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Continuity, Freeness, and Filtrations

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Abstract

The role played by continuous morphisms in propositional modal logic is investigated: it turns out that they are strictly related to filtrations and to suitable variants of the notion of a free algebra. We also employ continuous morphisms in the incremental construction of (standard) finitely generated free $\mathbb{S}4$-algebras.

This Technical Report reproduces the content of my talk given at the workshop on Topological Methods in Logic, held in Tbilisi, June 2010; it is also the preliminary version of a paper to appear in the 'birthday' special issue of the Journal of Applied Non Classical Logics.[1]

1 Introduction

In this paper, we shall basically try to understand whether a key mathematical concept like *continuity* can be influential within an area of logic, namely modal logic, one of whose main sources of inspiration, since Tarski-McKinsey times [19, 20], was precisely topology, namely the field of mathematics having continuity as its main subject. In fact, interior axioms for topology are precisely the axioms for the Lewis modal system S4 and completeness theorems can be established, both at an elementary and at a more advanced level (see [23] and the literature quoted therein). When we turn our attentions to morphisms, however, the situation drastically changes and the strict correspondence between logical and topological practice seems to be broken. In fact, on one side, continuity is the natural notion of a morphism in topology and despite the fact that continuity is very easy to express in terms of the interior/closure operators, this is not the notion of morphism used by logicians. Logicians in fact prefer to adopt the strict algebraic point of view and define morphisms just as functions that preserve all the operators, including the modal operators. This choice is shared by people

[1]I wish to thank the two referees of the JANCL for their accurate suggestions.

preferring to work at the semantic (instead of at the algebraic) level, where p-morphisms (and not just relation-preserving morphisms) are adopted as the notion of morphism between frames.

Relying on the topological intuition, we can define a continuous morphism between modal algebras as a Boolean morphism μ satisfying the inequation

$$\mu(\Box a) \leq \Box\mu(a).$$

It should be noticed that such morphism are ubiquitous in the mathematical applications of quantified modal logic: you can find them not only in topological models [14, 9], but also in models derived from topos-theoretical interpretations, as witnessed by papers like [12], [21] (see the survey [6] for a more complete picture). In propositional modal logic, the question of the employment of continuous morphisms is raised in [13], where it is shown that they play a fundamental role in establishing a proper adjointness between algebraic and semantic settings; a similar observation is contained in the more elaborated setting of [3]. Generally speaking, wherever subframes and relation-preserving morphisms (not just generated subframes and p-morphisms) are mentioned, there is a potential application area for continuity. The point however is to show the practical relevance of continuous morphisms, through algebraic characterizations explicitly involving them.

In this paper, we show the viability of this perspective, by supplying a substantial example. The example deals with 'old fashioned' modal logic and concerns filtrations: filtrations were introduced in the early sixties as a uniform method for establishing finite model property and decidability for various basic modal systems [17]. Computationally, they are not quite popular, however their simplicity makes them very appealing. Minimal filtration requires, to be defined, the use of a relation preserving map which is not a p-morphism, a fact making filtrations good candidates for our purposes. Indeed, in this paper, we get a quite nice algebraic characterization of filtrations in terms of suitable notion of freeness: whereas standard free algebras are the left adjoint to the forgetful functor from the category of modal algebras (with full morphisms) into **Set**, filtrations turn out to play a similar role with respect to the forgetful functor from the category of modal algebras with continuous morphisms into **Set**. To be precise, a little adjustement is neeeded in the above framework: we have to consider not just modal algebras, but slight equational extensions of the variety of modal algebras, namely the extensions with finitely many ground equations involving additional constants - this is the customary framework of word problems in computational algebra. (In fact, without this extension, universal objects would be rather uninteresting, they would correspond to filtrations over trivial sets of formulae). To get the above characterization, we revisit filtrations by introducing them from a purely algebraic point of view and we restate familiar definitions and results (see Section 3 below). In the final part of the paper, we show how to use filtrations to lift to transitive modal logics the step-by-step construction of free Heyting algebras given in [10], [4]. Step-by-step constructions of free algebras recently received considerable attention within the coalgebraic approach to modal logic; we underline that their relevance is not purely theoretical, as they retain a strict relationship to the theory of normal forms, see for instance [11].

The paper is self-contained, modulo very basic algebraic background (Section 2.1 of [15] contains for instance all what is needed, and even more).

2 Continuous Morphisms

For simplicity, we consider just single normal modal operators, although our framework can be easily generalized beyond this restriction. We shall directly introduce the algebraic settings and work only within the algebraic conceptualization of propositional modal logic.

A *modal algebra* $(B, \Box)$ is a Boolean algebra B endowed with a *hemimorphism*, i.e. with a finite meets preserving operator $\Box$. Preservation of finite meets means that the equations

$$\Box(a \wedge b) = \Box a \wedge \Box b, \qquad \Box\top = \top$$

are satisfied for all a, b from the support of B. We shall systematically confuse a Boolean algebra with its support set; also the Boolean operators and the $\Box$ operator are given names that do not depend on the algebraic structure they are part of (but in case we need a more precise notation, we use $\Box_B, \Box_{B'}$, etc. for this purpose). The logical connective names $\top, \bot, \wedge, \vee, \neg$ are used for the corresponding Boolean operations. The dual operator of $\Box$ is indicated with $\Diamond$ and is defined as $\neg\Box\neg$.

We shall fix in the following a variety $\mathbb{V}$ (i.e. an equational class) of modal algebras. Notable examples of such varieties are the variety $\mathbb{K}$ of all modal algebras and the variety $\mathbb{S}4$ of *topological boolean algebras*; the latter is axiomatized by the $S4$-inequations

$$\Box a \leq a, \qquad \Box a \leq \Box\Box a. \tag{1}$$

Adding the further inequation

$$a \leq \Box\Diamond a \tag{2}$$

we get the variety $\mathbb{S}5$, corresponding to one-variable monadic fragment of classical first-order logic.

It is useful to consider $\mathbb{V}, \mathbb{K}, \mathbb{S}4, \mathbb{S}5$ as categories; when we do that, we consider as arrows the *algebraic morphisms*, i.e. the Boolean morphisms $\mu : (B, \Box) \longrightarrow (B', \Box)$ satisfying the preservation equation

$$\mu(\Box\, a) = \Box\, \mu(a). \tag{3}$$

for all $a \in B$. The algebraic morphisms are also called *open* morphisms, in analogy to the topological case.

Besides algebraic morphisms, another class of morphisms will play a fundamental role in this paper, namely those which are only *continuous*, in the sense that they only satisfy the inequation

$$\mu(\Box\, a) \leq \Box\, \mu(a). \tag{4}$$

It is easy to see that the identity morphisms are continuous and that the composition of two continuous morphisms is continuous (in fact, $\nu(\mu(\Box\, a)) \leq \Box\, \nu(\mu(a))$ follows from (4) by applying ν to both inequality sides and then by using the fact that ν is continuous).

We use $\mathbb{V}_c$ (and similarly $\mathbb{K}_c, \mathbb{S}4_c, \mathbb{S}5_c, \dots$) to indicate the category having the same objects as $\mathbb{V}$, but as arrows the continuous morphisms. Clearly, $\mathbb{V}$ is a subcategory of $\mathbb{V}_c$; an easy but important fact is given by the following Proposition:

Proposition 2.1. *The inclusion functors* $\mathbb{V} \subseteq \mathbb{V}_c, \mathbb{K} \subseteq \mathbb{K}_c, \mathbb{S}4 \subseteq \mathbb{S}4_c, \dots$ *reflect object isomorphisms.*

Proof. The statement means that if we are given continuous morphisms $\mu : (B, \Box) \longrightarrow (B', \Box)$ and $\nu : (B', \Box) \longrightarrow (B, \Box)$ such that $\nu \circ \mu = id_B$ and $\mu \circ \nu = id_{B'}$, then μ and ν are also open (i.e. they come from $\mathbb{V}$). This is trivially established (e.g. for μ) by applying μ to the lattice inequation $\nu(\Box\mu(b)) \leq \Box(\nu(\mu(b))) = \Box\, b$, which holds by the continuity of ν. $\Box$

The relevance of the previous Proposition lies in the following observation; suppose we build an algebra by relying on a universal property which characterizes it uniquely (up to isomorphism) in $\mathbb{V}_c$: then, *the same algebra is determined uniquely (up to isomorphism) in* $\mathbb{V}$ *also.*[2] As an example, notice that there is a forgetful functor $\mathbb{V}_c \longrightarrow$ **Set**, where **Set** is the category of sets and functions (this is the functor associating with every algebra its carrier set and with every morphism the morphism itself seen as a plain function). The analogous (standard) forgetful functor $\mathbb{V} \longrightarrow$ **Set** has a left adjoint, and the left adjoint is the customary free algebra construction (in algebraic logic, free algebras can be described through the Lindenbaum algebras induced by logical calculi). Now, we are going to show (see Proposition 2.2 below) that, under mild assumptions, the left adjoint to $\mathbb{V}_c \longrightarrow$ **Set** exists

[2]Notice that this observation would not apply, for instance, if we considered instead of $\mathbb{V}_c$ just the less interesting category of modal algebras endowed with pure Boolean morphisms.

