$

Let B the Boolean subalgebra corresponding to the set $\{p, q, r\}$ and let i be the Boolean embedding dual to the function mapping p, q, r to themselves and q' to q. Only two filtrations exist in our framework. The filtration $\Box_1$ gives rise to the following graph dual to $(B, \Box_1)$

$$p \longrightarrow q \longrightarrow r$$

Using $\Box_0$, we get the dual of the transitive graph



§4. Free Algebras Step-by-Step

The free $\mathbb{V}$-algebra over a finite Boolean algebra B is $(\mathcal{F}_{\mathbb{V}}(P), \Box)$, where the presentation P is the multiplication table of B.

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An interesting task is to build $(\mathcal{F}_{\mathbb{V}}(P), \Box)$ step-by-step (the n-th step is the 'Lindembaum algebra' of terms of modal degree at most n). The task can be easily accomplished in case $\mathbb{V}$-axioms have rank 1, it is more involved otherwise.

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There exists a solution for the analogous problem of building the free Heyting algebra over a finite distributive lattice. We show how to lift it to $\mathbb{S}4$, by employing continuous morphisms and our filtrations theory.

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We exploit the parallelism

$$\mathbb{S}4 \rightsquigarrow \mathbb{HA} \qquad \mathbb{S}4_c \rightsquigarrow \mathbb{HA}_c$$

where $\mathbb{HA}_c$ are Heyting algebras endowed with distributive lattice morphisms.

§4. Free Algebras Step-by-Step

Let $Z \xrightarrow{f} W \xrightarrow{g} Z$ be continuous (i.e. order-preserving) maps among finite posets. We say that f is g-open iff the following holds for all p, q[a]

$$q \leq f(p) \quad \Rightarrow \quad \exists q' \leq p \; (g(f(q')) = g(q)).$$

g-openness means that the dual distributive lattice morphism f^* preserves implications of the kind $g^*(S_1) \to g^*(S_2)$.

[a] For short, ordering/preordering relations will always be ambiguously noted as $\leq$. In addition, the domain variables like $p, q, S, T \ldots$ range over is not written explicitly (it must be deduced from context).

§4. Free Algebras Step-by-Step

A subset $S \subseteq W$ is g-open iff the inclusion $S \subseteq W$ is g-open. We let W_g be the set of g-open rooted subsets of W and $\rho_g : W_g \longrightarrow W$ be the map that takes root; W_g is a poset (ordering is inclusion), ρ_g is continuous and g-open. It has the following

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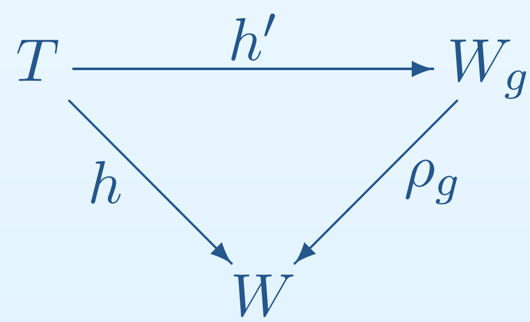
Universal Property *For every continuous and g-open $h : T \longrightarrow W$, there is a unique continuous ρ_g-open h' such that the triangle below commutes*



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Given a finite poset W, define an inverse chain

$$W_0 \xleftarrow{g_0} W_1 \xleftarrow{g_1} W_2 \xleftarrow{g_2} \cdots$$

by putting $g_0 : W_1 \longrightarrow W_0$ equal to the unique $W \longrightarrow \mathbf{1}$ and $g_i : W_{i+1} \longrightarrow W_i$ equal to $\rho_{g_i} : W_{g_{i-1}} \to W_i$.

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Theorem *The colimit* $\lim_i W_i^*$ *is the free Heyting algebra over the finite distributive lattice* W^*.

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Proof requires however additional care. In fact, the 'algebraic version' of the universal property of $\rho_g : W_g \longrightarrow W$ now sounds

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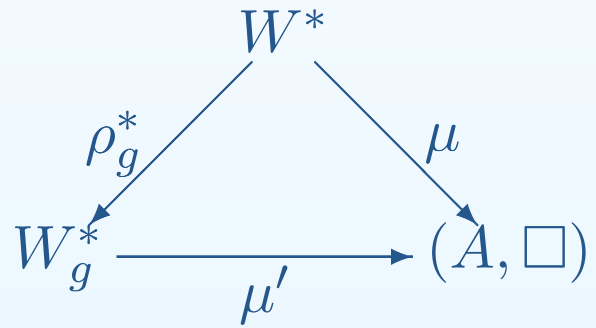

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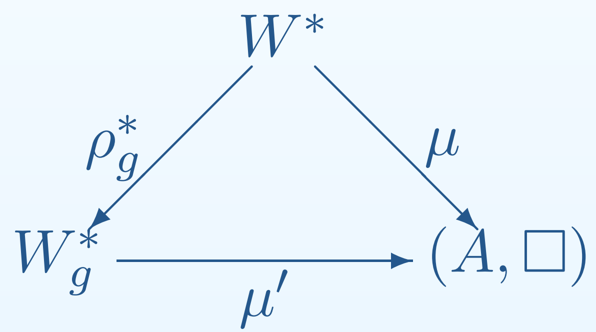

Getting this starred version of the universal property from the unstarred one requires an argument that replaces $(A, \Box)$ with a suitable finite subalgebra of it.

§4. Free Algebras Step-by-Step

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Filtrations can be used to this aim. Filtration Lemma guarantees what is needed for the proof, however one should take a filtration that produces a continuous factorization of μ. The filtration $\Box_T$ does the job.

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- the full $\mathbb{S}4$-algebraic structure is reached at each step W_i^*, not only in the colimit (fmp follows);
- each of the W_i^* is uniquely characterized by a universal property which is formulated in terms of continuous morphisms.

§5. Conclusions

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After all, modal logicians could care a bit more for continuity!

Thanks for Your Attention