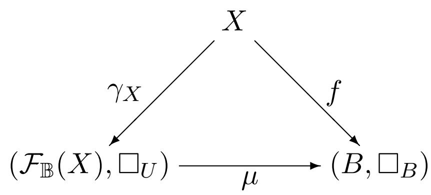
and gives a kind of 'free-continuous algebra'. Such algebra, by Proposition 2.1 above, is well defined (up to isomorphism) both with respect to the Boolean and to the modal components.

Given a Boolean algebra B, the universal modality over B is the hemimorphism defined as follows

$$\Box_U x = \top \text{ iff } x = \top, \qquad \Box_U x = \bot \text{ iff } x \neq \top.$$

It is well-known that, if $\mathbb{V} \supseteq \mathbb{S}5$, then $(B, \Box_U) \in \mathbb{V}$ for every Boolean algebra B (actually, $\mathbb{S}5$ could be equivalently defined as the variety generated by these algebras). Notice also that any Boolean morphism $(B, \Box_U) \longrightarrow (A, \Box)$ is continuous. These observations immediately yields the following

Proposition 2.2. *Suppose $\mathbb{S}5 \subseteq \mathbb{V}$; for a set X, let us denote by $\mathcal{F}_{\mathbb{B}}(X)$ the free Boolean algebra over X and by γ_X the injection of X into the support of $\mathcal{F}_{\mathbb{B}}(X)$. Then the following universal property holds: for every set-theoretic map $f : X \longrightarrow B$ into the support of a modal algebra $(B, \Box_B) \in \mathbb{V}$, there exists a unique continuous morphism $\mu : (\mathcal{F}_{\mathbb{B}}(X), \Box_U) \longrightarrow (B, \Box_B)$ such that the triangle*



commutes.

The above result resembles very much the (straightforward) fact that the trivial 'two-opens' topology gives the right adjoint to the forgetful functor from the category of topological spaces into the category of sets. In this sense, free-continuous modal algebras represent rather uninteresting objects. To get more significant constructions, we need to restate our definitions in a slightly more general framework, namely in the framework of freeness-modulo-a-finite-presentation (this is the customary framework for word problems in computational algebra).

2.1 Presentations

We call Σ the signature of modal algebras; a *finite presentation* (in Σ) is a pair

$$P = (X_P, T_P)$$

where X_P is a finite set of variables and T_P is a finite set of pairs of Σ-terms involving at most the variables from X_P (these variables will be indicated by the letters $x, y, \dots$). A modal algebra $(B, \Box)$ *together with an assignment* $\alpha : X_P \longrightarrow B$ *satisfies the presentation*

P iff for every $(t, u) \in T_P$, we have that $t = u$ is true in $(B, \Box)$ under α - we note this by $(B, \Box, \alpha) \models P$.[3]

Finite presentations have 'best solution' algebras in $\mathbb{V}$ (this is a general fact): such 'best solutions' are the algebras which are initial objects in the variety obtained from $\mathbb{V}$ by adding to the signature Σ the finite set of constants X_P and to any set of Σ-equations axiomatizing $\mathbb{V}$ the set of further equations T_P. We can rephrase this in our terms as follows:

Definition 2.3. Given a variety $\mathbb{V}$ and a finite presentation P, the *free* $\mathbb{V}$*-algebra over* P is any pair given by an algebra $(\mathcal{F}_{\mathbb{V}}(P), \Box)$ and an assignment $\alpha_P : X_P \longrightarrow \mathcal{F}_{\mathbb{V}}(P)$ such that:

(i) $(\mathcal{F}_{\mathbb{V}}(P), \Box) \in \mathbb{V}$;

(ii) $(\mathcal{F}_{\mathbb{V}}(P), \Box, \alpha_P) \models P$;

(iii) for any $(B, \Box) \in \mathbb{V}$ and any β such that $(B, \Box, \beta) \models P$, there exists a unique (open) morphism $\mu : (\mathcal{F}_{\mathbb{V}}(P), \Box) \longrightarrow (B, \Box)$ in $\mathbb{V}$ such that $\mu \circ \alpha_P = \beta$.

The pair $(\mathcal{F}_{\mathbb{V}}(P), \alpha_P)$ is unique (up to isomorphism); it always exists and can be built up by dividing the free $\mathbb{V}$-algebra over X_P by the obvious congruence relation. Notice that what we call here free $\mathbb{V}$-algebras over a (finite) presentation P is commonly called a finitely presented $\mathbb{V}$-algebra (with presentation P). In the following, for simplicity, we might use the same name $\mathcal{F}_{\mathbb{V}}(P)$ for the modal algebra $(\mathcal{F}_{\mathbb{V}}(P), \Box)$, its Boolean reduct and its support set.

Presentations can be flattened: flattening is a common practice in automated reasoning and it is reflected in the modal logician's work by the closure of a set of formulae under subformulae. A presentation $P = (X_P, T_P)$ is *flat* iff T_P contains only pairs of terms (t, u), where u is a variable and t is either a variable or of the kind $\neg x, \Box x, x \wedge y, x \vee y, \top, \bot$, where x, y are variables. Every presentation can be flattened by repeatedly replacing 'complex' terms t by variables x_t and by adding to T_P the pair (t, x_t). From now on, *we assume that our presentations are all flat.*

The conditional word problem for $\mathbb{V}$ is the following: "given a finite presentation $P = (X_P, T_P)$ and a pair of variables $x, y \in X_P$, to decide whether the quasi-equation

$$t_1 = u_1 \,\&\cdots\&\, t_k = u_k \to x = y \tag{5}$$

is valid in $\mathbb{V}$" (here we put $T_P = \{t_1 = u_1, \ldots, t_k = u_k\}$). Notice that there is no loss of generality in using a variable equation as the consequent of (5): in fact, flattening can be used to reduce the apparently more general case $t_1 = u_1 \,\&\cdots\&\, t_k = u_k \to v = w$ (where w, v are Σ-terms whatsoever) to the special case (5).

[3] Here, when we say that $t = u$ is true in $(B, \Box)$ under α, we mean that $\alpha(t) = \alpha(u)$, where α is extended from variables to terms by interpreting the logical connectives into the corresponding algebraic operations.

From the point of view of modal logic, the conditional word problem above corresponds to the problem of *deciding the global consequence relation*; from the computational algebra viewpoint instead, the conditional word problem is usually called just the 'word' problem, because it is implicit that a standard algebraic word problem is relative to a finite presentation (the presentation consists of generators and relations for groups, of polynomial bases for ideals in K-algebras, etc.).

Since the free algebra over any P always exists, the conditional word problem can be equivalently formulated as the problem of checking whether a variable equation is true in an algebra of the kind $\mathcal{F}_{\mathbb{V}}(P)$ under the assignment α_P. However, this observation is not of great help, because the algebra $\mathcal{F}_{\mathbb{V}}(P)$ is usually very complicated. What about then replacing open morphisms with continuous morphisms in Definition 2.3(iii)?

Definition 2.4. Given a variety $\mathbb{V}$ and a finite presentation P, the *c-free* $\mathbb{V}$*-algebra over* P (provided it exists) is any pair given by an algebra $(\mathcal{F}^c_{\mathbb{V}}(P), \Box)$ and an assignment $\alpha^c_P : X_P \longrightarrow \mathcal{F}^c_{\mathbb{V}}(P)$ such that:

(i) $(\mathcal{F}^c_{\mathbb{V}}(P), \Box) \in \mathbb{V}_c$;

(ii) $(\mathcal{F}^c_{\mathbb{V}}(P), \Box, \alpha^c_P) \models P$;

(iii) for any $(B, \Box) \in \mathbb{V}_c$ and any β such that $(B, \Box, \beta) \models P$, there exists a unique *continuous* morphism $\mu : (\mathcal{F}_{\mathbb{V}}(P), \Box) \longrightarrow (B, \Box)$ in $\mathbb{V}_c$ such that $\mu \circ \alpha^c_P = \beta$.

We now wonder whether all this can make any sense. Suppose, for the time being, that we are so lucky that c-free $\mathbb{V}$-algebras exist and are kinds of toys, e.g. that they are all finite (we say in this case that $\mathbb{V}$ is *c-locally finite*) . What can we get out of them? Quite a lot, indeed:

Proposition 2.5. *If* $\mathbb{V}$ *is axiomatized by a finite set of equations and is c-locally finite, then the conditional word problem (aka the global consequence relation) is decidable in* $\mathbb{V}$.

Proof. It is sufficient to show that if the conditional equation (5) fails in $\mathbb{V}$, then it fails in $\mathcal{F}^c_{\mathbb{V}}(P)$ (this yields finite model property and hence decidability by recursive enumerability of the set of valid formulae). If (5) fails in $\mathbb{V}$, there are an algebra $(B, \Box)$ and an assignment β such that $(B, \Box, \beta) \models P$ and $\beta(x) \neq \beta(y)$; from Definition 2.4, it follows that $\alpha^c_P(x) \neq \alpha^c_P(y)$ (because $\mu \circ \alpha^c_P = \beta$); since $(\mathcal{F}^c_{\mathbb{V}}(P), \Box, \alpha^c_P) \models P$, (5) fails in $\mathcal{F}^c_{\mathbb{V}}(P)$ as well. □

Remark Clearly, flattening is the key prerequisite for the above proof to work. We underline a hidden important point here: the unique continuous morphism μ required by Definition 2.3(iii) does a much bigger work than it might appear at a first glance. In fact, suppose that the flat pair $(y, \Box x)$ is in P; then, μ has to fully preserve $\Box\alpha^c_P(x)$, in the sense that we have not only $\mu(\Box\alpha^c_P(x)) \leq \Box\mu(\alpha^c_P(x))$, but precisely $\mu(\Box\alpha^c_P(x)) = \Box\mu(\alpha^c_P(x))$. This is seen as follows: since $(\mathcal{F}^c_{\mathbb{V}}(P), \Box, \alpha^c_P) \models P$ and $(B, \Box, \beta) \models P$ hold, $\Box x = y$ must be true both under α^c_P and under β, i.e. we must have both $\alpha^c_P(y) = \Box\alpha^c_P(x)$ and $\beta(y) = \Box(\beta(x))$; it follows that

$$\mu(\Box\alpha^c_P(x)) = \mu(\alpha^c_P(y)) = \beta(y) = \Box(\beta(x)) = \Box\mu(\alpha^c_P(x)).$$

Thus, in c-free algebras, operations corresponding to flat terms occurring in the presentation P are built up 'with precision', despite the fact that Definition 2.4(iii) requires only a 'loose' μ.

Remark Using the universal properties of Definitions 2.3(iii) and 2.4(iii), it is possible to establish an interesting connection between $\mathcal{F}_{\mathbb{V}}(P)$ and $\mathcal{F}^c_{\mathbb{V}}(P)$. Because of such universal properties, there must exist a continuous map $\mu : (\mathcal{F}^c_{\mathbb{V}}(P), \Box) \longrightarrow (\mathcal{F}_{\mathbb{V}}(P), \Box)$ and an open map $\nu : (\mathcal{F}_{\mathbb{V}}(P), \Box) \longrightarrow (\mathcal{F}^c_{\mathbb{V}}(P), \Box)$; by the uniqueness condition in Definition 2.4(iii), the composition $(\mathcal{F}^c_{\mathbb{V}}(P), \Box) \stackrel{\mu}{\longrightarrow} (\mathcal{F}_{\mathbb{V}}(P), \Box) \stackrel{\nu}{\longrightarrow} (\mathcal{F}^c_{\mathbb{V}}(P), \Box)$ must be identity. Thus μ is injective and ν is a quotient (in particular, in duality terms, the descriptive frame dual to $\mathcal{F}^c_{\mathbb{V}}(P)$ is a generated subframe of the descriptive frame dual to $\mathcal{F}_{\mathbb{V}}(P)$). This is an evidence that c-free algebras are much simpler than free algebras.

The big problem, however, is existence: in fact, since there are axiomatizable logics lacking finite model property, it is clear that existence and finiteness of c-free algebras cannot be guaranteed. Existence alone is problematic indeed. Notice that products exists in $\mathbb{V}_c$ and they are preserved by the forgetful functor into the category of Sets. One may try then to use some kind of adjoint functor Theorem, because (as we already observed) c-free algebras can be seen as initial objects in suitable categories. Theorem 1 from Chapter 5.6 of [18] could be a good candidate because (ordinary) free algebras can play the role of solution sets. However, existence and preservation of equalizers is a crucial ingredient that is missing (if they were to exist, c-free algebras could be built as equalizers of all continuous endomorphisms of free algebras). We leave the exploration of the viability of this strategy to future research and we take another approach, through old constructions which are very familiar to modal logicians since the early sixities.

3 Filtrations Revisited

Here we build filtrations in a language-free way, namely we won't use sets of formulae, models, and so on. We just filtrate over finite Boolean subalgebras.

Let us consider a modal algebra $(A, \Box_A)$ and a finite Boolean subalgebra B of A; let i be the inclusion Boolean morphism from B into A.

Definition 3.1. A *filtration* of $(A, \Box_A)$ over $B \overset{i}{\hookrightarrow} A$ is a hemimorphism $\Box_B : B \longrightarrow B$ such that:

(i) i is continuous, i.e. we have $i(\Box_B b) \leq \Box_A i(b)$, for all $b \in B$;

(ii) for every $b, c \in B$, it happens that

$$\Box_A i(b) = i(c) \quad \Rightarrow \quad c \leq \Box_B b.$$

The motivation for Condition (ii) is that it is needed to prove the following

Lemma 3.2 (Filtration Lemma). *Let $\Box_B$ be a filtration of $(A, \Box_A)$ over $B \overset{i}{\hookrightarrow} A$; then for every $b, c \in B$, it holds that*

$$\Box_A i(b) = i(c) \quad \Rightarrow \quad c = \Box_B b.$$

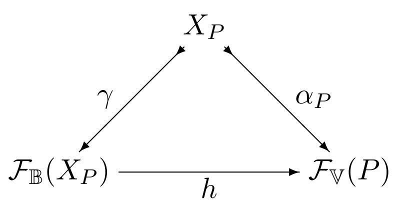
Proof. Assume $\Box_A i(b) = i(c)$; we need to prove only that $\Box_B b \leq c$, because the other side comes directly from Condition (ii) of Definition 3.1. But from $\Box_A i(b) \leq i(c)$ and continuity of i, we get $i(\Box_B b) \leq i(c)$, hence what we need by injectivity of i. $\Box$

The meaning of the Filtration Lemma is that $\Box_B$ is defined in such a way that it agrees with $\Box_A$ 'as far as possible', i.e. as long as $\Box_A i(b)$ is equal to an element coming from the Boolean subalgebra B, then $\Box_B b$ is defined to be equal to that element.

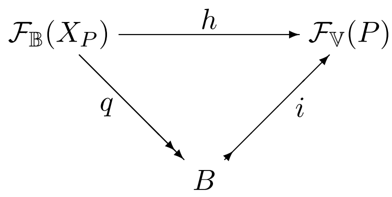
We say that $\mathbb{V}$ *admits filtrations* iff for every finite Boolean subalgebra B of a modal algebra $(A, \Box_A) \in \mathbb{V}$ there exists a $\mathbb{V}$-*filtration* of A over B, i.e. a filtration $\Box_B$ such that $(B, \Box_B) \in \mathbb{V}$.

Theorem 3.3. *If $\mathbb{V}$ admits filtrations, then $\mathbb{V}$ is c-locally finite.*

Proof. Let $P = (X_P, T_P)$ be a (finite, flat) presentation; let $\mathcal{F}_{\mathbb{B}}(X_P)$ be the free Boolean algebra over the finite set of free generators X_P and let $\gamma : X_P \longrightarrow \mathcal{F}_{\mathbb{B}}(X_P)$ be the embedding of the free generators into the support of $\mathcal{F}_{\mathbb{B}}(X_P)$. Consider the free $\mathbb{V}$-algebra over P given by Definition 2.3; by the universal property of free algebras, we get a Boolean morphism h making the triangle

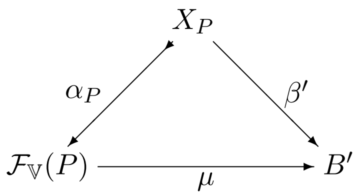


commute. Taking now the image factorization of h in the category of Boolean algebras, we get the commutative triangle



Notice that the Boolean algebra B is finite (it is a quotient of a finitely generated free Boolean algebra), hence there is a filtration $\Box_B$ of $\mathcal{F}_{\mathbb{V}}(P)$ over B such that the algebra $(B, \Box_B)$ is in $\mathbb{V}$. We show that this algebra (endowed with the composite map $q \circ \gamma : X_P \longrightarrow B$) fulfills the requirements of Definition 2.4.

We first need to prove that $(B, \Box_B, q \circ \gamma) \models P$. This is guaranteed by the Filtration Lemma: take in fact a pair $(\Box x, y) \in P$. Since we have $(\mathcal{F}_{\mathbb{V}}(P), \Box, \alpha_P) \models P$ by Definition 2.3(ii), we must have $\Box \alpha_P(x) = \alpha_P(y)$, that is $\Box i(q(\gamma(x))) = i(q(\gamma(y)))$ (because $\alpha_P = i \circ q \circ \gamma$). By the Filtration Lemma, we obtain $\Box_B q(\gamma(x)) = q(\gamma(y))$, yielding $(B, \Box_B, q \circ \gamma) \models P$.[4] Let now $(B', \Box', \beta')$ be such that $(B', \Box', \beta') \models P$, with $(B', \Box') \in \mathbb{V}$. By the universal property of $\mathcal{F}_{\mathbb{V}}(P)$, there exists an (open) morphism μ such that the triangle



commutes. The composition $(B, \Box_B) \xrightarrow{i} (\mathcal{F}_{\mathbb{V}}(P), \Box) \xrightarrow{\mu} (B', \Box')$ is a continuous (see Definition 3.1(i)) morphism such that $(\mu \circ i) \circ (q \circ \gamma) = \beta'$, as required. It remains to check uniqueness; uniqueness however is trivial, because there cannot exist two different continuous morphisms $\nu_1, \nu_2 : (B, \Box_B) \longrightarrow (B', \Box')$ with $\nu_1 \circ (q \circ \gamma) = \nu_2 \circ (q \circ \gamma)$ because the image of $q \circ \gamma$ generates B as a Boolean algebra. $\Box$

Remark The above proof does not depend on the particular $\mathbb{V}$-filtration adopted: this is not surprising, because we are basically filtering a $\mathbb{V}$-canonical model and the $\mathbb{V}$-filtration of $\mathbb{V}$-canonical models is unique (see [7]). In fact, in view of the Remark following Proposition 2.1, we can see the above proof as a proof of the uniqueness of $\mathbb{V}$-filtrations over free $\mathbb{V}$-algebras.

3.1 Basic Filtrations

This Subsection and the next one are very elementary in spirit: we investigate existence of filtrations and of $\mathbb{V}$-filtrations. We shall recover the situation depicted in standard modal logic

[4] Pairs like $(x_1 \wedge x_2, y), (x_1 \vee x_2, y), \ldots$ are trivially handled by the fact that i is an injective Boolean morphism.

textbooks [17], [22], [16], [7]. Fix a modal algebra $(A, \Box_A)$ and a finite boolean subalgebra $i : B \hookrightarrow A$. From Definition 3.1, it can be derived that there are a smallest and a biggest candidate filtrations. To show this, we make a couple of preliminary remarks.

- Condition (i) from Definition 3.1 is equivalent to asking that

$$\Box_B(b) \leq i_* \Box_A i(b) \tag{6}$$

holds for all $b \in B$, where i_* is the right adjoint to i; notice that the right adjoint exists for general reasons, because B is finite (hence complete as a lattice) and i preserves all joins, being a Boolean morphism.[5]

- Condition (ii) from Definition 3.1 is equivalent to asking that

$$\bigvee\{c \in B \mid \exists a \in B\ (a \leq b\ \&\ i(c) = \Box_A i(a))\} \leq \Box_B\, b \tag{7}$$

for all $b \in B$.[6]

In fact, Condition (ii) follows from (7): to see it, assume $\Box_A i(b) = i(c)$ and take $a := b$, so that $c \leq \Box_B b$ obtains from (7). Vice versa, if Condition (ii) holds, in order to show (7), pick a, b, c so that $a \leq b$ and $i(c) = \Box_A i(a)$ hold: from Condition (ii), it follows that $c \leq \Box_B a$, moreover $\Box_B a \leq \Box_B b$ follows from $a \leq b$, and finally $c \leq \Box_B b$ obtains by transitivity.

To summarize, let us define for $b \in B$:

$$\begin{aligned} \Box_1 b &:= i_* \Box_A i(b); \\ \Box_0 b &:= \bigvee\{c \in B \mid \exists a \in B\ (a \leq b\ \&\ i(c) = \Box_A i(a))\}. \end{aligned}$$

Proposition 3.4. *For every $b \in B$, we have that $\Box_0 b \leq \Box_1 b$. Moreover, a hemimorphism $\Box_B : B \longrightarrow B$ is a filtration iff we have*

$$\Box_0 b \leq \Box_B b \leq \Box_1 b \tag{8}$$

for every $b \in B$.

[5] Readers needing elementary details can consult e.g. Section 2.1 of [15].

[6] Notice that we are allowed to use the above infinite join as it is taken on a finite Boolean algebra. In the proofs below, we shall systematically use the standard property/definition of joins to be formulated in general terms as

$$\bigvee\{c \mid c \in I\} \leq d \quad \Leftrightarrow \quad (\forall c \in I)\ c \leq d,$$

where I is any index set.

Proof. To show that $\Box_0 b \leq \Box_1 b$ holds, pick a, c such that $a \leq b$ and $i(c) = \Box_A i(a)$: the goal is to show that $c \leq \Box_1 b = i_* \Box_A i(b)$. But $c \leq i_* \Box_A i(b)$ is equivalent to $i(c) \leq \Box_A i(b)$ which in turns follows from $i(c) = \Box_A i(a) \leq \Box_A i(b)$ (the latter is because $a \leq b$ and $i, \Box_A$ are both monotonic).

That the inequalities $\Box_0 b \leq \Box_B b \leq \Box_1 b$ characterize filtrations has been already established above. □

Proposition 3.5. *$\Box_0$ and $\Box_1$ are $\mathbb{K}$-filtrations, indeed they are the smallest and the biggest $\mathbb{K}$-filtrations.*[7]

Proof. In view of Proposition 3.4, we only need to show that $\Box_0$ and $\Box_1$ are hemimorphisms.

Let us first show it for $\Box_0$. Since $\Box_0 \top$ is equal to

$$\bigvee \{c \in B \mid \exists a \in B \ (a \leq \top \ \& \ i(c) = \Box_A i(a))\}$$

it is clear that $\Box_0 \top = \top$ (take $c := a := \top$). Also, the fact that

$$b_1 \leq b_2 \ \Rightarrow \ \Box_0 b_1 \leq \Box_0 b_2$$

is evident from the definition of $\Box_0$. To finally show that

$$\Box_0 b_1 \wedge \Box_0 b_2 \leq \Box_0 (b_1 \wedge b_2)$$

it is sufficient to check that for all a_1, a_2, c_1, c_2

$$a_1 \leq b_1 \ \& \ a_2 \leq b_2 \ \& \ i(c_1) = \Box_A i(a_1) \ \& \ i(c_2) = \Box_A i(a_2) \quad \Rightarrow \quad c_1 \wedge c_2 \leq \Box_0 (b_1 \wedge b_2).$$

From the antecedents, it follows that $a_1 \wedge a_2 \leq b_1 \wedge b_2$ and

$$i(c_1 \wedge c_2) = i(c_1) \wedge i(c_2) = \Box_A i(a_1) \wedge \Box_A i(a_2) = \Box_A i(a_1 \wedge a_2),$$

which means that $c_1 \wedge c_2$ belongs to the set $\{c \in B \mid \exists a \in B \ (a \leq b_1 \wedge b_2 \ \& \ i(c) = \Box_A i(a))\}$. This implies that $c_1 \wedge c_2 \leq \Box_0 (b_1 \wedge b_2)$.

The proof that $\Box_1$ is a hemimorphism is immediate, because $\Box_1$ is the composition of three hemimorphisms (recall that i_* preserves meets, being a right adjoint). □

[7] Since the order comparison is made on $\Box$ operators (and not on accessibility relations of Kripke models), the words 'smallest' and 'biggest' have dual meanings with respect to what the reader may expect.

3.2 Transitive Filtrations

We finally show that $\mathbb{S}4$ also admits filtrations; an $\mathbb{S}4$-filtration can be obtained for instance by taking the so-called reflexive-transitive closure of $\Box_1$. This is the hemimorphism defined as

$$\Box_T(b) \ := \ \bigwedge_{n\geq 0} \Box_1^n(b).$$

Proposition 3.6. *$\Box_T$ is an $\mathbb{S}4$-filtration.*

Proof. We use again Proposition 3.4. It is clear that $\Box_T$ is a hemimorphism satisfying the inequations (1). We only need to check that $\Box_0 b \leq \Box_T b$ (that $\Box_T b \leq \Box_1 b$ is also trivial). The proof is essentially included in the proof of Theorem 2.4 from [15]. We report it here. Unravelling the definitions, what we need is to check is that for all $n \geq 0$ and for all $a, b, c \in B$ we have

$$a \leq b \ \ \& \ \ i(c) = \Box_A i(a) \quad \Rightarrow \quad c \leq \Box_1^n b. \tag{9}$$

We show this by induction on n: the key ingredient are the $S4$ inequations (1). For $n = 0$, notice that the antecedents of (9) and the first inequation from (1) imply

$$i(c) = \Box_A i(a) \leq i(a) \leq i(b)$$

yielding $c \leq b = \Box_1^0 b$ because i is monic.

Suppose now that (9) holds for n and let us prove it for $n+1$. Assume $a \leq b$ and $i(c) = \Box_A i(a)$; from induction hypothesis, we get $c \leq \Box_1^n b$, i.e. (applying i) $\Box_A i(a) = i(c) \leq i(\Box_1^n b)$. By the second inequation from (1), we then obtain $i(c) = \Box_A i(a) \leq \Box_A i(\Box_1^n b)$ and by adjointness $c \leq i_*(\Box_A i(\Box_1^n b)) = \Box_1^{n+1} b$, as required. $\Box$

Another example of an $\mathbb{S}4$-filtration is the *Lemmon filtration* defined by

$$\Box_t b \ := \ \bigvee \{c \in B \mid \exists a \in B \ (a \wedge c \leq b \ \& \ i(c) = \Box_A i(a))\}. \tag{10}$$

Proposition 3.7. *$\Box_t$ is an $\mathbb{S}4$-filtration.*

Proof. The proof that $\Box_t$ is a hemimorphism is nearly identical to the above proof that $\Box_0$ is a hemimorphism. That $\Box_0 b \leq \Box_t b$ is also pretty clear. To show that $\Box_t b \leq \Box_1 b$, pick a, c such that $a \wedge c \leq b$ and $i(c) = \Box_A i(a)$: the goal is to prove that $c \leq \Box_1 b = i_* \Box_A i(b)$, i.e. that $i(c) \leq \Box_A i(b)$. From $a \wedge c \leq b$, we get $\Box_A i(a) \wedge \Box_A i(c) \leq \Box_A i(b)$, which gives the claim in view of the facts that $\Box_A$ satisfies the second inequation from (1) and $i(c) = \Box_A i(a)$.

We finally need to check that $\Box_t$ satisfies the inequations (1). To show that $\Box_t b \leq b$, pick as usual a, c such that $a \wedge c \leq b$ and $i(c) = \Box_A i(a)$. We must check that $c \leq b$ holds which follows from $i(c) = \Box_A i(a) \leq i(a) \wedge i(c) \leq i(b)$ and the injectivity of i (notice that we used the fact that $\Box_A$ satisfies the first inequation from (1)). To show that $\Box_t b \leq \Box_t \Box_t b$, pick again a, c such that $a \wedge c \leq b$ and $i(c) = \Box_A i(a)$. This time we must show that $c \leq \Box_t \Box_t b$, which is proved if $a \wedge c \leq \Box_t b$ holds (because then c would be part of the family whose join is $\Box_t \Box_t b$). But from $a \wedge c \leq b$ and $i(c) = \Box_A i(a)$, we get $c \leq \Box_t b$ (because c is part of the family whose join is $\Box_t b$) and a fortiori $a \wedge c \leq \Box_t b$. $\Box$

Since the boolean algebra B is finite, by the Duality Theorem between finite modal algebras and finite frames, we can in principle convert any filtration $\Box_B$ into a binary accessibility relation on a finite set. The conversion is in fact interesting in some cases. Let us give some more details in this respect. First we recall the needed background (see e.g. [15], Section 2.1): with a frame (W, R) (here W is a set and $R \subseteq W \times W$ is a relation), it is possible to associate the powerset modal algebra $(\mathcal{P}(W), \Box_R)$ where $\Box_R(S) := \{p \in W \mid \forall q \in W(pRq \Rightarrow q \in S\}$. Vice versa, to a *finite* modal algebra $(C, \Box_C)$ it is possible to associate the finite frame (W_C, R_C) where W_C is the set of atoms of C and W_C is defined as

$$pR_C q \quad :\Leftrightarrow \quad p \leq \Diamond_C q \quad \Leftrightarrow \quad \Box_C \neg q \leq \neg p.$$

It then turns out that $(C, \Box_C) \simeq (\mathcal{P}(W_C), \Box_{R_C})$.

Suppose now that B is a finite Boolean subalgebra of a modal algebra $(A, \Box_A)$; if we apply the above finite duality to the filtration $\Box_0$, it is not difficult to see that we get the relation R_0 between atoms of B which is so defined

$$pR_0 q \quad \Leftrightarrow \quad (\forall a, c \in B)\ [i(c) = \Box_A i(a) \Rightarrow (p \leq c \Rightarrow q \leq a)]. \tag{11}$$

To see this, notice that (by applying the definition of $\Box_0$ and the usual property of joins) $pR_0 q$ is equivalent to

$$\forall c, a \ \ (a \leq \neg q \ \&\ i(c) = \Box_A i(a) \ \Rightarrow \ c \leq \neg p);$$

taking the contrapositive (and recalling that p, q are atoms, so for instance $a \leq \neg q$ is the same as $q \leq \neg a$ and as $q \not\leq a$), one gets exactly (11). Similarly, the transitive Lemmon filtration gives rise to the relation R_t defined as

$$pR_t q \quad \Leftrightarrow \quad (\forall a, c \in B)\ [i(c) = \Box_A i(a) \Rightarrow (p \leq c \Rightarrow q \leq a \ \&\ q \leq c)]. \tag{12}$$

Remark We can now try to make a connection with traditional filtrations introduced in modal logic textbooks. In traditional filtrations, our framework $B \stackrel{i}{\hookrightarrow} (A, \Box_A)$ is obtained as follows: our $(A, \Box_A)$ is of the kind $(\mathcal{P}(W), \Box_R)$ for a frame (W, R) and we also have $B \simeq \mathcal{P}(Q)$, where Q is a quotient set of W. The quotient Q is defined as follows. A valuation

V on (W, R) and a set of formulae (closed under subformulae) Γ are given; two elements of W get identified in Q iff in the Kripke model (W, R, V) they force exactly the same formulae from Γ. Now, in our setting, we have neither V nor Γ (because we do not have a modal language at all), however we can somewhat recover Γ by thinking of the elements of B themselves to be the elements of Γ. Apparently, we do not have in this way the information about which elements from Γ are boxed formulae: to recover this information, we say that c 'is boxed' iff $i(c)$ is equal to $\Box_A i(a)$ for some $a \in B$. If we read formulae (11),(12) according to this intuition, their meaning becomes quite transparent and familiar. For instance, (11) says that q is R_0-accessible from p iff for every 'boxed' c, it happens that *if p forces c than q forces the corresponding 'unboxed' a*. The Lemmon trick making R_0 transitive is also recognizable from (12).

We continue the Section with a detailed example; the example will also show that not all traditional filtrations can be simulated by filtrations in the sense of the present paper.

Example To work out simple examples, it is better to adopt finite duality. According to finite duality, our framework $B \overset{i}{\hookrightarrow} (A, \Box_A)$, in case A is *finite*, can be reformulated as follows. We are given a Kripke frame (W, R), a set Q and a surjective map $\pi : W \longrightarrow Q$ (then $(A, \Box_A)$ is $(\mathcal{P}(W), \Box_R)$, B is $\mathcal{P}(Q)$ and i is inverse image under π). Let (W, R) be the reflexive graph depicted as

$$\boxed{p \longrightarrow q \qquad q' \longrightarrow r}$$

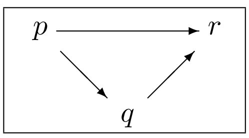
(here and below, we omit edges connecting a point to itself). We let $Q = \{p, q, r\}$ and we let π be the function mapping p, q, r to themselves and q' to q. The filtration $\Box_1$ gives rise to the following graph dual to $(B, \Box_1)$

$$\boxed{p \longrightarrow q \longrightarrow r}$$

This is evident from the fact that the relation dual to $\Box_1 = i_* \circ \Box_A \circ i$ is dual to the composition of the relations $\pi \circ R \circ \pi^{op}$, hence it is the image of R under π. To compute the relation dual to $\Box_0$, we use (11). To this end, observe that the only pairs (a, c) from B such that $\Box_A i(a) = i(c)$ (and $c \neq \emptyset$) are

$$(\{p, q, r\}, \{p, q, r\}), \quad (\{q, r\}, \{q, r\}), \quad (\{r\}, \{r\}), \quad (\{p, r\}, \{r\}).$$

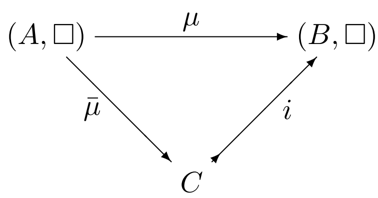
Applying (11), we get the transitive graph



A direct inspection, using Proposition 3.4, makes evident that no further filtration exists. Notice that, on the contrary, it is possible to obtain more 'language-dependent' traditional filtrations introducing a Kripke model over (W, R) and taking suitable filtering sets of formulae Γ (in particular, the total relation making Q a 3-clique can be obtained from a Γ not containing any $\Box$). This phenomenon is due to the fact that our 'recovery policy' for the missed Γ in the above Remark always produces some canonical maximal Γ.

Before concluding this section, we state a Lemma (to be used afterwards) relating filtrations and Boolean factorizations of continuous morphisms:

Lemma 3.8. *Let $\mu : (A, \Box) \longrightarrow (B, \Box)$ be a continuous morphism; suppose that μ, as a Boolean morphism, factorizes as*



where C is a finite Boolean algebra and i is injective. Let $\Box_C$ be a filtration on C. The following statements hold

(i) *let $a \in A$ be such that $\Box\mu(a) = \mu(\Box a)$: then we have also $\Box_C\bar{\mu}(a) = \bar{\mu}(\Box a)$;*

(ii) *if $\Box_C$ is the biggest filtration $\Box_1$, then $\bar{\mu}$ is continuous;*

(iii) *if $(A, \Box) \in \mathbb{S}4$ and $\Box_C$ is the biggest transitive filtration $\Box_T$, then $\bar{\mu}$ is continuous.*

Proof. (i) Let us put $c := \bar{\mu}(\Box a)$ (hence $i(c) = \mu(\Box a) = \Box\mu(a) = \Box i(\bar{\mu}(a))$); by the Filtration Lemma, we get $\Box_C\bar{\mu}(a) = c$. Replacing in the latter c by its definition, we get $\Box_C\bar{\mu}(a) = \bar{\mu}(\Box a)$.

(ii) For every $a \in A$, we have that $\bar{\mu}(\Box a) \leq \Box_C\bar{\mu}(a)$ is equivalent to $i(\bar{\mu}(\Box a)) \leq \Box i(\bar{\mu}(a))$ (use adjointness together with the definition of $\Box_C$ as $i_* \circ \Box \circ i$); the latter is trivially true by the facts that $i \circ \bar{\mu} = \mu$ and that μ is continuous.

(iii) For every $a \in A$, we show by induction on n that $\bar{\mu}(\Box a) \leq (i_* \circ \Box \circ i)^n\bar{\mu}(a)$. For $n = 0$, it is sufficient to use the reflexivity axiom. For the induction step, we have the following sequences of implications (read them from bottom to top)

$$\frac{\frac{\frac{\frac{\frac{\frac{\bar{\mu}(\Box a) \leq (i_* \circ \Box \circ i)^n \bar{\mu}(a)}{\Box i(\bar{\mu}(\Box a)) \leq \Box i((i_* \circ \Box \circ i)^n \bar{\mu}(a))}}{\Box \mu(\Box a) \leq \Box i((i_* \circ \Box \circ i)^n \bar{\mu}(a))}}{\mu(\Box\Box a) \leq \Box i((i_* \circ \Box \circ i)^n \bar{\mu}(a))}}{i(\bar{\mu}(\Box\Box a)) \leq \Box i((i_* \circ \Box \circ i)^n \bar{\mu}(a))}}{i(\bar{\mu}(\Box a)) \leq \Box i((i_* \circ \Box \circ i)^n \bar{\mu}(a))}}{\bar{\mu}(\Box a) \leq (i_* \circ \Box \circ i)(i_* \circ \Box \circ i)^n \bar{\mu}(a)}$$

(we used the transitivity axiom, continuity of μ, the fact that $i \circ \bar{\mu} = \mu$ and monotonicity of $\Box$, i). The last inequality holds by the induction hypothesis. $\Box$

4 Free Topological Boolean Algebras

We now use our filtration machinery to lift the incremental construction of finitely generated free Heyting algebras taken from [10] to the case of finitely generated free topological Boolean algebras. The construction from [10] has been recently carefully re-analyzed in a co-algebraic setting by N. Bezhanishvili and M. Gehrke in [4]. The case of topological Boolean algebras represents an interesting example in the co-algebraic perspective, because reflexivity and transitivity axioms for $\mathbb{S}4$ are simple but still beyond the well understood case of 1-rank equations, where co-algebraic methods have been successfully applied for the incremental description of free algebras [5].

We keep the paper self-contained, but at the same time we shall especially focus on proofs that require substantial adaptations from [10].

In this section, all modal algebras $(A, \Box)$ are implicitly assumed to be topological Boolean algebras, i.e. to belong to $\mathbb{S}4$. We deal with preordered sets $P, Q, \dots$: these are sets endowed with a reflexive and transitive binary relation (such a relation is always indicated with $\leq$, unless otherwise stated). For a preordered set P, we denote with P^* the set of subsets of P; such P^* is tacitly endowed with a $\mathbb{S}4$-algebra structure by setting[8]

$$\Box S \ := \ \{p \in P \mid \forall q \in Q \ (q \leq p \ \Rightarrow \ q \in S)\}.$$

A continuous map $f : P \longrightarrow Q$ is just a monotonic map; the inverse image along a continuous f turns out to be a continuous morphism $f^* : (Q^*, \Box) \longrightarrow (P^*, \Box)$. Notice that f^* is open iff f satisfies the well-know p-morphism condition

$$q \leq f(p) \quad \Rightarrow \quad \exists p' \in P \ (p' \leq p \ \& \ f(p') = q)$$

[8]Notice that this amounts to using $\geq$ (and not $\leq$) as the accessibility relation in the associated Kripke frame structure.

(below, we shall call this p-morphism condition directly 'openness condition'). The categories of finite preordered sets and continuous (open) maps is dual to the category of finite $\mathbb{S}4$-algebras and continuous (open) morphisms: this finite duality will play a central role in the following because it will allow us to move back and forth (from finite algebras to finite preordered sets and vice versa) the constructions we are interested in and their universal properties.

The main ingredient of our construction (like in [10]) is the relativization of the openness condition. Given two continuous morphisms $(A, \Box) \xrightarrow{\nu} (B, \Box) \xrightarrow{\mu} (C, \Box)$, we say that μ is *ν-open* iff we have

$$\mu(\Box\nu(a)) = \Box\mu(\nu(a))$$

for all $a \in A$ (that is, 'μ preserves Boxes of elements coming from A via ν'). Dually, if we are given continuous maps

$$P \xrightarrow{f} Q \xrightarrow{g} R,$$

we say that f is *g-open* iff for every $S \subseteq R$ it happens that

$$f^*(\Box g^*(S)) \ = \ \Box f^*(g^*(S)).$$

By standard correspondence machinery, it is easy to show that f is g-open iff for every $p \in P, q \in Q$ it happens that

$$q \leq f(p) \quad \Rightarrow \quad \exists p' \in P \ (p' \leq p \ \& \ g(f(p')) = g(q)).$$

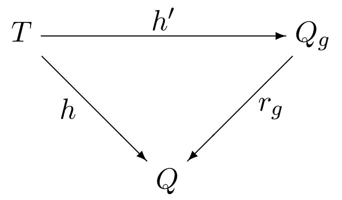
A subset $S \subseteq Q$ is g-open iff the inclusion map $S \subseteq Q$ is g-open; equivalently, this means that forall $s \in S$, $q \in Q$ (if $q \leq s$ then there exists $s' \in S$ with $s' \leq s$ and $g(s') = g(q)$). The following Remark is useful in calculations:

Remark *If S is g-open and $s \in S$, then $(\downarrow_Q s) \cap S$ is also g-open (here $\downarrow_Q s = \{q \in Q \mid q \leq s\}$).* To see why this is true, pick $\tilde{s} \in (\downarrow_Q s) \cap S$ and $q \in Q$, $q \leq \tilde{s}$; since $\tilde{s} \in S$ and S is g-open, there is $s' \leq \tilde{s}$ with $s' \in S$ and $g(s') = g(q)$. But then $s' \in (\downarrow_Q s) \cap S$ (because $s' \leq \tilde{s} \leq s$) and $(\downarrow_Q s) \cap S$ is g-open because $\tilde{s}, q$ were arbitrary.

Given a continuous map $g : Q \longrightarrow R$, we can form the preordered set Q_g defined as follows: (i) the underling set of Q_g is the set of pairs (ρ, S) such that S is g-open and $\rho \in S$ is such that $s \leq \rho$ holds for all $s \in S$; (ii) the preorder relation of Q_g is just set-theoretic inclusion on the second components.[9] The map $r_g : Q_g \to Q$ given by the projection to the first component is clearly continuous: in fact, if $(\rho, S) \leq (\rho', S')$, then $\rho \in S \subseteq S'$, hence $r_g(\rho, S) = \rho \leq \rho' = r_g(\rho', S')$. The universal property of the construction of Q_g is given by the following Proposition:

[9]Notice that the only difference with respect to [10] is that we do not have antisymmetry here, so ρ is not uniquely determined and hence must be explicitly mentioned in a $(\rho, S) \in Q_g$.

Proposition 4.1. *Let $g : Q \longrightarrow R$ be continuous; then the map r_g is g-open. Moreover, for every continuous and g-open map $h : T \longrightarrow Q$ there exists a unique continuous and r_g-open map h' such that the triangle*



commutes.

Proof. The proof does not differ from the analogous statement from [10] (we report it here just for the sake of completeness).

Let us first show that r_g is g-open; suppose that $q \leq r_g(\rho, S) = \rho$; since S is g-open and $\rho \in S$, there is $s \in S$ such that $g(s) = g(q)$. By the above Remark, $(\downarrow_Q s) \cap S$ is g-open and hence $(s, (\downarrow_Q s) \cap S) \in Q^g$ is smaller than (ρ, S) in the Q_g-preordering. In addition, we have $g(r_g(s, (\downarrow_Q s) \cap S)) = g(s) = g(q)$, as wanted.

To show the universal property, let $h : T \longrightarrow Q$ be continuous and g-open; we show that if there is a r_g open h' such that $h' \circ r_g = h$, then h' has the following definition for all $w \in T$

$$h'(w) \;=\; (h(w), \{h(w') \mid w' \in T,\ w' \leq w\}). \tag{13}$$

Suppose that h' exists. That the first component of $h'(w)$ is $h(w)$ follows from the fact that we must have $r_g(h'(w)) = h(w)$. Thus, supposing $h'(w) := (\rho, S)$, we must have $\rho = h(w)$. Let now $z \in S$; since h' is r_g-open and $(z, (\downarrow_Q z) \cap S) \leq (\rho, S)$, there must be $w' \leq w$ such that $r_g(h'(w')) = r_g(z, (\downarrow_Q z) \cap S) = z$. However, $r_g(h'(w')) = h(w')$, hence z is of the form $h(w')$ for some $w' \leq w$. Suppose, vice versa, that $w' \leq w$ and let $h'(w') := (\rho', S')$ (actually, we already know that $\rho' = h(w')$). Then we have $h'(w') \leq h'(w)$ and $h(w') \in S$, because $\rho' = h(w') \in S' \subseteq S$.

Thus, the desired h', if it exists, is given by (13). Now, we show that (13) is in fact a good definition for h'. The fact that h', defined in this way, is continuous is obvious; it is also clear that we have $r_g \circ h' = h$. Still, we need to show that (i) $h'(w) \in Q^g$ for all $w \in T$; (ii) h' is r_g-open.

The only non self-evident point in (i) is the fact that $\{h(w') \mid w' \in T,\ w' \leq w\}$ is g-open. To this aim, pick $w' \leq w$ and let $q \leq h(w')$; since h is g-open, there exists $\tilde{w} \leq w'$ such that $g(q) = g(h(\tilde{w}))$. But then $h(\tilde{w}) \in \{h(w') \mid w' \in T,\ w' \leq w\}$ is such that $h(\tilde{w}) \leq h(w)$ and $g(q) = g(h(\tilde{w}))$, which shows the claim.

Finally, to show (ii), take $(\rho, S) \leq h'(w) = (h(w), \{h(w') \mid w' \in T,\ w' \leq w\})$. It follows that $S \subseteq \{h(w') \mid w' \in T,\ w' \leq w\}$. Since $\rho \in S$, we have $\rho = h(\tilde{w})$ for some $\tilde{w} \leq w$. For this

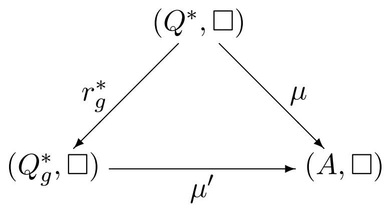
$\tilde{w}$, we have

$$r_g(h'(\tilde{w})) = h(\tilde{w}) = \rho = r_g(\rho, S),$$

as required by r_g-openness. $\square$

Now the problem is to dualize the above statement and to replace the preordered set T occurring in it by an arbitrary $\mathbb{S}4$-algebra $(A, \square)$. In the framework of [10], this operation is rather easy and depends on a straightforward property of Heyting implication. Looking more carefully at this property, one can realize that the Filtration Lemma is behind it: this observation is roughly what we need for the proof of the next Lemma.

Lemma 4.2. *Let Q, R be finite preordered sets and let $g : Q \longrightarrow R$ be an order-preserving map; for every topological Boolean algebra $(A, \square)$ and for every continuous and g^*-open morphism $\mu : Q^* \longrightarrow (A, \square)$, there exists a unique continuous and r_g^*-open morphism μ' such that the triangle*

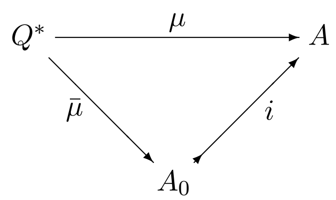


commutes.

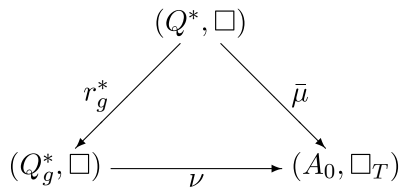
Proof. We let A_0 be the finite Boolean subalgebra of A generated by the set

$$\{\mu(S) \mid S \subseteq Q\} \cup \{\square\mu(S) \mid S \subseteq Q\}.$$

We also factorize μ as a composition of Boolean morphisms



where i is inclusion. We apply filtration mechanism and use the biggest transitive filtration $\square_T$ to endow A_0 with a $\mathbb{S}4$-algebra structure. By the factorization Lemma 3.8(iii), $\bar{\mu}$ is continuous. Since μ is g^*-open, we have $\mu(\square a) = \square\mu(a)$ for every $a \in Q^*$ which is of the kind $g^*(S)$ for some $S \in R^*$; by Lemma 3.8(i), the same is true for $\bar{\mu}$, which means that $\bar{\mu}$ is g^*-open as well. Thus, dualizing Proposition 4.1, we get that there exists a unique r_g^*-open and continuous morphism ν such that the triangle



commutes. We take μ' to be the composite morphism $i \circ \nu$ and we wish to show that it matches the desired requirements. Certainly, μ' is continuous, because it is the composite of continuous morphisms; to show that it is r_g^*-open, we need to prove that for every $S \subseteq Q$, we have

$$\mu'(\Box(r_g^*(S)) = \Box\mu'(r_g^*(S)). \tag{14}$$

Since ν is r_g^*-open, we have that

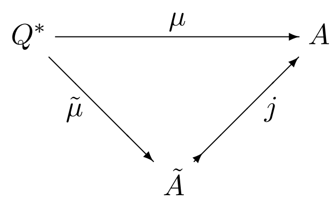
$$\mu'(\Box r_g^*(S)) = i(\nu(\Box r_g^*(S))) = i(\Box_T \nu(r_g^*(S))) = i(\Box_T \bar{\mu}(S)).$$

On the other hand

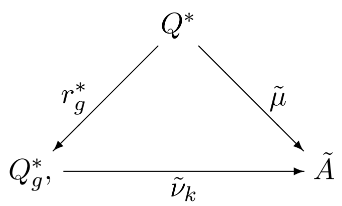
$$\Box\mu'(r_g^*(S)) = \Box i(\nu(r_g^*(S))) = \Box i(\bar{\mu}(S)) = \Box\mu(S).$$

According to the definition of A_0, there is $c \in A_0$ such that $\Box\mu(S) = i(c)$, so it is sufficient to show that $c = \Box_T \bar{\mu}(S)$. However, from $\Box i(\bar{\mu}(S)) = \Box\mu(S) = i(c)$, the identity $c = \Box_T \bar{\mu}(S)$ follows from the Filtrationa Lemma.

Thus we have established that there exists a continuous and r_g^*-open map μ' such that $\mu' \circ r_g^* = \mu$; it remains to prove that there cannot be two different morphisms ν_1, ν_2 with these properties. Suppose there are and take $\tilde{A}$ to be the finite Boolean subalgebra of A generated by the images of ν_1, ν_2 (notice that, since $\nu_1 \circ r_g^* = \mu$, this algebra contains also the image of μ). We can still factorize μ as $j \circ \tilde{\mu}$, where j is the inclusion of $\tilde{A}$ into A



Restricting ν_k in the codomain, we also have two commutative Boolean algebras triangles



(for $k = 1, 2$) where $j \circ \tilde{\nu}_k = \nu_k$. Let us endow $\tilde{A}$ with the biggest transitive filtration $\Box_T$. By Lemma 3.8(iii), $\tilde{\nu}_1, \tilde{\nu}_2, \tilde{\mu}$ are continuous (because so are $\nu_1 = j \circ \tilde{\nu}_1$, $\nu_2 = j \circ \tilde{\nu}_2$, $\mu = j \circ \tilde{\mu}$) and by Lemma 3.8(i), $\tilde{\nu}_1$ and $\tilde{\nu}_2$ are also r_g^*-open (because so are ν_1, ν_2): from Proposition 4.1, it follows that $\tilde{\nu}_1 = \tilde{\nu}_2$, hence also $\nu_1 = \nu_2$. $\Box$

Having established the above Lemma, the rest of the construction of finitely generated free algebras follows the same lines as [10]: we just build a colimit chain, by iterating the Q_g^*-constructions.

Given a direct chain

$$B^0 \xrightarrow{\varepsilon^1} B^1 \xrightarrow{\varepsilon^2} B^2 \xrightarrow{\varepsilon^3} \cdots \tag{15}$$

of Boolean algebras and Boolean monomorphisms, we can form the colimit Boolean algebra B^∞; this algebra, endowed with the canonical injections $\iota_n : B^n \longrightarrow B^\infty$, enjoys the following property:

- for every Boolean algebra A, the compositions with the ι_n's induces a bijection between Boolean morphisms $\mu : B^\infty \longrightarrow A$ and families $\{\mu^n : B^n \longrightarrow A\}$ of Boolean morphisms such that $\mu^{n+1} \circ \varepsilon^n = \mu^n$ (we write $\mu = [\mu_i]$ to specify that μ corresponds to the family $\{\mu_i\}$).

Suppose that now the chain (15) is a chain in $\mathbb{S}4_c$

$$(B^0, \Box) \xrightarrow{\varepsilon^1} (B^1, \Box) \xrightarrow{\varepsilon^2} (B^2, \Box) \xrightarrow{\varepsilon^3} \cdots \tag{16}$$

and that ε_{i+1} is ε_i-open. Then it is possible to introduce a hemimorphism in B^∞ by putting

$$\Box[a \in B^n] \;:=\; [\Box\varepsilon^n(a) \in B^{n+1}] \tag{17}$$

(we use the notation $[a \in B^n]$ to indicate the colimit equivalence class represented by $a \in B^n$). It is easily seen that the definition is well-given and that $(B^\infty, \Box) \in \mathbb{S}4$. Also the canonical injections $\iota_n : (B^n, \Box) \longrightarrow (B^\infty, \Box)$ are easily seen to be continuous (because the ε^n are continuous).[10]

Proposition 4.3. *With the above notation, the bijection*

$$\mu \longmapsto \{\mu_n := \mu \circ \iota_n\}$$

restricts to a bijection between open morphisms $\mu : (B^\infty, \Box) \longrightarrow (A, \Box)$ and families of continuous morphisms $\mu_n : (B^n, \Box) \longrightarrow (A, \Box)$ such that

[10] In more detail, we have

$$\iota_n(\Box a) = [\Box a \in B^n] = [\varepsilon^n(\Box a) \in B^{n+1}] \leq [\Box\varepsilon^n(a) \in B^{n+1}] = \Box[a \in B^n] = \Box\iota_n(a),$$

where we used the definition of ι_n together with identities of the kind $[c \in B^n] = [\varepsilon(c) \in B^{n+1}]$, coming from the colimit construction.

(i) $\mu_{n+1} \circ \varepsilon^n = \mu_n$*;*

(ii) μ_{n+1} *is* ε^n*-open.*

Proof. If (ii) holds, then μ is open, because

$$\Box\mu([a \in B^n]) = \Box\mu([\varepsilon^n(a) \in B^{n+1}]) = \Box\mu_{n+1}(\varepsilon^n(a)) =$$
$$= \mu_{n+1}(\Box\varepsilon^n(a)) = \mu([\Box\varepsilon^n(a) \in B^{n+1}]) = \mu(\Box[a \in B^n])$$

(we used identities of the kind $\mu([c \in B^n]) = \mu(\iota_n(c)) = \mu_n(c)$ together with (17)).

Vice versa, if μ is open, then $\mu_n = \iota_n \circ \mu$ is continuous (as a composition of continuous maps).[11] It remains to show that μ_{n+1} is ε^n-open. Notice that for $a \in B^n$, we have

$$\Box\mu([a \in B^n]) = \Box\mu_n(a) = \Box\mu_{n+1}(\varepsilon^n(a))$$

and also

$$\mu(\Box[a \in B^n]) = \mu([\Box\varepsilon^n(a) \in B^{n+1}]) = \mu_{n+1}(\Box\varepsilon^n(a)).$$

Since μ is open, we get that

$$\Box\mu_{n+1}(\varepsilon^n(a)) = \mu_{n+1}(\Box\varepsilon^n(a))$$

i.e. that μ_{n+1} is ε^n-open. $\Box$

Given a finite preordered set, let us build an inverse chain

$$P^0 \xleftarrow{f^1} P^1 \xleftarrow{f^2} P^2 \xleftarrow{f^3} \cdots \tag{18}$$

as follows. P^0 is the one-point poset, P^1 is P, f^1 is the unique map into P^0; recursively, for $n > 1$, P^{n+1} is $P^n_{f_n}$ and f^{n+1} is r_{f_n}. Taking duals, this inverse chain gives rise to a direct chain

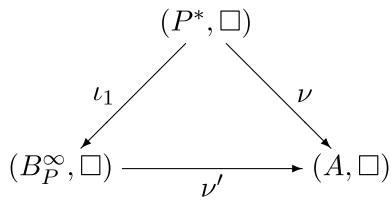
$$(B^0, \Box) \xrightarrow{\varepsilon^1} (B^1, \Box) \xrightarrow{\varepsilon^2} (B^2, \Box) \xrightarrow{\varepsilon^3} \cdots \tag{19}$$

where $B^n := (P^n)^*$ and $\varepsilon^n := (f^n)^*$. We let B_P^∞ be the direct limit of the Boolean algebras B^n; since each ε^{n+1} is ε^n-open,[12] we actually have a topological Boolean algebra structure $(B_P^\infty, \Box)$ on B_P^∞ given by (17). If we now put together Proposition 4.3 and Lemma 4.2, we get the following result:

Theorem 4.4. *For every finite preordered set* P *and for every continuous morphism* $\nu : (P^*, \Box) \longrightarrow (A, \Box)$ *there is a unique open morphism* ν' *such that the triangle*

[11] A side remark: it can be shown that the continuity of μ_n follows from (i)-(ii) and (17).

[12] For ε^2, observe that every morphism with domain B^1 is ε^1-open (notice also that there is only one possible topological Boolean algebra structure on B^0).



commutes.

To get a colimit description of finitely generated free topological Boolean algebras from Theorem 4.4 it is sufficient to combine it with Proposition 2.2.

5 Conclusions

We revisited filtrations from an algebraic point of view: filtrations have been introduced in a language-independent way and they have been shown to satisfy suitable universal properties formulated in terms of continuous morphims. We also presented an application to the incremental construction of free topological Boolean algebras along the lines of [10].

We feel the present work can be continued in various directions. On one hand, it would be nice to extend our approach to continuity, filtrations and finite model property beyond the simple logics considered here; in particular, it would be important to see whether it can be adapted to logics like dynamic logic or the μ-calculus, or to a coalgebraic framework. On the other hand, there are probably many other topics in modal logic where continuous morphisms could play a conceptually important and clarifying role: subframe logics [8], especially within some algebraic approach like in [1],[2], could for instance be a good candidate.

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